Science in China Series F: Information Sciences

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Connectivity and synchronization of Vicsek model

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The collective behavior of multi-agent systems is an important studying point for the investigation of complex systems, and a basic model of multi-agent systems is the so called Vicsek model, which possesses some key features of complex systems, such as dynamic behavior, local interaction, changing neighborhood, etc. This model looks simple, but the nonlinearly coupled relationship makes the theoretical analysis quite complicated. Jadbabaie et al. analyzed the linearized heading equations in this model and showed that all agents will synchronize eventually, provided that the neighbor graphs associated with the agents' positions satisfy a certain connectivity condition. Much subsequent research effort has been devoted to the analysis of the Vicsek model since the publication of Jadbabaie's work. However, an unresolved key problem is when such a connectivity is satisfied. This paper given a sufficient condition to guarantee the synchronization of the Vicsek model, which is imposed on the model parameters only. Moreover, some counterexamples are given to show that the connectivity of the neighbor graphs is not sufficient for synchronization of the Vicsek model if the initial headings are allowed to be in $[0, 2\pi)$, which reveals some fundamental differences between the Vicsek model and its linearized version.

Vicsek model, synchronization, stochastic matrix, connectivity, neighbor graph

1 Introduction

In recent years, increasing research attention has been paid to the collective behavior of multi-agent systems, e.g., schools of fish, flocks of birds, etc. In fact, the collective behavior arises from diverse systems, such as biological, physical, social, economic, and artificial systems. Biologists, physicists, mathematicians, computer scientists, and systems and control experts have tried to understand how the schools and flocks achieve synchronization without central control and global information, via modeling, simulation, and mathematical analysis.

Received July 23, 2006; accepted March 4, 2008
doi:10.1007/s11432-008-0077-2
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Supported by the National Natural Science Foundation of China (Grant Nos. 60221301 and 60334040)

Natural flocks and schools seem to consist of two balanced, opposing behaviors: a desire to stay close to the flock and a desire to avoid collisions within the flock[1]. The latter is to keep enough space for itself, for example, the birds need some space for flapping; and the former is the need of some social activities, such as protection from predators, migrating, and having more opportunity to find food. Reynolds[2] did the computer simulation according to the above features of the flocks and put forward a well-known model named Boid model. The simulation is carried out based on the following three rules: a) collision avoidance: avoiding collision with nearby flockmates; b) velocity matching: attempting to match velocity with nearby flockmates; c) flock centering: attempting to stay close to nearby flockmates.

A related but simpler model is the so-called Vicsek model, which was studied by Vicesk et al.[3] in 1995 from the viewpoint of statistical mechanics. Its purpose was to study gathering, transport, and phase transition in nonequilibrium systems. This model is composed of N autonomous agents in discrete-time, each agent is driven by a constant absolute velocity, assuming the average direction of motion of the agents in its neighborhood with some random noises added. The Vicsek model is a basic one for studying multi-agent systems, which possesses some key features, such as dynamic behavior, local interaction, changing neighborhood, etc. Through simulations, Vicsek et al.[3] found some interesting results: when the density of agents is large and the noise is small, all agents will move in the same direction, which is called synchronization. Following this, mathematicians and control theorists have tried to give a rigorous theoretical analysis for this synchronization phenomenon. Jadbabaie et al.[4] studied the linearized heading equation in the Vicsek model, and introduced a sequence of undirected neighbor graphs based on positions of the N agents. They proved that if the dynamical neighbor graphs are jointly connected in a certain "uniform" way, then the system will synchronize; after that, Savkin[6] considered the synchronization of the same linearized model but with discretized headings and showed that if the union of neighbor graphs is connected infinity times, then synchronization can also be achieved. However, an unresolved key problem is the following: under which conditions does the system dynamics satisfy the required connectivity? In fact, the verification of the connectivity condition turns out to be a difficult issue due to the complicated nonlinear interactions, as will be explained in the next section. One way to avoid such connectivity conditions is to modify the Vicsek model from local interactions to global ones with the interaction weights as a decreasing function of the agents' distances[5]. Another way to guarantee the required connectivity of the Vicsek model is to work in a random framework for large population, where the initial conditions are uniformly distributed, as was done in Tang and Guo[7].

In this paper, we will study the original Vicsek model in a deterministic framework and give a sufficient condition imposed only on the initial states and model parameters to guarantee the synchronization. To the authors knowledge, this is the first such kind of result on synchronization analysis of Vicsek model. Furthermore, we will give two counterexamples to show that the "uniform" connectivity of the associated neighbor graphs is not sufficient for synchronization of the Vicsek model, if the initial headings are allowed to be in $[0, 2\pi)$. This reveals some fundamental differences between Vicsek model and its linearized version. Partial results have been reported in ref. [8] without giving detailed proofs.

2 Main results

The Vicsek model is composed of N autonomous agents (or subsystems or particles) moving in the plane with the same absolute velocity; each agent's heading is updated according to the vector average of its neighbors. The neighbors of an agent $i(1 \leqslant i \leqslant N)$ at time t are those which lie within a circle of radius $r(r > 0)$ centered at the agent i's current position $(x_i(t), y_i(t))$. Denote the neighbors of the agent i at time t as $\mathcal{N}_i(t)$, i.e.,

$$\mathcal{N}_i(t) = \{j \big| \, d_{ij}(t) < r\},$$

where $d_{ij}(t) = \sqrt{(x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2}$. It is easy to see that each agent is a neighbor of itself. Each agent moves in the plane with a constant absolute velocity $v(v > 0)$, so its position is updated as follows:

$$\begin{cases} x_i(t+1) = x_i(t) + v\cos\theta_i(t) \\ y_i(t+1) = y_i(t) + v\sin\theta_i(t) \end{cases} \quad \forall i, \tag{2.1}$$

where $\theta_i(t)$ is the heading of agent i at time t, which is updated according to the following formula,

$$\theta_i(t+1) = \arctan \frac{\sum_{j\in\mathcal{N}_i(t)} \sin\theta_j(t)}{\sum_{j\in\mathcal{N}_i(t)} \cos\theta_j(t)}. \tag{2.2}$$

Note that the dynamical behavior of the above system is determined completely by the initial states, the moving velocity v, and the neighborhood radius r. Furthermore, the neighbors of each agent are determined by the positions of other agents, and the headings of each agent is determined by the neighbors' headings; at the same time, the headings will also affect the positions, so there is a complicated nonlinear relationship between positions and headings of all agents, which makes the complete theoretical analysis quite hard.

Obviously, the multi-agent system described by the Vicsek model is a dynamical network, and some basic concepts from graph theory will be useful in the analysis. Note that the neighbors of each agent will change over time, and Jadbabaie et al.[4] used an undirected graph sequence $\mathcal{G}_t = \{\mathcal{V}, \mathcal{E}_t\}$ to describe the interaction among agents, where $\mathcal{V} = \{1, 2, \cdots, N\}$ is the set of all agents, and $\mathcal{E}_t$ is the edge set which will change over time. Edges are formed in the following way: if the distance between two agents i and j at time t is less than r, then there exists an edge between i and j, $(i, j) \in \mathcal{E}_t$. A graph is called connected if there exists a path between any two vertexes of the graph. For convenience, we rewrite (2.2) into the following equivalent "quasi-linear" form:

$$\tan\theta_i(t+1) = \sum_{j\in\mathcal{N}_i(t)} \frac{\cos\theta_j(t)}{\sum_{k\in\mathcal{N}_i(t)} \cos\theta_k(t)} \tan\theta_j(t), \tag{2.3}$$

and put (2.3) to the following compact matrix form:

$$\tan\theta(t+1) = A(t)\tan\theta(t), \tag{2.4}$$

where $\tan\theta(t) \triangleq (\tan\theta_1(t), \cdots, \tan\theta_N(t))^\tau$, $A(t) \triangleq (a_{ij}(t))$ is the weighted average matrix of graph $\mathcal{G}_t$:

$$a_{ij}(t) = \begin{cases} \dfrac{\cos\theta_j(t)}{\sum_{k\in\mathcal{N}_i(t)} \cos\theta_k(t)}, & \text{if } (i,j) \in \mathcal{E}_t; \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

To analyze the synchronization of the Vicsek model, Jadbabaie et al.[4] studied the following linearized version of (2.2):

$$\theta_i(t+1) = \frac{1}{n_i(t)} \sum_{j \in \mathcal{N}_i(t)} \theta_j(t), \tag{2.6}$$

where $n_i(t)$ is the cardinality of $\mathcal{N}_i(t)$. Accordingly, the heading update equation (2.4) is replaced by the following formula:

$$\theta(t+1) = \widetilde{A}(t)\theta(t), \tag{2.7}$$

where $\theta(t) \triangleq (\theta_1(t), \cdots, \theta_N(t))^\tau$, and the entries of matrix $\widetilde{A}(t)$ are

$$\widetilde{a}_{ij}(t) = \begin{cases} \dfrac{1}{n_i(t)}, & \text{if } (i,j) \in \mathcal{E}_t; \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

The purpose of this paper is to study the synchronization of the Vicsek model together with its relationship with the connectivity of the associated neighbor graphs. We first need a definition of synchronization as follows (cf. ref. [4]).

Definition 2.1. The above multi-agent system is said to achieve synchronization if the headings of all agents satisfy

$$\lim_{t \to \infty} \theta_i(t) = \theta, \quad i = 1, \cdots, N,$$

where θ may depend on the initial states $\{\theta_i(0), x_i(0), y_i(0), i = 1, \cdots, N\}$ and system parameters v, r.

The following two theorems will establish the synchronization of both the original Vicsek model and its linearized version under conditions imposed only on the initial states and the model parameters r, v, and N.

Theorem 2.1. For the Vicsek model (2.1)(2.2), let $\{\theta_i(0) \in (-\frac{\pi}{2}, \frac{\pi}{2}), i = 1, \cdots, N, \}$, and the initial neighbor graph $\mathcal{G}_0 = \{\mathcal{V}, \mathcal{E}_0\}$ be connected. Then, the system will synchronize, if the absolute velocity v satisfies the following:

$$v \leqslant \frac{d}{\triangle_0} \left(\frac{\cos \overline{\theta}}{N} \right)^N, \tag{2.9}$$

where N is the number of agents, and

$$\overline{\theta} = \max_i |\, \theta_i(0)|, \quad d = r - \max_{i,j \in \mathcal{E}_0} d_{ij}(0),$$
$$\triangle_0 = \max_{i,j} \{\tan \theta_i(0) - \tan \theta_j(0)\}.$$

Remark 2.1. The restriction of the initial headings to $(-\frac{\pi}{2}, \frac{\pi}{2})$ cannot be extended to $[0, 2\pi)$ in general, as will be shown by Example 3.1 in the next section. However, for the linearized Vicsek model, such extension is feasible, as will be shown in the next theorem.

Theorem 2.2. For the linearized Vicsek model (2.1)(2.6), let $\theta_i(0) \in [0, 2\pi)$, and the initial neighbor graph be connected. Then, the system will synchronize, if the absolute velocity v satisfies the following:

$$v \leqslant \frac{d(\frac{1}{N})^N}{2\pi}, \tag{2.10}$$

where d is defined as in Theorem 2.1.

We will only give the proof of Theorem 2.1 in section 4, since the proof of Theorem 2.2 is similar and simpler, see ref. [11] for details. The above two theorems show that if the absolute velocity is suitably small, then the system will synchronize. It is worth mentioning that for systems described by the Vicsek model, a certain self-convergence property holds without any constraint on the model parameters, provided that $\theta_i(0) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. This phenomenon can be stated in the following proposition and was independently found in refs. [8, 9]. However, the proofs are different, since unlike ref. [9] our proof does not rely on the product of infinite matrices in ref. [10], which may be advantageous in dealing with nonlinear cases where matrix product does not work, see ref. [11].

Proposition 2.1. For the Vicsek model, let $\{\theta_i(0) \in (-\frac{\pi}{2}, \frac{\pi}{2}), i = 1...N\}$. Then, no matter what v and r are, there always exists $\theta_i, i \in \mathcal{V}$, such that

$$\lim_{t\to\infty} \theta_i(t) = \theta_i, \quad i = 1...N,$$

and if $\theta_i \neq \theta_j$, then agents i and j will not be neighbors after some time instant.

3 Counterexamples

Obviously, for the Vicsek model (2.1)(2.2) and its linearized version (2.1)(2.6), connectivity of the neighbor graphs is not necessary for synchronization. A simple example is the system where all agents have the same headings at $t = 0$, but the initial graph is not connected. It is easy to see that the system will synchronize, but the associated graphs will not be connected forever. Jadbabaie et al.[4] pointed out that if the neighbor graphs are jointly connected in a certain "uniform" way, then the linearized Vicsek model will synchronize. However, for the original Vicsek model, this result is no longer true for initial headings in $[0, 2\pi)$. Han et al.[12] gave a counterexample, but the velocity there is 0. For the Vicsek model with $v > 0$, this section will give two counterexamples to show some interesting phenomena concerning the relationship between the dynamical behavior of the neighbor graphs and synchronization.

Example 3.1. Let $N = 12$, and all agents be distributed on the unit circle uniformly with the headings being symmetric as shown in Figure 1 below. To be precise, we have

$$(x_i(0), y_i(0)) = \left(\cos\frac{(i-1)\pi}{6}, \sin\frac{(i-1)\pi}{6}\right); \tag{3.1}$$

$$\theta_i(0) = \left\{[16 - i + 3\cdot(-1)^i]\frac{\pi}{6}\right\} \bmod (2\pi), \quad i = 1, \cdots, 12. \tag{3.2}$$

Assume further that the absolute velocity satisfies $0 < v \leqslant 0.1$ and the neighborhood radius is taken as $r = 0.8$.

We now proceed to show the following facts for the Vicsek model: the neighbor graphs are connected at each time instant, but the heading of each agent changes from one angle to another, with the difference between these two angles being π, so the headings will not be convergent. Furthermore, whatever how small the velocity is, the system will never synchronize.

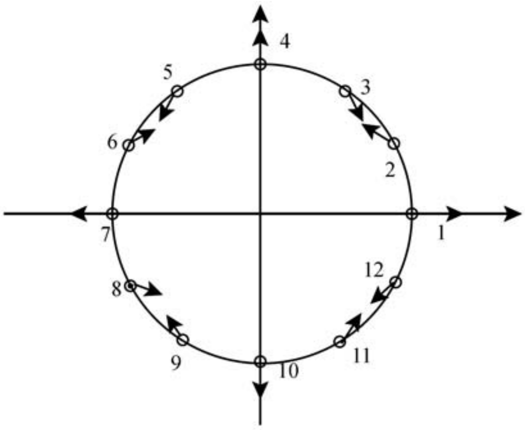

Figure 1.

First of all, at time $t = 0$, according to the initial positions determined by (3.1), we have

$$\begin{aligned} d_{ij}^2(0) &= (x_i(0) - x_j(0))^2 + (y_i(0) - y_j(0))^2 \\ &= \left(-2\sin\frac{(i+j-2)\pi}{12}\sin\frac{(i-j)\pi}{12}\right)^2 + \left(2\cos\frac{(i+j-2)\pi}{12}\sin\frac{(i-j)\pi}{12}\right)^2 \\ &= 4\sin^2\frac{(i-j)\pi}{12} \\ &= \begin{cases} \leqslant 4\sin^2\dfrac{\pi}{12} \approx 0.268 < r^2, & \text{if } |j-i| = 0, 1, 11; \\ \geqslant 4\sin^2\dfrac{\pi}{6} = 1 > r^2, & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, we know that each agent has three neighbors including itself, and thus the initial neighbor graph is connected.

Now, at $t = 1$ according to (2.2), the heading of each agent will be updated by

$$\begin{aligned} &\theta_i(1) \\ &= \arctan\frac{\sin\theta_{i-1}(0) + \sin\theta_i(0) + \sin\theta_{i+1}(0)}{\cos\theta_{i-1}(0) + \cos\theta_i(0) + \cos\theta_{i+1}(0)} \\ &= \arctan\frac{(1 - 2\cos\frac{\pi}{6})\sin\frac{[16-i+3\cdot(-1)^i]\pi}{6}}{(1 - 2\cos\frac{\pi}{6})\cos\frac{[16-i+3\cdot(-1)^i]\pi}{6}} \\ &= (\theta_i(0) + \pi) mod(2\pi), \end{aligned} \tag{3.3}$$

which shows that heading is reversed at $t = 1$. Moreover, by the position update law (2.1), we have

$$\begin{pmatrix} x_i(1) \\ y_i(1) \end{pmatrix} = \begin{pmatrix} \cos\dfrac{(i-1)\pi}{6} \\ \sin\dfrac{(i-1)\pi}{6} \end{pmatrix} + v \begin{pmatrix} \cos\dfrac{(16-i+3\cdot(-1)^i)\pi}{6} \\ \sin\dfrac{(16-i+3\cdot(-1)^i)\pi}{6} \end{pmatrix}.$$

Thus, at $t = 1$ the distance between agents i and $j(1 \leqslant i, j \leqslant 12)$ is as follows:

$$\begin{aligned} &d_{ij}^2(1) \\ &= \big\{x_i(0) - x_j(0) + v\big(\cos\theta_i(0) - \cos\theta_j(0)\big)\big\}^2 \end{aligned}$$

$$
\begin{aligned}
&+ \left\{y_i(0) - y_j(0) + v\big(\sin\theta_i(0) - \sin\theta_j(0)\big)\right\}^2 \\
&= 4\sin^2\frac{(i-j)\pi}{12} + 4v^2\sin^2\frac{\theta_i(0)-\theta_j(0)}{2} \\
&\quad - 8v\sin\frac{(i-j)\pi}{12}\sin\frac{\theta_i(0)-\theta_j(0)}{2}\cos\frac{[10-2(i+j)+3(-1)^i+3(-1)^j]\pi}{12}.
\end{aligned}
$$

From this and the assumption $0 < v \leqslant 0.1$, we proceed to show that the neighbor graphs will remain unchanged at $t = 1$. We consider three cases separately.

Case 1. If $|i-j| = 0, 1, 11$, then we have

$$
\begin{aligned}
d_{ij}^2(1) &\leqslant 4\sin^2\frac{\pi}{12} + 4v^2\sin^2\frac{5\pi}{12} + 8v\sin\frac{\pi}{12}\sin\frac{5\pi}{12}\cos 0 \\
&= 4\left(\sin\frac{\pi}{12} + v\sin\frac{5\pi}{12}\right)^2 \leqslant 0.51 < r^2.
\end{aligned}
$$

Case 2. If $|i-j|$ is even and $|i-j| \geqslant 2$, then we have

$$
\begin{aligned}
&d_{ij}^2(1) \\
&= 4(1+v^2)\sin^2\frac{(i-j)\pi}{12} + 8v\sin^2\frac{(i-j)\pi}{12}\cos\frac{[10-2(i+j)\pm 6]\pi}{12} \\
&\geqslant (4 - 8\times 0.1)\sin^2\frac{(i-j)\pi}{12} \geqslant (4-0.8)\sin^2\frac{\pi}{6} = 0.8 > r^2.
\end{aligned}
$$

Case 3. If $|i-j|$ is odd, and $|i-j| \neq 1, 11$, then we have

$$
\begin{aligned}
d_{ij}^2(1) &\geqslant 4\sin^2\frac{3\pi}{12} + 4v^2\cos^2\frac{5\pi}{12} - 8v\sin\frac{5\pi}{12}\cos\frac{3\pi}{12} \\
&\geqslant 2 - 0.4\sqrt{2} > r^2.
\end{aligned}
$$

Hence at $t = 1$, the neighbor graph is still connected. Repeating the above procedure, we know that the neighbor graphs will be connected at all time instants, but the headings will be reversed from one time to the next one, and so the system will not synchronize. At the same time,we can see from the proof that whatever how small the absolute velocity v is, the system will not synchronize.

Proposition 2.1 shows that when the initial headings belong to $(-\pi/2, \pi/2)$, the Vicsek model will always have some convergence properties regardless of the connectivity of the associated neighbor graphs; furthermore, if a certain connectivity does hold, then the Vicsek model will actually synchronize[8]. In this case, it is likely that the neighbor graphs will remain unchanged after a certain time instant. However, the next example will further show that when the initial headings are allowed to be in $[0, 2\pi)$, the neighbor graphs may change from time to time, while they are jointly connected uniformly.

Example 3.2. Let $N = 24, r = 0.3, v = 0.1$, and all agents be distributed on the unit circle uniformly with the headings being symmetric as shown in Figure 2 below. To be precise, we have

$$
\begin{aligned}
&(x_i(0), y_i(0)) = \left(\cos\frac{(i-1)\pi}{12}, \sin\frac{(i-1)\pi}{12}\right); \\
&\theta_i(0) = \left\{\frac{\pi}{12}[5 + 6\cdot(-1)^i + i]\right\}\mathrm{mod}(2\pi), \quad i = 1, \cdots, N.
\end{aligned}
$$

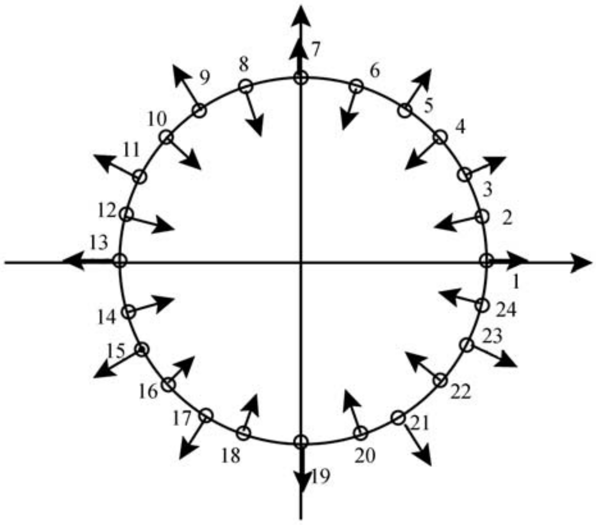

Figure 2.

Similar to Example 3.1, it is easy to show that, at $t = 2k$ (k is a positive integer), each agent has three neighbors including itself and the neighbor graphs are connected; however, at time $t = 2k+1$, each agent's neighbor is itself only and the graphs are no longer connected. Hence, the neighbor graphs change from time to time while keeping jointly connected. The system in this example does not synchronize either.

4 Proof of Theorem 2.1

To prove Theorem 2.1, we need to recall some preliminary results on stochastic matrices first (cf. refs. [10,13]).

A square matrix $A = (a_{ij})_{n\times n}$ is called nonnegative (positive) and denoted by $A \geqslant 0(> 0)$, if all its entries satisfy $a_{ij} \geqslant 0(> 0)$; and a nonnegative matrix A is called stochastic if the sum of each row satisfies $\sum_{j=1}^{n} a_{ij} = 1, i = 1, \cdots, n$. For two matrixes $A = (a_{ij})$ and $B = (b_{ij})$, $A \geqslant (>)B$ means $a_{ij} \geqslant (>)b_{ij}$. A matrix $A = (a_{ij})$ is said to have the property of connectivity if for different p and $q(1 \leqslant p, q \leqslant n)$, there always exists a sequence of integers $k_1 = p, k_2, \cdots, k_{m-1}, k_m = q, 1 \leqslant m \leqslant n$, such that the entries of the matrix $a_{k_1k_2}, a_{k_2k_3}, \cdots, a_{k_{m-1}k_m}$ are all nonzero. If a matrix A is nonnegative, then $(I + A)^n > 0$ is equivalent to the connectivity of the graph associated with A; if all diagonal entries of matrix A are greater than 0, then $A^n > 0$ is equivalent to the connectivity of the graph associated with A. A stochastic matrix A is called indecomposable and aperiodic (SIA), if $Q \triangleq \lim_{k\to\infty} A^k$ exists and all rows of Q are equal.

For Vicsek model with the initial headings in $(-\frac{\pi}{2}, \frac{\pi}{2})$, matrix $A(t)$ defined by (2.5) is a stochastic matrix, and all the diagonal entries are nonzero.

The Proof of Theorem 2.1

For any two agents i and j, we have

$$(\cos\theta_i(t) - \cos\theta_j(t))^2 + (\sin\theta_i(t) - \sin\theta_j(t))^2 = 4\left|\sin\frac{\theta_i(t) - \theta_j(t)}{2}\right|^2.$$

Thus, by the properties of Euclidean norm, we have

$$d_{ij}(t+1) \leqslant d_{ij}(t) + v|\theta_i(t) - \theta_j(t)|. \tag{4.1}$$

Next, by the fact that $\tan x - x$ is an increasing function of $x \in (-\pi/2, \pi/2)$, we have

$$\tan \max_i\{\theta_i(t)\} - \max_i\{\theta_i(t)\} \geqslant \tan \min_i\{\theta_i(t)\} - \min_i\{\theta_i(t)\},$$

i.e.,

$$\max_i \theta_i(t) - \min_i \theta_i(t) \leqslant \tan \max_i\{\theta_i(t)\} - \tan \min_i\{\theta_i(t)\},$$

which in conjunction with (4.1) yields

$$d_{ij}(t+1) \leqslant d_{ij}(t) + v\triangle_t, \quad t \geqslant 0, \tag{4.2}$$

where $\triangle_t = \max_{i,j}\{\tan\theta_i(t) - \tan\theta_j(t)\}$. By (2.2), it is easy to see that $\triangle_t$ is a non-increasing sequence of time t.

We will further show that $\triangle_t$ actually decreases exponentially fast, i.e., there exists a constant $0 < L < 1$, such that

$$\triangle_{kN} \leqslant L^k \triangle_0, \forall k \geqslant 0. \tag{4.3}$$

We use induction and first consider the case where $k = 1$.

For arbitrary $i, j \in \mathcal{E}_0$, using (4.2) and the monotonicity of $\triangle_t$, we have

$$d_{ij}(t+1) \leqslant d_{ij}(0) + v\sum_{l=0}^{t} \triangle_l \leqslant d_{ij}(0) + (t+1)v\triangle_0, \quad 0 \leqslant t \leqslant N-1.$$

Moveover, by (2.9), we can obtain

$$d_{ij}(t+1) \leqslant d_{ij}(0) + (t+1)\frac{d}{\triangle_0}\left(\frac{\cos\overline{\theta}}{N}\right)^N \triangle_0 < r, \quad 0 \leqslant t \leqslant N-1.$$

This means that those agents that are neighbors at $t = 0$ will still be neighbors at $t = 1, \cdots, N$, and hence the graphs $\mathcal{G}_t, t = 1, \cdots, N$ are connected. Because all nonzero entries of matrix $A(t)$ defined by (2.5) are not less than $\frac{\cos\overline{\theta}}{N}$, where $\overline{\theta} = \max_i |\,\theta_i(0)|$. Thus, $A(t) \geqslant A_N \triangleq (a'_{ij}), i, j = 1, \cdots, N$, where

$$a'_{ij} = \begin{cases} \dfrac{\cos\overline{\theta}(0)}{N}, & \text{if } (i,j) \in \mathcal{E}_0; \\ 0, & \text{otherwise.} \end{cases}$$

Next, we consider the heading update. By (2.4), we have

$$\tan\theta(N) = A(N-1)\cdots A(0)\tan\theta(0). \tag{4.4}$$

Denote $\mathbf{1} = (1, 1\cdots, 1)^\tau$, then by the above proven fact and the connectivity of the initial graph we have

$$A(N-1)\cdots A(0) \triangleq (a_{ij}^N(0)) \geqslant (A_N)^N \geqslant \left(\frac{\cos\overline{\theta}(0)}{N}\right)^N \mathbf{1}\mathbf{1}^\tau, \tag{4.5}$$

i.e., $a_{ij}^N(0) \geqslant (\frac{\cos\overline{\theta}(0)}{N})^N > 0$. Since $\{A(t), t = 1, 2, \cdots,\}$ are stochastic matrices and the product of stochastic matrices is still stochastic, it is clear that $\sum_{j=1}^N a_{ij}^N(0) = 1$. Hence, for arbitrary $i, i' = 1, \cdots, N$, we have $\sum_{j=1}^{N}(a_{ij}^N(0) - a_{i'j}^N(0)) = 0$. Set

$$L' = \sum_{j=1}^{N} \left(a_{ij}^N(0) - a_{i'j}^N(0)\right)^+ = \sum_{j=1}^{N} \left(a_{ij}^N(0) - a_{i'j}^N(0)\right)^- ,$$

where $(a)^+ = \max(a, 0), a^- = \max(-a, 0)$. We then have by (4.5),

$$L' \leqslant \sum_{j=1}^{N} \left(a_{ij}^N(0) - \left(\frac{\cos\overline{\theta}(0)}{N}\right)^N \right) = 1 - N\left(\frac{\cos\overline{\theta}(0)}{N}\right)^N \triangleq L < 1. \tag{4.6}$$

By (4.4), we know that there exist $i, i' \in \{1, 2, \cdots, N\}$, such that

$$\max_l (\tan\theta_l(N)) = \sum_{j=1}^{N} a_{ij}^N(0)\tan\theta_j(0);$$

$$\min_l (\tan\theta_l(N)) = \sum_{j=1}^{N} a_{i'j}^N(0)\tan\theta_j(0). \tag{4.7}$$

Thus, by (4.6) and (4.7), it follows that

$$\begin{aligned}
&\triangle_N = \max_i (\tan\theta_i(N)) - \min_i (\tan\theta_i(N)) \\
&= \sum_{j=1}^{N} \left(a_{ij}^N(0) - a_{i'j}^N(0)\right)\tan\theta_j(0) \\
&= \sum_{j=1}^{N} \left(a_{ij}^N(0) - a_{i'j}^N(0)\right)^+ \tan\theta_j(0) - \sum_{j=1}^{N} \left(a_{ij}^N(0) - a_{i'j}^N(0)\right)^- \tan\theta_j(0) \\
&\leqslant L'\{\max_i \tan\theta_i(0) - \min_i \tan\theta_i(0)\} \leqslant L\triangle_0,
\end{aligned}$$

so (4.3) holds when $k = 1$.

Next, assume (4.3) holds for all $k \leqslant K$ with some $K \geqslant 1$. By the monotonicity of $\triangle_t$, we have $\triangle_t \leqslant L^l\triangle_0$ for arbitrary $t \in [lN, (l+1)N)$ with $l \leqslant K$. Thus, for any $i, j \in \mathcal{E}_0$ and any $t \in [KN, (K+1)N)$, we have by (4.2)

$$\begin{aligned}
&d_{ij}(t+1) \leqslant d_{ij}(t) + v\triangle_t \leqslant d_{ij}(0) + v\sum_{l=0}^{t}\triangle_l \\
&\leqslant d_{ij}(0) + Nv(1 + L + L^2 + \cdots + L^K)\triangle_0 \\
&< d_{ij}(0) + vN\frac{1}{1-L}\triangle_0 \leqslant r.
\end{aligned}$$

Hence, those agents that are neighbors at $t = 0$ will still be neighbors for $t \in [KN, (K+1)N)$. Consequently, by the connectivity of the initial graph again, we have

$$A((K+1)N-1)\cdots A(KN) \triangleq (a_{ij}^N(K)) \geqslant (A_N)^N \geqslant \left(\frac{\cos\overline{\theta}(0)}{N}\right)^N \mathbf{1}\mathbf{1}^\tau.$$

Thus, by the same method as for the case $k = 1$, we can show that

$$\triangle_{(K+1)N} \leqslant L\triangle_{KN} \leqslant L^{K+1}\triangle_0.$$

Therefore, (4.3) is true when $k = K + 1$. Consequently, by the induction and the monotonicity of $\triangle_t$, we know that $\triangle_t \to 0$ exponentially fast, and hence the system will synchronize. This completes the proof of Theorem 2.1.

5 Concluding remarks

The main contributions of this paper are the following: i) the establishment of synchronization property for a basic class of multi-agent systems described by the now well-known Vicsek model, under conditions imposed only on the system initial conditions and the model parameters, which are obviously easy-to-verify in comparison with the previously used connectivity condition on the system trajectory itself; and ii) the construction of two counterexamples showing that the connectivity of the associated neighbor graphs is no longer sufficient for synchronization of the Vicsek model when the initial headings are allowed to be in $[0, 2\pi)$, revealing a fundamental difference between the Vicsek model and its linearized version. It is worth mentioning that the exponential convergence rate as proven in Theorem 2.1 allows us to investigate systems with noises[14], and if we work in a stochastic framework with random initial states uniformly distributed and with a large population, then the parameter conditions on both the velocity and the radius can be considerably relaxed or even removed[15]. Of course, many interesting problems still remain open, which call for further efforts on the theoretical study of multi-agent systems with local rules.

The authors would like to thank Mr. Tang Gongguo and Prof. Han Jing for their valuable discussions.

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