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Adaptive neural networks control for uncertain parabolic distributed parameter systems with nonlinear periodic time-varying parameter

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This paper studies the problem of adaptive neural networks control (ANNC) for uncertain parabolic distributed parameter systems (DPSs) with nonlinear periodic time-varying parameter (NPTVP). Firstly, the uncertain nonlinear dynamic and unknown periodic TVP are represented by using neural networks (NNs) and Fourier series expansion (FSE), respectively. Secondly, based on the ANNC and reparameterization approaches, two control algorithms are designed to make the uncertain parabolic DPSs with NPTVP asymptotically stable. The sufficient conditions of the asymptotically stable for the resulting closed-loop systems are also derived. Finally, a simulation is carried out to verify the effectiveness of the two control algorithms designed in this work.

nonlinear parabolic distributed parameter systems, adaptive neural networks control, Fourier series expansion, asymptotically stable

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1 Introduction

Distributed parameter systems (DPSs) can be used to simulate the state change of a large number of practical engineering systems. Such systems often have the characteristics of time-space evolution at the same time, that is, system variables and parameters change not only with time, but also with the change of space position. Parabolic DPSs is a typical DPSs. It is widely used in mathematical models in the fields of physics, chemistry, biology, ecology and control engineering. For example, it is applied to engineering practices such as modeling and control of infectious diseases, chemical reaction control, heat conduction control, pipeline flow control [1–4]. Consequently, it is very valuable and essential to study the control problem of parabolic DPSs with strong application background.

In recent years, the nonlinear structure of the actual industrial control system becomes more and more complex. In addition, people's requirements for control accuracy and system performance are getting higher and higher. However, the widespread existence of stochastic disturbance, multivariable coupling, various uncertainties, etc., poses a greater challenge to the study of control and synthesis of nonlinear systems [5–7], which has attracted the attention of many scholars and researchers. T-S fuzzy method is an effective method, which is widely used to deal with nonlinear systems. The design of fuzzy control algorithms for nonlinear ordinary differential equations (ODEs) based on T-S fuzzy model were studied in refs. [8–10]. Then, this method was extended to the nonlinear parabolic PDEs [11, 12]. In refs. [11, 12], the main idea was to reduce the nonlinear parabolic PDEs to a low dimensional nonlinear ODEs model by using Galerkin method, and then used the existing fuzzy control technology

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to design a suitable controller for the obtained low dimensional nonlinear ODEs model. However, because the system information is truncated before the design of control algorithms, some inherent characteristics of the original system may be lost, resulting in control overflow. Therefore, for the sake of more perfect performance and higher control accuracy, scholars are committed to directly studying the fuzzy controller design method of nonlinear parabolic DPSs based on fuzzy PDEs model. In ref. [13], a class of fuzzy boundary controllers for nonlinear DPSs was proposed. In ref. [14], a distributed fuzzy control algorithm was proposed for nonlinear parabolic DPSs. Recently, Wang et al. [15–18] have proposed many distributed control algorithms based on T-S fuzzy PDEs model. NNs [19] and fuzzy logic systems (FLSs) [20] are also effective tools to deal with nonlinear systems, which have been widely used in many researches [21–34]. In refs. [21–27], FLSs were mainly used to approximate nonlinear systems with unknown nonlinear function. However, the systems studied in refs. [21–27] were all systems modeled by ODEs. Wu et al. [32] and Luo et al. [35] studied the problem of adaptive control for uncertain nonlinear DPSs based on NNs. However, their main idea was to reduce the nonlinear parabolic PDEs to a low dimensional nonlinear ODEs model by Galerkin method. Then, the existing fuzzy control technology was used to design a suitable controller for the low dimensional nonlinear ODEs model. Most of the literatures mentioned above are concerned with unknown nonlinear dynamical systems with constant parameterization. Then, Li et al. [36–38] gave some results on the synchronization of nonlinear reaction-diffusion NNs with unknown TVP. However, in these results, the unknown TVPs in the studied systems were linearly ones rather than nonlinearly ones. Therefore, this paper is devoted to the study of uncertain parabolic DPSs with unknown NPTVP.

Based on the above description and summary of the existing results, we mainly give the research motivation behind this study from the following aspects. Firstly, this paper studies a class of parabolic PDEs with NPTVP. It is widely used in practical engineering research fields such as aerospace engineering and machine tool mechanical control. For example, for industrial robots, aerospace vehicles and CNC machine tools, interference usually exists and has periodicity due to the influence of the external environment. Secondly, with the development of science and technology, the nonlinear structure of most industrial control systems becomes more and more complex. Moreover, in practical applications, the periodic nonlinear disturbances in most systems can not be fully linearized. Therefore, the research on the stabilization of uncertain PDEs with NPTVP has important practical value and research significance. In addition, it should be emphasized that the existing researches on nonlinear DPSs with PTVP mainly consider linearly ones rather than nonlinearly ones. As far as the authors know, there are few researches on the control of uncertain parabolic DPSs with NPTVP, which motivates us to devote ourselves to the research of this work.

In this study, we mainly focus on the problem of ANNC for uncertain parabolic DPSs with NPTVP, which is mainly based on the theories of adaptive control, NNs and Fourier series expansion (FSE). Contributions of this study are stated as follows.

(i) Different from refs. [21–34], NNs or FLSs cannot be directly used to approximate uncertain PDEs with unknown NPTVP. If the NNs or FLSs are used to approximate the nonlinear system directly, there will be unmeasurable PTVP in the signal of the approximator. Therefore, NNs or FLSs can not directly approach unknown system with NPTVP. In order to solve this problem, we need to take the following two steps: first, the unmeasurable PTVP is reconstructed by FSE technique. Then, the reconstructed parameters are used as the new input signal of the NNs or FLSs to describe the nonlinear periodic time-varying uncertain DPSs.

(ii) Unlike PDEs with TVP studied in refs. [36–38], this paper gives an ANNC scheme for the uncertain parabolic DPSs with NPTVP, which has a wider application in actual engineering modeling.

(iii) Based on the ANNC technology and reparameterization approach, two control algorithms are designed to make the uncertain parabolic DPSs with NPTVP asymptotically stable.

The overall structure of the study is arranged as follows. Sect. 2 introduces the system formulation and some preliminaries. In Sect. 3, the main results are given. A simulation is carried out to verify the effectiveness of the two control algorithms in Sect. 4. Finally, give a conclusion for this paper.

Notations: $\mathcal{R}$ and $\mathcal{R}^n$ represent the set of the real numbers and n-dimensional Euclidean space, respectively. $\mathcal{H} \triangleq (\mathcal{L}_2[0,l];\mathcal{R})$ is a Hilbert space of square integrable vector function $\phi(\mathfrak{s}) : [0,l] \to \mathcal{R}$ with $\langle\phi_1(\cdot),\phi_2(\cdot)\rangle = \int_0^l \phi_1^{\mathrm{T}}(\mathfrak{s})\phi_2(\mathfrak{s})\mathrm{d}\mathfrak{s}$ and $\|\phi_1\|_2 = \sqrt{\langle\phi_1(\cdot),\phi_1(\cdot)\rangle}$. $\|E\| = \sqrt{E^{\mathrm{T}}E}$ and $\|E\|_1 = \sum_{i=1}^n |e_i|$, where $E = [e_1, e_2, \cdots, e_n]^{\mathrm{T}} \in \mathcal{R}^n$. The superscript T is used for the transpose of a vector or a matrix. $\chi_t(\mathfrak{s},t) = \frac{\partial\chi(\mathfrak{s},t)}{\partial t}$, $\chi_{\mathfrak{s}}(\mathfrak{s},t) = \frac{\partial\chi(\mathfrak{s},t)}{\partial \mathfrak{s}}$, $\chi_{\mathfrak{s}\mathfrak{s}}(\mathfrak{s},t) = \frac{\partial^2\chi(\mathfrak{s},t)}{\partial \mathfrak{s}^2}$.

2 Problem formulation and some preliminaries

Consider the following periodic time-varying PDEs:

$$\begin{cases} \chi_t(\mathfrak{s},t) = \chi_{\mathfrak{s}\mathfrak{s}}(\mathfrak{s},t) + f(\chi(\mathfrak{s},t),\varphi(t)) + u(\mathfrak{s},t), \mathfrak{s} \in (0,l), \\ \chi(\mathfrak{s},0) = \chi_0(\mathfrak{s}), \\ \chi(0,t) = \chi(l,t) = 0, \end{cases} \tag{1}$$

where $\chi(\mathfrak{s},t) \in \mathcal{H}$ is the state variable with $\mathfrak{s} \in (0,l)$ is the spatial coordinate variable and $t > 0$ is the time. $\varphi(t)$:

$[0,\infty)\to\mathcal{R}$ is an unknown perturbation parameter with a known period τ, namely, $\varphi(t)=\varphi(t+\tau)$. $f(\chi(\mathfrak{s},t),\varphi(t)):\mathcal{H}\times\mathcal{R}\to\mathcal{H}$ is an unknown continuous function and satisfies $f(0)=0$. $u(\mathfrak{s},t)$ is the control input.

Remark 1 In this work, the uncertain function $f(\chi,\varphi(t))$ needs to satisfy the following assumptions. Firstly, $f(\chi,\varphi(t))$ is a continuous function, which can be approximated by radial basis NNs $Q^{\mathrm{T}}\phi(\chi,\varphi(t))$ on a compact set Ω. Secondly, for the unknown nonlinear TVP $\varphi(t)$, it is assumed that it is a TVP with a known period τ. The assumption of function $f(\chi,\varphi(t))$ is to make the uncertain nonlinear function approximate by NNs, and then deal with the system with uncertain nonlinear terms. This assumption has been given in the existing literatures [30, 31]. In addition, for $\varphi(t)$, a TVP that changes periodically has been applied in actual engineering modeling. For example, for industrial robots, aerospace vehicles and CNC machine tools, interference usually exists and has periodicity due to the influence of the external environment. Its change cycle is closely related to the change cycle of the external environment.

Next, several Lemmas are introduced, which will be used later.

Lemma 1. (Wirtinger's inequality [16]) Let $\chi(\mathfrak{s},t)\in\mathcal{H}$ and it satisfies $\chi(0,t)=0$ or $\chi(l,t)=0$. Then, we get

$$\int_0^l \chi^2(\mathfrak{s},t)\mathrm{d}\mathfrak{s}\le\frac{4l^2}{\pi^2}\int_0^l\chi_{\mathfrak{s}}^2(\mathfrak{s},t)\mathrm{d}\mathfrak{s},\ t\ge 0. \tag{2}$$

Lemma 2. (Young's inequality [39]) Suppose $a,b\ge 0\in\mathcal{R}$, $\varrho_1>1$, $\frac{1}{\varrho_1}+\frac{1}{\varrho_2}=1$, then

$$ab\le\frac{a^{\varrho_1}}{\varrho_1}+\frac{b^{\varrho_2}}{\varrho_2},$$

if and only if $a^{\varrho_1}=b^{\varrho_2}$, the equality $ab=\frac{a^{\varrho_1}}{\varrho_1}+\frac{b^{\varrho_2}}{\varrho_2}$ holds.

Lemma 3. (Approximation theory of radial basis function NNs [40]) If the number of neural nodes m is large enough, $Q^{\mathrm{T}}\phi(\omega,g)$ can approximate any continuous function $f(\omega,g)$ on a compact set $\Omega\subset\mathcal{H}\times\mathcal{R}$, i.e.,

$$f(\omega,g)=Q^{\mathrm{T}}\phi(\omega,g)+\xi(\omega,g), \tag{3}$$

where $\xi(\omega,g)$ is the inherent approximation error of NNs and satisfies $|\xi(\omega,g)|<\xi^*$ with ξ^* is an unknown positive constant. $Q=[q_1,q_2,\cdots,q_m]\in\mathcal{R}^m$ is the optimal weight vector that defined as

$$Q:=\arg\min_{\hat{Q}\in\mathcal{R}^m}\{\sup_{(\omega,g)\in\Omega}|f(\omega,g)-\hat{Q}^{\mathrm{T}}\phi(\omega,g)|\}, \tag{4}$$

where $\hat{Q}(t)$ is the estimate of Q at t, $\phi(\omega,g)\triangleq[\phi_1(\omega,g),\phi_2(\omega,g),\cdots,\phi_m(\omega,g)]^{\mathrm{T}}:\Omega\to\mathcal{R}^m$ is the known NNs basis function vector defined as

$$\phi_j(\omega,g)=\exp\left\{-\frac{(\omega-s_j^1)^2+(g-s_j^2)^2}{c_j^2}\right\}, \tag{5}$$

with $s_j\triangleq[s_j^1,s_j^2]^{\mathrm{T}}$ and c_j are the center and width of the basis function, respectively.

Based on the above problem formulation and some preliminaries, we aim to design two ANNC algorithms to make the nonlinear periodic time-varying uncertain parabolic DPSs asymptotically stable. The architecture of the ANNC scheme is shown in Figure 1.

3 Main results

From Lemma 3, if the input variables $\chi\triangleq\chi(\mathfrak{s},t)$ and $\varphi(t)$ of NNs are both measurable, the unknown continuous function can be described directly as

$$f(\chi,\varphi(t))=Q^{\mathrm{T}}\phi(\chi,\varphi(t))+\xi(\chi,\varphi(t)). \tag{6}$$

However, in this work the variable $\varphi(t)$ is an unknown PTVP, which leads to $\phi(\chi,\varphi(t))$ unknown. Thus, we cannot use eq. (6) directly. We need to further construct a new equivalent form. Since $\varphi(t)$ is a TVP with a known period τ, it can be described by a linearized FSE [41] as

$$\varphi(t)=P^{\mathrm{T}}\rho(t)+\varepsilon_1(t), \tag{7}$$

where $P\triangleq[p_1,p_2,\cdots,p_n]^{\mathrm{T}}\in\mathcal{R}^n$ is the weight coefficient of Fourier expansion and $\rho(t)\triangleq[\rho_1(t),\rho_2(t),\cdots,\rho_n(t)]^{\mathrm{T}}\in\mathcal{R}^n$ is the basis function vector of Fourier series, where

$$\rho_1(t)=1,\ \rho_{(2i)}(t)=\sqrt{2}\sin(2i\pi t/\tau),$$
$$\rho_{(2i+1)}(t)=\sqrt{2}\cos(2i\pi t/\tau),\ \ i=1,2,\cdots,\frac{n-1}{2},$$

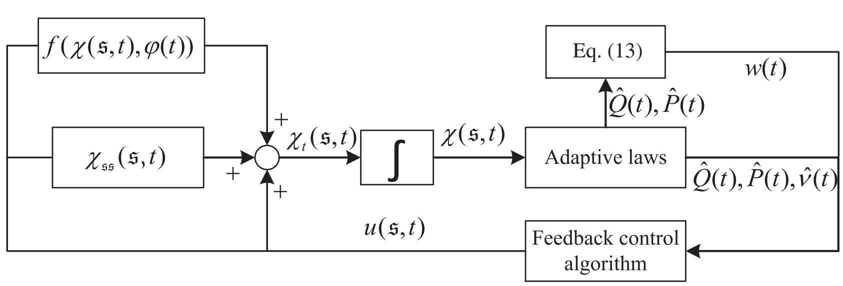

Figure 1 Scheme architecture for ANNC.

and $\varepsilon_1(t)$ is the inherent approximation error of FSE, which satisfies $|\varepsilon_1(t)| \le \varepsilon_1^*$ with ε_1^* is an unknown positive constant.

From eqs. (6) and (7), we can construct the following new equivalent form:

$$f(\chi, \varphi(t)) = Q^{\mathrm{T}}\phi(\chi, P^{\mathrm{T}}\rho(t) + \varepsilon_1(t)) + \xi(\chi, \varphi(t)), \tag{8}$$

where $|\xi(\chi, \varphi(t))| < \xi^*$ with ξ^* is an unknown positive constant.

A feedback control algorithm is designed as

$$u(\mathfrak{s}, t) = -\hat{Q}^{\mathrm{T}}(t)\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) - (\hat{\nu}(t) + w(t))\mathrm{sgn}(\chi(\mathfrak{s}, t)), \tag{9}$$

with the following adaptive laws:

$$\begin{aligned}\dot{\hat{Q}}(t) =&\varpi_1 \int_0^l \chi(\mathfrak{s}, t)[\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) \\ &- \phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\hat{P}^{\mathrm{T}}(t)\rho(t)]\mathrm{d}\mathfrak{s},\end{aligned} \tag{10}$$

$$\dot{\hat{P}}(t) = \varpi_2 \int_0^l \chi(\mathfrak{s}, t)\hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\rho(t)\mathrm{d}\mathfrak{s}, \tag{11}$$

and

$$\dot{\hat{\nu}}(t) = \varpi_3 \int_0^l |\chi(\mathfrak{s}, t)|\mathrm{d}\mathfrak{s}, \tag{12}$$

where

$$\begin{aligned}&\nu = \|P\|^2 + \|Q\|^2 + \|Q\|_1 + \lambda^*, \\ &w(t) = \|\hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\rho(t)\|^2 \\ &\qquad + \|\hat{P}^{\mathrm{T}}(t)\rho(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\|^2, \\ &\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) = \frac{\partial\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))}{\partial(\hat{P}^{\mathrm{T}}(t)\rho(t))},\end{aligned} \tag{13}$$

with $\hat{P}(t),\ \hat{Q}(t),\ \hat{\nu}(t)$ are the estimated values of $P,\ Q,\ \nu$, respectively, and $\varpi_1 > 0,\ \varpi_2 > 0,\ \varpi_3 > 0$ are constant parameters that can be adjusted.

Remark 2 Combining the NNs approximation method and reparameterization technology, the unknown nonlinear term $f(\chi, \varphi(t))$ is rewritten as eq. (8), where Q is the optimal weight vector of NNs and P is the weight vector of FSE. The adaptive parameters $\hat{Q}(t), \hat{P}(t)$ and $\hat{\nu}(t)$ in the control algorithm (9) are the estimated values of Q, P and ν at t. Through these estimates, the control algorithm (9) can be designed to make the PDEs system with NPTVP asymptotically stable. In addition, the parameters ϖ_1, ϖ_2 and ϖ_3 contained in the adaptive laws are adjustable parameters, which can affect the convergence speed of the closed-loop system.

Substituting eqs. (8) and (9) into eq. (1), we can derive the closed-loop system of eq. (1) as

$$\begin{cases}\chi_t(\mathfrak{s}, t) = \chi_{\mathfrak{s}\mathfrak{s}}(\mathfrak{s}, t) + Q^{\mathrm{T}}\phi(\chi, P^{\mathrm{T}}\rho(t) + \varepsilon_1(t)) + \xi(\chi, \varphi(t)) \\ -\hat{Q}^{\mathrm{T}}(t)\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) - (\hat{\nu}(t) + w(t))\mathrm{sgn}(\chi(\mathfrak{s}, t)), \\ \chi(\mathfrak{s}, 0) = \chi_0(\mathfrak{s}), \\ \chi(0, t) = \chi(l, t) = 0.\end{cases} \tag{14}$$

Theorem 1. There exists a feedback controller (9) and adaptive laws (10)–(12) to ensure that the solution of the system (14) is asymptotically stable, namely, $\lim_{t\to\infty} \|\chi(\mathfrak{s}, t)\|_2 = 0$. In addition, the control input u and estimations of adaptive parameters $\hat{Q}(t),\ \hat{P}(t),\ \hat{\nu}(t)$ are all bounded.

Proof. Construct the candidate Lyapunov function for eq. (14) as follows:

$$\mathbb{W}(t) = \mathbb{W}_1(t) + \mathbb{W}_2(t), \tag{15}$$

where

$$\mathbb{W}_1(t) = \frac{1}{2}\int_0^l \chi^2(\mathfrak{s}, t)\mathrm{d}\mathfrak{s}, \tag{16}$$

and

$$\mathbb{W}_2(t) = \frac{1}{2\varpi_1}\tilde{Q}^{\mathrm{T}}(t)\tilde{Q}(t) + \frac{1}{2\varpi_2}\tilde{P}^{\mathrm{T}}(t)\tilde{P}(t) + \frac{1}{2\varpi_3}\tilde{\nu}^2(t), \tag{17}$$

with $\tilde{Q}(t) = Q - \hat{Q}(t),\ \tilde{P}(t) = P - \hat{P}(t),\ \tilde{\nu}(t) = \nu - \hat{\nu}(t)$.

Because of the eq. (9), it can be known that the function $\mathbb{W}_1(t)$ is discontinuous in $\mathcal{R}$. The Dini derivative of function $\mathbb{W}_1(t)$ obtained along the solution of eq. (14) with the respect t is as follows:

$$\begin{aligned}D^+\mathbb{W}_1(t) = &\int_0^l \chi(\mathfrak{s}, t)\chi_{\mathfrak{s}\mathfrak{s}}(\mathfrak{s}, t)\mathrm{d}\mathfrak{s} + \int_0^l \chi(\mathfrak{s}, t)[Q^{\mathrm{T}}\phi(\chi, P^{\mathrm{T}}\rho(t)) \\ &- \hat{Q}^{\mathrm{T}}(t)\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))]\mathrm{d}\mathfrak{s} \\ &+ \int_0^l \chi(\mathfrak{s}, t)[Q^{\mathrm{T}}\phi(\chi, P^{\mathrm{T}}\rho(t) + \varepsilon_1(t)) \\ &- Q^{\mathrm{T}}\phi(\chi, P^{\mathrm{T}}\rho(t))]\mathrm{d}\mathfrak{s} + \int_0^l \chi(\mathfrak{s}, t)\xi(\chi, \varphi(t))\mathrm{d}\mathfrak{s} \\ &- (w(t) + \hat{\nu}(t))\int_0^l \chi(\mathfrak{s}, t)\mathrm{sgn}(\chi(\mathfrak{s}, t))\mathrm{d}\mathfrak{s} \\ &= \sum_{\iota=1}^{5} K_\iota.\end{aligned} \tag{18}$$

From Lemma 1, the boundary condition, and using the integral by parts, one has

$$\begin{aligned}K_1 &= \int_0^l \chi(\mathfrak{s}, t)\chi_{\mathfrak{s}\mathfrak{s}}(\mathfrak{s}, t)\mathrm{d}\mathfrak{s} = -\int_0^l \chi_{\mathfrak{s}}^2(\mathfrak{s}, t)\mathrm{d}\mathfrak{s} \\ &\le -\frac{\pi^2}{4l^2}\int_0^l \chi^2(\mathfrak{s}, t)\mathrm{d}\mathfrak{s}.\end{aligned} \tag{19}$$

From Taylor series expansion, one can get

$$\begin{aligned}&Q^{\mathrm{T}}\phi(\chi, P^{\mathrm{T}}\rho(t)) - \hat{Q}^{\mathrm{T}}(t)\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) \\ =&Q^{\mathrm{T}}[\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) + \phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\tilde{P}^{\mathrm{T}}(t)\rho(t) \\ &+ o((\tilde{P}^{\mathrm{T}}(t)\rho(t))^2)] - \hat{Q}^{\mathrm{T}}(t)\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) \\ =&(\hat{Q}^{\mathrm{T}}(t) + \tilde{Q}^{\mathrm{T}}(t))\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\tilde{P}^{\mathrm{T}}(t)\rho(t)\end{aligned}$$

$$
\begin{aligned}
&+ Q^{\mathrm{T}} o((\tilde{P}^{\mathrm{T}}(t)\rho(t))^2) + \tilde{Q}^{\mathrm{T}}(t)\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) \\
=&\hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\tilde{P}^{\mathrm{T}}(t)\rho(t) + \tilde{Q}^{\mathrm{T}}(t)[\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) \\
&- \phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\hat{P}^{\mathrm{T}}(t)\rho(t)] + M,
\end{aligned} \tag{20}
$$

where $M = Q^{\mathrm{T}} o((\tilde{P}^{\mathrm{T}}(t)\rho(t))^2) + \tilde{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))P^{\mathrm{T}}\rho(t)$ and $o((\tilde{P}^{\mathrm{T}}(t)\rho(t))^2)$ is a remainder term, which is a second higher order infinitesimal of $\tilde{P}^{\mathrm{T}}(t)\rho(t)$.

Remark 3 It should be noted that the Taylor series expansion of $\phi(\chi, P^{\mathrm{T}}\rho(t))$ with respect to $(\chi, \hat{P}^{\mathrm{T}}\rho(t))$ is used in eq. (20), which can be described as

$$
\begin{aligned}
\phi(\chi, P^{\mathrm{T}}\rho(t)) = \ &\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) + \phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\tilde{P}^{\mathrm{T}}(t)\rho(t) \\
&+ o((\tilde{P}^{\mathrm{T}}(t)\rho(t))^2).
\end{aligned} \tag{21}
$$

Using the relationships $\tilde{Q}(t) = Q - \hat{Q}(t)$, $\tilde{P}(t) = P - \hat{P}(t)$ and eq. (21), M can be rewritten as

$$
\begin{aligned}
M =&Q^{\mathrm{T}} o((\tilde{P}^{\mathrm{T}}(t)\rho(t))^2) + \tilde{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))P^{\mathrm{T}}\rho(t) \\
=&Q^{\mathrm{T}} o((\tilde{P}^{\mathrm{T}}(t)\rho(t))^2) + Q^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))P^{\mathrm{T}}\rho(t) \\
&- \hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))P^{\mathrm{T}}\rho(t) \\
=&Q^{\mathrm{T}}\phi(\chi, P^{\mathrm{T}}\rho(t)) - Q^{\mathrm{T}}\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) \\
&- Q^{\mathrm{T}}\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))(P^{\mathrm{T}} - \hat{P}^{\mathrm{T}}(t))\rho(t) \\
&+ Q^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))P^{\mathrm{T}}\rho(t) \\
&- \hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))P^{\mathrm{T}}\rho(t) \\
=&- \hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))P^{\mathrm{T}}\rho(t) \\
&+ Q^{\mathrm{T}}\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\hat{P}^{\mathrm{T}}(t)\rho(t) \\
&+ Q^{\mathrm{T}}[\phi(\chi, P^{\mathrm{T}}\rho(t)) - \phi(\chi, \hat{P}^{\mathrm{T}}\rho(t))].
\end{aligned} \tag{22}
$$

Substituting eq. (22) into eq. (20), one gets

$$
\begin{aligned}
&Q^{\mathrm{T}}\phi(\chi, P^{\mathrm{T}}\rho(t)) - \hat{Q}^{\mathrm{T}}(t)\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) \\
=&\hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\tilde{P}^{\mathrm{T}}(t)\rho(t) + \tilde{Q}^{\mathrm{T}}(t)[\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) \\
&- \phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\hat{P}^{\mathrm{T}}(t)\rho(t)] - \hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))P^{\mathrm{T}}\rho(t) \\
&+ Q^{\mathrm{T}}\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\hat{P}^{\mathrm{T}}(t)\rho(t) \\
&+ Q^{\mathrm{T}}[\phi(\chi, P^{\mathrm{T}}\rho(t)) - \phi(\chi, \hat{P}^{\mathrm{T}}\rho(t))] \\
\leq&\hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\tilde{P}^{\mathrm{T}}(t)\rho(t) + \tilde{Q}^{\mathrm{T}}(t)[\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) \\
&- \phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\hat{P}^{\mathrm{T}}(t)\rho(t)] \\
&+ \|P\|\|\hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\rho(t)\| \\
&+ \|Q\|\|\hat{P}^{\mathrm{T}}(t)\rho(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\| + \|Q\|_1.
\end{aligned} \tag{23}
$$

Noting that every element of $\phi(\chi, P^{\mathrm{T}}\rho(t)) - \phi(\chi, \hat{P}^{\mathrm{T}}\rho(t))$ is bounded by one, one has $Q^{\mathrm{T}}[\phi(\chi, P^{\mathrm{T}}\rho(t)) - \phi(\chi, \hat{P}^{\mathrm{T}}\rho(t))] \leq \|Q\|_1$, which has been used in the derivation of eq. (23).

From eq. (23) and using Lemma 2, K_2 can be deduced as

$$
\begin{aligned}
K_2 =& \int_0^l \chi(\mathfrak{s}, t)[Q^{\mathrm{T}}\phi(\chi, P^{\mathrm{T}}\rho(t)) - \hat{Q}^{\mathrm{T}}(t)\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))]\mathrm{d}\mathfrak{s} \\
\leq& \int_0^l \chi(\mathfrak{s}, t)\tilde{Q}^{\mathrm{T}}(t)[\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) \\
&- \phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\hat{P}^{\mathrm{T}}(t)\rho(t)]\mathrm{d}\mathfrak{s} \\
&+ \int_0^l \chi(\mathfrak{s}, t)\hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\tilde{P}^{\mathrm{T}}(t)\rho(t)\mathrm{d}\mathfrak{s} \\
&+ \|Q\|_1 \int_0^l |\chi(\mathfrak{s}, t)|\mathrm{d}\mathfrak{s} \\
&+ \|Q\|\|\hat{P}^{\mathrm{T}}(t)\rho(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\| \int_0^l |\chi(\mathfrak{s}, t)|\mathrm{d}\mathfrak{s} \\
&+ \|P\|\|\hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\rho(t)\| \int_0^l |\chi(\mathfrak{s}, t)|\mathrm{d}\mathfrak{s} \\
\leq& \int_0^l \chi(\mathfrak{s}, t)\tilde{Q}^{\mathrm{T}}(t)[\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) \\
&- \phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\hat{P}^{\mathrm{T}}(t)\rho(t)]\mathrm{d}\mathfrak{s} \\
&+ \int_0^l \chi(\mathfrak{s}, t)\hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\tilde{P}^{\mathrm{T}}(t)\rho(t)\mathrm{d}\mathfrak{s} \\
&+ (\|Q\|_1 + \|Q\|^2 + \|P\|^2) \int_0^l |\chi(\mathfrak{s}, t)|\mathrm{d}\mathfrak{s} \\
&+ (\|\hat{P}^{\mathrm{T}}(t)\rho(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\|^2 \\
&+ \|\hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\rho(t)\|^2) \int_0^l |\chi(\mathfrak{s}, t)|\mathrm{d}\mathfrak{s}.
\end{aligned} \tag{24}
$$

From $K_3 + K_4$, one can get

$$
K_3 + K_4 = \int_0^l \lambda(\chi, t)\chi(\mathfrak{s}, t)\mathrm{d}\mathfrak{s} \leq \lambda^* \int_0^l |\chi(\mathfrak{s}, t)|\mathrm{d}\mathfrak{s}, \tag{25}
$$

where $\lambda(\chi, t) = \xi(\chi, \varphi(t)) + Q^{\mathrm{T}}[\phi(\chi, P^{\mathrm{T}}\rho(t) + \varepsilon_1(t)) - \phi(\chi, P^{\mathrm{T}}\rho(t)]$ and $\lambda^* > |\lambda(\chi, t)|$ is an unknown constant.

Remark 4 It is necessary to state that $\lambda(\chi, t)$ is bounded. The specific reasons are as follows: on the one hand, from the differential mean value theorem of multi-variable function, one has

$$
\begin{aligned}
&\phi_j(\chi, P^{\mathrm{T}}\rho(t) + \varepsilon_1(t)) - \phi_j(\chi, P^{\mathrm{T}}\rho(t) \\
&= \varepsilon_1(t)\frac{\partial \phi_j(\chi, \varsigma)}{\partial \varsigma}|_{\varsigma = P^{\mathrm{T}}\rho(t) + \sigma\varepsilon_1(t)},\ \sigma \in (0, 1).
\end{aligned}
$$

From eq. (5), we can easily deduce $|\frac{\partial \phi_j(\chi,\varsigma)}{\partial \varsigma}| \leq \frac{\sqrt{2e}}{ec_j}$. On the other hand, $|\xi(\chi, \varphi)| < \xi^*$. Thus, we can obtain $\lambda(\chi, t)$ is bounded, i.e., $|\lambda(\chi, t)| < \lambda^*$ with λ^* is an unknown constant.

Substituting eqs. (19), (24) and (25) into eq. (18), one has

$$
\begin{aligned}
D^+\mathbb{W}_1(t) \leq& -\frac{\pi^2}{4l^2} \int_0^l \chi^2(\mathfrak{s}, t)\mathrm{d}\mathfrak{s} \\
&+ \int_0^l \chi(\mathfrak{s}, t)\hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\tilde{P}^{\mathrm{T}}(t)\rho(t)\mathrm{d}\mathfrak{s} \\
&+ \int_0^l \chi(\mathfrak{s}, t)\tilde{Q}^{\mathrm{T}}(t)[\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) \\
&- \phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\hat{P}^{\mathrm{T}}(t)\rho(t)]\mathrm{d}\mathfrak{s}
\end{aligned}
$$

$$
\begin{aligned}
&+ (\|\hat{P}^{\mathrm{T}}(t)\rho(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\|^2 \\
&+ \|\hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\rho(t)\|^2) \int_0^l |\chi(\mathfrak{s}, t)| \mathrm{d}\mathfrak{s} \\
&+ (\|Q\|_1 + \|Q\|^2 + \|P\|^2 + \lambda^*) \int_0^l |\chi(\mathfrak{s}, t)| \mathrm{d}\mathfrak{s} \\
&- (w(t) + \hat{\nu}(t)) \int_0^l \chi(\mathfrak{s}, t) \mathrm{sgn}(\chi(\mathfrak{s}, t)) \mathrm{d}\mathfrak{s}. \qquad (26)
\end{aligned}
$$

Differentiating $\mathbb{W}_2(t)$ with the respect t, one has

$$
\dot{\mathbb{W}}_2(t) = -\frac{1}{\varpi_1}\tilde{Q}^{\mathrm{T}}(t)\dot{\hat{Q}}(t) - \frac{1}{\varpi_2}\tilde{P}^{\mathrm{T}}(t)\dot{\hat{P}}(t) - \frac{1}{\varpi_3}\tilde{\nu}(t)\dot{\hat{\nu}}(t). \qquad (27)
$$

Combing with eqs. (26), (27) and (10)–(12), one deduces that

$$
\begin{aligned}
D^+\mathbb{W}(t) \leq & -\frac{\pi^2}{4l^2}\int_0^l \chi^2(\mathfrak{s}, t)\mathrm{d}\mathfrak{s} + \nu \int_0^l |\chi(\mathfrak{s}, t)|\mathrm{d}\mathfrak{s} \\
& + \int_0^l \chi(\mathfrak{s}, t)\hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\tilde{P}^{\mathrm{T}}(t)\rho(t)\mathrm{d}\mathfrak{s} \\
& + \int_0^l \chi(\mathfrak{s}, t)\tilde{Q}^{\mathrm{T}}(t)[\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) \\
& - \phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\hat{P}^{\mathrm{T}}(t)\rho(t)]\mathrm{d}\mathfrak{s} + w(t)\int_0^l |\chi(\mathfrak{s}, t)|\mathrm{d}\mathfrak{s} \\
& - (w(t) + \hat{\nu}(t))\int_0^l \chi(\mathfrak{s}, t)\mathrm{sgn}(\chi(\mathfrak{s}, t))\mathrm{d}\mathfrak{s} \\
& - \frac{1}{\varpi_1}\tilde{Q}^{\mathrm{T}}(t)\dot{\hat{Q}}(t) - \frac{1}{\varpi_2}\tilde{P}^{\mathrm{T}}(t)\dot{\hat{P}}(t) - \frac{1}{\varpi_3}\tilde{\nu}(t)\dot{\hat{\nu}}(t) \\
\leq & -\frac{\pi^2}{4l^2}\int_0^l \chi^2(\mathfrak{s}, t)\mathrm{d}\mathfrak{s} \leq 0. \qquad (28)
\end{aligned}
$$

From eq. (28) and the Lyapunov stability theory, we obtain that $\lim_{t\to\infty}\|\chi(\mathfrak{s}, t)\|_2 = 0$, which implies that there exists a feedback controller (9) and adaptive laws (10)–(12) to ensure that the solution of the system (14) is asymptotically stable. Moreover, from eq. (28), we have $\mathbb{W}(t) \leq \mathbb{W}(0)$ is uniformly bounded for any bounded initial condition $\mathbb{W}(0)$, which implies that the control input u and estimations of adaptive parameters $\hat{Q}(t)$, $\hat{P}(t)$, $\hat{\nu}(t)$ are all bounded.

It should be noted that the discontinuous symbolic function $\mathrm{sgn}(\cdot)$ is introduced into the design of the feedback controller (9). Its appearance may cause the control input signal to vibrate, which makes the control result vibrate. In order to avoid this phenomenon, we use continuous hyperbolic tangent function $\tanh(\cdot)$ to replace the discontinuous symbolic function $\mathrm{sgn}(\cdot)$ in the controller that we design next. Then, the new feedback control algorithm can be designed as follows:

$$
\begin{aligned}
u(\mathfrak{s}, t) = & -\hat{Q}^{\mathrm{T}}(t)\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) \\
& - (\hat{\nu}(t) + w(t))\tanh\left(\frac{(\hat{\nu}(t) + w(t))\chi(\mathfrak{s}, t)}{h(t)}\right), \qquad (29)
\end{aligned}
$$

with the following adaptive laws:

$$
\begin{aligned}
\dot{\hat{Q}}(t) = \varpi_1 \Big\{ & \int_0^l \chi(\mathfrak{s}, t)[\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) \\
& - \phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\hat{P}^{\mathrm{T}}(t)\rho(t)]\mathrm{d}\mathfrak{s} - \beta_1 h(t)\hat{Q}(t)\Big\}, \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\hat{P}}(t) = \varpi_2 \Big[& \int_0^l \chi(\mathfrak{s}, t)\hat{Q}^{\mathrm{T}}(t)\phi'(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t))\rho(t)\mathrm{d}\mathfrak{s} \\
& - \beta_2 h(t)\hat{P}(t)\Big], \qquad (31)
\end{aligned}
$$

and

$$
\dot{\hat{\nu}}(t) = \varpi_3\left[\int_0^l |\chi(\mathfrak{s}, t)|\mathrm{d}\mathfrak{s} - \beta_3 h(t)\hat{\nu}(t)\right], \qquad (32)
$$

where $h(t)$ is a positive function and satisfies $\int_0^\infty h(t)\mathrm{d}t < \infty$ and β_1, β_2, β_3 are positive constant parameters that can be adjusted.

Remark 5 It should be pointed out that in the design of the control algorithm (29), on the one hand, continuous hyperbolic tangent function $\tanh(\cdot)$ is used to replace the discontinuous symbol function $\mathrm{sgn}(\cdot)$ to avoid the tremor phenomenon of the control input signal. On the other hand, we introduce the positive integral bounded function $h(t)$, which can compensate the error between the symbol function $\mathrm{sgn}(\cdot)$ and the hyperbolic tangent function $\tanh(\cdot)$, so that the system under the control algorithm (29) can be asymptotically stable rather than ultimately uniformly bounded.

Substituting eqs. (8) and (29) into eq. (1), one gets the following closed-loop system:

$$
\begin{cases}
\chi_t(\mathfrak{s}, t) = \chi_{\mathfrak{s}\mathfrak{s}}(\mathfrak{s}, t) + Q^{\mathrm{T}}\phi(\chi, P^{\mathrm{T}}\rho(t) + \varepsilon_1(t)) \\
\qquad + \xi(\chi, \varphi(t)) - \hat{Q}^{\mathrm{T}}(t)\phi(\chi, \hat{P}^{\mathrm{T}}(t)\rho(t)) \\
\qquad - (\hat{\nu}(t) + w(t))\tanh\left(\frac{(\hat{\nu}(t)+w(t))\chi(\mathfrak{s},t)}{h(t)}\right), \\
\chi(\mathfrak{s}, 0) = \chi_0(\mathfrak{s}), \\
\chi(0, t) = \chi(l, t) = 0.
\end{cases} \qquad (33)
$$

Theorem 2. There exists a feedback controller (29) and adaptive laws (30)–(32) to ensure that the solution of system (33) is asymptotically stable, i.e., $\lim_{t\to\infty}\|\chi(\mathfrak{s}, t)\|_2 = 0$. In addition, the control input u and estimations of adaptive parameters $\hat{Q}(t)$, $\hat{P}(t)$, $\hat{\nu}(t)$ are all bounded.

Proof. Construct the candidate Lyapunov function for eq. (33) as follows:

$$
\begin{aligned}
\mathbb{W}(t) = & \frac{1}{2}\int_0^l \chi^2(\mathfrak{s}, t)\mathrm{d}\mathfrak{s} + \frac{1}{2\varpi_1}\tilde{Q}^{\mathrm{T}}(t)\tilde{Q}(t) + \frac{1}{2\varpi_2}\tilde{P}^{\mathrm{T}}(t)\tilde{P}(t) \\
& + \frac{1}{2\varpi_3}\tilde{\nu}^2(t). \qquad (34)
\end{aligned}
$$

Differentiating $\mathbb{W}(t)$ along eq. (33) with the respect t and combing with Theorem 1 and eqs. (30)–(32), one can deduce

$$\begin{aligned}\dot{\mathbb{W}}(t) \leq & -\frac{\pi^2}{4l^2}\int_0^l \chi^2(\mathfrak{s},t)\mathrm{d}\mathfrak{s} + (\hat{\nu}(t)+w(t))\int_0^l |\chi(\mathfrak{s},t)|\mathrm{d}\mathfrak{s} \\ & -(\hat{\nu}(t)+w(t))\int_0^l \chi(\mathfrak{s},t)\tanh(\frac{(\hat{\nu}(t)+w(t))\chi(\mathfrak{s},t)}{h(t)})\mathrm{d}\mathfrak{s} \\ & +\beta_1 h(t)\tilde{Q}^{\mathrm{T}}(t)\hat{Q}(t)+\beta_2 h(t)\tilde{P}^{\mathrm{T}}(t)\hat{P}(t)+\beta_3 h(t)\tilde{\nu}(t)\hat{\nu}(t).\end{aligned} \tag{35}$$

Using the relationships $\tilde{\alpha}^{\mathrm{T}}(t)\hat{\alpha}(t) = \frac{1}{2}\alpha^{\mathrm{T}}(t)\alpha(t) - \frac{1}{2}\hat{\alpha}^{\mathrm{T}}(t)\hat{\alpha}(t) - \frac{1}{2}\tilde{\alpha}^{\mathrm{T}}(t)\tilde{\alpha}(t)$ and $0 \leq |A| - A\tanh(\frac{A}{F}) \leq \varepsilon_3 F(\varepsilon_3 = 0.2785)$ [42] to eq. (35), one obtains

$$\begin{aligned}\dot{\mathbb{W}}(t) \leq & -\frac{\pi^2}{4l^2}\int_0^l \chi^2(\mathfrak{s},t)\mathrm{d}\mathfrak{s} + \varepsilon_3 h(t) + \frac{\beta_3}{2}h(t)\nu^2 \\ & +\frac{\beta_1}{2}h(t)Q^{\mathrm{T}}Q + \frac{\beta_2}{2}h(t)P^{\mathrm{T}}P \\ = & -\frac{\pi^2}{4l^2}\int_0^l \chi^2(\mathfrak{s},t)\mathrm{d}\mathfrak{s} + \mu h(t),\end{aligned} \tag{36}$$

where $\mu = \varepsilon_3 + \frac{1}{2}(\beta_3\nu^2 + \beta_1 Q^{\mathrm{T}}Q + \beta_2 P^{\mathrm{T}}P)$.

From eq. (36), one can further obtain

$$\mathbb{W}(t) + \frac{\pi^2}{4l^2}\int_0^t\int_0^l \chi^2(\mathfrak{s},t)\mathrm{d}\mathfrak{s}\mathrm{d}r \leq \mathbb{W}(0) + \mu\int_0^t h(r)\mathrm{d}r. \tag{37}$$

From the definition of $h(t)$, we obtain that $\mu\int_0^t h(r)\mathrm{d}r$ is bounded. According to Barbalats Lemma [43], we can have $\lim_{t\to\infty}\|\chi(\mathfrak{s},t)\|_2 = 0$, which means that the feedback controller (29) with adaptive laws (30)–(32) can make the solution of system (33) asymptotically stable. Moreover, from the boundedness of $\mathbb{W}(t)$, we can easily get the boundedness of $\hat{\nu}(t)$, $\hat{Q}(t)$, $\hat{P}(t)$ and u.

Remark 6 In this work, because the uncertain PDEs with unknown NPTVP are studied in this paper, there will be an unmeasurable PTVP in the signal of the approximator if the NNs are used to directly approximate the nonlinear system. Therefore, the NNs cannot be used to directly approximate the unknown system with NPTVP. In order to solve this problem, we need to take the following two steps: Firstly, we use FSE technology to reconstruct the unmeasurable PTVP. Then, the reconstruction parameter is used as the new input signal of the NNs to describe the nonlinear periodic time-varying uncertain DPSs.

Remark 7 From the aspect of the researched system, unlike most existing results, the unknown parameters in their studied systems are mostly constant or linear time-varying ones. However, this article studies the DPSs with NPTVP, which has a wider application in actual engineering modeling. For example, it has been widely used in practical engineering research fields such as aerospace engineering and machine tool mechanical control.

Remark 8 Due to the existence of nonlinear TVP, this work cannot directly use NNs or FLSs as an approximator to approximate nonlinear terms like traditional methods. Instead, the unmeasured PTVP is first reconstructed by FSE. Then, the reconstructed new parameter is used as the new input signal of NNs or FLSs to describe the term with NPTVP. Based on the NNs technology and reparameterization approach, two control algorithms are designed to make the uncertain parabolic DPSs with NPTVP asymptotically stable.

4 Simulation result

A numerical example is introduced to verify the effectiveness of the two previously designed algorithms. Consider the following nonlinear periodic time-varying DPSs described as

$$\begin{cases}\chi_t(\mathfrak{s},t) = \chi_{\mathfrak{s}\mathfrak{s}}(\mathfrak{s},t) + 10\chi(\mathfrak{s},t) + \sin(\chi(\mathfrak{s},t)) \\ \qquad + 2\cos(2\pi t)\chi(\mathfrak{s},t) + u(\mathfrak{s},t), \mathfrak{s} \in (0,l), \\ \chi(\mathfrak{s},0) = \chi_0(\mathfrak{s}), \\ \chi(0,t) = \chi(l,t) = 0,\end{cases} \tag{38}$$

where the initial state value is selected as $\chi_0(\mathfrak{s}) = 5\sin(\pi\mathfrak{s})$ and set $l = 1$. Then, we give the state evolution of the system (38) with $u(\mathfrak{s},t) = 0$ as Figure 2. Figure 2 clearly presents that the open-loop system of eq. (38) at the equilibrium plane $\chi(\mathfrak{s},t) = 0$ is unstable.

Next, through the system (38), we verify the effectiveness of the control algorithms (9) with eqs. (10)–(12) and (29) with eqs. (30)–(32), respectively.

Case I For the control algorithm (9) with eqs. (10)–(12), the Fourier series are chosen as

$$\rho_1(t) = 1,\ \rho_{(2i)}(t) = \sqrt{2}\sin(it),$$

$$\rho_{(2i+1)}(t) = \sqrt{2}\cos(it),\ i = 1,2,3,4,$$

and the NNs basic function is set as

$$\phi_j(\chi,\varphi) = \exp\{-(10\chi - s_j^1)^2 - 10(\varphi - s_j^2)^2\}, j = 1,2,\cdots,6,$$

where $s_1^1 = s_1^2 = 0.2$, $s_2^1 = s_2^2 = 0.3$, $s_3^1 = s_3^2 = 0.4$, $s_4^1 = s_4^2 = 0.6$, $s_5^1 = s_5^2 = 0.7$ and $s_6^1 = s_6^2 = 0.8$. Select the parameters $\varpi_1 = 4, \varpi_2 = 5, \varpi_3 = 0.1$. Then, we get the simulation results under the control algorithm (9) with eqs. (10)–(12), which are shown as Figures 3 and 4.

From Figure 3, it can be seen that the closed-loop system of eq. (38) achieves an asymptotically stable result under the control of algorithm (9). In other words, the control algorithm (9) with eqs. (10)–(12) can make the system (38) asymptotically stable. In addition, through the evolutions of the trajectories as in Figure 4(a)–(d), it can be found that the control input $u(\mathfrak{s},t)$ and the adaptive parameter estimates $\hat{\nu}(t)$, $\hat{Q}(t)$, $\hat{P}(t)$ are all bounded, which are consistent with the results of Theorem 1.

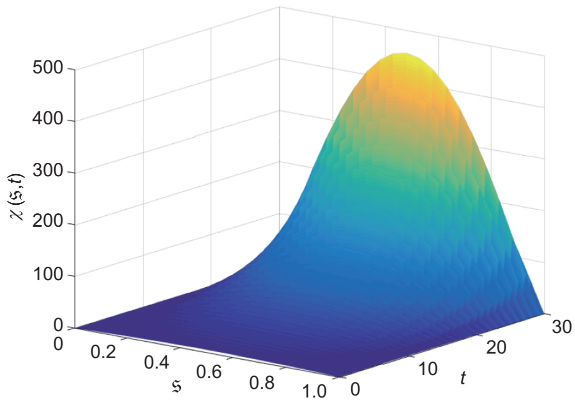

Figure 2 The evolution of open-loop system (38).

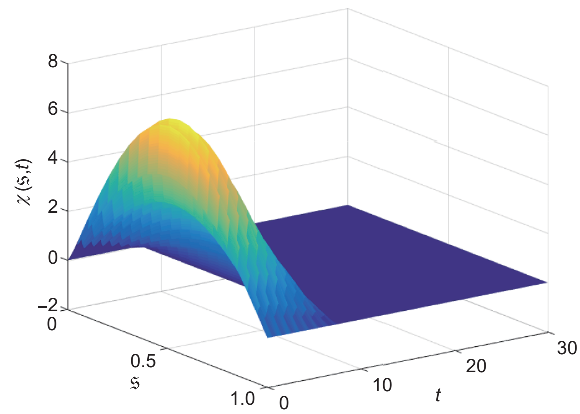

Figure 3 The evolution of closed-loop system (38) under the control algorithm (9).

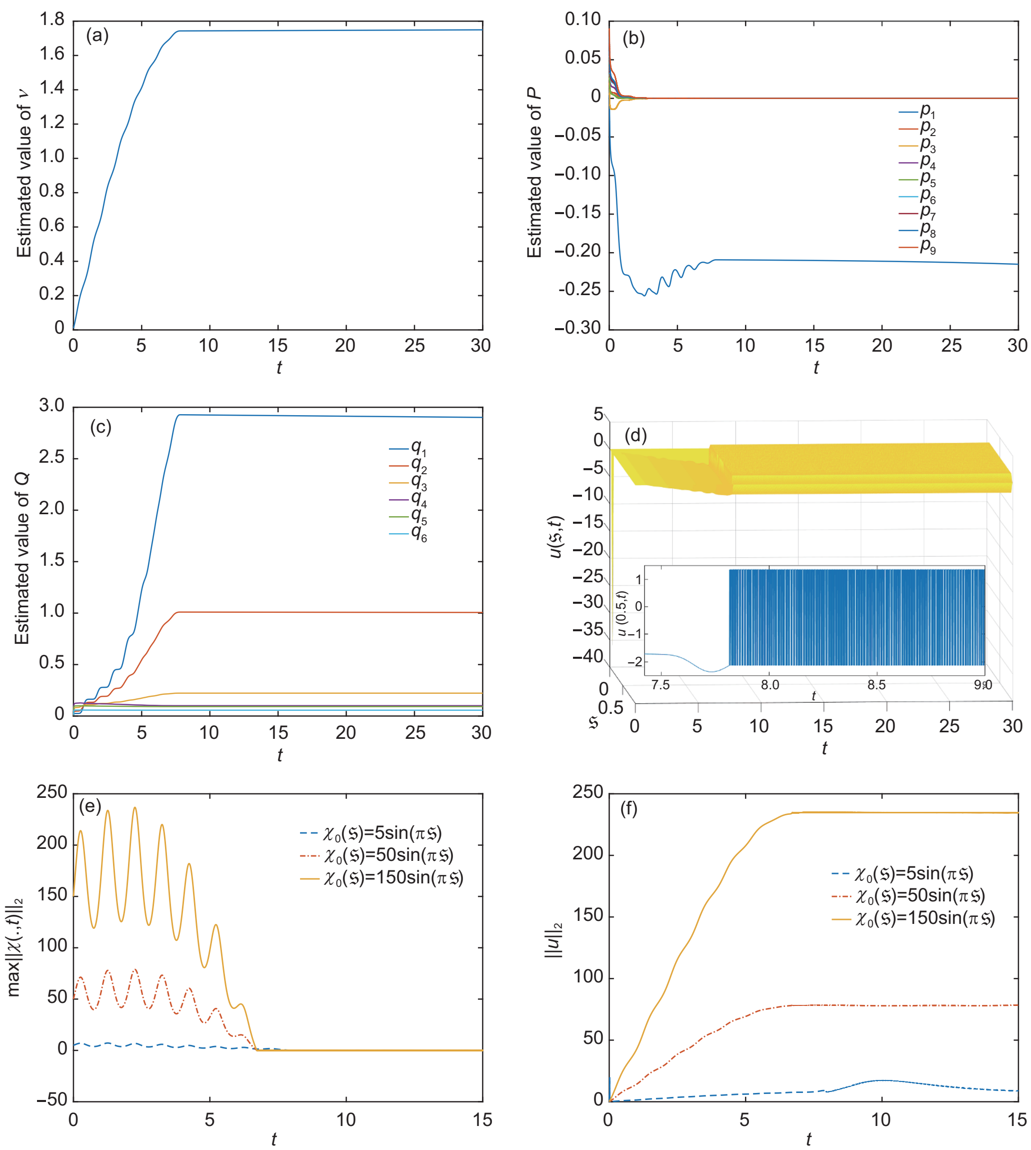

Figure 4 (a)–(c) Estimates of v, P, Q under the control algorithm (9); (d) the control input $u(\mathfrak{s}, t)$; (e), (f) the evolutions of $\max\|\chi(\cdot, t)\|_2$ and $\|u\|_2$ under the control algorithm (9).

From Figure 4(e), it can be clearly seen that under the control algorithm (9), the selection of the initial value of the state has little effect on the convergence speed of the closed-loop system. However, it can be clearly seen from Figure 4(f) that the larger the selected initial value of the state, the larger the control input to be applied.

Remark 9 From Figure 4(d), we can clearly find that the control input signal under the control algorithm (9) with eqs. (10)–(12) has a chattering phenomenon, which is caused by a discontinuous sign function sgn(·). In addition, in order to clearly see the chattering phenomenon of the control signal, we give the evolution trend of $u(0.5, t)$, which shows that the control signal appears obvious chattering phenomenon after about $t = 7.85$.

Case II For the control algorithm (29) with eqs. (30)–(32), the choice of Fourier series and NNs basic function are same as Case I and set $h(t) = \frac{1}{1+t^2}$. Select the parameters $\varpi_1 = 4, \varpi_2 = 5, \varpi_3 = 0.1$ and $\beta_1 = 1, \beta_2 = 1, \beta_3 = 0.01$. Then, we get the simulation results under the control algorithm (29) with eqs. (30)–(32), which are shown as Figures 5 and 6.

From Figure 5, it can be seen that the closed-loop system of (38) achieves an asymptotically stable result under the control of algorithm (29) with eqs. (30)–(32). In addition, through the evolutions of the trajectories as Figure 6(a)–(d), it can be found that the control input $u(\mathfrak{s}, t)$ and the adaptive parameter estimates $\hat{\nu}(t)$, $\hat{Q}(t)$, $\hat{P}(t)$ are all bounded, which are consistent with the results of Theorem 2.

From Figure 6(e), it can be clearly seen that under the control algorithm (29), the selection of the initial value of the state has little effect on the convergence speed of the closed-loop system. However, it can be clearly seen from Figure 6(f) that the larger the selected initial value of the state, the larger the control input to be applied.

Remark 10 Significantly, from Figure 6(d), we can

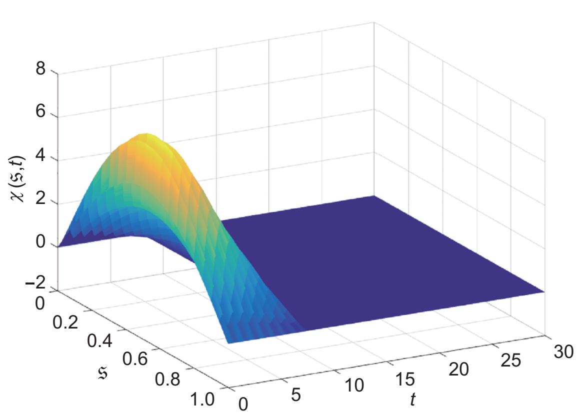

Figure 5 The evolution of closed-loop system (38) under the control algorithm (29).

see that the control algorithm (29) with (30)–(32) can effectively avoid the chattering phenomenon of control input signal $u(\mathfrak{s}, t)$, which is obtained by using continuous hyperbolic tangent function tanh (·) instead of discontinuous sign function sgn(·).

From Figures 3 and 5, we can get that both the control algorithm (9) with (10)–(12) and the control algorithm (29) with (30)–(32) can effectively make the system (38) asymptotically stable. In addition, through in Figures 4(d) and 6(d), we can find that compared with the control algorithm (9) with (10)–(12), the control algorithm (29) with (30)–(32) can effectively avoid control input signal $u(\mathfrak{s}, t)$ chattering, which is achieved by using continuous hyperbolic tangent function tanh (·) instead of discontinuous sign function sgn(·).

5 Conclusion

In this work, the problem of ANNC for uncertain parabolic DPSs with NPTVP has been studied, which is mainly based on the theories of adaptive control, NNs and FSE. Firstly, NNs and FSE are used to represent uncertain nonlinear dynamic system and unknown PTVP. Secondly, according to the ANNC approach and the reparameterization technique, a control algorithm is designed to make the uncertain parabolic DPSs with NPTVP asymptotically stable. However, due to the discontinuous sign function contained in the control algorithm, the chattering phenomenon of the control input signal may occur. Therefore, in order to avoid the above phenomenon, we further designed the other control algorithm. In this algorithm, on the one hand, we use continuous hyperbolic tangent function to replace discontinuous sign function. On the other hand, we introduce a positive integral bounded function in order to compensate the error between them and make the system asymptotically stable rather than ultimately uniformly bounded. Finally, a simulation is carried out to verify that both the two control algorithms can make the system asymptotically stable, and the second control algorithm can avoid the chattering phenomenon of the control input signal.

Based on the research in this work, we will mainly focus on the following two research directions in our future work. On the one hand, this work mainly studies the control problems of DPSs with a known period TVP. Thus, the authors can further consider the case of unknown period in future work. On the other hand, in this work, we only study the stabilization problem of a single system modeled by the uncertain parabolic DPS with nonlinear periodic time-varying parameter. The future work will be devoted to extending the method to the synchronization problem of complex dynamic networks or the consistency problem of multi-agent systems, which are modeled by DPSs.

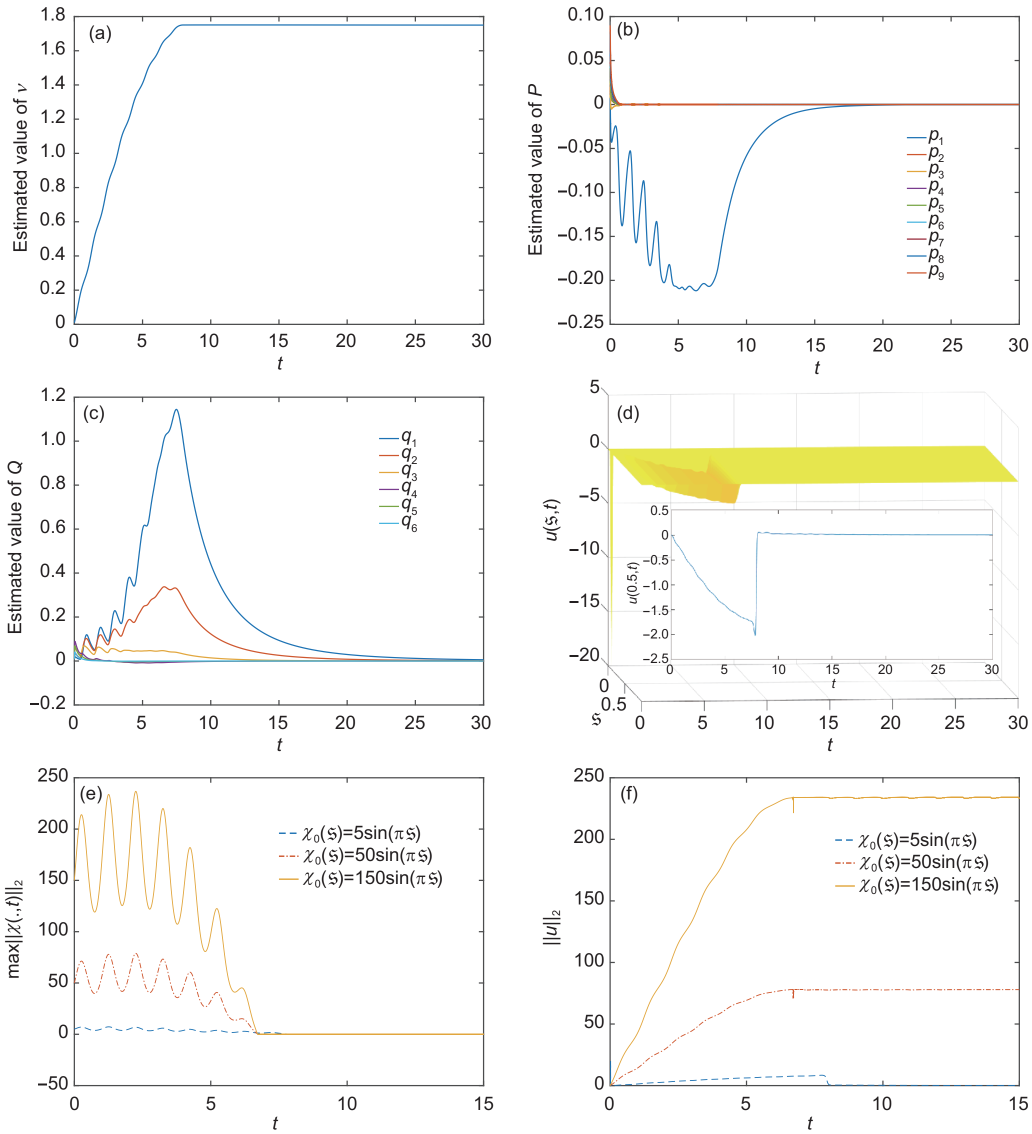

Figure 6 (a)–(c) Estimates of ν, P, Q under the control algorithm (29); (d) the control input $u(\mathfrak{s},t)$; (e), (f) the evolutions of $\max\|\chi(\cdot,t)\|_2$ and $\|u\|_2$ under the control algorithm (29).

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