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Mixed mode oscillations as well as the bifurcation mechanism in a Duffing's oscillator with two external periodic excitations

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The main purpose of the paper is to present an approach to account for the mechanism of bursting oscillations occurring in the systems with multiple periodic excitations. Since the traditional slow-fast analysis method can be used only for the systems with two scales in time domain, when there exists an order gap between the exciting frequencies and the natural frequency, how to explore the mechanism of the complicated dynamics remains an open problem, especially for the case when two exciting terms exist. To explain our approach, a relative simple Duffing's oscillator with two external periodic excitations is introduced as an example. For the case when one exciting frequency is integer times of the other exciting frequency, by employing Moivre's equation, the two exciting terms can be transformed into the functions of one basic periodic exciting term. Regarding the basic periodic exciting term as a slow-varying parameter, the two exciting terms can be changed into the functions of the slow-varying parameter, based on which the whole model can be transformed into a generalized autonomous system with one slow-varying parameter. Equilibrium branches as well as the related bifurcations of the generalized autonomous system can be derived with the variations of the amplitudes of the two excitations, which describes the relationship between the state variables of the generalized autonomous system and the slow-varying parameter. Considering the slow-varying parameter as a generalized state variable, one may obtain the so-called transformed phase portraits, which present the relationship between the state variable and the slow-varying parameter. The bifurcation mechanism of the mixed oscillations can be obtained by overlapping the equilibrium branches and the transformed phase portraits. Upon the approach, different types of bursting oscillations are presented. It is pointed that when the trajectory moves almost strictly along the segments of the stable equilibrium branches to the fold bifurcation points, jumping phenomena to other stable equilibrium segments can be observed, leading to the repetitive spiking oscillations, which can be approximated by the transient procedure from the bifurcation points to the stable equilibrium segments. Furthermore, because of the distributions of stable segments on the equilibrium branches of the generalized autonomous system may vary with the exciting amplitudes, the forms of bursting oscillations may change. When more fold bifurcations involve in the bursting attractors, more forms of quiescent states as well as spiking states exist in the oscillations, leading to different types of bursting oscillations.

multiple external periodic excitations, two scales in frequency domain, bursting oscillations, bifurcation mechanism

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1 Introduction

Since the relatively simple slow-fast H-H model was established, which may describe the behaviors of neuron activities, the dynamics of systems with multiple scales receives a lot of attentions [1–5], which often behaves in the combinations of relatively large-amplitude oscillations, called spiking states (SPs), and small-amplitude oscillations or at rest, known as quiescent states (QSs) [6]. The alternations between SPs and QSs form the special mixed-mode oscil-

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lations, called the bursting oscillations [7]. Because there exists no valid analytical method, most of the related reports were focused on the numerical simulations as well as the approximated solutions [8,9]. It was until the slow-fast analysis method was presented that researchers turn to the mechanism of the bursting oscillations [10,11]. The main idea of the method is to divide the slow-fast system into two subsystems, in which the fast subsystem dominates the states of QSs and SPs as well as the related bifurcations, while the slow subsystem moderates the associated movements [12,13]. Upon the method many types of bursting oscillations as well as the bifurcation mechanism were presented [1,14–16].

For a typical slow-fast dynamical system, described by

$$\dot{x} = f(x, y, \mu), \dot{y} = \varepsilon g(x, y, \mu), \tag{1}$$

where $x \in R^M$, $y \in R^N$, $\mu \in R^K$, when $0<\varepsilon<<1$, implying the state variables y changes on a much smaller time scales comparing with the state variables x, implying two scales exist in time domain.

Note that the slow-fast analysis method is valid only for the system with two scales in time domain and only one slow variable [17–19]. The bursting mechanism of the coupled system with two scales in frequency domain, implying an order gap exists between two frequencies related to the system, still remains a challenge [20]. Furthermore, the coexistences of multiple exciting terms with different frequencies may cause more complicated bursting behaviors [21].

In this paper, we try to present a new approach to investigate the bifurcation mechanism of the system. A relative simple Duffing's oscillator with two periodic external exciting terms is used as an example to show how to explore the bifurcation mechanism of the bursting oscillations when an order gap exists between the exciting frequencies and the natural frequency. By introducing Moivre's equation, it is pointed out that the two external exciting terms can be expressed in terms of one slow-varying parameter. Bifurcations of the generalized autonomous system with the variation of the slow-varying parameter can be used to account for different types of bursting oscillations appearing in the oscillator.

2 Mathematical model

The relatively simple Duffing's oscillator with two periodic exciting terms is introduced as an example. When the exciting frequencies are not far from the natural frequency, complicated phenomena may appear, such as chaotic oscillations and the time delay effect caused by the external excitations. In order to show more complicated behaviors, nonlinear terms up to fifth order are included in the oscillator, which can be written as

$$\ddot{x} + \mu\dot{x} + \alpha x + \beta x^3 + \lambda x^5 = w_1 + w_2, \tag{2}$$

where $w_1=A_1\cos(\omega_1 t)$ and $w_2=A_2\cos(k\omega_1 t)$ represent the two periodic exciting terms. For $\omega_1<<1$, $k\in N>>1$, while the other parameters are fixed at regular values, three scales in frequency domain can be found in the system, in which the largest scale correspond to the natural frequency Ω_N, $\Omega_N\in O(1.0)$.

Note that the state variables oscillate mainly according to the natural frequency, while in an arbitrary period related to the natural frequency, both the two exciting terms keeps almost constants, implying that all the two exciting terms can be regarded as slow-varying variables, denoted by w_1 and w_2.

It should be pointed out that for the case when only one exciting term corresponding to one slow-varying variable exist, our group developed a method to explore the mechanism of the oscillations with two scales in frequency domain, the results of which can be found in the series of publications [22,23]. The main contributions of the research work related to systems with two scales extend the traditional slow-fast method from time domain to frequency domain [24]. However, all the results are obtained with only one periodic external excitation.

A challenge arises that how to cope with the dynamics when two slowing varying variables exist in the oscillator, since the traditional slow-fast analysis method can be employed only for the case with one slow-varying variable.

Here we introduce a new scheme, the main idea of which is to express the two slowing varying variables in terms of one slow-varying variable by employing Moivre's equation. By regarding the slow-varying variable as a parameter, the original system can be considered as a generalized autonomous system. Based on the equilibrium branches as well as the related bifurcations of the generalized autonomous system, by employing the associated transformed phase portrait, different types of bursting oscillations as well as the mechanism are presented.

3 Description of the method

From Moivre's equation, which can be expressed as

$$(\cos x + \mathrm{i}\sin x)^n = \cos(nx) + \mathrm{i}\sin(nx), \tag{3}$$

where $\mathrm{i}=\sqrt{-1}$, one may obtain the following formula by balancing the real parts of two sides in eq. (3), written as

$$\begin{aligned}\cos(nx) = {} & C_n^0\cos^n x + C_n^2\cos^{(n-2)}x(\mathrm{i}\sin x)^2 \\ & + C_n^4\cos^{(n-4)}x(\mathrm{i}\sin x)^4 + \cdots \\ & + C_n^{2m}\cos^{(n-2m)}x(\mathrm{i}\sin x)^{2m},\end{aligned} \tag{4}$$

where $2m<n$. For example, when k=10, one may obtain

$$\cos(10\omega_1 t) = 512w^{10} - 1280w^8 + 1120w^6 \\ -400w^4 + 50w^2 - 1, \quad (5)$$

where $w=\cos(\omega_1 t)$. The two exciting terms in eq. (2) can be expressed in terms of one slow-varying parameter w, which implies that the only one slow-varying variable exists in the oscillator.

Regarding w as a parameter, one may derive the bifurcation details of the generalized autonomous system with the variation of w. In order to investigate the influence of the bifurcations on the dynamics, here we turn to the so-called transformed phase portrait.

For a traditional phase portrait, expressed as $\Pi : \{[x(t), \dot{x}(t)] | \forall t \in R\}$, which can show the relationships between different real state variables with the variation of t. However, the effect of the bifurcations related to w on the behaviors cannot be observed based on the traditional phase portrait. Therefore, the concept of transformed phase portrait, defined as

$\Pi_G : \{[x(t), \dot{x}(t), w] \equiv [x(t), \dot{x}(t), \cos(\omega_1 t)] | \forall t \in R\}$,

is introduced, which can describe the relationships between the real state variables and the slow-varying parameter w. The mechanism of different types of oscillations may be obtained via the combination of the transformed phase portraits with the bifurcations analysis associated.

4 Bifurcation of the generalized autonomous system

Regarding $w=\cos(\omega_1 t)$ as a bifurcation parameter, for high order external resonance with k=10, eq. (2) can be rewritten in the autonomous form with one slow-varying parameter as

$$\dot{x} = y, \dot{y} = -\mu y - \alpha x - \beta x^3 - \gamma x^5 + F, \quad (6)$$

where $F=A_1w+A_2(512w^{10}-1280w^8+1120w^6-400w^4+50w^2-1)$. The equilibrium of eq. (6) can be expressed as $E_0(x_0,0)$, where x_0 satisfies

$$-\alpha x_0 - \beta x_0^3 - \gamma x_0^5 + F = 0, \quad (7)$$

the stabilities of which can be determined by the associated characteristic equation, written in the form:

$$\lambda^2 + \mu\lambda + \alpha + 3\beta x_0^2 + 5\gamma x_0^4 = 0. \quad (8)$$

The equilibrium point E_0 is stable for $\alpha + 3\beta x_0^2 + 5d\gamma x_0^4 > 0$, which may lose the stability via fold bifurcation at

$$FB: \alpha + 3\beta x_0^2 + 5\gamma x_0^4 = 0, \quad (9)$$

leading to jumping phenomenon between different equilibrium points, while no Hopf bifurcation can be observed since the damping coefficient μ>0.

To reveal the fold bifurcation properties, we turn to the perturbation method. Assuming a fold bifurcation occurs at $w=w_0$, the following conditions are satisfied

$$\begin{cases} -\alpha x_0 - \beta x_0^3 - \gamma x_0^5 + F_0 = 0, \\ \alpha + 3\beta x_0^2 + 5\gamma x_0^4 = 0. \end{cases} \quad (10)$$

A small perturbation occurs on the slow-varying parameter w_0, implying $w=w_0+\kappa$, with $0<\kappa<<1$, resulting in $F_0 \to F_0 + \varepsilon$, which cause x_0 to change to $x_0+\delta$, where $F_0=F(w_0)$.

The first order approximation of ε can be expressed as $\varepsilon=\kappa \mathrm{d}F_0(w_0)/\mathrm{d}w$, while δ can be written in the form:

$$\delta = \varepsilon / (\alpha + 3\beta x_0^2 + 5\gamma x_0^4). \quad (11)$$

The first order approximation of the characteristic eq. (8) can be derived in the form:

$$\lambda^2 + \mu\lambda + (6\beta x_0 + 20\gamma x_0^3)\delta = 0. \quad (12)$$

Therefore, if $\left(6\beta x_0 + 20\gamma x_0^3\right)\delta > 0$, the equilibrium point is a node, while for $\left(6\beta x_0 + 20\gamma x_0^3\right)\delta < 0$, it is a saddle point. The fold bifurcation at $w=w_0$ is saddle-node bifurcation, resulting in the jumping phenomenon between different equilibrium points.

For the parameters fixed at μ=0.18, α=1.0, β=−2.0, γ=0.6, ω_1=0.001, one may obtain four special values of x_0 corresponding to the fold bifurcation from eq. (9), denoted by $X_{\pm 1}$=±0.4284 and $X_{\pm 2}$=±1.3478. Figure 1 gives the equilibrium branches as well as the related bifurcations with different values of exciting amplitudes at $A_1=A_2$=1.0, 2.0, 5.0, respectively.

Remark. Though the two exciting terms appear in linear forms in the oscillator, when the coupling of the two exciting terms is transformed in terms of one slow-varying parameter, denoted by F, it is nonlinear dependent on the slow-varying parameter w, which leads to different distribution of the equilibrium branches for different exciting amplitudes taken.

From Figure 1, one may find that with the increase of the exciting amplitude, the distribution of the equilibrium branches may change. For example, when $A_1=A_2$=1.0, shown in Figure 1(a), seven independent equilibrium branches, denoted by EB_i (i=1,2,...,7), can be observed with the variation of w. With the increase of the exciting amplitudes, they may interact with each other. When both the two exciting amplitudes increase to $A_1=A_2$=2.0, shown in Figure 1(b), the interaction between EB_2 and EB_3 in Figure 1(a) lead to a new EB_2 in Figure 1(b). Furthermore, EB_1 may interact with EB_4 in Figure 1(a), leading to a new form of EB_1 in Figure 1(b).

Further increase of the two exciting amplitudes may cause more equilibrium branches to combine together. When $A_1=A_2$=5.0, EB_1 merges not only with EB_2 but also with EB_3 in Figure 1(b), leading to a new equilibrium branch EB_1, which implies the equilibrium branches EB_i (i=1,2,...,5) in Figure 1(a) combines together, while only EB_6 and EB_7 keep

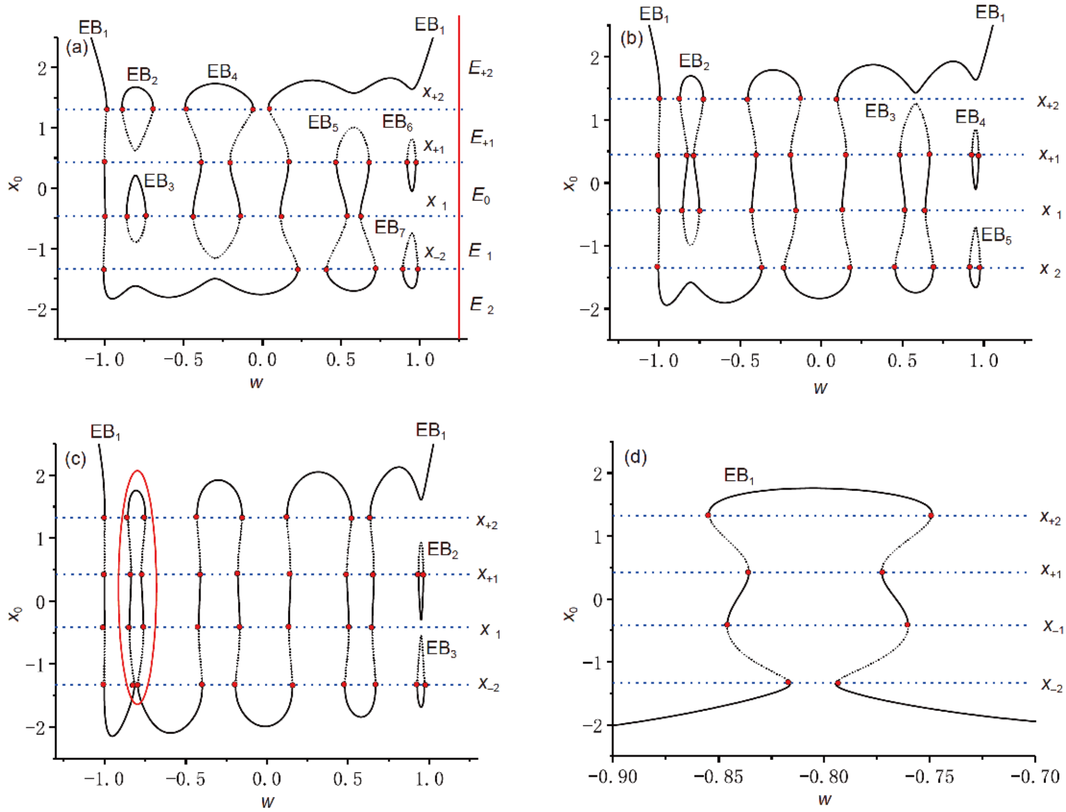

Figure 1 (Color online) Equilibrium branches as well as the bifurcations. (a) A_1=A_2=1.0; (b) A_1=A_2=2.0; (c) A_1=A_2=5.0; (d) The locally enlarged part of the equilibrium branch in (c) for A_1=A_2=5.0.

the original forms, denoted by EB_2 and EB_3 in Figure 1(c).

Remark. With the increase of the exciting amplitude, the equilibrium branches may change to pass across the lines x_0=$X_{\pm i}$ (i=1,2), corresponding to fold bifurcation values of x, which may lead to the combination between different equilibrium branches.

5 Evolution of the bursting oscillations

In the following, we focus on the influence of the equilibrium branches as well as the related bifurcations on the structures of attractors of the system with different values of exciting amplitudes.

5.1 Periodic 2-fold bursting oscillations

For A_1=A_2=1.0, the phase portraits of the oscillator on the (x, y) plane is presented in Figure 2(a), while the related time history of x is plotted in Figure 2(b), the local enlarged parts of which are shown in Figure 2(c) and 2(d), respectively.

From the phase portrait in Figure 2(a), one may find that the trajectory moves alternatively around two equilibrium points of the generalized autonomous system, while jumping phenomena occur at the two fold bifurcation points related to x=$X_{\pm 2}$. Three obvious scales, corresponding to three frequencies associated with T_i (i=0,1,2), respectively, which can be approximated at Ω_0=2π/T_0=0.001=ω_1, Ω_1=2π/T_1=0.01=10ω_1, Ω_2=2π/T_2=3.239 with T_0=6283.18, T_1=628.32, T_2=1.940.

To reveal the mechanism of the oscillations, we turn to transformed phase portrait, which is plotted in Figure 3(a) on the (w,x) plane, while the overlap of transformed phase portrait and the corresponding equilibrium branches as well as the related bifurcations is shown in Figure 3(b)–(d).

From the portrait on the (w,x) plane in Figure 3(a), one may find that the trajectory can be divided into four segments, corresponding to two quiescent states, denoted by $QS_\pm$, and two spiking states, represented by $SP_\pm$, respectively.

Assuming the trajectory starts at the point M_{-1}, at which the slow-varying variable w takes its minimum w=−1.0, it moves almost strictly along the stable equilibrium branch EB_1 in region E_{-2}, appearing in quiescent state QS_-. When the trajectory reaches the point M_{-2}, located on the line x=X_{-2}, fold bifurcation occurs, causing the trajectory to jump to the sole stable equilibrium branch EB_1 in region E_{+2}. Repetitive spiking oscillations SP_+ with relatively large amplitude take place. With further increase of w, the amplitudes of the oscillations decrease gradually and finally settle down to EB_1 at the point M_{+3}. The trajectory then

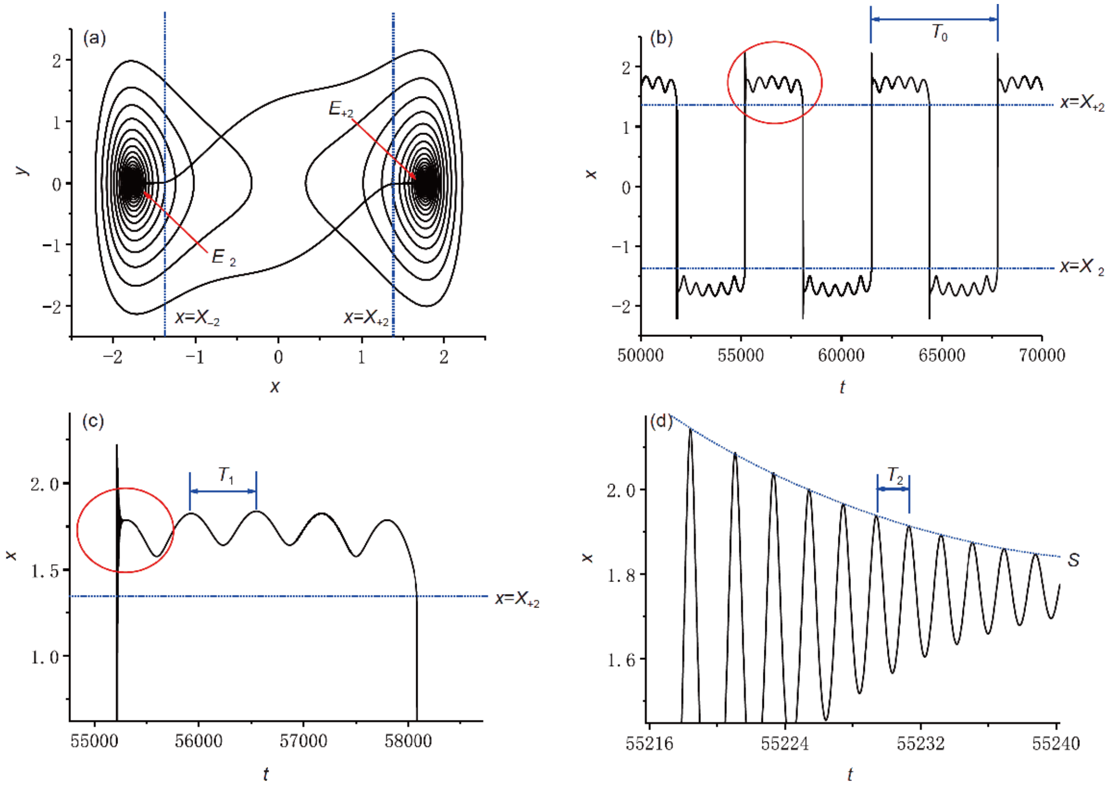

Figure 2 (Color online) Phase portrait on the (x,y) plane in (a) and related time history of x in (b) as well as the locally enlarged parts in (c) and (d).

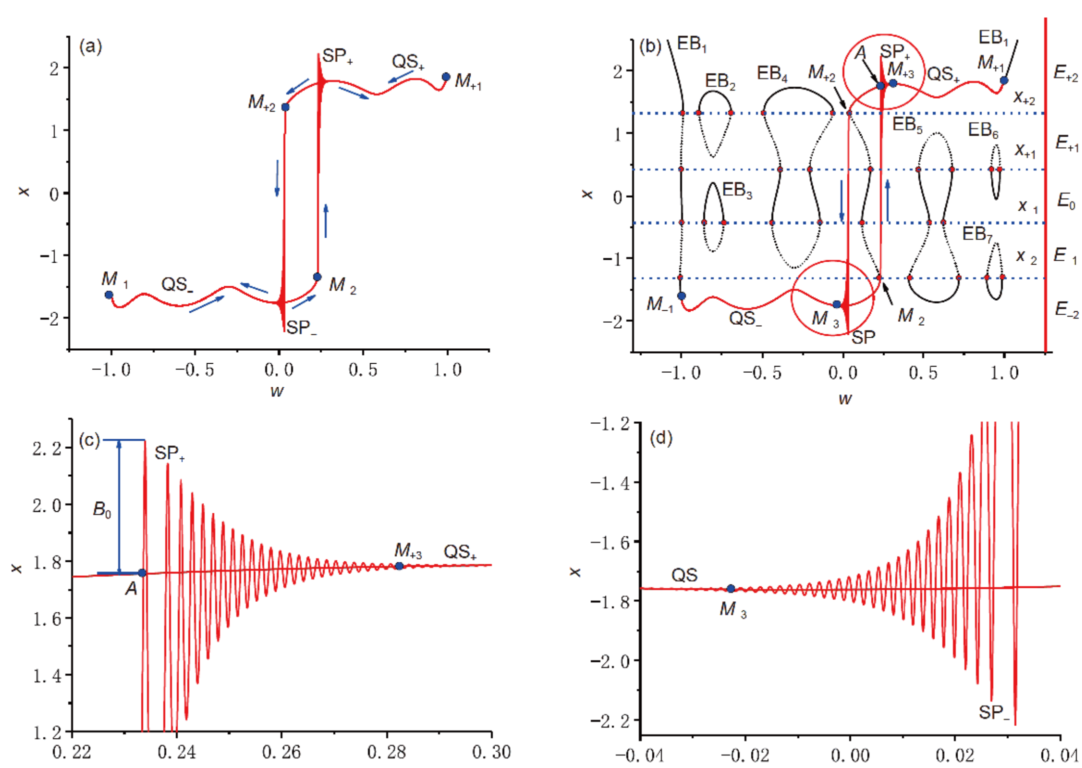

Figure 3 (Color online) Transformed phase portrait on the (w,x) plane in (a), the overlap of the phase portrait and equilibrium branches of the generalized autonomous system in (b), and the locally enlarged parts in (c) and (d).

moves almost strictly along the stable EB_1 in quiescent state QS_+ until it reaches the point M_{+1}, at which w takes its maximum value w=+1.0. Half period of the symmetric movement finishes.

With further increase of time, the trajectory turns to the left to begin the other half period of the movement, since the slow-varying variable w may decrease. The trajectory moves almost strictly along the stable equilibrium branch EB_1 in region E_{+2} until it reaches the point M_{+2}, at which fold bifurcation occurs, causing the trajectory to jump to the stable equilibrium branch EB_1 in region E_{-2}, leading to repetitive spiking state SP_{-}. The amplitudes of the oscillations decrease gradually and finally the trajectory settles down to the stable equilibrium branch EB_1 in region E_{-2} to begin the quiescent state QS_{-}. The trajectory moves almost strictly along EB_1 until it reaches the starting point M_{-1}, which finishes one period of the movement.

It can be found that the trajectory moves along the stable equilibrium branch EB_1 located in $E_{\pm 2}$, respectively, while two fold bifurcations cause the jumping phenomena between the equilibrium branch in the two different regions, leading to the alternation between quiescent states and spiking states. Therefore, it can be called symmetric periodic 2-fold bursting.

Remark. Note that the fold bifurcations cause the jumping phenomena between the two parts of the equilibrium branch, leading to spiking states. Therefore, the repetitive spiking oscillations can be approximated by the transient procedure from the bifurcation point to the associated equilibrium point. For example, the pair of complex conjugate eigenvalues of the equilibrium point $A(0.233, 1.755)$ (shown in Figure 3(c)) can be computed at $\lambda_{\pm}=-0.090\pm 3.312\mathrm{i}$, leading to that the transient procedure can be approximated by $X=B_0\exp(-0.090+3.312\mathrm{i})t$, with $B_0=0.465$. The oscillating frequency can be obtained at $\Omega_S=3.312$, which agrees very well with Ω_2, while the enveloping of the maximum amplitudes of the oscillations also agrees very well with the curve S, computed by $X=B_0\exp(-0.090)t$ in Figure 2(d).

5.2 Periodic 6-fold bursting oscillations

Now we increase the amplitudes of the two excitations to $A_1=A_2=2.0$, the equilibrium branches as well as the bifurcations of the corresponding generalized autonomous system can be observed in Figure 1(b). The related phase portrait on the (x,y) plane as well as the associated time history of x is presented in Figure 4, from which one may find that the trajectory also moves around two equilibrium point $E_{\pm 2}$ alternatively, connected by the jumping phenomena via fold bifurcations at $x=X_{\pm 2}$.

Three frequencies, corresponding to T_i (i=0,1,2) in Figure 4, involve the oscillations, which can also be computed at $\Omega_0=2\pi/T_0=0.001=\omega_1$, $\Omega_1=2\pi/T_1=0.01=10\omega_1$, $\Omega_2=2\pi/T_2=3.1765$, where Ω_0 and Ω_1 exactly equate to the two exciting frequency, while Ω_2 can be approximated by the imaginary parts of the pair of the complex conjugate eigenvalues related to equilibrium point $E_{\pm 2}$, computed at 3.1642.

We also use the transformed phase portrait on the (w,x) plane and the equilibrium branches as well as the related bifurcations of the generalized autonomous system to account

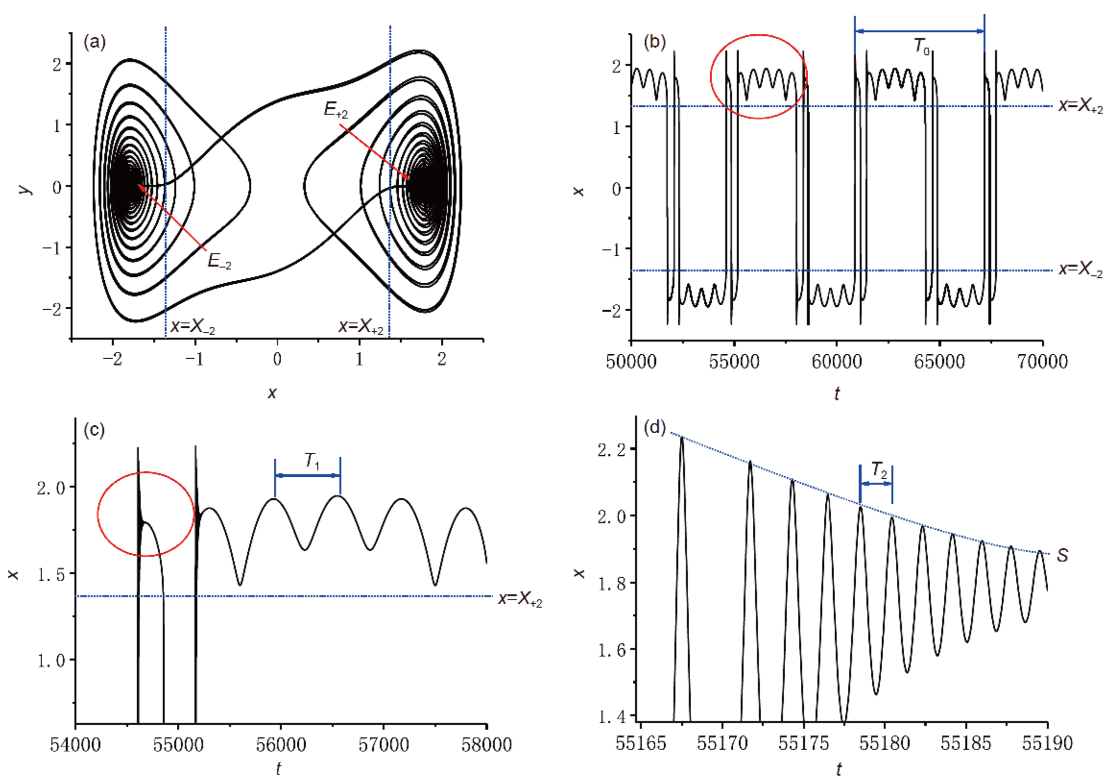

Figure 4 (Color online) Phase portrait on the (x,y) plane in (a) and related time history of x in (b) as well as the locally enlarged parts in (c) and (d).

for the mechanism of the oscillations, shown in Figure 5.

Unlike the bursting form in Figure 3, more quiescent states and spiking states as well as the connecting bifurcations appear in the attractor. The trajectory, starting from the point M_{-1}, shown in Figure 5(a), in quiescent state QS_{-1}, along the stable equilibrium branch EB_1 in E_{-2} region jump to the EB_1 in E_{+2} region at the fold bifurcation point M_{-2} with $x=X_{-2}$, leading to repetitive spiking state SP_{+1}. The trajectory finally settle down to the stable EB_1 in E_{+2} to start the quiescent state QS_{+1}, shown in Figure 5(c), which may jump to the stable equilibrium branch EB_1 in E_{-2} at the fold bifurcation point M_{+2}, leading to spiking state SP_{-2}. The trajectory finally settle down to stable EB_1 in E_{-2}, resulting in quiescent state QS_{-2}, shown in Figure 5(d), which jumps to the stable EB_1 in E_{+2}, yielding spiking state SP_{+3}. The trajectory finally settles down to EB_1 in E_{+2} to begin quiescent state QS_{+3}.

When the trajectory moves almost strictly along EB_1 in E_{+2} to the point M_{+1}, half period of the movement is finished. Further increase of time may cause the trajectory to begin the left half period of the movement, which is omitted here for simplicity, since the procedure is symmetric to the first half period of the movement.

It can be found that there exist six fold bifurcations, which cause the alternations between the quiescent states and spiking states. Therefore, the movement can be called symmetric periodic 6-fold bursting.

5.3 Periodic 14-fold bursting oscillations

Further increase of the exciting amplitudes may lead to more complicated structure of the bursting oscillations. Figure 6 gives the transformed phase portrait, and its overlap with the equilibrium branches for A_1=A_2=5.0, while the phase portrait on the (x,y) plane and the related time history is omitted here for simplicity.

Further investigation of the transformed phase portrait in Figure 6 reveals that the trajectory oscillates around the two parts of equilibrium branch EB_1, which are located in regions $E_{\pm 2}$, respectively. Fold bifurcations cause the trajectory moving almost strictly along the stable EB_1 in one region to jump to stable EB_1 in another region via repetitive spiking oscillations approximated by the transient procedure from the bifurcation point to the stable equilibrium branch.

For example, the trajectory starting from the point M_{-1}, shown in Figure 6(b), moves strictly along the stable equilibrium branch EB_1 in E_{-2} region, until it arrives at the point M_{-2}, at which fold bifurcation occurs, causing the trajectory to jump to the stable equilibrium branch EB_1 in E_{+2} region, leading to the repetitive spiking oscillations SP_{+1}, shown in Figure 7(a). The amplitudes of the oscillations gradually decrease with the evolution of time until the trajectory settles down to E_{+2} to start the quiescent state QS_{+1}. Then the trajectory moves almost strictly along E_{+2} until it reaches the

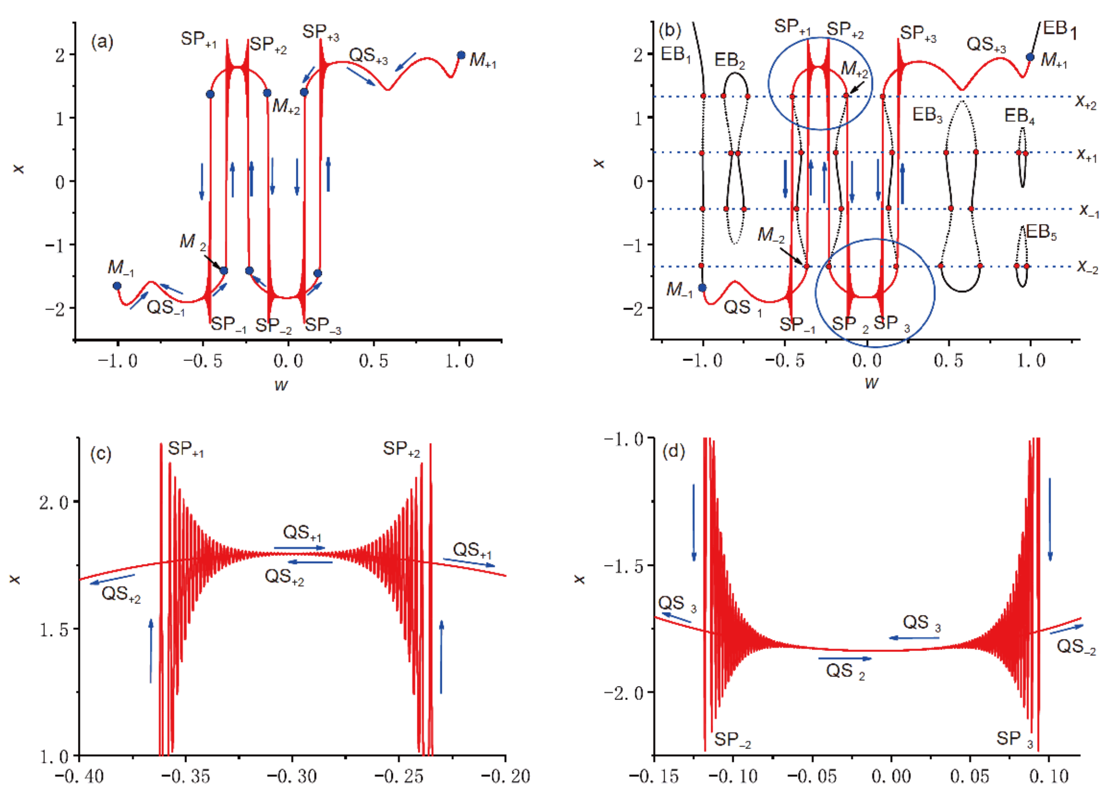

Figure 5 (Color online) Transformed phase portrait on the (w,x) plane in (a), the overlap of the phase portrait and equilibrium branches of the fast subsystem in (b), and the locally enlarged parts in (c) and (d).

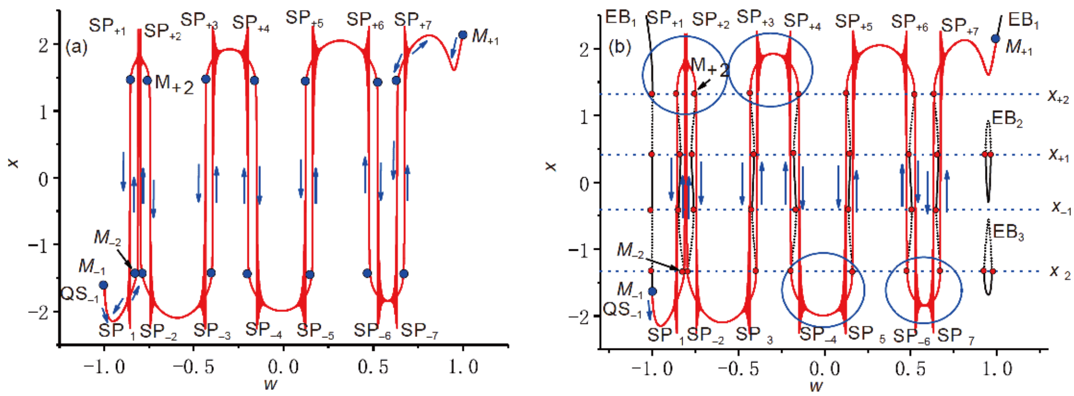

Figure 6 (Color online) Transformed phase portrait on the (w,x) plane in (a), together with its overlap with equilibrium branches as well as the bifurcations in (b).

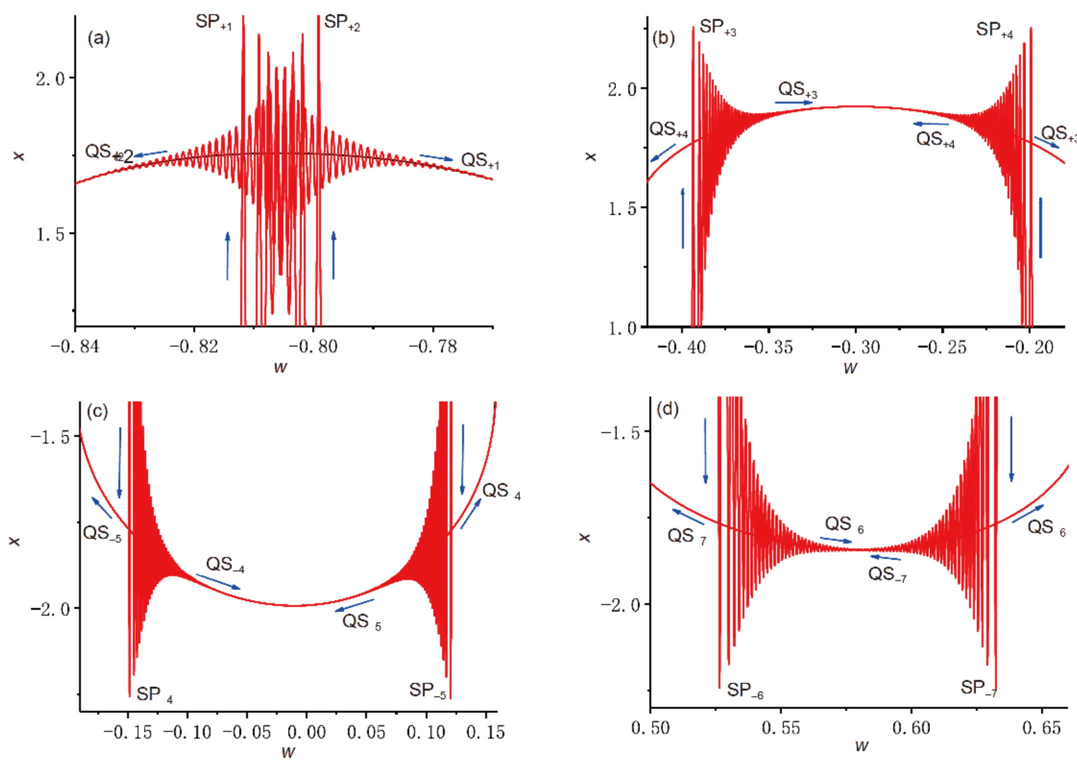

Figure 7 (Color online) Locally enlarged parts in Figure 6.

fold bifurcation point M_{+2}, at which the trajectory jumps to the stable equilibrium branch EB_1 in E_{-2} region.

The trajectory may jump between the E_{+2} and E_{-2} regions of the stable equilibrium branch EB_1 alternatively via fold bifurcations, causing the trajectory changes between different quiescent states and spiking states. Note that the trajectory in spiking oscillations caused by fold bifurcations may settle down to the stable equilibrium branch to begin the quiescent stages, the details can be found in Figure 7(b)–7(d).

Unlike the bursting oscillations above, there totally exist 14 fold bifurcations in the attractor, combining 7 forms of quiescent states and 7 types of spiking states, which may be called symmetric periodic 14-fold bursting.

Remark. (1) The different forms of bursting oscillations observed with different values of exciting amplitudes can be also understood from the equilibrium branches as well as the related bifurcations plotted in Figure 1. Note that all the bursting attractors may visit the stable equilibrium branch EB_1 in two regions $E_{\pm 2}$ in turn. With the increase of the exciting amplitudes, the equilibrium branch EB_1 may be divided into more stable segments, connecting with unstable segments. For example, when A_1=A_2=1.0, the stable EB_1 in the region E_{-2} appears in one segment, which may become two stable segments for A_1=A_2=2.0 and evolves to be four segments for A_1=A_2=5.0, leading to different number of fold bifurcations on the attractors.

(2) When the trajectory moves almost strictly along the

stable EB_1 in one region to the fold bifurcation points, jumping phenomena to the stable EB_1 in another region occur, leading to the repetitive spiking oscillations. Therefore, all the spiking states can be approximated by the transient procedures from the bifurcation points down to the related stable segments of the equilibrium branch EB_1, governed by the generalized autonomous system, from which one may obtain the evolutions of the amplitudes as well as the frequencies of the spiking oscillations.

6 Discussion and conclusions

An approach to account for the mechanism of bursting mechanism in the systems with two slow-varying periodic excitations is presented, the main idea of which is based on the scheme to transform the two exciting terms into functions of one basic exciting term. When an order gap between the exciting frequencies and the natural frequency exists, the effect of two time scales appears, leading to the bursting oscillations. By employing the approach described in the manuscript, a generalized autonomous system can be derived, regarding the basic exciting term as a slow-varying parameter. Equilibrium branches as well as bifurcations of the generalized autonomous system can be used to account for the mechanism of bursting oscillations. With the variation of the exciting amplitudes, the distributions of the equilibrium branches may change, resulting in different forms of bursting oscillations, the mechanism of which can be obtained based on the overlap of the transformed phase portraits and equilibrium branches of the generalized autonomous systems.

Unlike the bursting oscillations in the autonomous slow-fast system, when the frequencies of the exciting terms are far less than the natural frequency of the system, the exciting terms can be considered as slow-varying terms, which leads to the generalized autonomous system as the fast subsystem. New approaches can be presented based one the schemes to transform different slow-varying parameters into one slow-varying parameter, since the traditional slow-fast analysis method can be used only for the case with one slow-varying parameter.

Here we should point out that, the approach may be valid only for the cases when the two exciting frequencies behave in the exact resonant states. When perturbation occurs in the exciting frequencies, the time t may be considered as a slow-varying parameter, leading to another form of generalized autonomous system. Similarly, the mechanism of the bursting oscillations can be obtained accordingly.

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