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Synchronization criteria for multiple memristor-based neural networks with time delay and inertial term

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This present work uses different methods to synchronize the inertial memristor systems with linear coupling. Firstly, the mathematical model of inertial memristor-based neural networks (IMNNs) with time delay is proposed, where the coupling matrix satisfies the diffusion condition, which can be symmetric or asymmetric. Secondly, by using differential inclusion method and Halanay inequality, some algebraic self-synchronization criteria are obtained. Then, via constructing effective Lyapunov functional, designing discontinuous control algorithms, some new sufficient conditions are gained to achieve synchronization of networks. Finally, two illustrative simulations are provided to show the validity of the obtained results, which cannot be contained by each other.

memristor-based neural networks (MNNs), inertial term, synchronization, discontinuous control

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1 Introduction

To structure more better and more realistic model of neural networks, researchers have replaced the resistor with memristor to simulate the biological synapse. The results show that the memristive systems are more accurate models of artificial neural networks, since the memristors have properties of memory and nanoscale, just like the human brain. These all thanks to Prof. Chua and the researchers from Hewlett-Packard Laboratory, the former predicted the existence of memristor in ref. [1], and the latter established a prototype of memristor [2, 3] made by TiO_2 thin films. Therefore, the memristor has attracted widespread attentions. Due to the good properties of memristor, a variety of memristor-based neural networks (MNNs) have been constructed, which can mimic the biological brain as well as a variety of applications, such as chip-in-the-loop learning and image processing. In recent years, many scholar focused on the dynamic behaviors and circuit implementation of MNNs [4–9].

As we all know, synchronization, which can be seen as one of common and important phenomenon, has been widely applied in different areas including biology, ecology, sociology, and technology etc. In ref. [10], the authors had used fractional-order differential inequality and obtained the projective synchronization criteria for fractional-order MNNs which extended the synchronization results in ref. [11]. Ref. [12] investigated synchronization of multiple memristive neural networks, some algebraic synchronization criteria had been given and by designing novel adaptive controller, the adaptive synchronization of MNNs investigated in ref. [13]. The main important types of synchronization are drive-response synchronization [14] and self-synchronization of coupled networks. Pecora and Carrol [15] firstly proposed the drive-response law to synchronize chaos system. By

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designing a suitable feedback controller and adopting the theory of finite time stability theory, the upper bounds of the synchronization time of chaotic MNNs had been estimated [16]. Generally speaking, drive-response synchronization need to design suitable controllers [17–21], such as feedback controller, adaptive controller, intermittent controller, etc. Nowadays, most of scholars investigate the drive-response synchronization of MNNs, few works related to the coupled memristive model are investigated.

As it is pointed out in refs. [22, 23], semicircular canal of mammal and surface layer of hair cells could be achieved by integrated circuits, which include inductor. Koch [24] mentioned that neuron could be described as active membrane under some conditions, its behavior is like band pass filter, electric tuning, and its circuit implementation can be completed by adding an inductance. Hence, it is important to add inertial term (the second derivative of state) to neural networks, which not only can improve the performance of the optimized network, but also can reflect the characteristics of biological neural network. Babcock and Westervelt [25] firstly proposed the model of inertial neural networks, by introducing inertial term to the Hopfield neural networks with one or two neurons, and investigated the complicated behavior such as chaos. Hence, the neural networks with inertial term is an important research topic. Recently, many works related to the inertial neural networks had been reported [26–30]. In ref. [26], the time-varying delays was concerned, stability of delayed inertial BAM neural network have been conducted, and ref. [30] introduced the uncertain parameters to IMNNs, robust passivity was also studied by adopting inputs and outputs control.

Motivated by the aforementioned discussions, this paper deals with the synchronization problem for the coupled IMNNs with time delay. In this paper, two effective methods are utilized to synchronize the coupled IMNNs with time delay. The main novelty of this paper can be summarized as follows: (1) By introducing the second-order derivative of networks' states and linearly coupling topological structure into MNNs, a mathematical model of the IMNNs is established, which extends the traditional model; (2) The topology structure of the networks can be directed or undirected, and the coupling matrix could be symmetric or asymmetric; (3) Two different methods are considered, and different synchronization criteria are derived, respectively, in the form of matrix-measure and LMIs.

Notations. Throughout this paper, $\overline{\mathrm{co}}[\beth]$ is the closure of the convex hull of some set $\beth$. For any matrices A and B, $A<0$ represents real, symmetric and negative definite, and $A>0$ means positive definite respectively, $A \geq B$ ($A \leq B$) means that each element of A and B satisfies the inequality $a_{ij} \geqslant b_{ij} (a_{ij} \leqslant b_{ij})$. For scalar $\tau > 0$, $C([-\tau, 0]; \mathbb{R}^n)$ denotes the family of continuous functions φ from $[-\tau, 0]$ to $\mathbb{R}^n$. $\|\cdot\|_p$ means the p-norm of matrices, $p = 1, 2, \infty, \omega$. $B(x, \delta) = \{y : \|y - x\| \leqslant \delta\}$ is the ball with center x and radius δ.

2 Network model and preliminaries

Consider the isolated IMNNs with time delay, which can be described by the following equation:

$$\begin{aligned}\frac{\mathrm{d}^2 x_i(t)}{\mathrm{d}t^2} =& -D\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} - Cx_i(t) + A(x_i(t))f(x_i(t)) \\ &+ B(x_i(t))f(x_i(t-\tau(t))) + J(t),\end{aligned} \tag{1}$$

where $x_i(t) = (x_{i1}(t), \cdots, x_{in}(t))^{\mathrm{T}} \in \mathbb{R}^n$, $(i = 1, 2, \cdots, N)$ is the state variable of the ith dynamical node, the second derivative of $x_i(t)$ is called an inertial term of system eq. (1). $D = \mathrm{diag}(d_1, d_2, \cdots, d_n)$ and $C = \mathrm{diag}(c_1, c_2, \cdots, c_n) > 0$ are constant matrices, $A(x_i(t)) = [a_{kj}(x_{ij}(t))]_{n\times n}$ and $B(x_i(t)) = [b_{kj}(x_{ij}(t))]_{n\times n}$ denote feedback connection weight matrix and delayed connection memristive weight matrix, respectively. According to the feature of memristor, the memristor-based weights $a_{kj}(x_{ij}(t))$ and $b_{kj}(x_{ij}(t))$ satisfy the following conditions:

$$\begin{aligned}a_{kj}(x_{ij}(t)) &= \begin{cases}\acute{a}_{kj}, & |x_{ij}| < T_j,\\ \grave{a}_{kj}, & |x_{ij}| > T_j,\end{cases}\\ b_{kj}(x_{ij}(t)) &= \begin{cases}\acute{b}_{kj}, & |x_{ij}| < T_j,\\ \grave{b}_{kj}, & |x_{ij}| > T_j,\end{cases}\end{aligned} \tag{2}$$

in which $T_j > 0$ is the switching jump, $\acute{a}_{kj}$, $\grave{a}_{kj}$, $\acute{b}_{kj}$ and $\grave{b}_{kj}$ are all constants, $k, j = 1, 2, \cdots, n$. $J(t) = (J_1(t), \cdots, J_n(t))^{\mathrm{T}} \in \mathbb{R}^n$ represents the input vector, $\tau(t)$ corresponds to the time-varying transmission delay, which is nonnegative function with the upper bound τ. And $f(x_i(t)) = (f_1(x_{i1}(t)), \cdots, f_n(x_{in}(t)))^{\mathrm{T}}$ denotes the output of neuron unit, which are subject to the following assumption.

($\mathcal{H}_1$): For $\forall u \neq v \in \mathbb{R}$, there exist constants $l_i > 0$ and $k_i (i = 1, 2, \cdots, n)$ such that

$$0 \leqslant \frac{f_i(u) - f_i(v)}{u - v} \leqslant l_i, \quad |f_i(u)| \leqslant k_i$$

hold. In the sequel, denote $L = \mathrm{diag}(l_1, l_2, \cdots, l_n)$ for brevity. Moreover, in the switching point, it satisfies $f_i(\pm T_i) = 0$.

Since the memristive system eq. (1) is discontinuous, its solutions are considered in the Filippov's sense. Before proceeding, we will introduce the definitions about Filippov solution.

Definition 1. [31] Consider differential system $\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x)$, where $f(t, x)$ is not continuous. The Filippov solution of Cauchy problem for the discontinuous system is an absolutely continuous function, satisfies initial condition $x(0) =$

x_0 and the differential inclusion:

$$\frac{\mathrm{d}x}{\mathrm{d}t} \in F(t,x), \qquad \text{for a.e. } t \in [0,T],$$

where $F(t,x)$ is set-valued map of $f(t,x)$, which is defined as

$$F(t,x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{\mathrm{co}}[f(t, B(x,\delta)\backslash N)].$$

It should be noted that Aubin and Cellina [32] proposed a functional differential inclusion with memory defined as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t} \in F(t, x_t),$$

where $F : \mathbb{R}\times C([-\tau,0],\mathbb{R}^n) \to 2^{\mathbb{R}^n}$ is a given set-valued map, and $x_t(\theta) = x(t+\theta)$.

Lu and Chen [33] have extended the notion extend the Filippov solution to the case of delayed neural networks. By applying the theory, the IMNNs model eq. (1) can be rewritten as follows:

$$\begin{aligned}\frac{\mathrm{d}^2x_i(t)}{\mathrm{d}t^2} \in &- D\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} - Cx_i(t) + \overline{\mathrm{co}}[A(x_i(t))]f(x_i(t)) \\ &+ \overline{\mathrm{co}}[B(x_i(t))]f(x_i(t-\tau(t))) + J(t),\end{aligned} \tag{3}$$

where $\overline{\mathrm{co}}[A(x_i(t))] = [\overline{\mathrm{co}}[a_{kj}(x_{ij}(t))]]_{n\times n}$ and $\overline{\mathrm{co}}[B(x_i(t))] = [\overline{\mathrm{co}}[b_{kj}(x_{ij}(t))]]_{n\times n}$,

$$\overline{\mathrm{co}}[a_{kj}(x_{ij}(t))] = \begin{cases} \acute{a}_{kj}, & |x_{ij}| < T_j, \\ \left[\underline{a}_{kj}, \overline{a}_{kj}\right], & |x_{ij}| = T_j, \\ \grave{a}_{kj}, & |x_{ij}| > T_j, \end{cases}$$

$$\overline{\mathrm{co}}[b_{kj}(x_{ij}(t))] = \begin{cases} \acute{b}_{kj}, & |x_{ij}| < T_j, \\ \left[\underline{b}_{kj}, \overline{b}_{kj}\right], & |x_{ij}| = T_j, \\ \grave{b}_{kj}, & |x_{ij}| > T_j, \end{cases}$$

in which $\underline{a}_{kj} = \min\{\acute{a}_{kj}, \grave{a}_{kj}\}$, $\overline{a}_{kj} = \max\{\acute{a}_{kj}, \grave{a}_{kj}\}$, $\underline{b}_{kj} = \min\{\acute{b}_{kj}, \grave{b}_{kj}\}$, $\overline{b}_{kj} = \max\{\acute{b}_{kj}, \grave{b}_{kj}\}$, $a^+_{kj} = \max\{|\underline{a}_{kj}|, |\overline{a}_{kj}|\}$, $b^+_{kj} = \max\{|\underline{b}_{kj}|, |\overline{b}_{kj}|\}$. Or there exist $\pi_{kj}(x_{ij}(t)) \in \overline{\mathrm{co}}[a_{kj}(x_{ij}(t))]$ and $\varpi_{kj}(x_{ij}(t)) \in \overline{\mathrm{co}}[b_{kj}(x_{ij}(t))]$ such that

$$\begin{aligned}\frac{\mathrm{d}^2x_i(t)}{\mathrm{d}t^2} = &- D\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} - Cx_i(t) + \pi(x_i(t))f(x_i(t)) \\ &+ \varpi(x_i(t))f(x_i(t-\tau(t))) + J(t),\end{aligned} \tag{4}$$

where $\pi(x_i(t)) = [\pi_{kj}(x_{ij}(t))]_{n\times n}$ and $\varpi(x_i(t)) = [\varpi_{kj}(x_{ij}(t))]_{n\times n}$. Obviously, $\underline{A} \le \pi(x_i(t)) \le \overline{A}$, $\underline{B} \le \varpi(x_i(t)) \le \overline{B}$ with $\underline{A} = (\underline{a}_{kj})_{n\times n}$, $\underline{B} = (\underline{b}_{kj})_{n\times n}$, $\overline{A} = (\overline{a}_{kj})_{n\times n}$, $\overline{B} = (\overline{b}_{kj})_{n\times n}$.

Let $x_i(t)$ be the ith node, the multiple IMNNs with N coupled identical nodes can be composed as

$$\begin{aligned}\frac{\mathrm{d}^2x_i(t)}{\mathrm{d}t^2} = &- D\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} - Cx_i(t) + A(x_i(t))f(x_i(t)) \\ &+ B(x_i(t))f(x_i(t-\tau(t))) + J(t) \\ &+ c\sum_{j=1}^{N} G_{ij}\Gamma\left(\frac{\mathrm{d}x_j(t)}{\mathrm{d}t} + x_j(t)\right), \quad i = 1,2,\cdots,N,\end{aligned} \tag{5}$$

where c is a positive real number, denoting coupling strength, $\Gamma = \mathrm{diag}(\gamma_1,\gamma_2,\ldots,\gamma_n) \in \mathbb{R}^{n\times n}$ is the inner coupling matrix, $G = (G_{ij})_{N\times N}$ is the topological structure matrix, which is defined as: if there is no connection from node i to node j, then $G_{ij} = 0$, otherwise, $G_{ij} > 0 (j \neq i)$. Moreover, it satisfies the following condition:

$$G_{ii} = -\sum_{j=1, j\neq i}^{N} G_{ij}, \quad i = 1,2,\cdots,N.$$

The initial value associated with inertial network eq. (5) is given as $x_i(s) = \varphi_i(s)$, $\frac{\mathrm{d}x_i(s)}{\mathrm{d}s} = \psi_i(s)$, where $\varphi_i(s), \psi_i(s) \in C([-\tau,0];\mathbb{R}^n)$, and $i = 1,2,\ldots,N$.

From system eq. (3), one knows that the dynamics of the ith node in the coupled networks is given as

$$\begin{aligned}\frac{\mathrm{d}^2x_i(t)}{\mathrm{d}t^2} \in &- D\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} - Cx_i(t) + \overline{\mathrm{co}}[A(x_i(t))]f(x_i(t)) \\ &+ \overline{\mathrm{co}}[B(x_i(t))]f(x_i(t-\tau(t))) \\ &+ J(t) + c\sum_{j=1}^{N} G_{ij}\Gamma\left(\frac{\mathrm{d}x_j(t)}{\mathrm{d}t} + x_j(t)\right), \\ & i = 1,2,\cdots,N,\end{aligned} \tag{6}$$

or

$$\begin{aligned}\frac{\mathrm{d}^2x_i(t)}{\mathrm{d}t^2} = &- D\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} - Cx_i(t) + \pi(x_i(t))f(x_i(t)) \\ &+ \varpi(x_i(t))f(x_i(t-\tau(t))) + J(t) \\ &+ c\sum_{j=1}^{N} G_{ij}\Gamma\left(\frac{\mathrm{d}x_j(t)}{\mathrm{d}t} + x_j(t)\right), \quad i = 1,2,\cdots,N.\end{aligned} \tag{7}$$

By adopting the variable substitution

$$r_i(t) = \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} + x_i(t),$$

then the IMNNs model eq. (7) can be transformed as

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -x_i(t) + r_i(t), \\ \dfrac{\mathrm{d}r_i(t)}{\mathrm{d}t} = -\Theta x_i(t) - \Lambda r_i(t) + \pi(x_i(t))f(x_i(t)) \\ \qquad +\varpi(x_i(t))f(x_i(t-\tau(t))) + J(t) \\ \qquad +c\sum_{j=1}^{N} G_{ij}\Gamma r_j(t), \quad i = 1,2,\cdots,N, \end{cases} \tag{8}$$

where $\Theta = I + C - D$, $\Lambda = D - I$.

Denote $x(t) = (x_1^{\mathrm{T}}(t),\cdots,x_N^{\mathrm{T}}(t))^{\mathrm{T}}$, $r(t) = (r_1^{\mathrm{T}}(t),\cdots,r_N^{\mathrm{T}}(t))^{\mathrm{T}}$, $\boldsymbol{\Theta} = I_N \otimes \Theta$, $\mathbf{f}(x(t)) = [f^{\mathrm{T}}(x_1(t)),\cdots,$

$f^{\mathrm{T}}(x_N(t))]^{\mathrm{T}}$, $\mathbf{J}(t) = [J^{\mathrm{T}}(t),\cdots,J^{\mathrm{T}}(t)]^{\mathrm{T}}$, $\mathbf{\Lambda} = I_N \otimes \Lambda$, $\mathbf{A}(x(t)) = \mathrm{diag}(\pi(x_1(t)),\cdots,\pi(x_N(t)))$, $\mathbf{B}(x(t)) = \mathrm{diag}(\varpi(x_1(t)),\cdots,\varpi(x_N(t)))$, $\mathbf{G} = G\otimes\Gamma$, then the matrix form of system (8) can be written as

$$\begin{cases}\dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t)+r(t),\\ \dfrac{\mathrm{d}r(t)}{\mathrm{d}t} = -\mathbf{\Theta}x(t)-\mathbf{\Lambda}r(t)+\mathbf{A}(x(t))\mathbf{f}(x(t))\\ \qquad +\mathbf{B}(x(t))\mathbf{f}(x(t-\tau(t)))+J(t)+c\mathbf{G}r(t).\end{cases} \tag{9}$$

In order to exhibit the main results distinctly, some definitions and lemmas are introduced.

Definition 2. The coupled IMNNs model eq. (5) is said to be globally synchronized if $x_i(t)-x_j(t)\rightarrow 0$ as $t\rightarrow\infty$ holds, for any given initial conditions $\varphi_i(\cdot)$, $\psi_i(\cdot)$, where $i,j=1,2,\cdots,N$.

Definition 3. [34] For a real matrix $W=(w_{ij})_{n\times n}$, the matrix measure is denoted as

$$\mu_p(W)=\lim_{h\longrightarrow 0^+}\frac{\|I+hW\|_p-1}{h},$$

where $\|\cdot\|_p$ is the corresponding induced matrix norm, which has the nonnegative property, but matrix measure can be negative and $p=1,2,\infty$ or ω.

Remark 1. The corresponding matrix measures are obtained as: $\mu_1(W)=\max\limits_j\left\{a_{jj}+\sum\limits_{i=1,i\neq j}^{n}|a_{ij}|\right\}$, $\mu_2(W)=\frac{\lambda_{\max}(A^{\mathrm{T}}+A)}{2}$, $\mu_\infty(A)=\max\limits_i\left\{a_{ii}+\sum\limits_{j=1,j\neq i}^{n}|a_{ij}|\right\}$, $\mu_\omega(W)=\max\limits_j\left\{w_{jj}+\sum\limits_{i=1,i\neq j}^{n}\frac{\omega_i}{\omega_j}|w_{ij}|\right\}$, here, $w_i>0 (i=1,2,\ldots,n)$ are given any constant numbers.

Definition 4. [35] Suppose $G\in T(\mathbb{R},K)$ be a $N\times N$ matrix, then the $(N-1)\times(N-1)$ matrix H defined by $H=MGJ$ satisfies $MG=HM$, where M is the following matrix:

$$M=\begin{bmatrix}1 & -1 & 0 & \cdots & 0 & 0\\ 0 & 1 & -1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & -1\end{bmatrix}_{(N-1)\times N},$$

$$J=\begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ 0 & 1 & 1 & \cdots & 1\\ \vdots & \vdots & \ddots & \vdots & 1\\ 0 & 0 & \cdots & 1 & 1\\ 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix}_{N\times(N-1)},$$

and $T(\mathbb{R},K)$ is the set of matrices with entries in $\mathbb{R}$ such that the sum of the entries in each row is equal to K.

Lemma 1. The matrix norm $\|\cdot\|_p$ and matrix measure $\mu_p(\cdot)$ have the following basic properties:

(1) $-\|A\|_p\leqslant\mu_p(A)\leqslant\|A\|_p, \forall A\in\mathbb{R}^{n\times n}$;
(2) $\mu_p(\alpha A)=\alpha\mu_p(A), \forall\alpha>0, A\in\mathbb{R}^{n\times n}$;
(3) $\mu_p(A+B)\leqslant\mu_p(A)+\mu_p(B), \forall A,B\in\mathbb{R}^{n\times n}$.

Lemma 2. (Halanay inequality [36]) Let $u(t):[t_0-\tau,\infty)\rightarrow[0,\infty)$ be a continuous function, and for all $t\geqslant t_0$, we have

$$D^+u(t)\leqslant -au(t)+b\sup_{t-\tau\leqslant\theta\leqslant t}u(\theta).$$

If $a>b>0$, then

$$u(t)\leqslant\sup_{t_0-\tau\leqslant\theta\leqslant t_0}u(\theta)e^{-r(t-t_0)},\ t\geqslant t_0,$$

where $r>0$ is the unique positive solution of the equation $r-a+be^{r\tau}=0$.

Lemma 3. Given any real matrices X, Y and $Q>0$ with appropriate dimensions, then the following matrix inequality holds:

$$X^{\mathrm{T}}Y+Y^{\mathrm{T}}X\leqslant X^{\mathrm{T}}QX+Y^{\mathrm{T}}Q^{-1}Y.$$

Lemma 4. (Schur Complement [37]) For given matrix

$$\begin{bmatrix}S_{11} & S_{12}\\ S_{12}^{\mathrm{T}} & S_{22}\end{bmatrix}<0,$$

where $S_{11}=S_{11}^{\mathrm{T}}, S_{22}=S_{22}^{\mathrm{T}}$, which is equivalent to one of the following conditions:

(1) $S_{11}<0,\quad S_{11}-S_{12}S_{22}^{-1}S_{12}^{\mathrm{T}}<0$;
(2) $S_{22}<0,\quad S_{22}-S_{12}^{\mathrm{T}}S_{11}^{-1}S_{12}<0$.

3 Synchronization criteria by matrix measure method

In this section, by using the matrix measure method and Halanay Inequality, the synchronization criteria for coupled IMNNs are derived. For fluent presentation, some notations are given: $\mathbf{\Theta}_1=I_{N-1}\otimes\Theta$, $\mathbf{\Lambda}_1=I_{N-1}\otimes\Lambda$, $\mathbf{L}_1=I_{N-1}\otimes L$, $A^+=(a_{kj}^+)_{n\times n}$, $B^+=(b_{kj}^+)_{n\times n}$, $\mathbf{A}^+=I_N\otimes A^+$, $\mathbf{B}^+=I_N\otimes B^+$, $\mathbf{A}_1^+=I_{N-1}\otimes A^+$, $\mathbf{B}_1^+=I_{N-1}\otimes B^+$, $\mathbf{H}=H\otimes\Gamma$, $\mathbf{M}=M\otimes I_n$.

Theorem 1. Under Assumption $(\mathcal{H}_1)$, if the following equality holds:

$$0<\xi_2<-\xi_1,$$

where $\xi_1=\max\{-1+\|\mathbf{\Theta}_1\|_p+l\|\mathbf{A}_1^+\|_p, 1+\mu_p(-\mathbf{\Lambda}_1)+c\|\mathbf{H}\|_p\}$, $\xi_2=l\|\mathbf{B}_1^+\|_p$, and $p=1,2,\infty,\omega$, $l=\max_{1\leqslant i\leqslant n}(l_i)$. Then, the coupled IMNNs eq. (5) can reach the synchronization.

Proof. Consider the following Lyapunov function:

$$V(t)=\|\mathbf{M}x(t)\|_p+\|\mathbf{M}r(t)\|_p. \tag{10}$$

Via calculating the upper-right Dini derivative of eq. (10) along the trajectory of IMNNs eq. (9), one can get

$$
\begin{aligned}
D^+V(t) &= \lim_{h\to 0^+}\frac{\|\mathbf{M}x(t+h)\|_p-\|\mathbf{M}x(t)\|_p}{h}+\lim_{h\to 0^+}\frac{\|\mathbf{M}r(t+h)\|_p-\|\mathbf{M}r(t)\|_p}{h}\\
&= \lim_{h\to 0^+}\frac{\|\mathbf{M}x(t)+h\mathbf{M}\dot{x}(t)+o(h)\|_p-\|\mathbf{M}x(t)\|_p}{h}+\lim_{h\to 0^+}\frac{\|\mathbf{M}r(t)+h\mathbf{M}\dot{r}(t)+o(h)\|_p-\|\mathbf{M}r(t)\|_p}{h}\\
&= \lim_{h\to 0^+}\frac{\|\mathbf{M}x(t)+h\mathbf{M}(-x(t)+r(t))+o(h)\|_p-\|\mathbf{M}x(t)\|_p}{h}\\
&\quad +\lim_{h\to 0^+}\frac{\|\mathbf{M}r(t)+h\mathbf{M}(-\boldsymbol{\Theta}x(t)-\boldsymbol{\Lambda}r(t)+A(x(t))\mathbf{f}(x(t)))+o(h)\|_p-\|\mathbf{M}r(t)\|_p}{h}\\
&\quad +\lim_{h\to 0^+}\frac{\|h\mathbf{M}(B(x(t))\mathbf{f}(x(t-\tau(t)))+\mathbf{J}(t)+c\mathbf{G}r(t))\|_p}{h}.
\end{aligned}
\tag{11}
$$

We can derive from Definition 4, $\mathbf{M}\boldsymbol{\Theta}=\boldsymbol{\Theta}_1\mathbf{M}$, $\mathbf{M}\boldsymbol{\Lambda}=\boldsymbol{\Lambda}_1\mathbf{M}$, $\mathbf{MJ}=0$,

$$
\begin{aligned}
\mathbf{MG}r(t) &= (M\otimes I_n)(G\otimes\Gamma)r(t)\\
&= (H\otimes\Gamma)(M\otimes I_n)r(t)\\
&= \mathbf{HM}r(t).
\end{aligned}
$$

From the Assumption ($\mathcal{H}_1$), it gets

$$
\begin{aligned}
&\|\mathbf{MA}(x(t))\mathbf{f}(x(t))\|_p \leqslant l\|\mathbf{A}_1^+\|_p\|\mathbf{M}x(t)\|_p,\\
&\|\mathbf{MB}(x(t))\mathbf{f}(x(t-\tau(t)))\|_p \leqslant l\|\mathbf{B}_1^+\|_p\|\mathbf{M}x(t-\tau(t))\|_p.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
D^+V(t) &\leqslant \lim_{h\to 0^+}\frac{\|I_{n(N-1)}+h(-I_{n(N-1)})\|_p-1}{h}\|\mathbf{M}x(t)\|_p\\
&\quad +\|\mathbf{M}r(t)\|_p\\
&\quad +\lim_{h\to 0^+}\frac{\|I_{n(N-1)}+h(-\boldsymbol{\Lambda}_1)\|_p-1}{h}\|\mathbf{M}r(t)\|_p\\
&\quad +\|\boldsymbol{\Theta}_1\|_p\|\mathbf{M}x(t)\|_p+l\|\mathbf{A}_1^+\|_p\|\mathbf{M}x(t)\|_p\\
&\quad +l\|\mathbf{B}_1^+\|_p\|\mathbf{M}x(t-\tau(t))\|_p+c\|\mathbf{H}\|_p\|\mathbf{M}r(t)\|_p.
\end{aligned}
$$

It follows from the Definition 3 and Lemma 1, one has

$$
\begin{aligned}
D^+V(t) &\leqslant (\mu_p(-I_{n(N-1)})+\|\boldsymbol{\Theta}_1\|_p+l\|\mathbf{A}_1^+\|_p)\|\mathbf{M}x(t)\|_p\\
&\quad +(1+\mu_p(-\boldsymbol{\Lambda}_1)+c\|\mathbf{H}\|_p)\|\mathbf{M}r(t)\|_p\\
&\quad +l\|\mathbf{B}_1^+\|_p\|\mathbf{M}x(t-\tau(t))\|_p\\
&\leqslant (-1+\|\boldsymbol{\Theta}_1\|_p+l\|\mathbf{A}_1^+\|_p)\|\mathbf{M}x(t)\|_p\\
&\quad +(1+\mu_p(-\boldsymbol{\Lambda}_1)+c\|\mathbf{H}\|_p)\|\mathbf{M}r(t)\|_p\\
&\quad +l\|\mathbf{B}_1^+\|_p\|\mathbf{M}x(t-\tau(t))\|_p\\
&\leqslant \xi_1V(t)+\xi_2V(t-\tau(t)).
\end{aligned}
\tag{12}
$$

From the conditions in Theorem 1, we have $-\xi_1>\xi_2>0$, according to Lemma 2, it can be obtained that

$$
V(t)\leqslant \sup_{t-\tau\leqslant s\leqslant t}V(s)e^{-\rho(t-t_0)},
\tag{13}
$$

where $-\rho=\xi_1-\xi_2e^{\rho\tau}$, from eq. (13) and Definition 2, we conclude that system eq. (5) can reach exponentially synchronization. This completes the proof.

Remark 2. Synchronization criteria for IMNNs have been obtained by using the matrix-measure method and the Halanay inequality in Theorem 1. Different from the general mathematical model of MNNs [38, 39], the inertial term increases the order and dimension of neural network, by using suitable transformation and the topology structure of the neural network, without constructing the Lyapunov functional and designing complex controller, the synchronization criteria are easy to be obtained.

Remark 3. It should be noted that the algebra synchronization results via matrix-measure method are easy to verify, and matrix-measure can be positive or negative, have less conservativeness than matrix norm criteria, which ignore the excitatory and inhibitory effects of the neurons.

4 Synchronization criteria by the Lyapunov-Krasovskii method

In this section, we will construct new Lyapunov functional, design a discontinuous controller, some LMIs conditions are derived to ensure the synchronization of the coupled IMNNs.

By designing the discontinuous controller, the coupled IMNNs eq. (5) can be changed as

$$
\begin{aligned}
\frac{\mathrm{d}^2x_i(t)}{\mathrm{d}t^2} &= -D\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}-Cx_i(t)+A(x_i(t))f(x_i(t))\\
&\quad +B(x_i(t))f(x_i(t-\tau(t)))+J(t)\\
&\quad +c\sum_{j=1}^{N}G_{ij}\Gamma r_j(t)+\sum_{j=1}^{N}G_{ij}\Gamma\mathrm{sign}\left(r_j(t)-r_i(t)\right),\\
&\quad i=1,2,\cdots,N.
\end{aligned}
\tag{14}
$$

Similarly, the matrix form of system eq. (14) can be written as

$$
\begin{cases}
\dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t)+r(t),\\
\dfrac{\mathrm{d}r(t)}{\mathrm{d}t} = -\boldsymbol{\Theta}x(t)-\boldsymbol{\Lambda}r(t)+\mathbf{A}(x(t))\mathbf{f}(x(t))\\
\qquad +\mathbf{B}(x(t))\mathbf{f}(x(t-\tau(t)))+\mathbf{J}(t)+c\mathbf{G}r(t)+\Psi,
\end{cases}
\tag{15}
$$

where

$$\Psi=\begin{bmatrix}\sum_{j=1}^{N}G_{1j}\Gamma\mathrm{sign}(r_j-r_1)\\ \vdots\\ \sum_{j=1}^{N}G_{Nj}\Gamma\mathrm{sign}(r_j-r_N)\end{bmatrix}\in\mathbb{R}^{nN}.$$

Theorem 2. Under Assumption $(\mathcal{H}_1)$ and $\dot{\tau}(t)\leqslant\mu<1(\mu\geqslant 0)$, coupled IMNNs eq. (5) can reach to synchronization if there exist positive definite matrix $P\in\mathbb{R}^{n\times n}$, and positive diagonal matrices $Q=\mathrm{diag}\{q_1,q_2,\cdots,q_n\}$, T, $S_1\in\mathbb{R}^{n\times n}$, such that the following LMIs hold:

$$\Upsilon_1=\begin{bmatrix}-Q\Lambda-\Lambda^{\mathrm{T}}Q & P-Q\Theta & Q\overline{B} & Q\overline{A}\\ * & -P-P^{\mathrm{T}}+L^{\mathrm{T}}TL+L^{\mathrm{T}}S_1L & 0 & 0\\ * & * & -(1-\mu)T & 0\\ * & * & * & -S_1\end{bmatrix}<0, \tag{16}$$

and

$$\Upsilon_2=\mathbf{QH}+\mathbf{H}^{\mathrm{T}}\mathbf{Q}<0,$$
$$\alpha_j-\gamma_j\zeta_{i,i+1}\leqslant 0, \tag{17}$$

where $k=\max_{1\leqslant i\leqslant n}(k_i)$, $\alpha_j=2\sum_{s=1}^{n}k[(\overline{a}_{js}-\underline{a}_{js})+(\overline{b}_{js}-\underline{b}_{js})]$, $\zeta_{i,i+1}=G_{i,i+1}+G_{i+1,i}-\sum_{m\neq i,i+1}^{N}(G_{im}+G_{i+1,m})$ and $\overline{A}=(\overline{a}_{kj})_{n\times n}$, $\overline{B}=(\overline{b}_{kj})_{n\times n}$, $\mathbf{P}=I_{N-1}\otimes P$, $\mathbf{Q}=I_{N-1}\otimes Q$, $\mathbf{T}=I_{N-1}\otimes T$.

Proof. Consider the following Lyapunov-Krasovskii functional:

$$V(t)=x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{PM}x(t)+r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QM}r(t)+\int_{t-\tau(t)}^{t}\mathbf{f}^{\mathrm{T}}(x(s))\mathbf{M}^{\mathrm{T}}\mathbf{TMf}(x(s))\mathrm{d}s.$$

Calculating the derivative of $V(t)$ along the trajectories of system eq. (9), it yields that

$$\begin{aligned}\dot{V}(t)\leqslant&2x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{PM}\dot{x}(t)+2r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QM}\dot{r}(t)\\&+\mathbf{f}^{\mathrm{T}}(x(t))\mathbf{M}^{\mathrm{T}}\mathbf{TMf}(x(t))\\&-(1-\mu)\mathbf{f}^{\mathrm{T}}(x(t-\tau(t)))\mathbf{M}^{\mathrm{T}}\mathbf{TMf}(x(t-\tau(t)))\\&\leqslant 2x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{PM}[-x(t)+r(t)]-2r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QM\Theta}x(t)\\&-2r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QM\Lambda}r(t)+2r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QMA}(x(t))\mathbf{f}(x(t))\\&+2r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QMB}(x(t))\mathbf{f}(x(t-\tau(t)))\\&+2r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QMJ}(t)+2cr^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QMG}r(t)\\&+\mathbf{f}^{\mathrm{T}}(x(t))\mathbf{M}^{\mathrm{T}}\mathbf{TMf}(x(t))+2r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QM\Psi}\\&-(1-\mu)\mathbf{f}^{\mathrm{T}}(x(t-\tau(t)))\mathbf{M}^{\mathrm{T}}\mathbf{TMf}(x(t-\tau(t))).\end{aligned} \tag{18}$$

Obviously,

$$\begin{aligned}&r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QM\Theta}x(t)=r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{Q\Theta}_1\mathbf{M}x(t),\\&r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QM\Lambda}x(t)=r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{Q\Lambda}_1\mathbf{M}x(t),\\&r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QMG}r(t)=r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QHM}r(t),\end{aligned} \tag{19}$$

then, we get

$$\begin{aligned}\dot{V}(t)\leqslant&-2x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{PM}x(t)+2x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{PM}r(t)\\&-2r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{Q\Theta}_1\mathbf{M}x(t)-2r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{Q\Lambda}_1\mathbf{M}r(t)\\&+2r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QMA}(x(t))\mathbf{f}(x(t))\\&+2r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QMB}(x(t))\mathbf{f}(x(t-\tau(t)))\\&+2r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QMJ}(t)\\&+2cr^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QHM}r(t)+2r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QM\Psi}\\&+\mathbf{f}^{\mathrm{T}}(x(t))\mathbf{M}^{\mathrm{T}}\mathbf{TMf}(x(t))\\&-(1-\mu)\mathbf{f}^{\mathrm{T}}(x(t-\tau(t)))\mathbf{M}^{\mathrm{T}}\mathbf{TMf}(x(t-\tau(t))).\end{aligned} \tag{20}$$

Since

$$\begin{aligned}&\mathbf{MA}(x(t))\mathbf{f}(x(t))=\mathbf{M\overline{A}f}(x(t))+\mathbf{M}(\mathbf{A}(x(t))-\mathbf{\overline{A}})\mathbf{f}(x(t)),\\&\mathbf{MB}(x(t))\mathbf{f}(x(t-\tau(t)))=\mathbf{M\overline{B}f}(x(t-\tau(t)))\\&\qquad+\mathbf{M}(\mathbf{B}(x(t))-\mathbf{\overline{B}})\mathbf{f}(x(t-\tau(t))).\end{aligned} \tag{21}$$

Under the Assumption $(\mathcal{H}_1)$, it obtains

$$\begin{aligned}&r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QM}[(\mathbf{A}(x(t))-\mathbf{\overline{A}})\mathbf{f}(x(t))\\&+(\mathbf{B}(x(t))-\mathbf{\overline{B}})\mathbf{f}(x(t-\tau(t)))]\\=&\sum_{i=1}^{N}\sum_{j=1}^{n}q_jv_{ij}(t)\sum_{s=1}^{n}((a_{js}(x_{is})-\overline{a}_{js})f_s(x_{is}(t))\\&-(a_{js}(x_{i+1,s})-\overline{a}_{js})f_s(x_{i+1,s}(t))\\&+((b_{js}(x_{is})-\overline{b}_{js})f_s(x_{is}(t-\tau(t)))\\&-(b_{js}(x_{i+1,s})-\overline{b}_{js})f_s(x_{i+1,s}(t-\tau(t))))\\\leqslant&\sum_{i=1}^{N}\sum_{j=1}^{n}q_j\alpha_j|v_{ij}(t)|,\end{aligned} \tag{22}$$

where $v(t)=\mathbf{M}r(t)=[v_1^{\mathrm{T}}(t),v_2^{\mathrm{T}}(t),\cdots,v_{N-1}^{\mathrm{T}}(t)]^{\mathrm{T}}$, $v_i(t)=r_i(t)-r_{i+1}(t)$, $u(t)=\mathbf{M}x(t)=[u_1^{\mathrm{T}}(t),u_2^{\mathrm{T}}(t),\cdots,u_{N-1}^{\mathrm{T}}(t)]^{\mathrm{T}}$, $u_i(t)=x_i(t)-x_{i+1}(t)$.

$$\begin{aligned}&2r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QM\overline{A}f}(x(t))\\&\leqslant\sum_{i=1}^{N-1}|r_i(t)-r_{i+1}(t)|^{\mathrm{T}}Q\overline{A}S_1^{-1}(\overline{A})^{\mathrm{T}}Q|r_i(t)-r_{i+1}(t)|\end{aligned}$$

$$+\sum_{i=1}^{N-1}|f(x_i(t))-f(x_{i+1}(t))|^{\mathrm{T}}S_1|f(x_i(t))-f(x_{i+1}(t))|$$
$$\leqslant\sum_{i=1}^{N-1}|v_i(t)|^{\mathrm{T}}Q\overline{A}S_1^{-1}(\overline{A})^{\mathrm{T}}Q|v_i(t)|+\sum_{i=1}^{N-1}|u_i(t)|^{\mathrm{T}}L^{\mathrm{T}}S_1L|u_i(t)|, \tag{23}$$

and

$$\begin{aligned}
&r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QM\Psi}\\
&=\sum_{i=1}^{N-1}\sum_{j=1}^{n}q_j\gamma_j(r_{ij}(t)-r_{i+1,j}(t))\\
&\quad\times\Bigg[\sum_{m=1}^{N}G_{im}\mathrm{sign}(r_{mj}(t)-r_{i,j}(t))\\
&\quad-\sum_{m=1}^{N}G_{i+1,m}\mathrm{sign}(r_{mj}(t)-r_{i+1,j}(t))\Bigg]\\
&=\sum_{i=1}^{N-1}\sum_{j=1}^{n}q_j\gamma_j\Bigg[-(G_{i,i+1}+G_{i+1,i})|r_{ij}(t)\\
&\quad-r_{i+1,j}(t)|+(r_{ij}(t)-r_{i+1,j}(t))\\
&\quad\times\Bigg(\sum_{m\neq i+1}^{N}G_{im}\mathrm{sign}(r_{mj}(t)-r_{ij}(t))\\
&\quad-\sum_{m\neq i}^{N}G_{i+1,m}\mathrm{sign}(r_{mj}(t)-r_{i+1,j}(t))\Bigg)\Bigg]\\
&\leqslant-\sum_{i=1}^{N-1}\sum_{j=1}^{n}q_j\gamma_j\zeta_{i,i+1}|v_{ij}|.
\end{aligned}\tag{24}$$

Next, it is easy to verify that

$$\mathbf{f}^{\mathrm{T}}(x(t))\mathbf{M}^{\mathrm{T}}\mathbf{TMf}(x(t))\leqslant\sum_{i=1}^{N-1}|u_i(t))|^{\mathrm{T}}L^{\mathrm{T}}TL|u_i(t)|,$$
$$\begin{aligned}
&\mathbf{f}^{\mathrm{T}}(x(t-\tau(t)))\mathbf{M}^{\mathrm{T}}\mathbf{TMf}(x(t-\tau(t)))\\
&\leqslant\sum_{i=1}^{N-1}|f(x_i(t-\tau(t)))-f(x_{i+1}(t-\tau(t)))|^{\mathrm{T}}T|f(x_i(t-\tau(t)))\\
&\quad-f(x_{i+1}(t-\tau(t)))|.
\end{aligned}\tag{25}$$

Substituting eqs. (19)–(25) into eq. (18), according to Lemma 4, we have

$$\begin{aligned}
\dot{V}(t)\leqslant&\sum_{i=1}^{N-1}\{|u_i(t)|^{\mathrm{T}}[-P-P^{\mathrm{T}}+L^{\mathrm{T}}S_1L+L^{\mathrm{T}}TL]|u_i(t)|\\
&+2|u_i(t)|^{\mathrm{T}}[P-Q\Theta]|v_i(t))|+|v_i(t)|^{\mathrm{T}}[-Q\Lambda\\
&-\Lambda^{\mathrm{T}}Q+Q\overline{A}S_1^{-1}(\overline{A})^{\mathrm{T}}Q]|v_i(t))|\\
&-(1-\mu)|f(x_i(t-\tau(t)))\\
&-f(x_{i+1}(t-\tau(t)))|^{\mathrm{T}}T|f(x_i(t-\tau(t)))\\
&-f(x_{i+1}(t-\tau(t)))|+2|v_i(t)|^{\mathrm{T}}Q\overline{B}|f(x_i(t-\tau(t)))\\
&-f(x_{i+1}(t-\tau(t)))|\}+2cr^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{QHM}r(t)\\
\leqslant&\sum_{i=1}^{N-1}\xi_i^{\mathrm{T}}\Upsilon_1\xi_i+\varpi^{\mathrm{T}}(t)\Upsilon_2\varpi(t),
\end{aligned}\tag{26}$$

where $\xi_i=(|v_i(t)|^{\mathrm{T}},|u_i(t)|^{\mathrm{T}},|f(x_i(t-\tau(t)))-f(x_{i+1}(t-\tau(t)))|^{\mathrm{T}})^{\mathrm{T}}$, $\varpi(t)=(r^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}})^{\mathrm{T}}$. From conditions in Theorem 2, one can get $\dot{V}(t)<0$, hence, we can draw the conclusion, the coupled IMNNs can reach synchronization under conditions in eqs. (16) and (17). The proof of Theorem 2 is completed.

Remark 4. In this paper, consider the synchronization problem for the coupled IMNNs [40] with or without designing controller. It should be noted that the topology structure can be directed or undirected, which means the configuration coupling matrix can be symmetric or asymmetric, and the derived conditions are delay-dependent, which means that the results in this paper are more less conservative.

5 Two illustrative examples

In this section, two illustrative examples are given to check the validity of the results obtained in Theorem 1 and Theorem 2.

Example 1. Consider the following IMNNs with time delay and the nearest neighboring topology:

$$\begin{aligned}
\frac{\mathrm{d}^2x_i(t)}{\mathrm{d}t^2}=&-D\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}-Cx_i(t)+A(x_i(t))f(x_i(t))\\
&+B(x_i(t))f(x_i(t-\tau(t)))+J(t)\\
&+c\sum_{j=1}^{5}G_{ij}\Gamma\left(\frac{\mathrm{d}x_j(t)}{\mathrm{d}t}+x_j(t)\right),\ i=1,2,3,4,5,
\end{aligned}\tag{27}$$

where the activation function $f(x_i(t))=\tanh(|x_i|-0.1)$, $\tau(t)=\frac{e^{\mathrm{T}}}{e^{\mathrm{T}}+1}$, the coupling strength $c=1$, and the external input vector $J(t)=[0,0]^{\mathrm{T}}$. Obviously, Assumption $(\mathcal{H})_1$ is satisfied with $L=I$. The network parameters and configuration matrix are set as

$$D=\begin{pmatrix}8&0\\0&8\end{pmatrix},\ C=\begin{pmatrix}7.2&0\\0&7.2\end{pmatrix},$$
$$G=\begin{pmatrix}-2&1&0&0&1\\1&-2&1&0&0\\0&1&-2&1&0\\0&0&1&-2&1\\1&0&0&1&-2\end{pmatrix},\ \Gamma=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$
$$A(x_i(t))=\begin{pmatrix}a_{11}(x_{i1})&a_{12}(x_{i2})\\a_{21}(x_{i1})&a_{22}(x_{i2})\end{pmatrix},$$

$$B(x_i(t)) = \begin{pmatrix} b_{11}(x_{i1}) & b_{12}(x_{i2}) \\ b_{21}(x_{i1}) & b_{22}(x_{i2}) \end{pmatrix},$$

with

$$a_{11}(x) = \begin{cases} 0.02, & |x| \leqslant 0.1, \\ -0.02, & |x| > 0.1, \end{cases}$$

$$a_{12}(x) = \begin{cases} 0.04, & |x| \leqslant 0.1, \\ -0.04, & |x| > 0.1, \end{cases}$$

$$a_{21}(x) = \begin{cases} 0.01, & |x| \leqslant 0.1, \\ -0.01, & |x| > 0.1, \end{cases}$$

$$a_{22}(x) = \begin{cases} 0.03, & |x| \leqslant 0.1, \\ -0.03, & |x| > 0.1, \end{cases}$$

$$b_{11}(x) = \begin{cases} 0.05, & |x| \leqslant 0.1, \\ -0.05, & |x| > 0.1, \end{cases}$$

$$b_{12}(x) = \begin{cases} 0.01, & |x| \leqslant 0.1, \\ -0.01, & |x| > 0.1, \end{cases}$$

$$b_{21}(x) = \begin{cases} 0.02, & |x| \leqslant 0.1, \\ -0.02, & |x| > 0.1, \end{cases}$$

$$b_{22}(x) = \begin{cases} 0.06, & |x| \leqslant 0.1, \\ -0.06, & |x| > 0.1. \end{cases}$$

According to the above parameter matrices, let $p = 1$. By computing, we can get $\xi_1 = -0.73$, $\xi_2 = 0.07$ and $\xi_1 + \xi_2 = -0.66 < 0$, which mean that the condition in Theorem 1 is satisfied, then the coupled IMNNs can reach synchronization.

To make numerical simulation for the coupled IMNNs, the initial values are randomly choose in the set $[0, 1] \times [0, 1]$ with step $h = 0.01$. The undirected topology structure is shown in Figures 1–3 depict the switching trajectories of the memristor parameters $a_{11}(x_{11})$ and $b_{11}(x_{11})$. Figures 4 and 5 display the synchronization error trajectories of $x_{i1} - x_{11}$ and $x_{i2} - x_{12}$, respectively, which indicate that the synchronization can be reached. This is in accordance with the conclusion of Theorem 1.

Example 2. Consider the coupled IMNNs eq. (27) with the nearest neighboring topology shown in Figure 1, and the system parameters are taken as

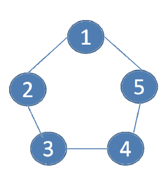

Figure 1 (Color online) The nearest neighbor networks with five nodes.

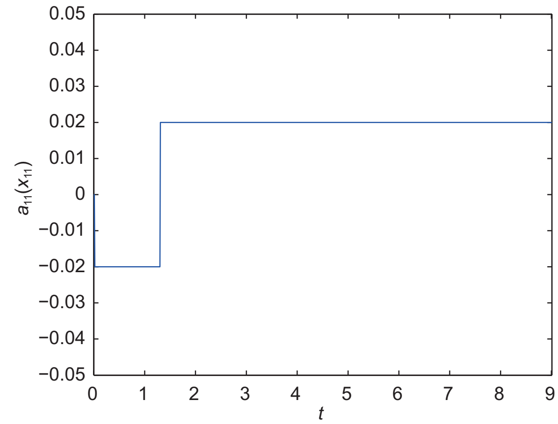

Figure 2 (Color online) The trajectories of connection weight coefficient $a_{11}(x_{11})$.

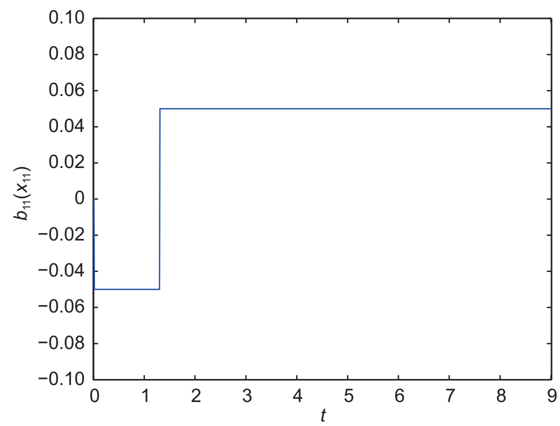

Figure 3 (Color online) The trajectories of connection weight coefficient $b_{11}(x_{11})$.

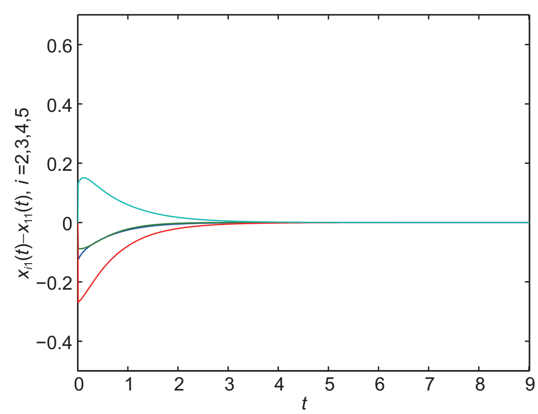

Figure 4 (Color online) The error trajectories of $x_{i1} - x_{11}$, $i = 2, 3, 4, 5$.

$$D = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}, \ C = \begin{pmatrix} 14 & 0 \\ 0 & 14 \end{pmatrix}, \ \Gamma = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix},$$

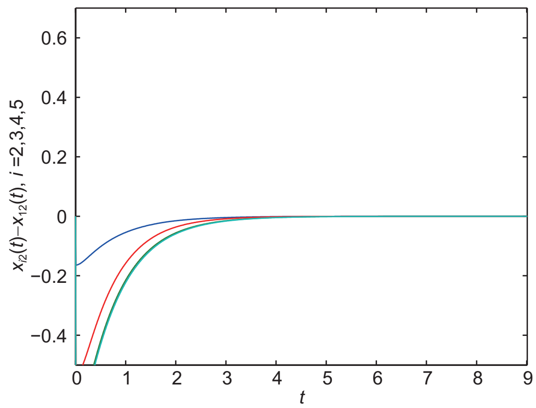

Figure 5 (Color online) The error trajectories of $x_{i2} - x_{12}$, $i = 2, 3, 4, 5$.

$$G = \begin{pmatrix} -1.2 & 1 & 0 & 0 & 0.2 \\ 1 & -2.2 & 1.2 & 0 & 0 \\ 0 & 2 & -3.5 & 1.5 & 0 \\ 0 & 0 & 3 & -4.4 & 1.4 \\ 0.2 & 0 & 0 & 4 & -4.2 \end{pmatrix},$$

$$a_{11}(x) = \begin{cases} 1.78, & |x| \leqslant 0.05, \\ 2.04, & |x| > 0.05, \end{cases}$$

$$a_{12}(x) = \begin{cases} 2.03, & |x| \leqslant 0.05, \\ 1.58, & |x| > 0.05, \end{cases}$$

$$a_{21}(x) = \begin{cases} 0.11, & |x| \leqslant 0.05, \\ 0.12, & |x| > 0.05, \end{cases}$$

$$a_{22}(x) = \begin{cases} 1.52, & |x| \leqslant 0.05, \\ 1.61, & |x| > 0.05, \end{cases}$$

$$b_{11}(x) = \begin{cases} -2.21, & |x| \leqslant 0.05, \\ -1.34, & |x| > 0.05, \end{cases}$$

$$b_{12}(x) = \begin{cases} 0.12, & |x| \leqslant 0.05, \\ 0.11, & |x| > 0.05, \end{cases}$$

$$b_{21}(x) = \begin{cases} 0.15, & |x| \leqslant 0.05, \\ 0.14, & |x| > 0.05, \end{cases}$$

$$b_{22}(x) = \begin{cases} -1.92, & |x| \leqslant 0.05, \\ -1.71, & |x| > 0.05, \end{cases}$$

where the activation function $f(x_i(t)) = \tanh(|x_i| - 0.05)$, the other parameters not mentioned are the same as defined in Example 1. From the given parameters, it is easy to verify that $\zeta_{i,i+1}(i = 1, 2, 3, 4)$ make 15 hold. Then, solving the LMIs in eqs. (14) and (15)) in Theorem 2, by using the LMI solver in Matlab, feasible positive definite matrices P, Q, T, S_1 could be found as

$$P = \begin{pmatrix} 101.1 & -0.5 \\ -0.5 & 101.9 \end{pmatrix}, \; Q = \begin{pmatrix} 10.7 & 0 \\ 0 & 10.7 \end{pmatrix},$$

$$T = \begin{pmatrix} 70.8 & 0 \\ 0 & 70.8 \end{pmatrix}, \; S_1 = \begin{pmatrix} 71.3 & 0 \\ 0 & 71.3 \end{pmatrix}.$$

For making numerical simulation for the coupled IMNNs with time delay, the trajectories of the connection weight parameters $a_{11}(x_{11})$ and $b_{11}(x_{11})$ are shown in Figures 6–9 show the synchronization errors of $x_{i1} - x_{11}$ and $x_{i2} - x_{12}$, which indicate $x_{i1} - x_{11} \to 0$ and $x_{i2} - x_{12} \to 0$ as $t \to \infty$, hence, synchronization can be reached. This is in accordance with the conclusion in Theorem 2.

6 Conclusions

In this paper, we investigate the multiple MNNs with coupling and inertial term. Firstly, by choosing appropriate variable transformation, networks model with inertial term

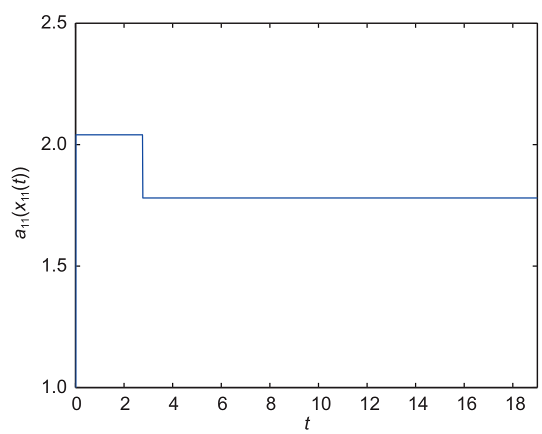

Figure 6 (Color online) The trajectories of connection weight coefficient $a_{11}(x_{11})$.

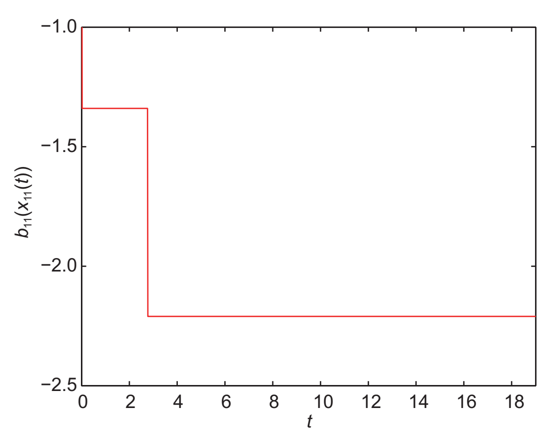

Figure 7 (Color online) The trajectories of connection weight coefficient $b_{11}(x_{11})$.

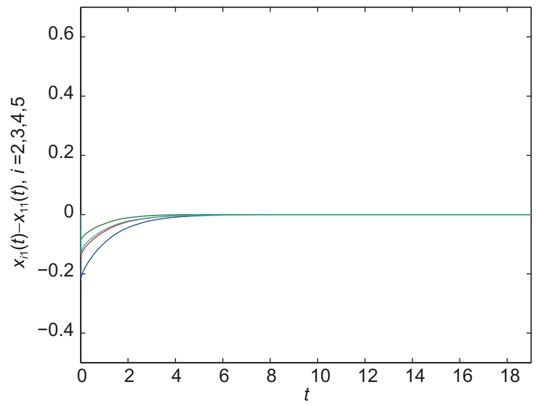

Figure 8 (Color online) The error trajectories of $x_{i1} - x_{11}$, $i = 2, 3, 4, 5$.

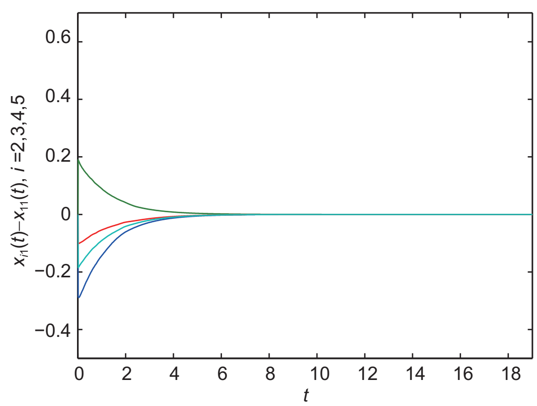

Figure 9 (Color online) The error trajectories of $x_{i2} - x_{12}$, $i = 2, 3, 4, 5$.

are rewritten as first-order differential equations. Secondly, by using differential inclusion and matrix measure method, some algebra conditions have been derived to guarantee the self-synchronization for multiple MNNs. Furthermore, via designing discontinuous controller, different synchronization criteria have been presented by LMIs. Finally, two illustrative examples have also been provided to demonstrate the validity of the proposed algebraic and LMIs synchronization criteria. In the future, the passivity and control issues of the coupled IMNNs will be conducted, which form some interesting research topics.

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