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A robust and efficient structural reliability method combining radial-based importance sampling and Kriging

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Simulation based structural reliability analysis suffers from a heavy computational burden, as each sample needs to be evaluated on the performance function, where structural analysis is performed. To alleviate the computational burden, related research focuses mainly on reduction of samples and application of surrogate model, which substitutes the performance function. However, the reduction of samples is achieved commonly at the expense of loss of robustness, and the construction of surrogate model is computationally expensive. In view of this, this paper presents a robust and efficient method in the same direction. The present method uses radial-based importance sampling (RBIS) to reduce samples without loss of robustness. Importantly, Kriging is fully used to efficiently implement RBIS. It not only serves as a surrogate to classify samples as we all know, but also guides the procedure to determine the optimal radius, with which RBIS would reduce samples to the highest degree. When used as a surrogate, Kriging is established through active learning, where the previously evaluated points to determine the optimal radius are reused. The robustness and efficiency of the present method are validated by five representative examples, where the present method is compared mainly with two fundamental reliability methods based on active learning Kriging.

structural reliability, simulation, radial-based importance sampling, Kriging, active learning

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1 Introduction

Input parameters of a structural model have an uncertain nature from a realistic point of view. It has been widely recognized that these uncertainties should be taken into account for a more reasonable decision. In this context, the structural reliability theory is a powerful tool. It provides a rational treatment of the uncertainties and allows assessment of the structural safety in the presence of such uncertain parameters. The assessment mainly lies in calculating the structural failure probability, which is expressed as a probability integral over all regions associated with structural failure in the design space. The probability integral is

$$P_f = \int_{G(\boldsymbol{x})\le 0} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \tag{1}$$

where $\boldsymbol{x}$ is the vector consisting of all basic random variables, which are simplifications of the uncertain parameters; $f(\boldsymbol{x})$ is the joint probability density function of the basic random variables; $G(\boldsymbol{x})$ called performance function, characterizes structural state according to a specified safety requirement. $G(\boldsymbol{x}) = 0$ represents the limit state surface separating the failure domain $G(\boldsymbol{x}) < 0$ and the safe domain $G(\boldsymbol{x}) > 0$ in the design space [1].

Simulation methods are oriented to solve this integral and the fundamental one is Monte-Carlo simulation (MCS). It provides an asymptotically unbiased and convergent failure probability estimator, but it needs a large number of samples to estimate small failure probabilities [2]. Therefore direct MCS is inefficient for practical structures, since they are

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generally well designed with a large safety margin. To improve the efficiency of direct MCS, one important direction is to reduce samples. In this direction, we can see various kinds of variance reduction techniques. Typically, they concentrate sampling in the important region, which contributes significantly to eq. (1) [3]. As a result, they are inapplicable to problems with multiple important regions. A robust sampling strategy is proposed by Harbitz [4]. That is RBIS, which restricts Monte-Carlo sampling to the tail part of the joint distribution of the random variables by excluding a central hyper sphere from the sampling domain in standard normal space. Recently, Grooteman [5] designed an adaptive scheme to implement RBIS efficiently (the corresponding reliability method is abbreviated to ARBIS). In this paper, a more efficient RBIS is developed based on Kriging.

Another important direction is to replace the performance function by a simple surrogate model (also known as metamodel), so that it is no longer necessary to run a time demanding structural model for each sample [6]. Popular surrogate models for this purpose include response surface [7,8], neural networks [9], support vector machines [10,11], and Kriging [12]. Recently, Kriging has gained increasing considerations mainly due to its stochastic properties. With these properties, one can derive useful information such as the probability of being negative and the probability of false sign prediction at any point. Dubourg [13] used the former to approximate the indicator function in the expression of optimal importance sampling density. However, more researches focus on the latter since it provides an efficient way to construct Kriging. This way is active learning, which means that Kriging is iteratively refined by actively selecting a new point to the design of experiments (DoE) each time [14].

The combination of Kriging and active learning is introduced to the field of structural reliability by Echard et al. [15]. They built a Kriging through active learning and then used it to classify Monte-Carlo samples (AK-MCS). In particular for problems with small failure probabilities, they replaced the Monte-Carlo sampling by importance sampling (AK-IS) [16]. As AK-IS is based on the non-robust first order reliability method (FORM) [17], Cadini et al. [18] adopted a K-means clustering algorithm instead of the FORM, while Zhao et al. [19] used Markov chain metropolis algorithm. Tong et al. [20] combined the active learning Kriging with subset importance sampling, providing a more efficient method than AK-IS. In addition, Zhang et al. [21] took account of the samples probability density during active learning to improve the accuracy. As to system reliability problems, Fauriat and Gayton [22] considered only the component performance function with significant influence on system failure probability in each learning step. However, this method is efficient when estimating low failure probabilities and lacks adaptivity when the initial samples are insufficient.

On the whole, existing reliability methods based on active learning Kriging can be considered as efficient for only specific types of problems (e.g., AK-MCS for problems with large failure probabilities and AK-IS for problems with one important region). In view of this, RBIS is introduced in this paper for its robustness. We skillfully use Kriging to fast determine the optimal radius of excluded hyper sphere, so the samples can be reduced to the most extent. Subsequently, the obtained Kriging continues to be used to classify samples after active learning.

2 Fundamental theory

2.1 Radial-based importance sampling (RBIS)

RBIS [4] aims to reduce Monte-Carlo samples in standard normal space. To this aim, it restricts sampling from a central hyper sphere that is inside the safe domain. As the central hyper sphere carries large probability weight in low dimensional space, the reduction of samples should be remarkable. When all samples are classified, the conditional failure probability outside the hyper sphere can be estimated by

$$\widehat{P}_{\text{fcon}} = P(G \le 0 \| \boldsymbol{u} | > r) \approx \frac{n_{\text{fail}}}{n_{\text{RBIS}}}, \tag{2}$$

where r, n_{fail} and n_{RBIS} denote the radius of the excluded sphere, number of the failure samples and total samples number, respectively.

The failure probability can be rewritten in terms of the conditional failure probability, yielding

$$\widehat{P}_f = \widehat{P}_{\text{fcon}} P(|\boldsymbol{u}| > r) = \frac{n_{\text{fail}}}{n_{\text{RBIS}}}\left[1 - \chi_n^2(r^2)\right], \tag{3}$$

where χ_n^2 is the chi-square distribution function with n degrees of freedom equal to the dimension of the problem. Correspondingly, the coefficient of variation on the estimator $\widehat{P}_f$ is given by

$$\text{COV}_{\widehat{P}_f} \approx \sqrt{\frac{1 - \widehat{P}_{\text{fcon}}}{\widehat{P}_{\text{fcon}} n_{\text{RBIS}}}}. \tag{4}$$

This equation indicates that RBIS requires fewer samples than MCS to achieve the same level of accuracy, because $\widehat{P}_{\text{fcon}}$ is larger than $\widehat{P}_f$. Moreover, the required number of samples will decrease to the minimum when the excluded sphere reaches the largest. Radius of the largest sphere is thus the optimal radius, which is equal to the shortest distance from the limit state surface to the space origin. However, it is not easy to determine the optimal radius with a little computational effort.

2.2 Kriging model

The unique feature of Kriging is that it assumes a deterministic response as Gaussian process. This stochastic property

provides rich information for structural reliability analysis. Kriging expresses the deterministic response $y(x)$, for an n dimensional input $x \in \mathbf{R}^n$, by a realization of a regression model and a random function [23],

$$y(\boldsymbol{x}) = \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\beta} + Z(\boldsymbol{x}), \tag{5}$$

where $\boldsymbol{f}(\boldsymbol{x}) = [f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), ..., f_p(\boldsymbol{x})]$ is p chosen functions, $\boldsymbol{\beta} = [\beta_1, \beta_2, ..., \beta_p]$ is their regression coefficients, $Z(\boldsymbol{x})$ is assumed to be a Gaussian process with zero mean and covariance

$$\mathrm{E}(Z(\boldsymbol{w}), Z(\boldsymbol{x})) = \sigma^2 R(\boldsymbol{\theta}, \boldsymbol{w}, \boldsymbol{x}), \tag{6}$$

between $Z(\boldsymbol{w})$ and $Z(\boldsymbol{x})$, where σ^2 is the process variance and $R(\boldsymbol{\theta}, \boldsymbol{w}, \boldsymbol{x})$ is the spatial correlation function with parameter $\boldsymbol{\theta}$. The correlation function is specified by the user, and several correlation functions exist in the literature. Among them the Gauss correlation function is mostly used

$$R(\boldsymbol{\theta}, \boldsymbol{w}, \boldsymbol{x}) = \exp\left(-\sum_{k=1}^{n} \theta_k |w_k - x_k|^2\right). \tag{7}$$

An interpretation of Kriging is that the term $\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\beta}$ in eq. (5) indicates a global model of the design space, which is similar to the polynomial model in a RSM (response surface method), and the second part in eq. (5) is used to model the deviation from $\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\beta}$.

Given m experimental points $\boldsymbol{X} = [\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_m]^{\mathrm{T}}$ with corresponding responses $\boldsymbol{Y} = [y_1, y_2, ..., y_m]^{\mathrm{T}}$, consider the linear Kriging predictor

$$\hat{y}(\boldsymbol{x}) = \boldsymbol{c}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{Y}, \tag{8}$$

where $\boldsymbol{c}(\boldsymbol{x}) \in \mathbf{R}^m$. From the consideration to keep the predictor unbiased and to minimize its mean squared error (MSE), one can deduce

$$\hat{y}(\boldsymbol{x}) = \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\beta}^* + \boldsymbol{r}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\gamma}^*, \tag{9}$$

where

$$\boldsymbol{\beta}^* = (\boldsymbol{F}^{\mathrm{T}}\boldsymbol{R}^{-1}\boldsymbol{F})^{-1}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{R}^{-1}\boldsymbol{Y}, \tag{10}$$

$$\boldsymbol{r}(\boldsymbol{x}) = [R(\boldsymbol{\theta}, \boldsymbol{x}_1, \boldsymbol{x}), R(\boldsymbol{\theta}, \boldsymbol{x}_2, \boldsymbol{x}), ..., R(\boldsymbol{\theta}, \boldsymbol{x}_m, \boldsymbol{x})]^{\mathrm{T}}, \tag{11}$$

$$\boldsymbol{\gamma}^* = \boldsymbol{R}^{-1}(\boldsymbol{Y} - \boldsymbol{F}\boldsymbol{\beta}^*), \tag{12}$$

where $\boldsymbol{F} = [\boldsymbol{f}(\boldsymbol{x}_1), \boldsymbol{f}(\boldsymbol{x}_2), ..., \boldsymbol{f}(\boldsymbol{x}_m)]^{\mathrm{T}}$; $\boldsymbol{R}$ is the matrix of stochastic-process correlations between different experimental points $R_{ij} = R(\boldsymbol{\theta}, \boldsymbol{x}_i, \boldsymbol{x}_j)$, $i, j = 1, ..., m$; $\boldsymbol{r}(\boldsymbol{x})$ is the vector of stochastic-process correlations between the untried point $\boldsymbol{x}$ and experimental points. Eq. (10) is also the generalized least squares solution of $\boldsymbol{F}\boldsymbol{\beta} \cong \boldsymbol{Y}$.

MSE of the predictor is

$$\phi(\boldsymbol{x}) = \sigma^2(1 + \boldsymbol{u}^{\mathrm{T}}(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{R}^{-1}\boldsymbol{F})^{-1}\boldsymbol{u} - \boldsymbol{r}^{\mathrm{T}}\boldsymbol{R}^{-1}\boldsymbol{r}), \tag{13}$$

where $\boldsymbol{u} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{R}^{-1}\boldsymbol{r} - \boldsymbol{f}$, the maximum likelihood estimate of σ^2 in the framework of generalized least squares fit is

$$\sigma^2 = \frac{1}{m}(\boldsymbol{Y} - \boldsymbol{F}\boldsymbol{\beta}^*)^{\mathrm{T}}\boldsymbol{R}^{-1}(\boldsymbol{Y} - \boldsymbol{F}\boldsymbol{\beta}^*). \tag{14}$$

Note that for a fixed set of experimental points the matrices $\boldsymbol{\beta}^*$ and $\boldsymbol{\gamma}^*$ are fixed. So for an untried point x we just need to compute $\boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{r}(\boldsymbol{x})$ to obtain $\hat{y}(\boldsymbol{x})$ and $\phi(\boldsymbol{x})$. The predictor $\hat{y}(\boldsymbol{x})$ and its MSE $\phi(\boldsymbol{x})$ can be regarded as mean and variance of the Gaussian process of the response, respectively.

3 The proposed method (AK-RBIS)

3.1 Outline of the proposed method

Note that the most probable failure point (MPP) is the closest limit state point to the origin (a limit state point refers to a point on the limit state surface); hence its distance to the origin equals the optimal radius. AK-RBIS is designed according to the framework of RBIS, which implies two stages to be performed: search MPP to determine the optimal radius (Stage 1) and then classify samples to compute the failure probability (Stage 2). In Stage 1, a progressive scheme is proposed to search MPP. The scheme searches a failure point in currently defined search domain. The searched failure point is then used to give a closer limit state point to the origin, which in turn is used to update the current search domain towards the origin. The scheme ends with the last limit state point as the MPP when no failure point can be found. In Stage 2, a Kriging is established by active learning, and serves as a surrogate to classify the samples which are generated according to the optimal radius obtained in Stage 1. Then the failure probability is estimated with eq. (3).

3.2 Search domain: Initialization and update

The search domains are defined as the inner space between two central spheres, as shown in Figure 1, so that the distance of a searched failure point to the origin can be controlled. For initial search domain, its lower radius $r_{\mathrm{low}}^{(0)}$ and upper radius $r_{\mathrm{up}}^{(0)}$ are specified by

$$\begin{cases} 1 - p_1 = \chi_n^2\left(\left(r_{\mathrm{low}}^{(0)}\right)^2\right), \\ 1 - p_2 = \chi_n^2\left(\left(r_{\mathrm{up}}^{(0)}\right)^2\right), \end{cases} \tag{15}$$

where p_1 and p_2 are the probability contents outside the $r_{\mathrm{low}}^{(0)}$ sphere and the $r_{\mathrm{up}}^{(0)}$ sphere, respectively. Usually, failure probabilities involved are larger than 10^{-6}. It is thus adequate to set p_1 and p_2 equal to 10^{-4} and 10^{-6} respectively.

Following the adaptive idea of ARBIS, later search domains are restricted by their respective limit state points [5]. Taking the ith search domain as an example, as shown in Figure 1, the outer sphere (with radius $r_{\mathrm{up}}^{(i)}$) passes through the limit state point, and radius of the inner sphere $r_{\mathrm{low}}^{(i)}$ is defined by

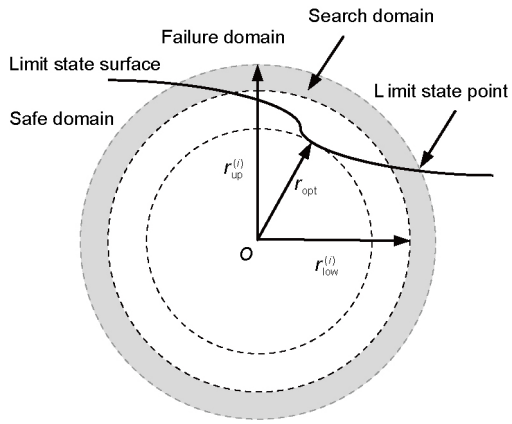

Figure 1 2-D illustration for the *i*th search domain.

$$1-\chi_n^2\left(\left(r_{\text{low}}^{(i)}\right)^2\right)=\left[1-\chi_n^2\left(\left(r_{\text{up}}^{(i)}\right)^2\right)\right]/t, \tag{16}$$

where the parameter t controls the ratio of the probability content in the search domain and that outside the outer sphere. t is recommended to be 0.8 [5].

According to the definition, a search domain can be updated towards the origin once a closer limit state point to the origin is obtained, and any failure point in a search domain is closer to the origin than the limit state point defining the search domain. Note that there exists at least one limit state point on the line segment connecting a failure point and the origin, because in a general case the origin is a safe point and the performance function is continuous. Therefore the update can be made if we can search a failure point in current search domain and then locate the limit state point on the line segment. Along this line of thought, we propose an efficient algorithm to search a failure point in each search domain, which would be explained in Section 3.3. Fortunately, it is an integrant procedure in directional sampling [24] to locate the limit state point on the line segment between a failure point and the origin. This procedure can be easily implemented through a linear fit and several quadratic fits afterwards, till an error under 0.01 [5]. The obtained limit state point is then used to define the next search domain. With this initialization and update strategy, the search domains will keep shrinking towards the origin.

3.3 Search failure point

As Stage 1 needs to search a failure point in a series of search domains, an efficient search algorithm is thus important in terms of the whole efficiency of AK-RBIS. Without knowledge about where is more probable to fail in a specified search domain beforehand, we uniformly distribute candidate failure points in the domain, and try to identify a real one among them. To do the former, the following high dimensional polar expressions are employed, as they are proper to generate uniform points in a domain bounded by two concentric spheres:

$$\begin{cases} u_1=\rho\cos\theta_1, \\ u_2=\rho\sin\theta_1\cos\theta_2, \\ \quad\vdots \\ u_{n-1}=\rho\sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \\ u_n=\rho\sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1}, \end{cases} \tag{17}$$

where ρ is a random number in the interval $\left(\left(r_{\text{low}}^{(i)}\right)^n,\left(r_{\text{up}}^{(i)}\right)^n\right)$, while $\theta_1, \theta_2, \ldots, \theta_{n-2}$ and θ_{n-1} are all independent random numbers in $(0, 2\pi)$.

Straightway, categories of the candidate points are verified through performance function evaluations successively, until a failure point is identified. The order is from high to low in terms of the probability of being negative. Thanks to Kriging, the probability of being negative for an unevaluated point can be obtained by

$$\pi(\boldsymbol{u})=P(\widehat{G}(\boldsymbol{u})\le 0)=\Phi\left(\frac{0-\mu_{\widehat{G}}(\boldsymbol{u})}{\sigma_{\widehat{G}}(\boldsymbol{u})}\right),\quad \boldsymbol{u}\notin \text{DoE}, \tag{18}$$

where $\pi(\boldsymbol{u})$ denotes the probability of being negative and is also called probabilistic classification function [13]; $\Phi(\cdot)$ is the standard normal distribution function; Kriging treats the performance function $G(\boldsymbol{u})$ as a realization of Gaussian stochastic field $\widehat{G}(\boldsymbol{u})$ and provides corresponding mean $\mu_{\widehat{G}}(\boldsymbol{u})$ and variance $\sigma_{\widehat{G}}^2(\boldsymbol{u})$. With the MATLAB tool box DACE, the construction and prediction of Kriging can be conveniently done.

In this sense, the candidate point with the largest π value is most probable to fail among the candidate population. This point is henceforth picked for failure verification, where its performance function is evaluated. The newly evaluated point is added to the DoE and then the Kriging is updated. If the point turns out to be safe, π values of all the candidate points are recalculated based on the predictions of the updated Kriging, and then another candidate point is selected for verification. The verification is repeated until a failure point is identified, or a convergence condition is satisfied. To save computational cost, all DoE points are reserved for constructing later Krigings through out the whole procedure of AK-RBIS. It is worthy of note that the issue of numerical singularity commonly encountered during the construction of Kriging can be avoided in the progressive scheme. This is because the candidate points are uniformly distributed, so that any two points in a DoE keep a considerable distance.

3.4 End condition of the progressive scheme

An end condition should be defined for the progressive scheme. In general, the situation that no failure region exists in a search domain fully demonstrates the associated limit state point to be the MPP. However, this is hard to verify in practical application. As a compromise, an approximate convergence condition is adopted that all candidate points fail with a probability below a specified level. This condition

is easily implemented with the probabilistic classification function. The condition requires only to compare the largest π value with a threshold p_{fth}. If the former is smaller, the convergence condition is satisfied. Considering the meaning of π, assignment for p_{fth} with 0.1 is enough.

Nevertheless, it is tricky to determine the number of candidate points in a search domain, since insufficient points may result in a premature convergence, whereas excessive points bring unbearable computational burden. Importantly, the premature convergence gives a larger radius, which will produce an erroneous result of failure probability. One way to solve this tricky problem is to generate fewer points first, and then add much more points after the first convergence (if occurs). The whole progressive scheme ends only when the convergence condition is satisfied twice in one search domain. In our experience, a failure point is identified quickly without occurrence of a convergence, except for very few (generally 1 or 2) search domains near the end of the scheme. From this viewpoint, the solution of the tricky problem is also an effective approach to improve the efficiency of the progressive scheme. When the problem dimension increases, it is suggested to increase number of the candidate points.

3.5 Classification of samples

Stage 2 of AK-RBIS is to classify samples outside the hyper sphere with the optimal radius obtained in Stage 1. To complete this task with least performance function evaluations, a Kriging is built through active learning, and then serves as a surrogate to predict signs of all the samples. This is a general procedure for active learning Kriging based reliability methods, and the originality here lies in the source of initial DoE. The final updated Kriging in Stage 1 is directly adopted as the initial Kriging in Stage 2, so that all DoE points are reserved for the purpose of cost saving.

Active learning aims to build a Kriging with least cost but guarantee high classification accuracy with respect to a given sample population. To this aim, it iteratively refines the Kriging by enriching the DoE with a point from the sample population, until a specific accuracy is reached. The point is selected for the greatest improvement on the classification accuracy of the Kriging. The point is recognized by a function, which is called learning function. An appropriate learning function can be $U(\boldsymbol{u}) = \left|\mu_{\hat{G}}(\boldsymbol{u})\right| / \sigma_{\hat{G}}(\boldsymbol{u})$ here. Because, according to the Gaussian Process assumption of Kriging, $\Phi\left(\left|\mu_{\hat{G}}(\boldsymbol{u})\right| / \sigma_{\hat{G}}(\boldsymbol{u})\right)$ is the probability that the sample $\boldsymbol{u}$ is correctly classified according to the predicted value $\mu_{\hat{G}}(\boldsymbol{u})$; in this vein, the sample with the smallest U value is most probable to be wrongly classified; as a consequence, its addition to the DoE is hoped to improve the current Kriging most in terms of the overall classification accuracy. Therefore, at each leaning step, U values of all samples are calculated based on the predictions of the current Kriging, and then the sample with the smallest U value is selected for the performance function evaluation; following this, the current Kriging is refined based on the enriched DoE. These are repeated until U values of all samples are above a predefined threshold, which indicates high overall classification accuracy.

3.6 Implementation of the proposed method

The implementation procedure of AK-RBIS is illustrated in Figure 2 and is explained in detail as follows.

(1) Stage 1: Determination of the optimal radius

Step 1. Transform original design space into standard normal space by Nataf transformation [25]. Then define initial search domain according to eq. (15). Next, establish initial Kriging with least uniform points in the initial search domain. If a failure point occurs in this step, go to Step 4.

Step 2. Fill current search domain with a certain number (say 10^3) of uniform points by eq. (17).

Step 3. Select the point with the largest π value to evaluate the performance function, and update the Kriging once the DoE is enriched. Repeat these until a failure point is found or the convergence condition is satisfied. Go to Step 5 for the first convergence. For the second convergence in the same search domain, end Stage 1 with the current $r_{\text{up}}^{(i)}$ as the optimal radius.

Step 4. Determine the limit state point on the line segment between the failure point and the origin. Then update the

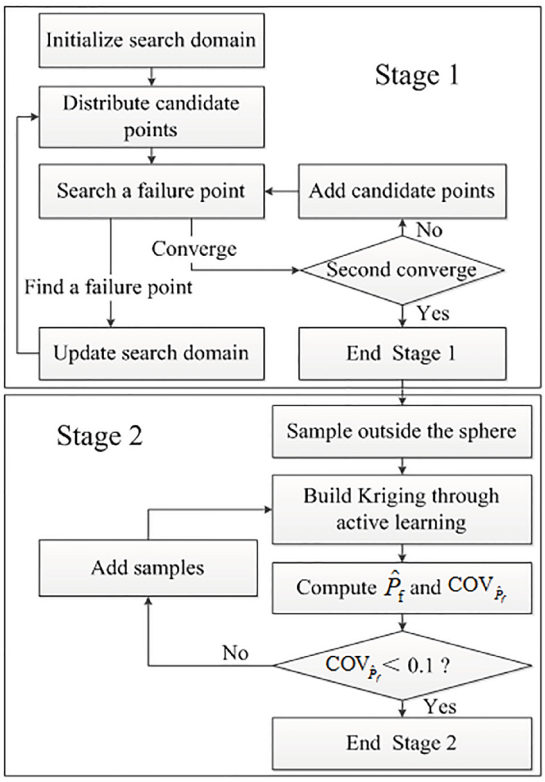

Figure 2 Flowchart of AK-RBIS.

search domain based on the limit state point according to eq. (16). Next, return to Step 2.

Step 5. Increase the number of points in current search domain to 10^4 by eq. (17). Then go back to Step 3.

(2) Stage 2: Classification of samples

Step 1. Generate Monte-Carlo samples outside the hyper sphere with the optimal radius obtained in Stage 1. To be simple, name the population S.

Step 2. Starting from the final updated Kriging in Stage 1 and for S, perform active learning by adding the sample with the smallest U value to the DoE in each learning step, until all U values are above 2 (corresponding to a probability of making a mistake on the sign of $\Phi(-2) = 0.023$) [15].

Step 3. Classify S by means of the well-learned Kriging surrogate. Based on the classification results, compute the failure probability and corresponding coefficient of variation according to RBIS principles. Then compare the coefficient of variation with an up limit. If the former is smaller, end AK-RBIS.

Step 4. Add samples to S by the same way as Step 1 and then go back to Step 2.

4 Validation of the proposed method

Five representative examples are presented to validate the robustness and efficiency of AK-RBIS. They cover typical features such as highly non-linear limit state surface, multiple important regions, small failure probability, structural system reliability, and non-normal random variables. They are solved by AK-RBIS and three related methods, ARBIS, AK-IS and AK-MCS, with the latter three in contrast to AK-RBIS. Among the latter three methods, ARBIS is robust and accurate, whereas AK-MCS and AK-IS, as two active learning Kriging based reliability methods, are efficient. Hence the comparison with the three methods is expected to give rise to a relatively convincing conclusion for AK-RBIS.

In addition, for all the examples: (i) Kriging employs constant model as the regression model and Gaussian model as the correlation function. (ii) To ensure comparable results, the up limit of the coefficient of variation on the failure probability estimator is set to 0.1, if there is no special note (however, relative errors of different methods are still difficult to precisely compare since all are computed from a specific Monte-Carlo or importance sampling population). (iii) As evaluations of the performance function govern the computational cost for practical structures, the number of calls to the performance function (N_{call}) is used to measure the computational cost. For AK-IS and AK-RBIS, N_{call} is presented as sum of the number in Stage 1 (searching MPP) and that in Stage 2 (classifying samples).

4.1 Example 1: Highly non-linear limit state surface

This example [26] is studied due to its highly non-linear limit state surface, as shown in Figure 3. Its performance function reads

$$G = 3 - u_2 + (4u_1)^4, \tag{19}$$

where u_1 and u_2 are standard normal random variables. Figure 3 also shows that the example involves a low conditional failure probability and a small important region.

Table 1 lists the results, which show that AK-RBIS, AK-MCS and AK-IS exhibit a great superiority upon ARBIS for order reduction of the magnitude of computational cost. Among these three methods, AK-RBIS and AK-IS are more accurate and slightly more efficient than AK-MCS. Actually, this example exactly belongs to the type of problems for which AK-IS is competitive. AK-IS is efficient in particular for problems with a unique and small important region, since fewer importance samples are able to cover a small region. However, this is not the case with AK-RBIS, as this example has a very small conditional failure probability outside the optimal sphere. According to eq. (4), a small conditional failure probability implies a large number of samples to be simulated. Nonetheless, AK-RBIS turns out to behave slightly better than AK-IS in terms of both accuracy and efficiency. This strongly proves the efficiency of the algorithm of AK-RBIS. One can also find that Stage 2 of AK-RBIS evaluates only three samples on the performance function, which should attribute to the reuse of the evaluated points in Stage 1.

4.2 Example 2: Series system with four branches

A reliability problem of series structural system is presented as the second example. The example originates from literature [27] and is modified here to have a small failure probability. Figure 4 shows that the problem has four important regions. The problem is formulated as

Table 1 Reliability results of Example 1

Method	P_f [a] (Relative error)	β	N_{call}
ARBIS	2.03×10^{-4} (12.8%)	2.925	4867
AK-MCS	1.66×10^{-4} (7.8%)	–	31
AK-IS	1.68×10^{-4} (6.7%)	3.000	6+22
AK-RBIS	1.84×10^{-4} (2.2%)	3.004	24+3

a) The reference P_f value is 1.80×10^{-4} [26]; results of ARBIS are from reference [5].

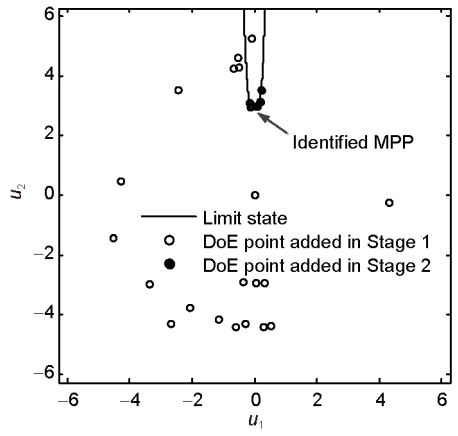

Figure 3 Illustration for the implementation of AK-RBIS in Example 1.

$$\begin{cases} g_1 = 0.1(u_1/1.5 - u_2/1.5)^2 \\ \quad -(u_1/1.5 + u_2/1.5)/\sqrt{2} + 3, \\ g_2 = 0.1(u_1/1.5 - u_2/1.5)^2 \\ \quad +(u_1/1.5 + u_2/1.5)/\sqrt{2} + 3, \\ g_3 = u_1/1.5 - u_2/1.5 + 3.5\sqrt{2}, \\ g_4 = -u_1/1.5 + u_2/1.5 + 3.5\sqrt{2}, \\ G = \min(g_1, g_2, g_3, g_4), \end{cases} \tag{20}$$

where u_1 and u_2 are standard normal random variables.

As shown in Table 2, AK-RBIS outperforms the other three methods. AK-MCS expends too much computational cost because it requires a large number of samples to simulate such a small failure probability. AK-IS yields erroneous results even though we have enlarged the sample population to decrease $\mathrm{COV}_{\hat{P}_f}$ to 0.02. This is because the population is centered at one point, thus unable to cover all important regions. As a result, AK-IS converges early to a wrong solution only accounting for part of all the important regions. But with regard to AK-RBIS, it only restricts Monte-Carlo sampling from a hyper sphere in the safe domain. So it can reach all important regions like MCS. On the other hand, as shown in Figure 4, AK-RBIS makes most DoE points near the limit state surface (limit state curve for this 2-D problem). This is important to avoid waste of computational cost, because only points near the limit state surface play an important role to improve the accuracy of the surrogate.

Contrary to Example 1, this example shows a large conditional failure probability. Then a small number of samples are feasible for AK-RBIS without loss of accuracy. In fact, structural system reliability problems commonly have multiple important regions, thus they are more likely to show a large conditional failure probability. As a result, AK-RBIS is more efficient for general system reliability problems.

4.3 Example 3: Creep-fatigue reliability problem

This section deals with a creep-fatigue reliability problem [28] from engineering environment. The problem involves exponential function, moderate dimensions and non-normal random variables. Its performance function is

$$\begin{aligned} &G(N_c, N_f, n_c, n_f, \theta_1, \theta_2) \\ &= 2 - \exp(\theta_1 D_c) + \frac{(\exp(\theta_1) - 2)(\exp(-\theta_2 D_c) - 1)}{\exp(-\theta_2) - 1} - D_f, \end{aligned} \tag{21}$$

where $D_c = n_c / N_c$ and $D_f = n_f / N_f$. All basic random variables are listed in Table 3.

Table 4 presents the results of ARBIS, AK-IS, AK-MCS and AK-RBIS. As shown in this table, on the whole, all methods except ARBIS obtain accurate results with a little computational cost. More specially, AK-RBIS expends the least computational cost, but reaches a similar level of accuracy compared with AK-IS and AK-MCS. Besides, Table 4 indicates that the proposed progressive scheme is more efficient than FORM for this example, as Stage 1 of AK-RBIS needs

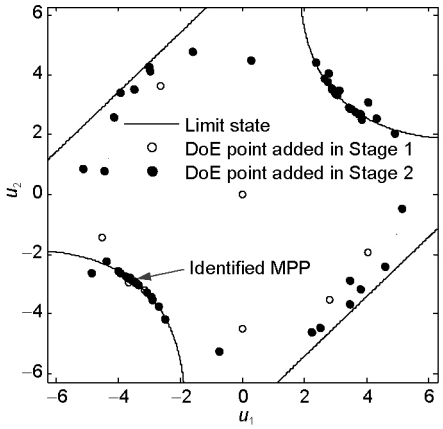

Figure 4 Illustration for the implementation of AK-RBIS in Example 2.

Table 2 Reliability results of Example 2

Method	P_f [a] (Relative error)	β	N_{call}
ARBIS	4.87×10^{-6} (8.9%)	4.505	918
AK-MCS	4.69×10^{-6} (4.9%)	–	197
AK-IS	2.23×10^{-6} (50.1%)	4.500	6+11
AK-RBIS	4.81×10^{-6} (7.6%)	4.509	14+52

a) The reference P_f value is 4.47×10^{-6}, obtained by MCS with 1×10^{8} samples.

Table 3 Distributions of the basic random variables in Example 3

Random variable	Distribution	Mean	Standard deviation
N_c	Log-normal	5490	1098
N_f	Log-normal	17100	3420
n_c	Log-normal	549	109.8
n_f	Log-normal	4000	800
θ_1	Normal	0.42	0.084
θ_2	Normal	6.0	1.2

Table 4 Reliability results of Example 3

Method	P_f [a] (Relative error)	β	N_{call}
ARBIS	1.64×10^{-4} (15.5%)	3.734	23175
AK-MCS	1.49×10^{-4} (4.9%)	–	90
AK-IS	1.38×10^{-4} (2.8%)	3.652	42+38
AK-RBIS	1.35×10^{-4} (4.9%)	3.656	26+35

a) The reference P_f value is 1.42×10^{-4}, obtained by MCS with 1×10^8 samples.

less computational cost than the first stage of AK-IS (the first stage of AK-IS is the application of FORM). From all the examples, one can find that the progressive scheme is roughly as efficient as FORM. However, in view of the non-robustness of FORM, the progressive scheme is considered to be more robust.

4.4 Example 4: Dynamic response of a non-linear oscillator

This example is a non-linear undamped dynamic system with single degree of freedom (Figure 5) [29]. Its performance function reads

$$G(c_1, c_2, m, r, t_1, F_1) = 3r - \left| \frac{2F_1}{m\omega_0^2} \sin\left(\frac{\omega_0 t_1}{2}\right) \right|, \tag{22}$$

where $\omega_0 = \sqrt{\frac{c_1 + c_2}{m}}$. Table 5 gives the distributions of the basic random variables.

As shown in Table 6, both AK-MCS and AK-RBIS produce accurate results with a little computational cost. Again, ARBIS produces accurate results but with an unacceptable computational burden. This can be explained by the absence

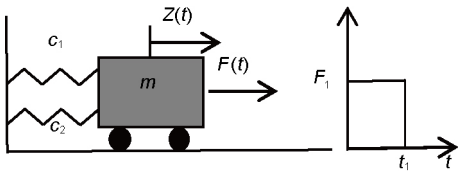

Figure 5 The non-linear oscillator.

Table 5 Distributions of the basic random variables in Example 4

Random variable	Distribution	Mean	Standard deviation
c_1	Normal	1	0.1
c_2	Normal	0.1	0.01
M	Normal	1	0.05
R	Normal	0.5	0.05
t_1	Normal	1	0.2
F_1	Normal	1	0.2

Table 6 Reliability results of Example 4

Method	P_f (Relative error)	β	N_{call}
ARBIS	2.82×10^{-2} (1.1%)	1.963	50775
AK-MCS	2.83×10^{-2} (0.7%)	–	58
AK-IS	2.52×10^{-2} (11.6%)	1.865	21+32
AK-RBIS	2.89×10^{-2} (1.4%)	1.918	14+51

a) For all the methods, coefficients of variation on the failure probability estimators are about 0.022; the reference P_f value is 2.85×10^{-2}, obtained by MCS with 5×10^6 samples; results of AK-MCS are from literature [15].

of a surrogate. Thus ARBIS is efficient only in traditional sense. In this example, AK-IS is efficient, but the accuracy is not satisfactory.

AK-RBIS and AK-MCS take the same way to classify samples, but adopt different sampling strategies. AK-RBIS neglects Monte-Carlo samples in the optimal excluded sphere, but needs additional effort to determine the optimal radius in advance. Accordingly, AK-RBIS is more efficient for problems with more probability content in the optimal sphere. However, the probability content in the optimal sphere of this example is only 0.27, so the reduction of samples is rather limited for AK-RBIS. AK-RBIS is therefore less efficient for this example, compared with a more general case. Nonetheless, AK-RBIS still expends roughly equivalent computational cost with AK-MCS. Without doubt, this should attribute to the inherent efficient algorithm of AK-RBIS, especially the progressive scheme.

Generally, the optimal excluded sphere contains much probability content for low dimensional problems, because it locates in the central part of the joint probability distribution function of the standard normal random variables. Hence AK-RBIS will do better for a more general case. Theoretically, in contrast with AK-MCS, AK-RBIS is robust against the variation of failure probability, because, according to eq. (4), the required number of samples depends on the conditional failure probability instead of the failure probability. Consequently, the superiority of AK-RBIS upon AK-MCS should be more significant for problems with small failure probabilities.

4.5 Example 5: Three-bay twelve-story frame

Finally, a frame structure (Figure 6) [30] is used to validate AK-RBIS with regard to real structures. The corresponding reliability problem is characterized by six basic random variables which are completely uncorrelated. They are member cross-section areas A_1, A_2, A_3, A_4, A_5, and the wind load P, respectively. Here, subscript i refers to members with label i as shown in Figure 5, i = 1, 2, ..., 5. The distributions of the basic random variables are given in Table 7. Elastic modulus of all the members equals 2.0×10^7 N m^{-2}. The sectional moments of inertia are expressed as

$$I_i = \alpha_i A_i^2,\ i = 1, 2, ..., 5, \tag{23}$$

where I_i are the sectional moments of inertia, α_i are the coefficients shown in Table 7.

The concerned performance function is

$$G(\boldsymbol{x}) = 0.096 - u_{\max}(\boldsymbol{x}), \tag{24}$$

where $u_{\max}(\boldsymbol{x})$ represents the maximum horizontal displacement of the frame.

The results are given in Table 8. Again, AK-IS is inaccurate. It can be seen that AK-RBIS and AK-MCS show similar performance for this example; they obtain sufficiently accurate result with equivalent number of calls to the performance function, and the numbers are small. In this example the probability content in the optimal sphere is as low as 0.0594. Therefore, this example is a more extreme case where AK-RBIS requires almost the same amount of samples with

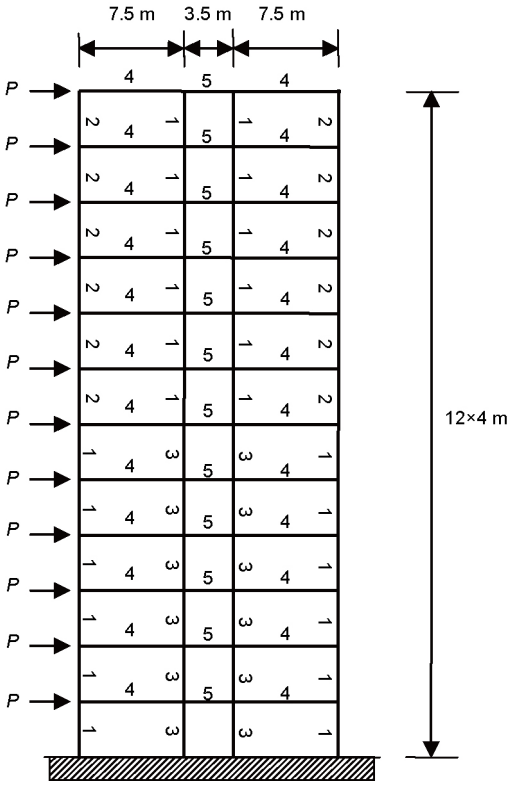

Figure 6 The three-bay twelve-story frame structure.

Table 7 Distributions of the basic random variables and the sectional coefficients in Example 5

Random variable (unit)	Distribution type	Mean	Standard deviation	Coefficient α_i
A_1 (m^2)	Lognormal	0.25	0.025	0.08333
A_2 (m^2)	Lognormal	0.16	0.016	0.08333
A_3 (m^2)	Lognormal	0.36	0.036	0.08333
A_4 (m^2)	Lognormal	0.20	0.020	0.26670
A_5 (m^2)	Lognormal	0.15	0.015	0.20000
P (N)	Gumbel	3×10^4	7.5×10^3	–

Table 8 Reliability results of Example 5 [a)]

Method	P_f (Relative error)	β	N_{call}
ARBIS	7.76×10^{-2} (3.3%)	1.386	1210
AK-MCS	7.59×10^{-2} (1.1%)	–	41
AK-IS	6.22×10^{-2} (17.2%)	1.325	28+23
AK-RBIS	7.57×10^{-2} (0.8%)	1.326	28+19

a) The reference failure probability is 7.51×10^{-2}, obtained by importance sampling with 2000 samples [30].

AK-MCS. It then seems that Stage 1 of AK-RBIS is useless. Note that all the points evaluated in Stage 1 are reused in the following active learning procedure. So the computational cost expended in Stage 1 is not wasted. In essence, the selection between AK-RBIS and AK-MCS is a trade-off between taking part of the cost to reduce samples first (and then classifying the reduced population) and directly taking all the cost to classify samples. For this extreme example, AK-RBIS still performs as well as AK-MCS. It then can be inferred that AK-RBIS would be more efficient than AK-MCS for general cases. The exception is highly dimensional problem, where the excluded hyper sphere possibly carries little failure probability content.

5 Summary and conclusions

This study focuses on two aspects to improve the efficiency of a simulation method for structural reliability analysis. They are the reduction of samples and the use of a surrogate model to classify samples, respectively. Kriging is adopted to efficiently implement both aspects. Based on Kriging, a progressive scheme is proposed to determine the optimal radius, with which RBIS would reduce samples to the highest degree while maintaining its robustness; by active learning an accurate Kriging is established with least number of performance function evaluations. As Kriging is updated throughout the whole procedure of the proposed method, evaluated points in the progressive scheme can also play an important role during active learning.

The proposed method (AK-RBIS) achieves sufficiently accurate failure probability with a little computational cost for all the five representative examples, which include typical features that may be encountered in real engineering context. This demonstrates that AK-RBIS is robust and efficient. With regard to problems for which AK-MCS or AK-IS is particularly suitable, AK-RBIS can also performs as well as (or even better than) it. With respect to the robustness of AK-RBIS, three critical points are noteworthy. First, AK-RBIS is still efficient for problems with small conditional failure probabilities outside the optimal excluded sphere, although these problems needs a large number of samples; second, the performance of AK-RBIS is not affected by the level of failure probability; third, AK-RBIS can solve problems with one important region satisfactorily, and could do even better for problems with multiple important regions, which are generally the cases with system reliability.

However, it still should be pointed out that the progressive scheme can possibly neglect a closed failure region near the origin if it exists, hence produces a smaller failure probability than the exact one. More work remains to be done on this aspect. Besides, as all the surrogates suffer the so-called "curse of dimensionality" and radial-based importance sampling is designed specially for low dimensional space, AK-RBIS can deal efficiently with only low dimensional problem. And it can be concluded from the examples that the highest applicable dimension can be at least 6. A more specific applicable dimension range of AK-RBIS would be targeted in our next work.

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