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Exponential H_∞ filtering analysis for discrete-time switched neural networks with random delays using sojourn probabilities

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This paper is concerned with the exponential H_∞ filtering problem for a class of discrete-time switched neural networks with random time-varying delays based on the sojourn-probability-dependent method. Using the average dwell time approach together with the piecewise Lyapunov function technique, sufficient conditions are proposed to guarantee the exponential stability for the switched neural networks with random time-varying delays which are characterized by introducing a Bernoulli stochastic variable. Based on the derived H_∞ performance analysis results, the H_∞ filter design is formulated in terms of Linear Matrix Inequalities (LMIs). Finally, two numerical examples are presented to demonstrate the effectiveness of the proposed design procedure.

switched neural networks, average dwell time, sojourn probability method, exponential stability

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1 Introduction

The last few decades have witnessed the development of neural networks in various areas ranging from associative memory, static image treatment, image processing and pattern recognition to medical diagnosis and data mining [1–7]. These applications purely depend on the dynamic behaviours of the underlying neural networks. The majority of the existing research has been concerned with the discrete time neural networks rather than their continuous-time counterparts in today's digital world.

Time-delay phenomena are commonly encountered in neural networks due to the limited switching speed of data processing to the inherent communication time of neurons and the existence of a delay in a system may induce oscillations or poor performances leading to instability [8–15]. In some neural networks these time delays often exist in a stochastic manner and its probabilistic characteristics such as Binomial distribution or normal distribution, can be regularly obtained by using the statistical methods [16]. It has been known that a neural network could be stabilized or destabilized by certain stochastic inputs. Hence the stability analysis problem for discrete-time stochastic neural networks has begun to attract the research interests [17–19].

In recent years, hybrid systems have attracted extensive attention. As a special class of hybrid systems, switching systems are composed of a family of continuous-time or discrete-time subsystems described by differential or difference equation which are organized by a switching rule that orchestrates switching. Switching systems have been found in many applications such as chemical processing, communication networks, power systems, traffic control, automotive engine control, aircraft control, etc. [20–22]. Switching between the systems may be arbitrary or there is a class of switched systems where the switching among different subsystems can be defined by the trajectory. The individual subsystems in the switched neural networks which are mathematically modeled have found their applications in the field

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of artificial intelligence and high speed signal processing. Based on this aspect, the model of a large class of switched systems is used to formulate a general stability analysis and the controller synthesis framework for switched systems.

Stability of the subsystems themselves is not sufficient for the stability of the overall system. We utilize the average dwell time method to examine the attenuation properties of switched systems [23, 24]. This method controls the dwell time of the switching system such that even if one or more subsystems are unstable, the overall switched system still remains stable. In [25], this method is also applied to study the model reduction problem for switched systems with time-varying delay. The fact of switching between unstable subsystems to make the switched system exponentially stable is remarkable [26–30]. By using the average dwell time approach and the novel Lyapunov-Krasovskii Functional (LKF), exponential stability problem can be solvable, if a set of linear matrix inequalities are solvable.

On the other hand, neuron state estimation problem has gained considerable amount of interest. It has wide applications in various fields such as secure communication, biological neural networks, gene regulatory networks, digital communication, etc. [31]. In [32], the state estimation problem for a class of discrete-time stochastic neural networks with random delays is characterized by a Bernoulli stochastic variable. In many applications, it is impossible or very expensive to acquire the state information of all neurons in neural networks because of their complicated structure. Therefore, one often needs to estimate the neuron states through the available measurements and then utilize the estimated neuron states to achieve certain design objectives. Through the available output measurements and by using the LMI technique, it can be employed to carry out the specific tasks such as dynamical performance analysis and synchronization issues for the controlled system. Based on different performance indexes and different physical implications, an exponential H_∞ filtering is designed for discrete-time switched neural networks [33–35].

Furthermore, a network exhibits a special characteristic called network mode switching where the network switches from one mode to the other mode with uncertain transition probabilities. A switching process in which the switching probabilities depend on a random sojourn time is a class of semi-Markov processes and is encountered in target tracking, systems subject to failures. In such a system, knowledge of the sojourn time is needed for the computation of the conditional transition probabilities. We usually do not know the transition probability between the subsystems so a new approach for the stability analysis is developed for switched linear discrete system based on the sojourn probabilities [36, 37]. A sojourn probability of a system is the probability of a switched system staying in a particular subsystem which is assumed to be known in prior. It has also been known that switching law of the sojourn-probability-dependent does not change the rules of the original switched systems but it does take some existing information that was not concerned before. Problems in random delay, packet loss and message missing of networked switched systems are some of the applications of the switched system model using sojourn-probability method. To the best of our knowledge, the exponential H_∞ filtering problem for a class of discrete-time switched neural networks with known sojourn probabilities and random time-varying delays has never been addressed in the literature yet, which also motivates the work of this paper. By using the developed system model, the exponential mean square stability of switched systems is obtained.

Summarizing the above discussions, this paper addresses the design of an exponential H_∞ state estimator for a class of discrete-time switched neural networks with random time-varying delays based on the sojourn probability approach. The main contributions of this paper are as follows: (1) By using sojourn probability information, a new kind of switching law is proposed for the switched systems and the proposed sojourn probability for linear discrete switched systems is measured in a statistical way; (2) A new type of switched system model is constructed, which is the generalization of many practical systems; (3) By using a stochastic variable in time delay which satisfies the Bernoulli random binary distribution, a guaranteed H_∞ state estimator for a class of discrete-time neural networks is designed; (4) Specifically, we use the the piecewise Lyapunov function technique and the average dwell time approach to obtain a new set of sufficient conditions to ensure the mean-square exponential stability of the resulting error system. The derived set of sufficient conditions for ensuring exponential stability is formulated in terms of the LMIs. Finally, numerical simulations verify the effectiveness of the theoretical results.

Notations: Throughout this paper, the superscript T and (-1) stand for matrix transposition and matrix inverse, respectively. $l_2[0,\infty)$ is the space of square integrable vectors. $\mathfrak{R}^n$ denotes the n-dimensional Euclidean space. $\mathfrak{R}^{n\times m}$ is the set of all $n\times m$ real matrices. $P>0$ means that P is positively definite. I_n and 0_n stand for the $n\times n$ identity matrix and zero matrix of appropriate dimensions, respectively and $diag\{\cdots\}$ denotes a block-diagonal matrix. $\|x\|$ denotes the Euclid norm vector x. Prob(.) means the occurrence probability of the event ".". $E\{x\}$ stands for the expectation of x and $E\{x/y\}$ for the expected value of x condition on y. The notation "$\star$" always denotes the symmetric block in the symmetric matrix.

2 Problem description and preliminaries

Consider the following nonlinear discrete time switched neural network with time varying delays

$$\begin{aligned} x(k+1) &= C_{s(k)}x(k) + B_{s(k)}f(x(k)) + A_{s(k)}f(x(k-d_{s(k)}(k))) \\ &\quad + D_{s(k)}w(k), \\ x(k) &= \phi(k)\ \ \forall\ \ k\in[k_0-d_M,k_0], \end{aligned} \tag{1}$$

where $x(k) \in Z^+ \rightarrow \Re^n$ is the state vector of the neural network; $f(x(k)) = [f_1(x_1(k), f_2(x(k)), \cdots, f_n(x(k))]^{\mathrm{T}}$ is the neuron activation function; $w(k) \in \Re^m$ is the exogenous disturbance signal belonging to $l_2[0,\infty)$; $s(k): Z^+ = 0,1,2,\cdots \rightarrow 1,2,\cdots,N = \Omega$ is the switching actions independent of the state; $C_i = diag\{c_{i1}, c_{i2}, \cdots c_{in}\}$ is positive diagonal matrices that represent the self-feedback term with entries $|c_{ij}| < 1$; $A_i, B_i \in \Re^{n\times n}, D_i \in \Re^{n\times q}(i \in \Omega)$ represents the connection weight, delayed connection weight and the disturbance weight respectively for the ith subsystem; $\phi(k)$ is the initial condition, which is continuous and defined on $[k_0 - d_M, k_0]$. The probability of $s(k) = i$ is assumed to be known, i.e.,

$$\mathrm{Prob}\{s(k) = i\} = \beta_i, \quad \sum_{i=1}^{N} \beta i = 1, \tag{2}$$

where $\beta_i \in (0,1)$ is the sojourn probability of the switched system staying in the ith subsystem. It is easier to obtain the statistic information β_i through the simple statistical way as

$$\beta_i = \lim_{k\to\infty} \frac{k_i}{k},$$

where k_i is the times of $s(k) = i$ in the interval $[1,k]$, $k \in Z^+$. A set of random variables $\delta_i(k): Z^+ \rightarrow \{0,1\}$ is used,

$$\delta_i(k) = \begin{cases} 1, & s(k) = i, \\ 0, & s(k) \neq i, \end{cases} \quad i \in \Omega, \quad k \in Z^+, \tag{3}$$

and for any $k \in Z^+$

$$\sum_{i=1}^{N} \delta_i(k) = 1, \quad E\{\delta_i(k)\} = \beta_i, \quad \sum_{i=1}^{N} \beta_i = 1. \tag{4}$$

The first equation in (4) is to guarantee that there is only one active subsystem at any time. Based on (1)–(4), the switched system at $s(k) = i$ means that the ith subsystem is activated and the global model of the ith switched neural network is given by

$$\begin{aligned} x(k+1) = \sum_{i=1}^{N} &\delta_i(k)\{C_i x(k) + B_i f(x(k)) \\ &+ A_i f(x(k - d_i(k))) + D_i w(k)\}, \\ x(k) = \phi(k),& \quad k = -d^M, -d^M + 1, \cdots, 0. \end{aligned} \tag{5}$$

where $d^M = \max\{d_i^M, i \in \Omega\}$, $\phi(k)$ is the initial state of $x(k)$.

Moreover, the activation functions satisfy the following assumption [38].

Assumption 1. For $j \in 1,2,\cdots,n$, there exist constants E_j^- and E_j^+ such that

$$E_j^- \leqslant \frac{f_j(x) - f_j(y)}{x - y} \leqslant E_j^+, \quad \forall\ x, y\ \in R, \quad x \neq y.$$

For convenience of presentation, in the following, we denote

$$E_1 = diag\{E_1^- E_1^+, \cdots, E_n^- E_n^+\},$$

$$E_2 = diag\left\{\frac{E_1^- + E_1^+}{2}, \cdots, \frac{E_n^- + E_n^+}{2}\right\}.$$

In this paper, the time-varying delay is assumed to satisfy the following assumption. The time-varying delay $d_i(k)$ is bounded and for the ith subsystem it satisfies the condition $d_i^m \leqslant d_i(k) \leqslant d_i^M$ for all $k \in N$, where d_i^m and d_i^M are constant positive scalars representing the minimum and maximum delays, respectively. There exists a constant d_i^0 satisfying $d_i^m \leqslant d_i^0 \leqslant d_i^M$ such that $d_i(k)$ takes values in $[d_i^m, d_i^0]$ or $(d_i^0, d_i^M]$. Its probability distribution can be observed as Prob $\{d_i(k) \in [d_i^m, d_i^0]\} = \rho_0$ and Prob $\{d_i(k) \in (d_i^0, d_i^M]\} = 1 - \rho_0$, where $0 \leqslant \rho_0 \leqslant 1$ and this depends upon the values of d_i^m, d_i^0, d_i^M. The probability distribution of the time-varying delay is described by defining two sets:

$$\mathfrak{D}_1 = \left\{k | d_i(k) \in [d_i^m, d_i^0]\right\}, \quad \mathfrak{D}_2 = \left\{k | d_i(k) \in (d_i^0, d_i^M]\right\}. \tag{6}$$

We define the mapping functions:

$$d_{1i}(k) = \begin{cases} d_i(k) & k \in \mathfrak{D}_1, \\ d_i^m & \text{else}, \end{cases} \qquad d_{2i}(k) = \begin{cases} d_i(k) & k \in \mathfrak{D}_2, \\ d_i^0 & \text{else}. \end{cases}$$

It follows from (6) that $\mathfrak{D}_1 \cup \mathfrak{D}_2 = Z \geqslant 0$, $\mathfrak{D}_1 \cap \mathfrak{D}_2 = \Phi$ where Φ is the empty set. From these mapping functions we can see that if $k \in \mathfrak{D}_1$ then the event $d_i(k) \in [d_i^m, d_i^0]$ and if $k \in \mathfrak{D}_2$ implies that $d_i(k) \in (d_i^0, d_i^M]$.

We also define the stochastic variable as

$$\rho(k) = \begin{cases} 1, & k \in \mathfrak{D}_1, \\ 0, & k \in \mathfrak{D}_2. \end{cases}$$

The variable $\rho(k)$ is a Bernoulli distributed white sequence with Prob $\{\rho(k) = 1\}$ = Prob $\{d_i(k) \in= (d_i^m, d_i^0]\} = E[\rho(k)] = \rho_0$ and Prob $\{\rho(k) = 0\}$ = Prob $\{d_i(k) \in (d_i^0, d_i^M]\} = 1 - E[\rho(k)] = 1 - \rho_0$. Furthermore, we can show that $E[\rho(k) - \rho_0] = 0$ and $E[\rho(k) - \rho_0]^2 = \rho_0\bar{\rho_0}$ where $\bar{\rho_0} = 1 - \rho_0$.

Based on the above assumptions the discrete-time switched neural network (5) which depends upon the distributed sequence can be written as

$$\begin{aligned} x(k+1) = \sum_{i=1}^{N} &\beta_i(k)\{C_i x(k) + B_i f(x(k)) \\ &+ \rho(k) A_i f(x(k - d_{1i}(k))) \\ &+ (1 - \rho(k)) A_i f(x(k - d_{2i}(k))) + D_i w(k)\}. \end{aligned} \tag{7}$$

By incorporating the randomly occurring time-varying delays $d_{1i}(k)$ and $d_{2i}(k)$ to the system we get

$$\begin{aligned} x(k+1) = \sum_{i=1}^{N} &\beta_i(k)\{C_i x(k) + B_i f(x(k)) + \rho_0 A_i f(x(k - d_{1i}(k))) \\ &+ (1 - \rho_0) A_i f(x(k - d_{2i}(k))) \\ &+ (\rho(k) - \rho_0) A_i [f(x(k - d_{1i}(k))) \end{aligned}$$

$$- f(x(k - d_{2i}(k)))] + D_i w(k)\}. \quad (8)$$

The neuron states are not fully obtainable in the network outputs. As we need an efficient estimate, we assume the network output $y(k)$ and the signal $z(k)$ to be estimated as

$$y(k) = \sum_{i=1}^{N} \beta_i(k) \{F_i x(k) + H_i f(x(k)) + G_i w(k)\},$$
$$z(k) = \sum_{i=1}^{N} \beta_i(k) \{M_{zi} x(k)\}, \quad (9)$$

where F_i, H_i, G_i, M_{zi} are known real constant matrices with appropriate dimensions.

This paper designs an efficient filter to estimate the neuron state from the available network output. The filtering problem consists of obtaining the estimate $\hat{z}(k)$ of the signal $z(k)$ with the estimation error $\hat{z}(k) - z(k)$, for all non-zero $w(k) \in l_2[0, \infty)$. For the estimation of $z(k)$, we consider the full order filter for the neural network (8) as

$$\hat{x}(k+1) = C_{di} x(k) + B_{di} y(k),$$
$$\hat{z}(k) = M_{di} x(k),$$
$$\hat{x}(k_0) = 0, \quad (10)$$

where $\hat{x}(k) \in R^n$ is the state vector and $\hat{z}(k) \in R^p$ is the output signal of the filter. C_{di}, B_{di}, M_{di} are the filter parameters to be designed.

We define the new state vector $\tilde{x}(k+1) = [x^{\mathrm{T}}(k) \;\; \hat{x}^{\mathrm{T}}(k)]^{\mathrm{T}}$ and the filtering error vector as $\tilde{z} = z(k) - \hat{z}(k)$, then from eqs. (8)–(10), the augmented system can be obtained as follows:

$$\begin{aligned}\tilde{x}(k+1) = \sum_{i=1}^{N} \beta_i(k)\{&\mathcal{C}_i \tilde{x}(k) + \mathcal{B}_i f(\tilde{x}(k)) \\ &+ \rho_0 \mathcal{A}_i f(\tilde{x}(k - d_{1i}(k))) \\ &+ \bar{\rho}_0 \mathcal{A}_i f(\tilde{x}(k - d_{2i}(k))) \\ &+ (\rho(k) - \rho_0)\mathcal{A}_i [f(\tilde{x}(k - d_{1i}(k))) \\ &- f(\tilde{x}(k - d_{2i}(k)))] + \mathcal{D}_i w(k)\}, \end{aligned} \quad (11)$$
$$\tilde{z}(k) = \sum_{i=1}^{N} \beta_i(k) \mathcal{M}_i \tilde{x}(k),$$
$$\tilde{x}(k_0) = \tilde{x}_0,$$

where $\tilde{x}_0 = [x(k_0) \;\; 0]^{\mathrm{T}}$,

$$\mathcal{C}_i = \begin{bmatrix} C_i & 0 \\ B_{di} E_i & C_{di} \end{bmatrix}, \mathcal{B}_i = \begin{bmatrix} B_i \\ B_{di} H_i \end{bmatrix}, \mathcal{A}_i = \begin{bmatrix} A_i \\ 0 \end{bmatrix},$$
$$\mathcal{D}_i = \begin{bmatrix} D_i \\ B_{di} G_i \end{bmatrix}, \mathcal{M}_i = [M_{zi} \;\; -M_{di}].$$

The objective of this paper is to focus on the sufficient conditions for the filtering problem (11) such that the filtering error is exponentially mean square stable and the H_∞ performance constraint is satisfied under known sojourn probabilities.

To obtain the main results of the paper, we need to introduce the following lemma and definition.

Definition 1 [30]. The switched neural network (11) with $w(k) = 0$ is said to be exponentially mean-square stable under switching signal $s(k)$, if there exist scalars $\Re > 0$ and $0 < \chi < 1$, such that the solution $\tilde{x}(k)$ satisfies $\mathrm{E}[\|\tilde{x}(k)\|^2] < \Re \chi^{(k-k_0)} \mathrm{E}\|\tilde{x}(k_0)\|_L^2, k \geqslant k_0$, where $\|\tilde{x}(k_0)\|_L = sup_{k_0 - d_M \leqslant \theta \leqslant k_0} \|\tilde{x}(\theta)\|$, and χ is called the decay rate.

Definition 2. For any $T_2 > T_1 \geqslant 0$, let $N_s(T_1, T_2)$ denote the number of switching numbers of $s(k)$ over (T_1, T_2). If $N_s(T_1, T_2) \leqslant N_0 + (T_2 - T_1)/T_a$ holds for $T_a > 0$, $N \geqslant 0$, then T_a is called the average dwell time and N_0 is the chatter bound.

Definition 3. For given scalars γ and $0 < \alpha < 1$, the switched neural network (11) is said to be exponentially mean-square stable with guaranteed H_∞ disturbance attenuation level γ, if it is exponentially mean-square stable under zero initial conditions and satisfies $\sum_{s=k_0}^{\infty} (1-\alpha)^s \tilde{z}^{\mathrm{T}}(s) \tilde{z}(s) \leqslant \sum_{s=k_0}^{\infty} \gamma^2 v^{\mathrm{T}}(s) v(s)$ for every non-zero $v(k) \in l_2[0, \infty)$.

Lemma 1 [39]. For any constant positive-definite matrix $S \in \Re^{n \times n}$, $S = S^{\mathrm{T}}$, two scalars $M \geqslant N > 0$, such that the sums concerned are well defined, then

(1) $$\left[\sum_{i=1}^{N} x(i)\right]^{\mathrm{T}} S \left[\sum_{i=1}^{N} x(i)\right] \leqslant N \left[\sum_{i=1}^{N} x^{\mathrm{T}}(i) S x(i)\right].$$

(2) $$\left[\sum_{i=k-M}^{k-N-1} \sum_{j=i}^{k-N-1} x(j)\right]^{\mathrm{T}} S \left[\sum_{i=k-M}^{k-N-1} \sum_{j=i}^{k-N-1} x(j)\right] \leqslant \frac{(M-N)(M-N+1)}{2} \sum_{i=k-M}^{k-N-1} \sum_{j=i}^{k-N-1} x^{\mathrm{T}}(j) S x(j).$$

(3) $$\left[\sum_{i=-M}^{-N-1} \sum_{j=k+i}^{k-1} x(j)\right]^{\mathrm{T}} S \left[\sum_{i=-M}^{-N-1} \sum_{j=k+i}^{k-1} x(j)\right] \leqslant \frac{(M-N)(M+N+1)}{2} \sum_{i=-M}^{-N-1} \sum_{j=k+i}^{k-1} x^{\mathrm{T}}(j) S x(j).$$

Lemma 2 [40]. For any vectors δ_1, δ_2, if constant matrices R, S and real scalars $\varsigma_1 \geqslant 0$, $\varsigma_2 \geqslant 0$ satisfying that $\left[\begin{smallmatrix} R & S \\ * & R \end{smallmatrix}\right] \geqslant 0$ and $\varsigma_1 + \varsigma_2 = 1$, then the following inequality holds:

$$-\frac{1}{\varsigma_1} \delta_1^{\mathrm{T}} R \delta_1 - \frac{1}{\varsigma_2} \delta_2^{\mathrm{T}} R \delta_2 \leqslant - \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R & S \\ * & R \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}.$$

Lemma 3 [41]. Suppose that Ω_1, Ξ_1, Ξ_2 are the constant matrices of appropriate dimensions, $\alpha \in [0, 1]$, then $\Omega_1 + [\alpha \Xi_1 + (1-\alpha) \Xi_2] \leqslant 0$ holds, if the inequalities $\Omega_1 + \Xi_1 \leqslant 0$ and $\Omega_1 + \Xi_2 \leqslant 0$ hold simultaneously.

Lemma 4 [42]. For the symmetric matrices $R > 0$, Ξ and matrix Γ, the following statements are equivalent:

(1) $\Xi - \Gamma^{\mathrm{T}} R \Gamma < 0$.

(2) There exists an appropriate dimensional matrix Π such that

$$\begin{bmatrix} \Xi + \Gamma^{\mathrm{T}}\Pi + \Pi^{\mathrm{T}}\Gamma & \Pi^{\mathrm{T}} \\ * & -R \end{bmatrix} < 0.$$

Lemma 5 [43]**.** Given matrices A, $P_0 = P_0^{\mathrm{T}}$ and $P_1 > 0$, then $A^{\mathrm{T}}P_1A - P_0 < 0$ holds if and only if there exists a matrix Y such that

$$\begin{bmatrix} -P_0 & A^{\mathrm{T}}Y \\ \star & P_1 - Y - Y^{\mathrm{T}} \end{bmatrix} < 0.$$

3 Main results

In this section, our aim is to deal with the exponential stability results under known sojourn probabilities of the subsystems with random time-varying delays.

Theorem 1. Under Assumption 1, for given scalars $\mu \geqslant 1, 0 \leqslant \alpha \leqslant 1$, the filtering error system (11) is said to be exponentially mean square stable with a H_∞ performance level $\gamma > 0$, if there exist symmetric positive definite matrices $P_i, Q_{1i}, R_{1i}, S_{1i}, T_{1i}, Z_{ui}, G_{ui}, H_{ui} (u = 1, 2)$, positive diagonal matrices $\Lambda_{1i}, \Lambda_{2i}, \Lambda_{3i}$, matrices $W, L_{1i}, \Pi = [\Pi_1 \ \ \Pi_2]$ of appropriate dimensions and for any switching signal $s(k)$ with average dwell time satisfying $T_a \geqslant T_a^* = \frac{ln\mu}{(ln(1-\alpha))}$, such that the following LMI holds for all $i, m \in \Omega, i \neq m$ and for $i \in \Omega$:

$$\tilde{\Xi} = \begin{bmatrix} \Xi_{11} & 0 & \tilde{\Psi}_i P_i \\ \star & -\gamma^2 I & \mathcal{D}_i^{\mathrm{T}} P_i \\ \star & \star & \Xi_{33} \end{bmatrix} < 0, \tag{12}$$

where

$$P_i \leqslant \mu P_m, Q_{1i} \leqslant \mu Q_{1m}, R_{1i} \leqslant \mu R_{1m}, S_{1i} \leqslant \mu S_{1m}, T_{1i} \leqslant \mu T_{1m},$$

$$Z_{ui} \leqslant \mu Z_{um}, G_{ui} \leqslant \mu G_{um}, H_{ui} \leqslant \mu H_{um}, u = 1, 2, \tag{13}$$

$$\begin{aligned} \tilde{\Psi}_i = [\, & \sqrt{\beta_i} C_i^{\mathrm{T}} \cdots \ 0 \ \ 0 \ \cdots \sqrt{\beta_i}\mathcal{B}_i^{\mathrm{T}} \ \cdots \sqrt{\beta_i}\rho_0\mathcal{A}_i^{\mathrm{T}} \cdots \\ & \sqrt{\beta_i}\bar{\rho}_0\mathcal{A}_i^{\mathrm{T}} \cdots \ 0 \ \ 0 \ \ 0 \ \cdots]^{\mathrm{T}}, \end{aligned} \tag{14}$$

$$\Xi_{11} = \begin{bmatrix} \tilde{\Phi}_{11} + \Gamma_i^{\mathrm{T}}\Pi + \Pi\Gamma_i^{\mathrm{T}} & \Pi_1 & \Pi_2 \\ \star & \tilde{\Phi}_{22} & L_{1i} \\ \star & \star & \tilde{\Phi}_{33} \end{bmatrix},$$

$$\Xi_{33} = diag\{-P_1, \cdots, -P_1, \cdots, -P_N, \cdots, -P_N\},$$

$$\tilde{\Phi}_{11} = \begin{bmatrix}
\Phi_1 & 0 & \Phi_2 & \Phi_3 & 0 & 0 & 0 & 0 & \Phi_4 & 0 & 0 & 0 & 0 & 0 \\
\star & \Phi_5 & 0 & 0 & \Phi_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \Phi_7 & 0 & 0 & \Phi_8 & \Phi_9 & \Phi_{10} & 0 & 0 & 0 & 0 & \Phi_{11} & 0 \\
\star & \star & \star & \Phi_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \Phi_{13} & \Phi_{14} & 0 & 0 & \Phi_{15} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \Phi_{16} & 0 & 0 & \Phi_{17} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \Phi_{18} & 0 & \Phi_{19} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \Phi_{20} & 0 & 0 & 0 & 0 & \Phi_{21} & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \Phi_{22} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \Phi_{23} & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \Phi_{24} & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \Phi_{25} & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \Phi_{26} & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \Phi_{27}
\end{bmatrix}$$

with

$$\Phi_1 = diag\{\Phi_{11}, \cdots, \Phi_{11}, \cdots, \Phi_{1N}, \cdots, \Phi_{1N}\},$$

$$\begin{aligned} \Phi_{1i} = & -\alpha_1 P_i + d_i^{\mu} Q_{1i} + d_i^{\gamma} R_{1i} + d_i^{r} Z_{1i} \\ & + d_i^{M} Z_{2i} - \alpha_i^{dM} G_{2i} - E_{1i}\Lambda_{1i} + \mathcal{M}_i^{\mathrm{T}}\mathcal{M}_i, \end{aligned}$$

$$\Phi_2 = \left[\alpha_1^{dM} G_{21} \ \cdots \ \alpha_N^{dM} G_{2N}\right],$$

$$\Phi_3 = [E_{21}\Lambda_{11} \ \cdots \ E_{2N}\Lambda_{1N}],$$

$$\Phi_4 = \sum_{j=1}^{N} \sqrt{\beta_j}\left(C_j - I\right)^{\mathrm{T}} W,$$

$$\Phi_5 = diag\{-\Phi_{51}, \cdots, -\Phi_{51}, \cdots, -\Phi_{5N}, \cdots, -\Phi_{5N}\},$$

$$\Phi_{5i} = \alpha_i^{do} Q_{1i} + E_{2i}\Lambda_{2i},$$

$$\Phi_6 = [E_{21}\Lambda_{21} \ \cdots \ E_{2N}\Lambda_{2N}],$$

$$\Phi_7 = diag\{-\Phi_{71}, \cdots, -\Phi_{71}, \cdots, -\Phi_{7N}, \cdots, -\Phi_{7N}\},$$

$$\Phi_{7i} = \alpha_i^{dM} R_{1i} + \alpha_i^{dr} G_{2i} + \alpha_i^{dM} G_{2i} + \alpha_i^{dM} H_{2i} + E_{1i}\Lambda_{3i},$$

$$\Phi_8 = [E_{21}\Lambda_{31} \ \cdots \ E_{2N}\Lambda_{3N}],$$

$$\Phi_9 = \left[\alpha_1^{dr} G_{11} \ \cdots \ \alpha_N^{dr} G_{1N}\right],$$

$$\Phi_{10} = \left[\alpha_1^{dr} G_{21} \ \cdots \ \alpha_N^{dr} G_{2N}\right],$$

$$\Phi_{11} = \left[\frac{\alpha_1^M}{d_1^{2M}} H_{21} \ \cdots \ \frac{\alpha_N^M}{d_N^{2M}} H_{2N}\right],$$

$$\Phi_{12} = diag\{\Phi_{121}, \cdots, \Phi_{121}, \cdots, \Phi_{12N}, \cdots, \Phi_{12N}\},$$

$$\Phi_{12i} = d_i^{\mu} S_{1i} + d_i^{\gamma} T_{1i} - \Lambda_{1i},$$

$$\Phi_{13} = diag\{\Phi_{131}, \cdots, \Phi_{131}, \cdots, \Phi_{13N}, \cdots, \Phi_{13N}\},$$

$$\Phi_{13i} = -\alpha_i^{d0} S_{1i} + \sum_{j=1}^{N} \beta_j \rho_0 \bar{\rho}_0 \mathcal{A}_j^{\mathrm{T}} P_i \mathcal{A}_j - \Lambda_{2i},$$

$$\Phi_{14} = \left[\sum_{j=1}^{N} \beta_j \rho_0 \bar{\rho}_0 \mathcal{A}_j^{\mathrm{T}} P_1 \mathcal{A}_j \ \cdots \ \sum_{j=1}^{N} \beta_j \rho_0 \bar{\rho}_0 \mathcal{A}_j^{\mathrm{T}} P_N \mathcal{A}_j\right],$$

$$\Phi_{15} = \sum_{j=1}^{N} \sqrt{\beta_j}\mathcal{B}_j^{\mathrm{T}} W,$$

$$\Phi_{16} = diag\{\Phi_{161}, \cdots, \Phi_{161}, \cdots, \Phi_{16N}, \cdots, \Phi_{16N}\},$$

$$\Phi_{16i} = -\alpha_i^{M} T_{1i} + \sum_{j=1}^{N} \beta_j \rho_0 \bar{\rho}_0 \mathcal{A}_j^{\mathrm{T}} P_i \mathcal{A}_j - \Lambda_{3i},$$

$$\Phi_{17} = \sum_{j=1}^{N} \sqrt{\beta_j} C_j^{\mathrm{T}},$$

$$\Phi_{18} = diag\{-\Phi_{181}, \cdots, -\Phi_{181}, \cdots, -\Phi_{18N}, \cdots, -\Phi_{18N}\},$$

$$\Phi_{18i} = \alpha_i^{dr} G_{1i} + \alpha_i^{dr} G_{2i},$$

$$\Phi_{19} = \sum_{j=1}^{N} \sqrt{\beta_j} C_j^{\mathrm{T}} W,$$

$$\Phi_{20} = diag\{-\Phi_{201}, \cdots, -\Phi_{201}, \cdots, -\Phi_{20N}, \cdots, -\Phi_{20N}\},$$

$$\Phi_{20i}=\alpha_i^{dr}G_{1i}+\alpha_i^{M}H_{2i},$$

$$\Phi_{21}=\left[\frac{\alpha_1^M}{d_1^{2M}}H_{21}\ \cdots\ \frac{\alpha_N^M}{d_N^{2M}}H_{2N}\right],$$

$$\Phi_{22}=diag\{\Phi_{221},\cdots,\Phi_{221},\cdots,\Phi_{22N},\cdots,\Phi_{22N}\},$$

$$\Phi_{22i}=d_i^rG_{1i}+d_i^MG_{2i}+d_i^{*2}H_{1i}+\frac{d_i^r(d_i^r+1)}{4}H_{2i}-W-W^{\mathrm{T}},$$

$$\Phi_{23}=diag\{-\Phi_{231},\cdots,-\Phi_{231},\cdots,-\Phi_{23N},\cdots,-\Phi_{23N}\},$$

$$\Phi_{23i}=\alpha_i^{dr}Z_{1i}+\alpha_i^{dr}Z_{2i}+\alpha_i^MH_{1i},$$

$$\Phi_{24}=diag\{-\Phi_{241},\cdots,-\Phi_{241},\cdots,-\Phi_{24N},\cdots,-\Phi_{24N}\},$$

$$\Phi_{24i}=\alpha_i^{dr}Z_{1i}+\alpha_i^MH_{1i},$$

$$\Phi_{25}=diag\{-\Phi_{251},\cdots,-\Phi_{251},\cdots,-\Phi_{25N},\cdots,-\Phi_{25N}\},$$

$$\Phi_{25i}=\frac{\alpha_i^M}{(d_i^{2M})^2}H_{2i},$$

$$\Phi_{26}=diag\{-\Phi_{261},\cdots,-\Phi_{261},\cdots,-\Phi_{26N},\cdots,-\Phi_{26N}\},$$

$$\Phi_{26i}=\frac{\alpha_i^M}{(d_i^{2m})^2}H_{2i},$$

$$\Phi_{27}=diag\{-\Phi_{271},\cdots,-\Phi_{271},\cdots,-\Phi_{27N},\cdots,-\Phi_{27N}\},$$

$$\Phi_{27i}=\frac{\alpha_i^M}{d_i^M}Z_{2i},$$

$$\tilde{\Phi}_{22}=\tilde{\Phi}_{33}=diag\{-H_{11},\cdots,-H_{11},\cdots,-H_{1N},\cdots,-H_{1N}\},$$

$$\Gamma_1=\begin{bmatrix} I_n & \cdots & 0_{n\times 11n} & \cdots & -I_n & \cdots & 0_n & \cdots\\ 0_n & \cdots & 0_{n\times 11n} & \cdots & 0_n & \cdots & 0_n & \cdots\end{bmatrix},$$

$$\Gamma_2=\begin{bmatrix} 0_n & \cdots & 0_{n\times 12n} & \cdots & 0_n & \cdots & 0_n\\ I_n & \cdots & 0_{n\times 12n} & \cdots & -I_n & \cdots & 0_n\end{bmatrix},$$

$$\alpha_1=1-\alpha,\ \alpha_i^m=(1-\alpha)^{d_i^m},\ \alpha_i^M=(1-\alpha)^{d_i^M},$$

$$\alpha_i^{d0}=(1-\alpha)^{d_i^0},\ \alpha_i^{dm}=\frac{(1-\alpha)^{d_i^m}}{d_i^m},$$

$$\alpha_i^{dM}=\frac{(1-\alpha)^{d_i^M}}{d_i^M},\ \alpha_i^{dr}=\frac{(1-\alpha)^{d_i^M}}{d_i^r},$$

$$d_i^*=\frac{(d_i^r)(d_i^M+d_i^m+1)}{2},\ d_i^r=d_i^M-d_i^m,$$

$$d_i^\mu=1+d_i^0-d_i^m,\quad d_i^\gamma=1+d_i^M-d_i^0,$$

$$d_i^{2m}=d_{2i}(k)-d_i^m,\quad d_i^{2M}=d_i^M-d_{2i}(k),$$

$$\varepsilon_i=d_i^M-d_{1i}(k),\quad \varpi_i=d_{1i}(k)-d_i^m.$$

Moreover, an estimate of the state decay is given by

$$E\left[\|\tilde{x}(k)\|^2\right]\leqslant\sqrt{\frac{\chi_2}{\chi_1}}\psi^{k-k_0}E\|\tilde{x}(k_0)\|_L^2,$$

where

$$\chi_1=\min_{\forall i\in N}\lambda_{\min}(P_i),$$

$$\begin{aligned}\chi_2=&\max_{\forall i\in N}\lambda_{\max}(P_i)+d_i^\mu(\max_{\forall i\in N}\lambda_{\max}(Q_{1i})+\hat{f}^2(\max_{\forall i\in N}\lambda_{\max}(S_{1i}))\\&+d_i^\gamma(\max_{\forall i\in N}\lambda_{\max}(R_{1i})+\hat{f}^2(\max_{\forall i\in N}\lambda_{\max}(T_{1i}))\\&+4d_i^r\max_{\forall i\in N}\lambda_{\max}(Z_{1i})+4d_i^M\max_{\forall i\in N}\lambda_{\max}(Z_{2i})\\&+4d_i^r\max_{\forall i\in N}\lambda_{\max}(G_{1i})+4d_i^M\max_{\forall i\in N}\lambda_{\max}(G_{2i})\\&+d_i^{*2}\max_{\forall i\in N}\lambda_{\max}(H_{1i})+\frac{d_i^r(d_i^r+1)}{4}\max_{\forall i\in N}\lambda_{\max}(H_{2i}),\end{aligned}$$

$$\hat{f}=\max_{1\leqslant s\leqslant n}\{|E_s^-|,|E_s^+|\}.$$

Proof. The Lyapunov-Krasovskii functional candidate is constructed in order to find the exponential mean square stability of filtering error system:

$$V_i(k)=\sum_{r=1}^{9}V_{ri}(k),\tag{15}$$

where

$$V_{1i}(k)=\sum_{i=1}^{N}\tilde{x}^{\mathrm{T}}(k)P_i\tilde{x}(k),$$

$$\begin{aligned}V_{2i}(k)=\sum_{i=1}^{N}\Bigg[&\sum_{s=k-d_{1i}(k)}^{k-1}(1-\alpha)^{(k-s-1)}\tilde{x}^{\mathrm{T}}(s)Q_{1i}\tilde{x}(s)\\&+\sum_{j=-d_i^0+1}^{-d_i^m}\sum_{s=k+j}^{k-1}(1-\alpha)^{(k-s-1)}\tilde{x}^{\mathrm{T}}(s)Q_{1i}\tilde{x}(s)\Bigg],\end{aligned}$$

$$\begin{aligned}V_{3i}(k)=\sum_{i=1}^{N}\Bigg[&\sum_{s=k-d_{2i}(k)}^{k-1}(1-\alpha)^{(k-s-1)}\tilde{x}^{\mathrm{T}}(s)R_{1i}\tilde{x}(s)\\&+\sum_{j=-d_i^M+1}^{-d_i^0-1}\sum_{s=k+j}^{k-1}(1-\alpha)^{(k-s-1)}\tilde{x}^{\mathrm{T}}(s)R_{1i}\tilde{x}(s)\Bigg],\end{aligned}$$

$$\begin{aligned}V_{4i}(k)=\sum_{i=1}^{N}\Bigg[&\sum_{s=k-d_{1i}(k)}^{k-1}(1-\alpha)^{(k-s-1)}f^{\mathrm{T}}(\tilde{x}(s))S_{1i}f(\tilde{x}(s))\\&+\sum_{j=-d_i^0+1}^{-d_i^m-1}\sum_{s=k+j}^{k-1}(1-\alpha)^{(k-s-1)}f^{\mathrm{T}}(\tilde{x}(s))S_{1i}f(\tilde{x}(s))\Bigg],\end{aligned}$$

$$\begin{aligned}V_{5i}(k)=\sum_{i=1}^{N}\Bigg[&\sum_{s=k-d_{2i}(k)}^{k-1}(1-\alpha)^{(k-s-1)}f^{\mathrm{T}}(\tilde{x}(s))T_{1i}f(\tilde{x}(s))\\&+\sum_{j=-d_i^M+1}^{-d_i^0-1}\sum_{s=k+j}^{k-1}(1-\alpha)^{(k-s-1)}f^{\mathrm{T}}(\tilde{x}(s))T_{1i}f(\tilde{x}(s))\Bigg],\end{aligned}$$

$$\begin{aligned}V_{6i}(k)=\sum_{i=1}^{N}\Bigg[&\sum_{j=-d_i^M}^{-d_i^m-1}\sum_{s=k+j}^{k-1}(1-\alpha)^{(k-s-1)}\tilde{x}^{\mathrm{T}}(s)Z_{1i}\tilde{x}(s)\\&+\sum_{j=-d_i^M}^{-1}\sum_{s=k+j}^{k-1}(1-\alpha)^{(k-s-1)}\tilde{x}^{\mathrm{T}}(s)Z_{2i}\tilde{x}(s)\Bigg],\end{aligned}$$

$$V_{7i}(k) = \sum_{i=1}^{N}\left[\sum_{j=-d_i^M}^{-d_i^m-1}\sum_{s=k+j}^{k-1}(1-\alpha)^{(k-s-1)}\eta^{\mathrm{T}}(s)G_{1i}\eta(s) + \sum_{j=-d_i^M}^{-1}\sum_{s=k+j}^{k-1}(1-\alpha)^{(k-s-1)}\eta^{\mathrm{T}}(s)G_{2i}\eta(s)\right],$$

$$V_{8i}(k) = \sum_{i=1}^{N}\left[d_i^*\sum_{l=-d_i^M}^{-d_i^m-1}\sum_{j=l}^{-1}\sum_{s=k+j}^{k-1}(1-\alpha)^{(k-s-1)}\eta^{\mathrm{T}}(s)H_{1i}\eta(s)\right],$$

$$V_{9i}(k) = \frac{1}{2}\sum_{i=1}^{N}\left[\sum_{l=-d_i^M}^{k-d_i^m-1}\sum_{j=l}^{k-d_i^m-1}\sum_{s=j}^{k-1}(1-\alpha)^{(k-s-1)}\eta^{\mathrm{T}}(s)H_{2i}\eta(s)\right].$$

where $\eta(k) = \tilde{x}(k+1) - \tilde{x}(k)$.

Calculating the difference $\Delta V_i(k) = V_i(k+1) - V_i(k)$ along the solutions and taking the mathematical expectations, we get

$$\begin{aligned}
&E\{\Delta V_{1i}(k) + \alpha V_{1i}(k)\}\\
&=E\left\{\sum_{i=1}^{N}\beta_i[\tilde{x}^{\mathrm{T}}(k+1)P_i\tilde{x}(k+1)-\tilde{x}^{\mathrm{T}}(k)P_i\tilde{x}(k)+\alpha\tilde{x}^{\mathrm{T}}(k)P_i\tilde{x}(k)]\right\}\\
&=E\left\{\sum_{i=1}^{N}\beta_i[C_i\tilde{x}(k) + B_i(f(\tilde{x}(k)) + \rho_0A_if(\tilde{x}(k-d_{1i}(k)))\right.\\
&\quad+\bar{\rho_0}A_if(\tilde{x}(k-d_{2i}(k)) + D_iw(k)]^{\mathrm{T}}P_i[C_i\tilde{x}(k) + B_i(f(\tilde{x}(k))\\
&\quad+\rho_0A_if(\tilde{x}(k-d_{1i}(k))) + \bar{\rho_0}A_if(\tilde{x}(k-d_{2i}(k)))) + D_iw(k)]\\
&\quad+\rho_0\bar{\rho_0}\,[f(\tilde{x}(k-d_{1i}(k)) - f(\tilde{x}(k-d_{2i}(k)))]^{\mathrm{T}}A_i^{\mathrm{T}}P_iA_i\\
&\quad\left.\times f(\tilde{x}(k-d_{1i}(k)) - f(\tilde{x}(k-d_{2i}(k)))] - \alpha_1\tilde{x}(k)P_i\tilde{x}(k)\right\}.
\end{aligned} \tag{16}$$

$$E\{\Delta V_{2i}(k) + \alpha V_{2i}(k)\} \leqslant E\left\{\sum_{i=1}^{N}\left[(d_i^{\mu})\tilde{x}^{\mathrm{T}}(k)Q_{1i}\tilde{x}(k) - \alpha_i^{d0}\tilde{x}^{\mathrm{T}}(k-d_{1i}(k))Q_{1i}\tilde{x}(k-d_{1i}(k))\right]\right\}. \tag{17}$$

$$E\{\Delta V_{3i}(k) + \alpha V_{3i}(k)\} \leqslant E\left\{\sum_{i=1}^{N}[(d_i^{\gamma})\tilde{x}^{\mathrm{T}}(k)R_{1i}\tilde{x}(k) - \alpha_i^M\tilde{x}^{\mathrm{T}}(k-d_{2i}(k))R_{1i}\tilde{x}(k-d_{2i}(k))]\right\}. \tag{18}$$

$$E\{\Delta V_{4i}(k) + \alpha V_{4i}(k)\} \leqslant E\left\{\sum_{i=1}^{N}[(d_i^{\mu})f^{\mathrm{T}}(\tilde{x}(k))S_{1i}f(\tilde{x}(k))) - \alpha_i^{d0}f^{\mathrm{T}}(\tilde{x}(k-d_{1i}(k)))S_{1i}f(\tilde{x}(k-d_{1i}(k)))]\right\}. \tag{19}$$

$$E\{\Delta V_{5i}(k) + \alpha V_{5i}(k)\} \leqslant E\left\{\sum_{i=1}^{N}[(d_i^{\gamma})f^{\mathrm{T}}(\tilde{x}(k))T_{1i}f(\tilde{x}(k)) - \alpha_i^Mf^{\mathrm{T}}(\tilde{x}(k-d_{2i}(k)))T_{1i}f(\tilde{x}(k-d_{2i}(k)))]\right\}. \tag{20}$$

$$\begin{aligned}
&E\{\Delta V_{6i}(k) + \alpha V_{6i}(k)\}\\
&= E\left\{\sum_{i=1}^{N}\left\{\sum_{j=-d_i^M}^{-d_i^m-1}\left[\sum_{s=k+1+j}^{k}(1-\alpha)^{k-s}\tilde{x}^{\mathrm{T}}(s)Z_{1i}\tilde{x}(s) - \sum_{s=k+j}^{k-1}(1-\alpha)^{k-s}\tilde{x}^{\mathrm{T}}(s)Z_{1i}\tilde{x}(s)\right]\right.\right.\\
&\quad\left.\left.+\sum_{j=-d_i^M}^{-1}\left[\sum_{s=k+1+j}^{k}(1-\alpha)^{k-s}\times\tilde{x}^{\mathrm{T}}(s)Z_{2i}\tilde{x}(s) - \sum_{s=k+j}^{k-1}(1-\alpha)^{k-s}\tilde{x}^{\mathrm{T}}(s)Z_{2i}\tilde{x}(s)\right]\right\}\right\}
\end{aligned}$$

$$\begin{aligned}
&E\{\Delta V_{6i}(k) + \alpha V_{6i}(k)\},\\
&= E\left\{\sum_{i=1}^{N}\left\{\tilde{x}^{\mathrm{T}}(k)(d_i^rZ_{1i} + d_i^MZ_{2i})\tilde{x}(k) - \alpha_i^M\sum_{k-d_i^M}^{k-d_i^m-1}\tilde{x}^{\mathrm{T}}(s)Z_{1i}\tilde{x}(s) - \alpha_i^M\sum_{s=k-d_i^M}^{k-1}\tilde{x}^{\mathrm{T}}(s)Z_{2i}\tilde{x}(s)\right\}\right\}.
\end{aligned} \tag{21}$$

By using Lemma 2, we have

$$\begin{aligned}
&-\sum_{s=k-d_i^M}^{k-d_i^m-1}\tilde{x}^{\mathrm{T}}(s)Z_{1i}\tilde{x}(s)\\
&= -\sum_{s=k-d_i^M}^{k-d_{1i}(k)-1}\tilde{x}^{\mathrm{T}}(s)Z_{1i}\tilde{x}(s) - \sum_{s=k-d_{1i}(k)}^{k-d_i^m-1}\tilde{x}^{\mathrm{T}}(s)Z_{1i}\tilde{x}(s)\\
&\leqslant -\frac{1}{d_i^r}\left\{\sum_{s=k-d_i^M}^{k-d_{1i}(k)-1}\tilde{x}^{\mathrm{T}}(s)Z_{1i}\sum_{s=k-d_i^M}^{k-d_{1i}(k)-1}\tilde{x}(s) + \sum_{s=k-d_{1i}(k)}^{k-d_i^m-1}\tilde{x}^{\mathrm{T}}(s)Z_{1i}\sum_{s=k-d_{1i}(k)}^{k-d_i^m-1}\tilde{x}(s)\right\},
\end{aligned} \tag{22}$$

$$\begin{aligned}
&-\sum_{s=k-d_i^M}^{k-1}\tilde{x}^{\mathrm{T}}(s)Z_{2i}\tilde{x}(s)\\
&= -\sum_{s=k-d_i^M}^{k-d_{1i}(k)-1}\tilde{x}^{\mathrm{T}}(s)Z_{2i}\tilde{x}(s) - \sum_{s=k-d_{1i}(k)}^{k-1}\tilde{x}^{\mathrm{T}}(s)Z_{2i}\tilde{x}(s)\\
&\leqslant -\frac{1}{d_i^r}\sum_{s=k-d_i^M}^{k-d_{1i}(k)-1}\tilde{x}^{\mathrm{T}}(s)Z_{2i}\sum_{s=k-d_i^M}^{k-d_{1i}(k)-1}\tilde{x}(s) - \frac{1}{d_i^M}\sum_{s=k-d_{1i}(k)}^{k-1}\tilde{x}^{\mathrm{T}}(s)Z_{2i}\sum_{s=k-d_{1i}(k)}^{k-1}\tilde{x}(s).
\end{aligned} \tag{23}$$

$$\begin{aligned}
&E\{\Delta V_{7i}(k) + \alpha V_{7i}(k)\}\\
&= E\left\{\sum_{i=1}^{N}\left\{\sum_{j=-d_i^M}^{-d_i^m-1}\left[\sum_{s=k+1+j}^{k}(1-\alpha)^{k-s}\eta^{\mathrm{T}}(s)G_{1i}\eta(s)\right.\right.\right.
\end{aligned}$$

$$- \sum_{s=k+j}^{k-1} (1-\alpha)^{k-s}\eta^{\mathrm{T}}(s)G_{1i}\eta(s)\Bigg]$$

$$+ \sum_{j=-d_i^M}^{-1}\left[\sum_{s=k+1+j}^{k}(1-\alpha)^{k-s}\eta^{\mathrm{T}}(s)G_{2i}\eta(s)\right.$$

$$\left.\left.\left. - \sum_{s=k+j}^{k-1}(1-\alpha)^{k-s}\eta^{\mathrm{T}}(s)G_{2i}\eta(s)\right]\right\}\right\}, \tag{24}$$

$$E\{\Delta V_{7i}(k)+\alpha V_{7i}(k)\}$$

$$= E\Bigg\{\sum_{i=1}^{N}\Bigg\{\eta^{\mathrm{T}}(k)(d_i^r G_{1i}+d_i^M G_{2i})\eta(k)$$

$$- \alpha_i^M \sum_{s=k+1-d_i^M}^{k-d_i^m-1}\eta^{\mathrm{T}}(s)G_{1i}\eta(s) - \alpha_i^M \sum_{s=k-d_i^M}^{k-1}\eta^{\mathrm{T}}(s)G_{2i}\eta(s)\Bigg\}\Bigg\},$$

$$- \sum_{s=k+1-d_i^M}^{k-d_i^m-1}\eta^{\mathrm{T}}(s)G_{1i}\eta(s)$$

$$= - \sum_{s=k+1-d_i^M}^{k-d_{2i}(k)-1}\eta^{\mathrm{T}}(s)G_{1i}\eta(s) - \sum_{s=k-d_{2i}(k)}^{k-d_i^m-1}\eta^{\mathrm{T}}(s)G_{1i}\eta(s). \tag{25}$$

Also we have,

$$- \sum_{s=k+1-d_i^M}^{k-d_i^m-1}\eta^{\mathrm{T}}(s)G_{1i}\eta(s)$$

$$\leqslant -\frac{1}{d_i^r}\sum_{s=k+1-d_i^M}^{k-d_{2i}(k)-1}\eta^{\mathrm{T}}(s)G_{1i}\sum_{s=k+1-d_i^M}^{k-d_{2i}(k)-1}\eta(s)$$

$$- \frac{1}{d_i^r}\sum_{s=k-d_{2i}(k)}^{k-d_i^m-1}\eta^{\mathrm{T}}(s)G_{1i}\sum_{s=k-d_{2i}(k)}^{k-d_i^m-1}\eta(s)$$

$$\leqslant -\frac{1}{d_i^r}\left\{\left[\tilde{x}(k-d_{2i}(k))-\tilde{x}(k-d_i^M)\right]^{\mathrm{T}}G_{1i}\left[\tilde{x}(k-d_{2i}(k))\right.\right.$$

$$\left.\left.-\tilde{x}(k-d_i^M)\right]\right\} - \frac{1}{d_i^r}\left\{\left[\tilde{x}(k-d_i^m(k)-\tilde{x}(k-d_{2i}(k))\right]^{\mathrm{T}}\right.$$

$$\times G_{1i}\left[\tilde{x}(k-d_i^m)-\tilde{x}(k-d_{2i}(k))\right]\Big\}, \tag{26}$$

$$- \sum_{s=k-d_i^M}^{k-1}\eta^{\mathrm{T}}(s)G_{2i}\eta(s)$$

$$= - \sum_{s=k-d_i^M}^{k-d_{2i}(k)-1}\eta^{\mathrm{T}}(s)G_{2i}\eta(s) - \sum_{s=k-d_{2i}(k)}^{k-1}\eta^{\mathrm{T}}(s)G_{2i}\eta(s)$$

$$\leqslant -\frac{1}{d_i^r}\sum_{s=k-d_i^M}^{k-d_{2i}(k)-1}\eta^{\mathrm{T}}(s)G_{2i}\sum_{s=k-d_i^M}^{k-d_{2i}(k)-1}\eta(s)$$

$$- \frac{1}{d_i^M}\sum_{s=k-d_{2i}(k)}^{k-1}\eta^{\mathrm{T}}(s)G_{2i}\sum_{s=k-d_{2i}(k)}^{k-1}\eta(s)$$

$$\leqslant -\frac{1}{d_i^r}\left\{\left[\tilde{x}(k-d_{2i}(k))-\tilde{x}(k-d_i^M)\right]^{\mathrm{T}}G_{2i}\left[\tilde{x}(k-d_{2i}(k))\right.\right.$$

$$\left.\left.-\tilde{x}(k-d_i^M)\right]\right\} - \frac{1}{d_i^M}\left[\tilde{x}(k)-\tilde{x}(k-d_{2i}(k))\right]^{\mathrm{T}}$$

$$\times G_{2i}\left[\tilde{x}(k)-\tilde{x}(k-d_{2i}(k))\right], \tag{27}$$

$$E\{\Delta V_{8i}(k)+\alpha V_{8i}(k)\}$$

$$= E\left\{\sum_{i=1}^{N}\left\{d_i^{\star 2}\eta^{\mathrm{T}}(k)H_{1i}\eta(k) - \alpha_i^M d_i^{\star}\sum_{s=-d_i^M}^{-d_i^m-1}\sum_{j=k+s}^{k-1}\eta^{\mathrm{T}}(j)H_{1i}\eta(j)\right\}\right\}.$$

Also,

$$- d_i^{\star}\sum_{s=-d_i^M}^{-d_i^m-1}\sum_{j=k+s}^{k-1}\eta^{\mathrm{T}}(j)H_{1i}\eta(j)$$

$$\leqslant -\frac{1}{\varsigma_1}\left[\sum_{s=-d_i^M}^{-d_{1i}(k)-1}\sum_{j=k+s}^{k-1}\eta(j)\right]^{\mathrm{T}}H_{1i}\left[\sum_{s=-d_i^M}^{-d_{1i}(k)-1}\sum_{j=k+s}^{k-1}\eta(j)\right]$$

$$- \frac{1}{\varsigma_2}\left[\sum_{s=-d_{1i}(k)}^{-d_i^m-1}\sum_{j=k+s}^{k-1}\eta(j)\right]^{\mathrm{T}}H_{1i}\left[\sum_{s=-d_{1i}(k)}^{-d_i^m-1}\sum_{j=k+s}^{k-1}\eta(j)\right],$$

where

$$\varsigma_1 = \frac{(d_i^M-d_{1i}(k))(d_i^M+d_{1i}(k)+1)}{(d_i^r)(d_i^M+d_i^m+1)},$$

$$\varsigma_2 = \frac{(d_{1i}(k)-d_i^m)(d_{1i}(k)+d_i^m+1)}{(d_i^r)(d_i^M+d_i^m+1)},$$

$$\varsigma_1+\varsigma_2 = 1.$$

$$- d_i^{\star}\sum_{s=-d_i^M}^{-d_i^m-1}\sum_{j=k+s}^{k-1}\eta^{\mathrm{T}}(j)H_{1i}\eta(j)$$

$$\leqslant -\left[\sum_{s=-d_i^M}^{-d_{1i}(k)-1}\sum_{j=k+s}^{k-1}\eta(j)\right]^{\mathrm{T}}H_{1i}\left[\sum_{s=-d_i^M}^{-d_{1i}(k)-1}\sum_{j=k+s}^{k-1}\eta(j)\right]$$

$$- \left[\sum_{s=-d_{1i}(k)}^{-d_i^m-1}\sum_{j=k+s}^{k-1}\eta(j)\right]^{\mathrm{T}}L_{1i}\left[\sum_{s=-d_i^M}^{-d_{1i}(k)-1}\sum_{j=k+s}^{k-1}\eta(j)\right]$$

$$- \left[\sum_{s=-d_{1i}(k)}^{-d_i^m-1}\sum_{j=k+s}^{k-1}\eta(j)\right]^{\mathrm{T}}H_{1i}\left[\sum_{s=-d_{1i}(k)}^{-d_i^m-1}\sum_{j=k+s}^{k-1}\eta(j)\right].$$

$$\leqslant -\left[d_i^M(k)\tilde{x}(k) - \sum_{s=k-d_i^M}^{k-d_{1i}(k)-1}\tilde{x}(s)\right]^{\mathrm{T}}H_{1i}\left[d_i^M(k)\tilde{x}(k)\right.$$

$$\left. - \sum_{s=k-d_i^M}^{k-d_{1i}(k)-1}\tilde{x}(s)\right] - \left[d_i^m(k)\tilde{x}(k) - \sum_{s=k-d_{1i}(k)}^{k-d_i^m-1}\tilde{x}(s)\right]^{\mathrm{T}}$$

$$\times 2L_{1i}\left[d_i^M(k)\tilde{x}(k) - \sum_{s=k-d_i^M}^{k-d_{1i}(k)-1}\tilde{x}(s)\right] - \left[d_i^m(k)\tilde{x}(k)\right.$$

$$\left. - \sum_{s=k-d_{1i}(k)}^{k-d_i^m-1}\tilde{x}(s)\right]^{\mathrm{T}}H_{1i}\left[d_i^m(k)\tilde{x}(k) - \sum_{s=k-d_{1i}(k)}^{k-d_i^m-1}\tilde{x}(s)\right]. \tag{28}$$

Also, we have

$$E\left\{\Delta V_{9i}(k)+\alpha V_{9i}(k)\right\}=E\left\{\sum_{i=1}^{N}\left\{\frac{d_i^r(d_i^r+1)}{4}\eta^{\mathrm{T}}(k)H_{2i}\eta(k)\right.\right.$$
$$\left.\left.-\frac{\alpha_i^M}{2}\sum_{s=k-d_i^M}^{k-d_i^m-1}\sum_{j=s}^{k-d_i^m-1}\eta^{\mathrm{T}}(j)H_{1i}\eta(j)\right\}\right\}.$$

Again from Lemma 2, we get

$$-\frac{1}{2}\sum_{s=k-d_i^M}^{k-d_i^m-1}\sum_{j=s}^{k-d_i^m-1}\eta^{\mathrm{T}}(j)H_{2i}\eta(j)$$
$$=-\frac{1}{2}\sum_{s=k-d_i^M}^{k-d_{2i}(k)-1}\sum_{j=s}^{k-d_{2i}(k)-1}\eta^{\mathrm{T}}(j)H_{2i}\eta(j)$$
$$-\frac{1}{2}\sum_{s=k-d_i^M}^{k-d_{2i}(k)-1}\sum_{j=k-d_{2i}(k)}^{k-d_i^m-1}\eta^{\mathrm{T}}(j)H_{2i}\eta(j)$$
$$-\frac{1}{2}\sum_{s=k-d_{2i}(k)}^{k-d_i^m-1}\sum_{j=s}^{k-d_i^m-1}\eta^{\mathrm{T}}(j)H_{2i}\eta(j)$$
$$\leqslant-\frac{1}{(d_i^M-d_{2i}(k))(d_i^M-d_{2i}(k)+1)}\left(\sum_{s=k-d_i^M}^{k-d_{2i}(k)-1}\sum_{j=s}^{k-d_{2i}(k)-1}\eta(j)\right)^{\mathrm{T}}$$
$$\times H_{2i}\left(\sum_{s=k-d_i^M}^{k-d_{2i}(k)-1}\sum_{j=s}^{k-d_{2i}(k)-1}\eta(j)\right)$$
$$-\frac{1}{(d_{2i}(k)-d_i^m)(d_{2i}(k)-d_i^m+1)}\left(\sum_{s=k-d_{2i}(k)}^{k-d_i^m-1}\sum_{j=s}^{k-d_i^m-1}\eta(j)\right)^{\mathrm{T}}$$
$$\times H_{2i}\left(\sum_{s=k-d_{2i}(k)}^{k-d_i^m-1}\sum_{j=s}^{k-d_i^m-1}\eta(j)\right)$$
$$\leqslant-\left[\tilde{x}(k-d_i^m)-\xi_1(k)\right]^{\mathrm{T}}H_{2i}\left[\tilde{x}(k-d_i^m)-\xi_1(k)\right]$$
$$-\left[\tilde{x}(k-d_{2i}(k))-\xi_2(k)\right]^{\mathrm{T}}H_{2i}\left[\tilde{x}(k-d_{2i}(k)-\xi_2(k)\right],\quad(29)$$

where

$$\xi_1(k)=\frac{1}{d_i^{2m}}\left[\sum_{s=k-d_{2i}(k)}^{k-d_i^m-1}x(s)\right],\ \xi_2(k)=\frac{1}{d_i^{2M}}\left[\sum_{s=k-d_i^M}^{k-d_{2i}(k)-1}x(s)\right].$$

We find the exponential stability of the filtering error system by taking $w(k)=0$. Adding all the above inequalities in (16)–(29) we get

$$E\left[\Delta V_i(k)+\alpha V_i(k)\right]\leqslant E[\zeta^{\mathrm{T}}(k)\Theta_i\zeta(k)],\qquad(30)$$

$$\Theta_i=\begin{bmatrix}\tilde{\Theta}_{11} & \tilde{\Psi}_i\\ \star & -P_i\end{bmatrix},\tilde{\Theta}_{11}=\begin{bmatrix}\hat{\Theta}_{11}+\Gamma_i^{\mathrm{T}}\Pi+\Pi\Gamma_i^{\mathrm{T}} & \Pi_1 & \Pi_2\\ \star & \tilde{\Phi}_{22} & L_{1i}\\ \star & \star & \tilde{\Phi}_{33}\end{bmatrix},$$

$$\hat{\Theta}_{11}=\begin{bmatrix}
\Theta_1 & 0 & \Phi_2 & 0 & 0 & 0 & 0 & 0 & \Phi_4 & 0 & 0 & 0 & 0 & 0\\
\star & \Theta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\star & \star & \Theta_3 & 0 & 0 & 0 & \Phi_9 & \Phi_{10} & 0 & 0 & 0 & 0 & \Phi_{11} & 0\\
\star & \star & \star & \Theta_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\star & \star & \star & \star & \Theta_5 & \Phi_{14} & 0 & 0 & \Phi_{15} & 0 & 0 & 0 & 0 & 0\\
\star & \star & \star & \star & \star & \Theta_6 & 0 & 0 & \Phi_{17} & 0 & 0 & 0 & 0 & 0\\
\star & \star & \star & \star & \star & \star & \Phi_{18} & 0 & \Phi_{19} & 0 & 0 & 0 & 0 & 0\\
\star & \star & \star & \star & \star & \star & \star & \Phi_{20} & 0 & 0 & 0 & 0 & \Phi_{21} & 0\\
\star & \star & \star & \star & \star & \star & \star & \star & \Phi_{22} & 0 & 0 & 0 & 0 & 0\\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \Phi_{23} & 0 & 0 & 0 & 0\\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \Phi_{24} & 0 & 0 & 0\\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \Phi_{25} & 0 & 0\\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \Phi_{26} & 0\\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \Phi_{27}
\end{bmatrix},$$

$$\Theta_1=diag\{\Theta_{11},\cdots,\Theta_{11},\cdots,\Theta_{1N},\cdots,\Theta_{1N}\},$$
$$\Theta_{1i}=-\alpha_1P_i+d_i^{\mu}Q_{1i}+d_i^{\gamma}R_{1i}+d_i^rZ_{1i}$$
$$+d_i^MZ_{2i}-\alpha_i^{dM}G_{2i},\quad\Theta_2=\{\alpha_1^{do}Q_{11}\ \cdots\ \alpha_N^{do}Q_{1N}\},$$
$$\Theta_3=diag\{\Theta_{31},\cdots,\Theta_{41},\cdots,\quad\Theta_{3N},\cdots,\Theta_{3N}\},$$
$$\Theta_{3i}=\alpha_i^{dM}R_{1i}+\alpha_i^{dr}G_{2i}+\alpha_i^{dM}G_{2i}+\alpha_i^{dM}H_{2i},$$
$$\Theta_4=diag\{\Theta_{41},\cdots,\Theta_{41},\cdots,\Theta_{4N},\cdots,\Theta_{4N}\},$$
$$\Theta_{4i}=d_i^{\mu}S_{1i}+d_i^{\gamma}T_{1i},$$
$$\Theta_5=diag\{\Theta_{51},\cdots,\Theta_{51},\cdots,\Theta_{5N},\cdots,\Theta_{5N}\},$$
$$\Theta_{5i}=-\alpha_i^{d0}S_{1i}+\sum_{j=1}^{N}\beta_j\rho_0\bar{\rho}_0\mathcal{A}_j^{\mathrm{T}}P_i\mathcal{A}_j,$$
$$\Theta_6=diag\{\Theta_{61},\cdots,\Theta_{61},\cdots,\Theta_{6N},\cdots,\Theta_{6N}\},$$
$$\Theta_{6i}=-\alpha_i^MT_{1i}+\sum_{j=1}^{N}\beta_j\rho_0\bar{\rho}_0\mathcal{A}_j^{\mathrm{T}}P_i\mathcal{A}_j,$$
$$\zeta(k)=\Bigg[\tilde{x}^{\mathrm{T}}(k)\ \tilde{x}^{\mathrm{T}}(k-d_{1i}(k))\ \tilde{x}^{\mathrm{T}}(k-d_{2i}(k))\ \ f^{\mathrm{T}}(\tilde{x}(k))$$
$$f^{\mathrm{T}}(\tilde{x}(k-d_{1i}(k)))\ \ f^{\mathrm{T}}(\tilde{x}(k-d_{2i}(k)))$$
$$\tilde{x}^{\mathrm{T}}(k-d_i^M)\ \ \tilde{x}^{\mathrm{T}}(k-d_i^m)\ \ \eta^{\mathrm{T}}(k)$$
$$\sum_{s=k-d_i^M}^{k-d_{1i}(k)-1}\tilde{x}^{\mathrm{T}}(s)\quad\sum_{s=k-d_{1i}(k)}^{k-d_i^m-1}\tilde{x}^{\mathrm{T}}(s)\quad\sum_{s=k-d_i^M}^{k-d_{2i}(k)-1}\tilde{x}^{\mathrm{T}}(s)$$
$$\sum_{s=k-d_{2i}(k)}^{k-d_i^m-1}\tilde{x}^{\mathrm{T}}(s)\quad\sum_{s=k-d_{1i}(k)}^{k-1}\tilde{x}^{\mathrm{T}}(s)\Bigg].$$

From Assumption 1 on the neuron activation functions, it is easy to see that that the following inequalities hold for any $\Lambda_{1i}=diag\{\upsilon_{1i},\upsilon_{2i},\cdots,\upsilon_{ni}\}>0,\Lambda_{2i}=diag\{\bar{\upsilon}_{1i},\bar{\upsilon}_{2i},\cdots,\bar{\upsilon}_{ni}\}>0,\Lambda_{3i}=diag\{\tilde{\upsilon}_{1i},\tilde{\upsilon}_{2i},\cdots,\tilde{\upsilon}_{ni}\}>0$,

$$\begin{bmatrix}x(k)\\ f(x(k))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}-E_{1i}\Lambda_{1i} & E_{2i}\Lambda_{1i}\\ \star & -\Lambda_{1i}\end{bmatrix}\begin{bmatrix}x(k)\\ f(x(k))\end{bmatrix}\geqslant0,$$
$$\begin{bmatrix}x(k-d_{1i}(k))\\ f(x(k-d_{1i}(k)))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}-E_{1i}\Lambda_{2i} & E_{2i}\Lambda_{2i}\\ \star & -\Lambda_{2i}\end{bmatrix}\begin{bmatrix}x(k-d_{1i}(k))\\ f(x(k-d_{1i}(k)))\end{bmatrix}\geqslant0,$$
$$\begin{bmatrix}x(k-d_{2i}(k))\\ f(x(k-d_{2i}(k)))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}-E_{1i}\Lambda_{3i} & E_{2i}\Lambda_{3i}\\ \star & -\Lambda_{3i}\end{bmatrix}\begin{bmatrix}x(k-d_{2i}(k))\\ f(x(k-d_{2i}(k)))\end{bmatrix}\geqslant0.$$

Since $\eta(k)=\tilde{x}(k+1)-\tilde{x}(k)$, we obtain the zero equation by introducing the relaxation matrix W with appropriate dimension,

$$
\begin{aligned}
&\sum_{i=1}^{N} 2\eta^{\mathrm{T}}(k)W^{\mathrm{T}}[\mathcal{C}_i\tilde{x}(k) + \mathcal{B}_i f(\tilde{x}(k)) + \rho_0\mathcal{A}_i f(\tilde{x}(k-d_{1i}(k))) \\
&\quad + \bar{\rho_0}\mathcal{A}_i f(\tilde{x}(k-d_{2i}(k))) + (\rho(k)-\rho_0)\mathcal{A}_i \\
&\quad \times [f(\tilde{x}(k-d_{1i}(k))) - f(\tilde{x}(k-d_{2i}(k)))] \\
&\quad + \mathcal{D}_i w(k) - \tilde{x}(k) - \eta(k)] = 0. \qquad (31)
\end{aligned}
$$

Combining the results and applying the Schur complement, we get

$$
\begin{aligned}
&\mathrm{E}[\Delta V_i(k) + \alpha V_i(k)] \\
&\leqslant \mathrm{E}\left[\zeta^{\mathrm{T}}(k)[\Sigma - \Gamma(k)\begin{bmatrix} H_{1i} & L_{1i} \\ \star & H_{1i}\end{bmatrix}\Gamma^{\mathrm{T}}(k)]\zeta(k)\right], \qquad (32)
\end{aligned}
$$

where Σ is given by (33) and

$$
\Gamma^{\mathrm{T}}(k) = \begin{bmatrix} \varepsilon_i I_n & \cdots & 0_{n\times 11n} & \cdots & -\varpi_i I_n & \cdots & 0 & \cdots \\ \varpi_i I_n & \cdots & 0_{n\times 12n} & \cdots & -\varepsilon_i I_n & \cdots & 0 & \cdots \end{bmatrix}.
$$

The LMI results in (12) mean that the inequalities $\Sigma + \Gamma_i^{\mathrm{T}}\Pi + \Pi\Gamma_i^{\mathrm{T}} \leqslant 0$.

$$
\Sigma = \begin{bmatrix} \Sigma_{11} & \tilde{\Psi}_i \\ \star & -P_i \end{bmatrix}, \qquad (33)
$$

$$
\Sigma_{11} = \begin{bmatrix} \tilde{\Sigma}_{11} + \Gamma_i^{\mathrm{T}}\Pi + \Pi\Gamma_i^{\mathrm{T}} & \Pi_1 & \Pi_2 \\ \star & \tilde{\Phi}_{22} & L_{1i} \\ \star & \star & \tilde{\Phi}_{33} \end{bmatrix},
$$

$$
\tilde{\Sigma}_{11} = \begin{bmatrix}
\Sigma_1 & 0 & \Phi_2 & \Phi_3 & 0 & 0 & 0 & 0 & \Phi_4 & 0 & 0 & 0 & 0 & 0 \\
\star & \Phi_5 & 0 & 0 & \Phi_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \Phi_7 & 0 & 0 & \Phi_8 & \Phi_9 & \Phi_{10} & 0 & 0 & 0 & 0 & \Phi_{11} & 0 \\
\star & \star & \star & \Phi_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \Phi_{13} & \Phi_{14} & 0 & 0 & \Phi_{15} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \Phi_{16} & 0 & 0 & \Phi_{17} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \Phi_{18} & 0 & \Phi_{19} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \Phi_{20} & 0 & 0 & 0 & 0 & \Phi_{21} & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \Phi_{22} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \Phi_{23} & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \Phi_{24} & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \Phi_{25} & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \Phi_{26} & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \Phi_{27}
\end{bmatrix},
$$

$$
\begin{aligned}
\Sigma_1 &= diag\{\Sigma_{11}, \cdots, \Sigma_{11}, \cdots, \Sigma_{1N}, \cdots, \Sigma_{1N}\}, \\
\Sigma_{1i} &= -\alpha_1 P_i + d_i^{\mu} Q_{1i} + d_i^{\gamma} R_{1i} + d_i^{r} Z_{1i} + d_i^{M} Z_{2i} \\
&\quad - \alpha_i^{dM} G_{2i} - E_{1i}\Lambda_{1i}.
\end{aligned}
$$

By using Lemma 2, we can guarantee that the inequality $\Sigma - \Gamma^{\mathrm{T}}(k)\Pi + \Pi\Gamma^{\mathrm{T}}(k) < 0$. Thus, it is easy to get that $\Delta V_i(k) - V_i(k) \leqslant -\alpha V_i(k)$ for all k.

This implies that

$$
V_{s(k)}(k) \leqslant (1-\alpha)^{k-k_t} V_{s(k_t)}(k_t),
$$

and it is easy to obtain that

$$
\begin{aligned}
V_{s(k)}(k) &\leqslant (1-\alpha)^{k-k_t} V_{s(k_t)}(k_t) \\
&\leqslant (1-\alpha)^{k-k_t}\mu V_{s(k_{t-1})}(k_t) \\
&\leqslant \mu(1-\alpha)^{k-k_t}(1-\alpha)^{k_t-k_{t-1}} V_{s(k_{t-1})}(k_{t-1}) \\
&= \mu(1-\alpha)^{k-k_{t-1}} V_{s(k_{t-1})}(k_{t-1}) \\
&\leqslant \cdots \leqslant \mu^{N_s(k_0,k)}(1-\alpha)^{k-k_0} V_{s(k_0)}(k_0). \qquad (34)
\end{aligned}
$$

We know that $N_{s(k)}(k) \leqslant (k-k_0)/T_a$, then (34) becomes

$$
V_{s(k)}(k) \leqslant ((1-\alpha)\mu^{\frac{1}{T_a}})^{k-k_0} V_{s(k_0)}(k_0). \qquad (35)
$$

Also there exist two scalars χ_1 and χ_2 such that

$$
V_{s(k_0)}(k_0) \leqslant \chi_2 \|x(k_0)\|_L^2
$$

and

$$
V_{s(k_0)}(k_0) \geqslant \chi_1 \|x(k_0)\|_L^2,
$$

where χ_1 and χ_2 are given in the statement of Theorem 1.

Using the above inequalities in (35) and taking the mathematical expectations of these we obtain

$$
\begin{aligned}
\chi_1 E[\|x(k)\|^2] &\leqslant ((1-\alpha)\mu^{\frac{1}{T_a}})^{k-k_0}\chi_2 E\|x(k_0)\|_L^2, \\
E[\|x(k)\|^2] &\leqslant \frac{\chi_2}{\chi_1}\chi^{2(k-k_0)} E\|x(k_0)\|_L^2, \qquad (36)
\end{aligned}
$$

where $\chi^2 = (1-\alpha)\mu^{\frac{1}{T_a}}$. Then, by using T_a we can easily obtain $\chi < 1$. Hence from Definition 1, the augmented system (11) is exponentially mean square stable. This completes the proof that the filtering error system (10) is exponentially mean square stable when $w(k) = 0$.

Also, for all nonzero $w(k)$, by using (33) we get

$$
\begin{aligned}
&\Delta V_i(k) + \alpha V_i(k) + \tilde{z}^{\mathrm{T}}(k)\tilde{z}(k) - \gamma^2 w^{\mathrm{T}}(k)w(k) \\
&\leqslant \zeta^{\mathrm{T}}(k)\Sigma\zeta(k) + \tilde{x}^{\mathrm{T}}(k)\mathcal{M}_i^{\mathrm{T}}\mathcal{M}_i\tilde{x}(k) - \gamma^2 w^{\mathrm{T}}(k)w(k) \\
&\leqslant \zeta_1(k)^{\mathrm{T}}\tilde{\Xi}\zeta_1(k),
\end{aligned}
$$

where

$$
\zeta_1(k) = [\zeta^{\mathrm{T}}(k) \ \ w^{\mathrm{T}}(k)],
$$

and $\tilde{\Xi}$ is given in the LMI (12). Therefore, it follows from the LMI (12) that

$$
\Delta V_i(k) + \alpha V_i(k) + \tilde{z}^{\mathrm{T}}(k)\tilde{z}(k) - \gamma^2 w^{\mathrm{T}}(k)w(k) < 0.
$$

It can be easily deduced that

$$
\Delta V_i(k_1) \leqslant (1-\alpha)V_i(k_0) + \tilde{z}^{\mathrm{T}}(k_0)\tilde{z}(k_0) - \gamma^2 w^{\mathrm{T}}(k_0)w(k_0).
$$

On iteration, we obtain the inequality as

$$
\begin{aligned}
\Delta V_i(k) \leqslant (1-\alpha)^{k-k_0}V_i(k_0) - \sum_{s=k_0}^{k-1}(1-\alpha)^{k-s-1}(\tilde{z}^{\mathrm{T}}(s)\tilde{z}(s) \\
- \gamma^2 w^{\mathrm{T}}(s)w(s)). \qquad (37)
\end{aligned}
$$

In order to establish the exponential H_∞ performance for the system, we consider the performance index given by

$$J = \sum_{s=k_0}^{\infty}(1-\alpha)^s \tilde{z}^{\mathrm{T}}(s)\tilde{z}(s) - \gamma^2 w^{\mathrm{T}}(s)w(s).$$

From (12) to (37) we get

$$\begin{aligned}\Delta V_{s(k)}(k_1) \leqslant & (1-\alpha)^{k-k_t} V_{s(k)}(k_t) \\ & - \sum_{s=k_t}^{k-1}(1-\alpha)^{k-s-1}\left(\tilde{z}^{\mathrm{T}}(s)\tilde{z}(s) - \gamma^2 w^{\mathrm{T}}(s)w(s)\right).\end{aligned}$$

It is also easy to obtain from [44]

$$\begin{aligned}&\sum_{s=k_0}^{k-1}(1-\alpha)^s(1-\alpha)^{k-s-1}\tilde{z}^{\mathrm{T}}(s)\tilde{z}(s) \\ &\leqslant \sum_{s=k_0}^{k-1}(1-\alpha)^{k-s-1}\gamma^2 w^{\mathrm{T}}(s)w(s)).\end{aligned}$$

which implies that

$$\sum_{s=k_0}^{\infty}(1-\alpha)^s \tilde{z}^{\mathrm{T}}(s)\tilde{z}(s) \leqslant \sum_{s=k_0}^{\infty}\gamma^2 w^{\mathrm{T}}(s)w(s)).$$

Therefore, from Definition 2, we conclude that the switched system (10) with given attenuation level $\gamma > 0$ is exponentially mean square stable under sojourn probability.

Now we are in a position to make use of LMI based sufficient conditions established in Theorem 1 and design the parameters of the filter in (10) by using Lemma 5 to construct the main results.

Theorem 2. Suppose Assumption 1 holds. For given scalars $\mu \geqslant 1, 0 \leqslant \alpha \leqslant 1$, the filtering error system (11) is said to be exponentially mean square stable with a H_∞ norm bound $\gamma > 0$, if there exist symmetric positive definite matrices $P_i = \left[\begin{smallmatrix} P_{1i} & P_{2i} \\ \star & P_{3i}\end{smallmatrix}\right], Q_{1i}, R_{1i}, S_{1i}, T_{1i}, Z_{ui}, G_{ui}, H_{ui}\,(u=1,2)$, positive diagonal matrices $\Lambda_{1i}, \Lambda_{2i}, \Lambda_{3i}$, matrices $W, L_{1i}, \Pi = [\Pi_1 \quad \Pi_2]$ of appropriate dimensions and for any switching signal $s(k)$ with average dwell time satisfying $T_a \geqslant T_a^* = \frac{ln\mu}{(ln(1-\alpha))}$, such that the following LMIs hold for all $i, m \in \Omega, i \neq m$ and for $i \in \Omega$:

$$\tilde{\Xi} = \begin{bmatrix} \tilde{\Xi}_{11} & 0 & \Upsilon_{1i} & \Upsilon_{2i} & \Upsilon_{3i} \\ \star & \star & \star & \star & -\gamma^2 I \\ \star & \star & \Upsilon_{4i} & \Upsilon_{5i} & 0 \\ \star & \star & \star & \Upsilon_{6i} & 0 \\ \star & \star & \star & \star & -I \end{bmatrix} < 0, \tag{38}$$

$P_i \leqslant \mu P_m, Q_{1i} \leqslant \mu Q_{1m}, R_{1i} \leqslant \mu R_{1m}, S_{1i} \leqslant \mu S_{1m}, T_{1i} \leqslant \mu T_{1m}, Z_{ui} \leqslant \mu Z_{um}, G_{ui} \leqslant \mu G_{um}, H_{ui} \leqslant \mu H_{um}, u = (1,2)$.

$$\tilde{\Xi}_{11} = \begin{bmatrix} \hat{\Omega}_{11} + \Gamma_i^{\mathrm{T}}\Pi + \Pi\Gamma_i^{\mathrm{T}} & \Pi_1 & \Pi_2 \\ \star & \tilde{\Phi}_{22} & L_{1i} \\ \star & \star & \tilde{\Phi}_{33} \end{bmatrix},$$

$$\hat{\Omega}_{11} = \begin{bmatrix}
\phi_1 & \phi_2 & 0 & \phi_3 & \phi_4 & 0 & 0 & 0 & 0 & \phi_5 & 0 & 0 & 0 & 0 & 0 \\
\star & \phi_6 & 0 & 0 & 0 & \phi_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \phi_8 & 0 & 0 & 0 & \phi_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \phi_{10} & 0 & 0 & 0 & \phi_{11} & \phi_{12} & 0 & 0 & 0 & 0 & \phi_{13} & 0 \\
\star & \star & \star & \star & \phi_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \phi_{15} & \phi_{16} & 0 & 0 & \phi_{17} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \phi_{18} & 0 & 0 & \phi_{19} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \phi_{20} & 0 & \phi_{21} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \phi_{22} & 0 & 0 & 0 & 0 & \phi_{23} & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \phi_{24} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \phi_{25} & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \phi_{26} & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \phi_{27} & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \phi_{28} & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \phi_{29}
\end{bmatrix},$$

$$\begin{aligned}
\phi_1 &= diag\{\phi_{11}, \cdots, \phi_{11}, \cdots, \phi_{1N}, \cdots, \phi_{1N}\}, \\
\phi_{1i} &= -\alpha_1 P_{1i} + d_i^{\mu} Q_{1i} + d_i^{\gamma} R_{1i} + d_i^{r} Z_{1i} \\
&\quad + d_i^{M} Z_{2i} - \alpha_i^{dM} G_{2i} - E_{1i}\Lambda_{1i} + \mathcal{M}_i^{\mathrm{T}}\mathcal{M}_i, \\
\phi_2 &= [-\alpha_1 P_{21} \cdots - \alpha_1 P_{21} \cdots - \alpha_1 P_{2N} \ \cdots - \alpha_1 P_{2N}], \\
\phi_3 &= \left[\alpha_1^{dM} G_{21} \ \cdots \ \alpha_1^{dM} G_{21} \ \cdots \alpha_N^{dM} G_{2N} \ \cdots \ \alpha_N^{dM} G_{2N}\right], \\
\phi_4 &= [E_{11}\Lambda_{11} \ \cdots \ E_{1N}\Lambda_{1N}], \\
\phi_5 &= \sum_{j=1}^{N} \sqrt{\beta_j}\{(C_j - I)^{\mathrm{T}} W\}, \\
\phi_6 &= [-\alpha_1 P_{31} \cdots - \alpha_1 P_{31} \ \cdots - \alpha_1 P_{3N} \cdots - \alpha_1 P_{3N}], \\
\phi_7 &= [E_{21}\Lambda_{21} \ \cdots \ E_{2N}\Lambda_{2N}], \\
\phi_8 &= diag\{-\phi_{81}, \cdots, -\phi_{81}, \cdots, -\phi_{8N}, \cdots, -\phi_{8N}\}, \\
\phi_{8i} &= \alpha_i^{do} Q_{1i} + E_{2i}\Lambda_{2i}, \\
\phi_9 &= [E_{31}\Lambda_{31} \ \cdots \ E_{3N}\Lambda_{3N}], \\
\phi_{10} &= diag\{-\phi_{101}, \cdots, -\phi_{101}, \cdots, -\phi_{10N}, \cdots, -\phi_{10N}\}, \\
\phi_{10i} &= \alpha_i^{dM} R_{1i} + \alpha_i^{dr} G_{2i} + \alpha_i^{dM} G_{2i} + \alpha_i^{dM} H_{2i} + E_{1i}\Lambda_{3i}, \\
\phi_{11} &= [\alpha_1^{dr} G_{11} \ \cdots \ \alpha_N^{dr} G_{1N}], \\
\phi_{12} &= [\alpha_1^{dr} G_{21} \ \cdots \ \alpha_N^{dr} G_{2N}], \\
\phi_{13} &= \left[\frac{\alpha_1^M}{d_1^{2M}} H_{21} \ \cdots \ \frac{\alpha_i^M}{d_i^{2M}} H_{2i}\right], \\
\phi_{14} &= diag\{\phi_{141}, \cdots, \phi_{141}, \cdots, \phi_{14N}, \cdots, \phi_{14N}\}, \\
\phi_{14i} &= d_i^{\mu} S_{1i} + d_i^{\gamma} T_{1i} - \Lambda_{1i}, \\
\phi_{15} &= diag\{\phi_{151}, \cdots, \phi_{151}, \cdots, \phi_{15N}, \cdots, \phi_{15N}\}, \\
\phi_{15i} &= -\alpha_i^{d0} S_{1i} + \sum_{j=1}^{N} \beta_j \rho_0 \bar{\rho}_0 \mathcal{A}_j^{\mathrm{T}} P_i \mathcal{A}_j - \Lambda_{2i}, \\
\phi_{16} &= \left[\sum_{j=1}^{N} \beta_j \rho_0 \bar{\rho}_0 \mathcal{A}_j^{\mathrm{T}} P_1 \mathcal{A}_j \cdots \sum_{j=1}^{N} \beta_j \rho_0 \bar{\rho}_0 \mathcal{A}_j^{\mathrm{T}} P_N \mathcal{A}_j\right], \\
\phi_{17} &= \sum_{j=1}^{N} \sqrt{\beta_j} \mathcal{B}_j^{\mathrm{T}} W, \\
\phi_{18} &= diag\{\phi_{181}, \cdots, \phi_{181}, \cdots, \phi_{18N}, \cdots, \phi_{18N}\}, \\
\phi_{18i} &= -\alpha_i^M T_{1i} + \sum_{j=1}^{N} \beta_j \rho_0 \bar{\rho}_0 \mathcal{A}_j^{\mathrm{T}} P_i \mathcal{A}_j - \Lambda_{3i},
\end{aligned}$$

$$\phi_{19} = \sum_{j=1}^{N} \sqrt{\beta_j} C_j^{\mathrm{T}} [W_1 \ \cdots \ W_N],$$

$$\phi_{20} = diag\{-\phi_{201}, \cdots, -\phi_{201}, \cdots, -\phi_{20N}, \cdots, -\phi_{20N}\},$$

$$\phi_{20i} = \alpha_i^{dr} G_{1i} + \alpha_i^{dr} G_{2i},$$

$$\phi_{21} = \sum_{j=1}^{N} \sqrt{\beta_j} C_j^{\mathrm{T}} W,$$

$$\phi_{22} = diag\{-\phi_{221}, \cdots, -\phi_{221}, \cdots, -\phi_{22N}, \cdots, -\phi_{22N}\},$$

$$\phi_{22i} = \alpha_i^{dr} G_{1i} + \alpha_i^{M} H_{2i},$$

$$\phi_{23} = \left[\frac{\alpha_1^M}{d_1^{2M}} H_{21} \ \cdots \ \frac{\alpha_i^M}{d_i^{2M}} H_{2i}\right],$$

$$\phi_{24} = diag\{\phi_{241}, \cdots, \phi_{241}, \cdots, \phi_{24N}, \cdots, \phi_{24N}\},$$

$$\phi_{24i} = d_i^r G_{1i} + d_i^M G_{2i} + d_i^{*2} H_{1i} + \frac{d_i^r(d_i^r + 1)}{4} H_{2i} - W - W^{\mathrm{T}},$$

$$\phi_{25} = diag\{-\phi_{251}, \cdots, -\phi_{251}, \cdots, -\phi_{25N}, \cdots, -\phi_{25N}\},$$

$$\phi_{25i} = \alpha_i^{dr} Z_{1i} + \alpha_i^{dr} Z_{2i} + \alpha_M H_{1i},$$

$$\phi_{26} = diag\{-\phi_{261}, \cdots, -\phi_{261}, \cdots, -\phi_{26N}, \cdots, -\phi_{26N}\},$$

$$\phi_{26i} = \alpha_i^{dr} Z_{1i} + \alpha_i^{M} H_{1i},$$

$$\phi_{27} = diag\{-\phi_{271}, \cdots, -\phi_{271}, \cdots, -\phi_{27N}, \cdots, -\phi_{27N}\},$$

$$\phi_{27i} = \frac{\alpha_i^M}{(d_i^{2M})^2} H_{2i},$$

$$\phi_{28} = diag\{-\phi_{281}, \cdots, -\phi_{281}, \cdots, -\phi_{28N}, \cdots, -\phi_{28N}\},$$

$$\phi_{28i} = \frac{\alpha_i^M}{(d_i^{2m})^2} H_{2i},$$

$$\phi_{29} = diag\{-\phi_{291}, \cdots, -\phi_{291}, \cdots, -\phi_{29N}, \cdots, -\phi_{29N}\},$$

$$\phi_{29i} = \frac{\alpha_i^M}{d_i^M} Z_{2i},$$

$$\tilde{\Phi}_{22} = \tilde{\Phi}_{33} = diag\{-H_{11}, \cdots, -H_{11}, \cdots, -H_{1N}, \cdots, -H_{1N}\},$$

$$\begin{aligned}\Upsilon_{1i} = \Big[& \sqrt{\beta_i}\{Y_{1i}^{\mathrm{T}} C_i + B_{Di} F_i\} \ \sqrt{\beta_i} C_{Di} \cdots \ 0 \ 0 \\ & \cdots \sqrt{\beta_i}\{Y_{1i}^{\mathrm{T}} B_i + B_{Di} H_i\} \cdots \sqrt{\beta_i}\rho_0 Y_{1i}^{\mathrm{T}} A_i \cdots \\ & \sqrt{\beta_i}\bar{\rho}_0 Y_{1i} A_i \ \cdots \ 0 \ 0 \ \cdots \sqrt{\beta_i}\left\{Y_{1i}^{\mathrm{T}} D_i + B_{Di} G_i\right\}\Big]^{\mathrm{T}},\end{aligned}$$

$$\begin{aligned}\Upsilon_{2i} = \Big[& \sqrt{\beta_i}\left\{Y_{2i}^{\mathrm{T}} C_i + B_{Di} F_i\right\} \ \sqrt{\beta_i} C_{Di} \cdots \ 0 \ 0 \ \cdots \\ & \sqrt{\beta_i}\left\{Y_{2i}^{\mathrm{T}} B_i + B_{Di} H_i\right\} \cdots \sqrt{\beta_i}\rho_0 Y_{2i}^{\mathrm{T}} A_i \cdots \\ & \sqrt{\beta_i}\bar{\rho}_0 Y_{2i} A_i \cdots \ 0 \ 0 \ \cdots \sqrt{\beta_i}\left\{Y_{2i}^{\mathrm{T}} D_i + B_{Di} G_i\right\}\Big]^{\mathrm{T}},\end{aligned}$$

$$\Upsilon_{3i} = [M_i \ M_{Di} \ 0 \cdots \ 0 \cdots \ 0]^{\mathrm{T}}, \quad \Upsilon_{4i} = P_{1i} - Y_{1i} - Y_{1i}^{\mathrm{T}},$$

$$\Upsilon_{5i} = P_{2i} - Y_{2i} - Y_{2i}^{\mathrm{T}}, \quad \Upsilon_{6i} = P_{3i} - Y_{3i} - Y_{3i}^{\mathrm{T}},$$

with an estimate of the state decay given in Theorem 1. Moreover the filter parameters are given by $C_{di} = Y_{3i}^{-T} C_{Di}, B_{di} = Y_{3i}^{-T} B_{Di}, M_{di} = M_{Di}$.

Proof. Define matrices Y_i and introduce P_i as follows:

$$Y_i = \begin{bmatrix} Y_{1i} & Y_{2i} \\ \star & Y_{3i} \end{bmatrix}, P_i = \begin{bmatrix} P_{1i} & P_{2i} \\ \star & P_{3i} \end{bmatrix}.$$

Applying Lemma 2, we get

$$\Xi = \begin{bmatrix} \Xi_{11} & 0 & \tilde{\Psi}_i P_i \\ \star & -\gamma^2 I & \mathcal{D}_i^{\mathrm{T}} P_i \\ \star & \star & P_i - Y_i - Y_i^{\mathrm{T}} \end{bmatrix} < 0. \tag{39}$$

By letting $C_{Di} = Y_{3i}^{\mathrm{T}} C_{di}, B_{Di} = Y_{3i}^{\mathrm{T}} B_{di}, M_{di} = M_{Di}$ and using the Schur complement, it is easy to see that the inequality (39) is equivalent to the LMI (38). Thus the proof is completed.

Remark 1. If there is no sojourn probability with $d_{1i}(k) = d_1(k)$, $d_{2i}(k) = d_2(k), d_i^M = d^M, d_i^m = d^m, d_i^0 = d^0$, then the filtering error system is reduced to

$$\begin{aligned}\tilde{x}(k+1) = & \, C_i\tilde{x}(k) + \mathcal{B}_i f(\tilde{x}(k)) + \rho_0 \mathcal{A}_i f(\tilde{x}(k - d_1(k))) \\ & + \bar{\rho}_0 \mathcal{A}_i f(\tilde{x}(k - d_2(k))) + (\rho(k) - \rho_0) \\ & \times \mathcal{A}_i[f(\tilde{x}(k - d_1(k))) - f(\tilde{x}(k - d_2(k)))] \\ & + \mathcal{D}_i w(k),\end{aligned} \tag{40}$$

where $C_i, \mathcal{B}_i, \mathcal{A}_i, \mathcal{D}_i$ are given in (11). Then as an immediate consequence of Theorem 2, the exponential filter for (40) will be designed by the following corollary by choosing the LKF (15).

Corollary 1. Suppose Assumption 1 holds, for given scalars $\mu \geqslant 1, 0 \leqslant \alpha \leqslant 1$, the filtering error system (40) is said to be exponentially mean square stable with a H_∞ norm bound $\gamma > 0$, if there exist symmetric positive definite matrices $P_i = \left[\begin{smallmatrix} P_{1i} & P_{2i} \\ \star & P_{3i} \end{smallmatrix}\right], Q_{1i}, R_{1i}, S_{1i}, T_{1i}, Z_{ui}, G_{ui}, H_{ui} (u = 1, 2)$, positive diagonal matrices $\Lambda_{1i}, \Lambda_{2i}, \Lambda_{3i}$, matrices $W, L_{1i}, \Pi = [\Pi_1 \ \ \Pi_2]$ of appropriate dimensions and for any switching signal $s(k)$ with average dwell time satisfying $T_a \geqslant T_a^* = \frac{ln\mu}{(ln(1-\alpha))}$, such that the following LMIs hold for all $i, m \in \Omega, i \neq m$ and for $i \in \Omega$:

$$\tilde{\Delta} = \begin{bmatrix} \tilde{\Delta}_{11} & 0 & \tilde{\Pi}_{1i} & \tilde{\Pi}_{2i} & \tilde{\Pi}_{3i} \\ \star & \star & \star & \star & -\gamma^2 I \\ \star & \star & \tilde{\Pi}_{4i} & \tilde{\Pi}_{5i} & 0 \\ \star & \star & \star & \tilde{\Pi}_{6i} & 0 \\ \star & \star & \star & \star & -I \end{bmatrix} < 0. \tag{41}$$

$P_i \leqslant \mu P_m, Q_{1i} \leqslant \mu Q_{1m}, R_{1i} \leqslant \mu R_{1m}, S_{1i} \leqslant \mu S_{1m}, T_{1i} \leqslant \mu T_{1m}, Z_{ui} \leqslant \mu Z_{um}, G_{ui} \leqslant \mu G_{um}, H_{ui} \leqslant \mu H_{um}, u = (1, 2)$,

$$\tilde{\Delta}_{11} = \begin{bmatrix} \hat{\Psi}_{11} + \Gamma_i^{\mathrm{T}}\Pi + \Pi\Gamma_i^{\mathrm{T}} & \Pi_1 & \Pi_2 \\ \star & \hat{\Psi}_{22} & L_{1i} \\ \star & \star & \hat{\Psi}_{33} \end{bmatrix},$$

$$
\hat{\Psi}_{11} = \begin{bmatrix}
\psi_1 & \psi_2 & 0 & \psi_3 & \psi_4 & 0 & 0 & 0 & 0 & \psi_5 & 0 & 0 & 0 & 0 & 0 \\
\star & \psi_6 & 0 & 0 & 0 & \psi_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \psi_8 & 0 & 0 & 0 & \psi_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \psi_{10} & 0 & 0 & 0 & \psi_{11} & \psi_{12} & 0 & 0 & 0 & 0 & \psi_{13} & 0 \\
\star & \star & \star & \star & \psi_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \psi_{15} & \psi_{16} & 0 & 0 & \psi_{17} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \psi_{18} & 0 & 0 & \psi_{19} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \psi_{20} & 0 & \psi_{21} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \psi_{22} & 0 & 0 & 0 & 0 & \psi_{23} & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \psi_{24} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \psi_{25} & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \psi_{26} & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \psi_{27} & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \psi_{28} & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \psi_{29}
\end{bmatrix},
$$

$$
\begin{aligned}
\psi_1 &= -\alpha_1 P_{1i} + \hat{d}^{\mu} Q_{1i} + \hat{d}^{\gamma} R_{1i} + \hat{d}^{r} Z_{1i} + d^M Z_{2i} \\
&\quad - \frac{(1-\alpha)^{d^M}}{d^M} G_{2i} - E_{1i}\Lambda_{1i} + \mathcal{M}_i^{\mathrm{T}} \mathcal{M}_i, \\
\psi_2 &= -\alpha_1 P_{2i}, \quad \psi_3 = \frac{(1-\alpha)^{d^M}}{d^M} G_{2i}, \quad \psi_4 = E_{1i}\Lambda_{1i}, \\
\psi_5 &= (C_i - I)^{\mathrm{T}} W_i, \quad \psi_6 = -\alpha_1 P_{3i}, \quad \psi_7 = E_{2i}\Lambda_{2i}, \\
\psi_8 &= -(1-\alpha)^{d^0} Q_{1i} - E_{2i}\Lambda_{2i}, \quad \psi_9 = E_{3i}\Lambda_{3i}, \\
\psi_{10} &= \frac{(1-\alpha)^{d^M}}{d^M} R_{1i} + \frac{(1-\alpha)^{d^M}}{\hat{d}^r} G_{2i} \\
&\quad + \frac{(1-\alpha)^{d^M}}{d^M} G_{1i} \frac{(1-\alpha)^{d^M}}{d^M} H_{1i} + E_{1i}\Lambda_{3i}, \\
\psi_{11} &= \frac{(1-\alpha)^{d^M}}{\hat{d}^r} G_{1i}, \quad \psi_{12} = \frac{(1-\alpha)^{d^M}}{\hat{d}^r} G_{2i}, \\
\psi_{13} &= \frac{(1-\alpha)^{d^M}}{d^{2M}} H_{1i}, \quad \psi_{14} = \hat{d}^{\mu} S_{1i} + \hat{d}^{\gamma} T_{1i} - \Lambda_{1i}, \\
\psi_{15} &= -(1-\alpha)^{d^0} S_{1i} + \rho_0 \bar{\rho}_0 \mathcal{A}_i^{\mathrm{T}} P_i \mathcal{A}_i - \Lambda_{2i}, \\
\psi_{16} &= \rho_0 \bar{\rho}_0 \mathcal{A}_i^{\mathrm{T}} P_{1i} \mathcal{A}_i, \quad \psi_{17} = \mathcal{B}_i^{\mathrm{T}} W, \\
\psi_{18} &= -(1-\alpha)^{d^M} T_{1i} + \rho_0 \bar{\rho}_0 \mathcal{A}_i^{\mathrm{T}} P_i \mathcal{A}_i - \Lambda_{3i}, \quad \psi_{19} = C_i^{\mathrm{T}} W, \\
\psi_{20} &= -\frac{(1-\alpha)^{d^M}}{\hat{d}^r} G_{2i}, \quad \psi_{21} = C_i^{\mathrm{T}} W, \\
\psi_{22} &= -\frac{(1-\alpha)^{d^M}}{\hat{d}^r} G_{1i} - (1-\alpha)^{d^M} H_{2i}, \quad \psi_{23} = \frac{(1-\alpha)^{d^M}}{d^{2M}} H_{2i}, \\
\psi_{24} &= \hat{d}^r G_{1i} + d^M G_{2i} + \hat{d}^{*2} H_{1i} + \frac{\hat{d}^r(\hat{d}^r + 1)}{4} H_{2i} - W - W^{\mathrm{T}}, \\
\psi_{25} &= -\frac{(1-\alpha)^{d^M}}{\hat{d}^r} Z_{1i} - \frac{(1-\alpha)^{d^M}}{d^M} Z_{2i} - (1-\alpha)^M H_{1i}, \\
\psi_{26} &= -\frac{(1-\alpha)^{d^M}}{\hat{d}^r} Z_{1i} - (1-\alpha)^{d^M} H_{1i}, \psi_{27} = -\frac{(1-\alpha)^M}{(d^{2M})^2} H_{2i}, \\
\psi_{28} &= -\frac{(1-\alpha)^M}{(d^{2m})^2} H_{2i}, \\
\psi_{29} &= -\frac{(1-\alpha)^M}{d^M} Z_{2i}, \hat{\Psi}_{22} = \hat{\Psi}_{33} = -H_{1i}, \\
\Pi_{1i} &= \Big[Y_{1i}^{\mathrm{T}} C_i + B_{Di} F_i \quad C_{Di} \quad 0_{n,2n} \quad Y_{1i}^{\mathrm{T}} B_i + B_{Di} H_i \\
&\qquad \rho_0 Y_{1i}^{\mathrm{T}} A_i \quad \bar{\rho}_0 Y_{1i} A_i \quad 0_{n,8n} \quad Y_{1i}^{\mathrm{T}} D_i + B_{Di} G_i \Big]^{\mathrm{T}}, \\
\Pi_{2i} &= \Big[Y_{2i}^{\mathrm{T}} C_i + B_{Di} F_i \quad C_{Di} \quad 0_{n,2n} \quad Y_{2i}^{\mathrm{T}} B_i + B_{Di} H_i \\
&\qquad \rho_0 Y_{2i}^{\mathrm{T}} A_i \quad \bar{\rho}_0 Y_{2i} A_i \quad 0_{n,8n} \quad Y_{2i}^{\mathrm{T}} D_i + B_{Di} G_i \Big]^{\mathrm{T}}, \\
\Pi_{3i} &= [M_i \;\; M_{Di} \;\; 0_{n,14n}]^{\mathrm{T}}, \Pi_{4i} = P_{1i} - Y_{1i} - Y_{1i}^{\mathrm{T}}, \\
\Pi_{5i} &= P_{2i} - Y_{2i} - Y_{2i}^{\mathrm{T}}, \Pi_{6i} = P_{3i} - Y_{3i} - Y_{3i}^{\mathrm{T}},
\end{aligned}
$$

and ensures an estimate of the state decay given in Theorem 1 with $\hat{d}^* = \frac{(\hat{d}^r)(d^M + d^m + 1)}{2}$, $\hat{d}^r = d^M - d^m$, $\hat{d}^{\mu} = 1 + d^0 - d^m$, $\hat{d}^{\gamma} = 1 + d^M - d^0$, $d^{2m} = d_2(k) - d^m$, $d^{2M} = d^M - d_2(k)$.

Remark 2. The H_∞ estimator design problem is solved in Theorem 1 for the addressed random delay neural network under sojourn probabilities. We derive an LMI-based sufficient condition for the existence of the full-order estimators that ensure the mean square exponential stability of the resulting estimation error system and reduce the effect of the disturbance input on the estimated signal to a prescribed level. The feasibility of the estimator design problem can be readily checked by the solvability of an LMI, which is dependent on the lower bound and upper bound of the time-varying delays.

Remark 3. The stability problems for the switched systems can be analyzed by two types of Lyapunov-Krasovskii functional methods, namely the common Lyapunov function method and the piecewise Lyapunov function method. In practical implementation, piecewise method is chosen to deal with the switching phenomenon, as it is difficult for the subsystems to share a common LKF. Based on this, Theorem 2 provides the delay-distribution-dependent for the exponential H_∞ filter design for the discrete-time switched neural networks with random delays.

Remark 4. Some new methods, such as the delay-partitioning approach [45] and free weighting matrix method [46] reduce the conservatism, but by doing so, many weighting matrices are added and the analysis and stability of the system become more complex in nature. Therefore, in this paper the use of triple Lyapunov functional terms and the reciprocal convex approach has reduced the conservatism more greatly and has given more tight upper bounds to estimate their time differences in the estimation.

Remark 5. It should be noted that the delay-distribution-dependent conditions proposed in this paper are dependent on μ and α. If $\mu = 1$, then $P_i \leqslant P_m, Q_{1i} \leqslant Q_{1m}, R_{1i} \leqslant R_{1m}, S_{1i} \leqslant S_{1m}, T_{1i} \leqslant T_{1m}, Z_{ui} \leqslant Z_{um}, G_{ui} \leqslant G_{um}, H_{ui} \leqslant H_{um}, u = (1,2), \forall i, m \in N$ which implies $P_i = P_m = P, Q_{1i} = Q_{1m} = Q, R_{1i} = R_{1m} = R, S_{1i} = S_{1m} = S, T_{1i} = T_{1m} = T, Z_{1i} = Z_{1m} = Z_1, Z_{2i} = Z_{2m} = Z_2, G_{1i} = G_{1m} = G_1, G_{2i} = G_{2m} = G_2, H_{1i} = H_{1m} = H_1, H_{2i} = H_{2m} = H_2, \forall i, m \in N$ which means that all the system share a common LKF. But by Remark 3, in order to use the piecewise LKF, the value of μ should always be greater than 1 needed.

Remark 6. When the sojourn probabilities are not completely known, that is, not all the sojourn probabilities $\beta_i (i = 1, \cdots, N)$ can be measured, without loss of generality, it is assumed that sojourn probabilities $\beta_1, \cdots, \beta_l (l < N)$ are completely known and $\beta_{l+1}, \cdots, \beta_N$ are completely unknown. In this case the system (8)–(9) can be written as

$$
x(k+1) = \sum_{i=1}^{N} \beta_i(k) \lambda_{1i}(k) \zeta(k),
$$

$$y(k) = \sum_{i=1}^{N} \beta_i(k)\lambda_{2i}(k)\zeta(k),$$
$$z(k) = \sum_{i=1}^{N} \beta_i(k)\lambda_{3i}(k)\zeta(k),$$

where

$$\lambda_{1i} = [C_i(k) \cdots\ 0\ \ 0\ \cdots \mathcal{B}_i(k)\ \cdots$$
$$\rho_0\mathcal{A}_i(k) \cdots\ \bar{\rho}_0\mathcal{A}_i(k) \cdots\ 0\ \ 0\ \ 0\ \cdots],$$
$$\lambda_{2i} = [F_i(k)\ \ 0\ \ 0\ \ H_i(k)\ \ 0 \cdots\ 0\ \ G_i(k)],$$
$$\lambda_{3i} = [M_{zi}(k)\ \ 0 \cdots\ 0\ \ 0\ \ 0 \cdots\ 0\ \ 0].$$

On the other hand, if it is partially known i.e, $\beta_i(i = 1, \cdots, l)$ are completely known and $\beta_i(i = l+1, \cdots, N)$ are completely unknown then the system can be rewritten as

$$x(k+1) = \sum_{i=1}^{l} \beta_i(k)\lambda_{1i}(k)\zeta(k) + \delta(k)\left\{\sum_{i=l+1}^{N} \lambda_{1i}(k)\zeta(k)\right\},$$
$$y(k) = \sum_{i=1}^{l} \beta_i(k)\lambda_{2i}(k)\zeta(k) + \delta(k)\left\{\sum_{i=l+1}^{N} \lambda_{2i}(k)\zeta(k)\right\},$$
$$z(k) = \sum_{i=1}^{l} \beta_i(k)\lambda_{3i}(k)\zeta(k) + \delta(k)\left\{\sum_{i=l+1}^{N} \lambda_{3i}(k)\zeta(k)\right\},$$

where $\delta(k)$ is the abbreviation of $\delta_{s(k)\in l+1,\cdots,N}$.

$$E\{\beta_i(k)\} = \beta_i, \quad E\{\delta(k)\} = \delta,$$
$$\sum_{i=1}^{l} \beta_i(k) + \delta(k) = 1, \quad \delta = 1 - \sum_{i=1}^{l} \beta_i.$$

4 Numerical simulation

In this section, two numerical examples are provided to demonstrate the effectiveness of the developed method on the design of the H_∞ filter for the discrete-time switched neural networks with random delays.

Example 1. Consider the discrete-time switched neural network (11) with two subsystems and two neurons which have the following parameters:

$$C_1 = \begin{bmatrix} -1.31 & -0.27 \\ -0.9 & -1.0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -1.48 & -0.91 \\ -0.9 & -0.8 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -1.2 & 0.6 \\ -0.8 & -1.0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.6 & 0.7 \\ -0.7 & -0.67 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} -0.28 & -0.09 \\ 0.09 & -0.8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.26 & -0.06 \\ -0.65 & 0.6 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} -0.26 & -0.05 \\ -0.09 & -0.8 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -0.29 & -0.12 \\ -0.08 & -0.9 \end{bmatrix}.$$

Let us choose the neuron activation functions as $f(x(k)) = [\tanh(0.44x_1(k))\ \ \tanh(0.2x_2(k))]^{\mathrm{T}}$.

Then from Assumption 1, it is easy to see that $E_1 = diag\{0, 0\}$ and $E_2 = diag\{0.22, 0.1\}$.

Also choose the parameters of measurements as $G_1 = -0.67, G_2 = 0.67, F_1 = [0.42\ \ -0.53], F_2 = [0.79\ \ -0.82], H_1 = [-0.45\ \ 0.86], H_2 = [0.89\ \ 0.32]$. Furthermore, the weight matrices of the output signal are taken as $M_{z1} = [0.63\ \ 0.61], M_{z2} = [0.57\ \ 0.18]$.

Solving the LMIs in Theorem 2 using Matlab LMI toolbox, it is found that LMI (38) is feasible for the given values $\rho_0 = 0.8, \mu = 1.25$ and $\beta_1 = \beta_2 = 0.5$. The minimum H_∞ performance level γ for different values of α are presented in Table 1 with time delay bounds $d_1^m = d_2^m = 1$, $d_1^0 = d_2^0 = 3$, $d_1^M = d_2^M = 7$ along with the time-varying delay $d_{1i}(k), d_{2i}(k)$ considered as $d_{11}(k) = d_{12}(k) = 2 + \sin((\pi k)/2)$ and $d_{21}(k) = d_{22}(k) = 4 + \sin((\pi k)/2)$ respectively. The optimized H_∞ performance level is attained at $\gamma_{\min} = 0.1768$ corresponding to the value of $\alpha = 0.01$. Setting $\mu = 1.25$ the average dwell time $T_a > T_a^* = 22.0891$. If we take $T_a = 23$, then we obtain the decay rate as $\chi = 0.9998 < 1$ and solving the estimate of the state decay we get

$$E[\|x(k)\|^2] \leqslant 0.0714\mathrm{e}^{-0.0002(k-k_0)}E\|\tilde{x}(k)\|_L^2, \quad \forall k \geqslant k_0.$$

The trajectory of the estimation error $x(k)$ is depicted in Figure 1(a). The calculated values of $\gamma_{\min}$ for different values of α are given in Table 1. It is confirmed from the simulation results and the convergence dynamics that the design of H_∞ filter is performed better over the random delays using sojourn probabilities in Theorem 2, than the one in the Corollary 1 [47]. This example demonstrates that less conservative results can be obtained for larger upper bounds and it can be concluded that the filtering error system (11) is exponentially mean square stable with a H_∞ performance attenuation level.

Example 2. Considering the discrete-time switched neural network (11) with two subsystems and $B_1 = B_2 = 0$, then we have

$$\begin{aligned}\tilde{x}(k+1) = \sum_{i=1}^{N} \beta_i(k)\,\{&C_i\tilde{x}(k) + +\rho_0\mathcal{A}_i f(\tilde{x}(k - d_{1i}(k))) \\ &+ \bar{\rho}_0\mathcal{A}_i f(\tilde{x}(k - d_{2i}(k))) + (\rho(k) - \rho_0) \\ &\times \mathcal{A}_i[f(\tilde{x}(k - d_{1i}(k))) - f(\tilde{x}(k - d_{2i}(k)))] \\ &+\mathcal{D}_i w(k)\}. \end{aligned} \tag{42}$$

The following parameters for the above system are given as

$$C_1 = \begin{bmatrix} -1.52 & -0.13 \\ -0.96 & -1.0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -1.28 & -0.51 \\ -0.9 & -0.8 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} -0.28 & -0.49 \\ 0.09 & -0.36 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.36 & -0.36 \\ -0.5 & -0.7 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} -0.26 & -0.05 \\ -0.29 & -0.8 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -0.59 & -0.12 \\ -0.58 & -0.9 \end{bmatrix}.$$

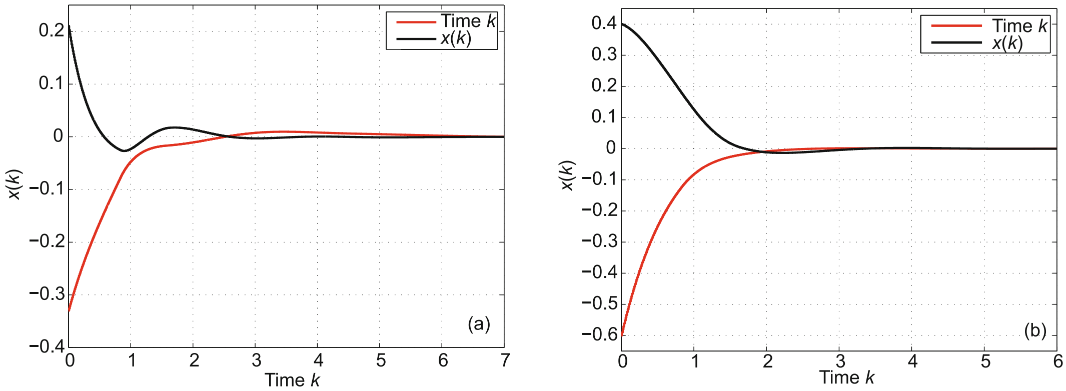

Figure 1 (a) The State trajectories of system (11); (b) the State trajectories of system (42).

Table 1 Optimized $\gamma_{\min}$ for various values of α

α		0.001	0.005	0.01	0.05	0.1
Value of Ta	Theorem 2	224	45	23	5	3
$\gamma_{\min}$		0.0071	0.0283	0.1768	0.3466	0.7072
Value of Ta	Corollary 1 [47]	196	38	17	4	2
$\gamma_{\min}$		0.0371	0.1936	0.3872	0.5346	0.9263

Let us choose the neuron activation functions as $f(x(k)) = [\tanh(0.66x_1(k)) \quad \tanh(0.44x_2(k))]^{\mathrm{T}}$. Then from Assumption 1, it is easy to see that $E_1 = diag\{0,0\}$ and $E_2 = diag\{0.22, 0.1\}$. Also choose the parameters of measurements as $G_1 = -0.86, G_2 = 0.87, F_1 = [0.42 \quad -0.43], F_2 = [0.87 \quad -0.84], H_1 = [-0.46 \quad 0.78], H_2 = [0.82 \quad 0.23]$. Furthermore, the weight matrices of the output signal are taken as $M_{z1} = [0.83 \quad 0.52], M_{z2} = [0.98 \quad 0.78]$. For the given values $\rho_0 = 0.9, \mu = 1.15, \beta_1 = 0.4, \beta_2 = 0.6, \alpha = 0.01$, and time delay bounds $d_1^m = d_2^m = 2$, $d_1^0 = d_2^0 = 4$, $d_1^M = d_2^M = 8$, feasibility for the given problem is attained by taking time-varying delay $d_{1i}(k)$ and $d_{2i}(k)$ considered as $d_{11}(k) = d_{12}(k) = 3 + \sin((\pi k)/2), d_{21}(k) = d_{22}(k) = 5 + \sin((\pi k)/2)$.

The optimized H_∞ performance level is attained at $\gamma_{\min} = 0.0283$ corresponding to the value of $\alpha = 0.01$. Setting $\mu = 1.15$ the average dwell time $T_a > T_a^* = 13.8416$. If we take $T_a = 14$, then we obtain the decay rate as $\chi = 0.9999 < 1$ and the estimate of the state decay is given by

$$E[||x(k)||^2] \leqslant 0.1128\mathrm{e}^{-0.0001(k-k_0)} E||\tilde{x}(k)||_L^2, \quad \forall k \geqslant k_0.$$

The state trajectory of $x(k)$ and its estimation are presented in Figure 1(b). It is confirmed from the simulation results that the error system (42) is exponentially mean square stable .

5 Conclusion

The problem of exponential H_∞ filter design for a class of discrete-time switched neural networks with sojourn probabilities has been investigated. The neural network under study involves the random time-varying delays characterized by introducing a Bernoulli stochastic variable. The probabilities of a system staying in each subsystem are assumed to be known in prior. By using these probability information, a new model of the switched system is proposed and sufficient conditions for the existence of the full order filter are established by using a piecewise LKF together with average dwell time method to ensure the exponential mean-square stability criteria with an H_∞ performance index γ. Finally, two numerical examples with simulations are employed to illustrate the effectiveness of this method.

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