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A direct-variance-analysis method for generalized stochastic eigenvalue problem based on matrix perturbation theory

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It has been extensively recognized that the engineering structures are becoming increasingly precise and complex, which makes the requirements of design and analysis more and more rigorous. Therefore the uncertainty effects are indispensable during the process of product development. Besides, iterative calculations, which are usually unaffordable in calculative efforts, are unavoidable if we want to achieve the best design. Taking uncertainty effects into consideration, matrix perturbation methodpermits quick sensitivity analysis and structural dynamic re-analysis, it can also overcome the difficulties in computational costs. Owing to the situations above, matrix perturbation method has been investigated by researchers worldwide recently. However, in the existing matrix perturbation methods, correlation coefficient matrix of random structural parameters, which is barely achievable in engineering practice, has to be given or to be assumed during the computational process. This has become the bottleneck of application for matrix perturbation method. In this paper, we aim to develop an executable approach, which contributes to the application of matrix perturbation method. In the present research, the first-order perturbation of structural vibration eigenvalues and eigenvectors is derived on the basis of the matrix perturbation theory when structural parameters such as stiffness and mass have changed. Combining the first-order perturbation of structural vibration eigenvalues and eigenvectors with the probability theory, the variance of structural random eigenvalue is derived from the perturbation of stiffness matrix, the perturbation of mass matrix and the eigenvector of baseline-structure directly. Hence the Direct-Variance-Analysis (DVA) method is developed to assess the variation range of the structural random eigenvalues without correlation coefficient matrix being involved. The feasibility of the DVA method is verified with two numerical examples (one is truss-system and the other is wing structure of MA700 commercial aircraft), in which the DVA method also shows superiority in computational efficiency when compared to the Monte-Carlo method.

matrix perturbation theory, generalized stochastic eigenvalue problem, structure with random parameter, direct variance analysis

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1 Introduction

Alongside the dramatically rapid development of science and technology, people are getting increasingly interested in the content of two aspects in the research area of modern engineering structure system. On one hand, engineering structure systems (e.g. aircraft-carrier and hypersonic-vehicle) are becoming more and more complex and extensive [1], but their serving environment is usually very hostile [2]. The situation mentioned above leads to the fact that the nonlinear, strong-coupled [3], time-varying and high-sensitive characteristics of engineering structure system are becoming extraordinarily significant [4]. On the other hand, engineering structure systems have to cope with various uncertain factors in their whole life cycle from design to retirement. It has been realized gradually that the various

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uncertainties are objective. For example, different technical merit designers and operators that enrolled, errors caused by equipment in the process of manufacture, differences in raw materials [5–7], differences in specification and implementation, variations in usage condition of structure system can all play an essential role in structural performance [8–12].

The so-called uncertainty effects are usually treated as modifications of structural parameters caused by the cross-coupled various uncertain factors. These uncertainty impacts will be amplified remarkably under the influence of the nonlinear, strong-coupled, time-varying and high-sensitive characteristics of large complex structures [10]. Consequently, it will lead to the reduction of structural performance or even structural failure. Considering the situation mentioned above, people will benefit from researching on the quick structural re-analysis method, which could be applied to the modification in structural parameters resulting from random perturbation of uncertain factors. This kind of investigation will make contributions to estimating the propagation and influence of uncertain factors precisely. Additionally, the standard of design, analysis and optimization in engineering structure will be enhanced by avoiding reduction of structural performance and structural failure caused by uncertainty effects [9].

In the design and analysis of realistic engineering structure, iterative calculation is inevitable if we want to obtain the best design when structural mechanical behaviors (such as vibration property) have changed. However, the time consumption of iterative calculation (e.g. the Monte-Carlo simulation, the response surface approximation) is far too much for large and complex engineering structure [13]. Obviously, it is unacceptable in the fierce market and military competition currently. Hence, matrix perturbation method which permits quick sensitivity analysis and structural re-analysis is booming and is widely regarded as a vital issue in recent years. High computational efficiency and less time consumption are the conspicuous features of matrix perturbation method, therefore numerous researchers have been enthusiastic about this topic [9,10,14]. By introducing the uncertain-but-nonrandom theory into the structural vibration problem, Qiu [15] analyzed the influence of uncertain parameters on structural natural frequency. Yang and Chen [16] applied the Pade approximation to the matrix perturbation theory and the expressions of eigenvalue and eigenvector were deduced. Kleiber and Hien [17] made contributions to the development of stochastic finite element method and made an implementation in computational program. Graham and Deodatis [18] studied the responses and eigenvalue analysis of stochastic finite element system under multi-correlativity. Pradlwarter and Schuëller [19] did some researches on the computation of large structural random eigenvalue with Monte-Carlo simulation method. Ref. [20] expanded the structural eigenvalue and eigenvector with polynomial chaos expansion method, presenting the theory of forced vibration response analysis in linear random system. In ref. [21], the reliability optimization of truss-structure was proposed with stochastic finite element theory which is built upon generalized perturbation theory.

Concerning the engineering structure system, we can see that the random properties of stiffness matrix and mass matrix result from the random properties of structural parameters. In current work, a common method of calculating the structural random eigenvalue statistics is to expand the stiffness matrix and mass matrix near the mean value of random structural parameter with Taylor-series expansion method [14] (which is named Taylor-series-perturbation method). In Taylor-series-perturbation approach, the variance of eigenvalue is derived from, namely, standard deviation of random structural parameters, correlation coefficient matrix of random structural parameters and sensitivity matrix of eigenvalue. Accordingly, in order to determine the variance of random structural eigenvalue, we have to know or to assume the correlation coefficient matrix of structural parameters. Unfortunately, the correlation coefficient matrix of structural parameters is barely obtainable in realistic engineering problem. Therefore, the application of matrix perturbation method in solving the problem of random eigenvalue variance analysis is restricted within narrow limits. From the perspectives above, a novel method that is more convenient in application is imperative.

This paper aims to develop a new method which is application-oriented and executable in engineering problem. In this proposed Direct-Variance-Analysis (DVA) method, we only need to know the variance of random structural parameter instead of the correlation coefficient matrix in calculating the variance of eigenvalue. So it is unnecessary for us to acquire or to assume the correlation coefficient matrix of structural parameters. Hence, this paper has made processing the statistical properties of random structural eigenvalue more convenient, and this paper has laid the foundation of widely applying the matrix-perturbation-based DVA method in engineering. The remainder of this paper is arranged as follows. Initially, the eigenproblem of structural vibration is systematically discussed. Secondly, the matrix perturbation method in dealing with the eigenproblem of structural vibration is introduced. Thirdly, a novel DVA method is developed based on the matrix perturbation method, and the correlative computational equations are deduced in detail. Fourthly, two numerical examples (one is planar truss-system and the other is wing structure of MA700 commercial aircraft) are investigated in order to validate the proposed method. Lastly, the conclusions are drawn. For the purpose of making the logic and venation of this paper understandable, the structural flowchart of this paper is presented in Figure 1.

2 Eigenproblem of structural vibration

Ignore the damping effect and consider the natural vibration

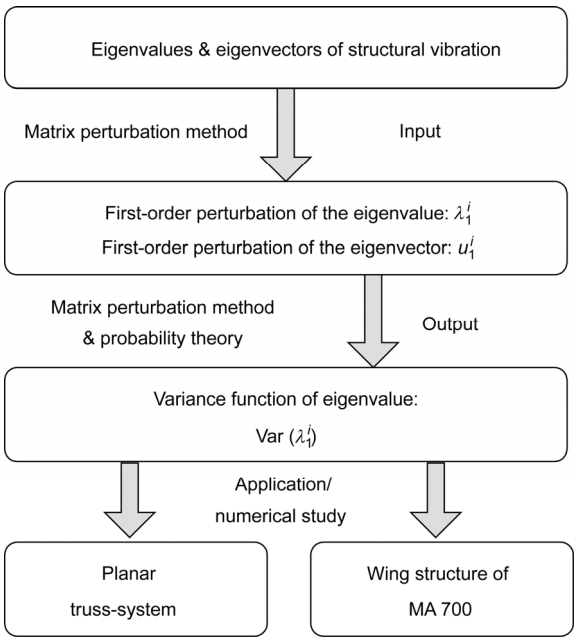

Figure 1 Structural flowchart of this paper.

equation of linear structure as

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = 0, \tag{1}$$

where $\ddot{\boldsymbol{q}}$ denotes a generalized acceleration vector, $\boldsymbol{q}$ denotes a generalized displacement. $\boldsymbol{M}$ and $\boldsymbol{K}$ represent mass matrix and stiffness matrix, respectively.

By considering that the natural vibration of structure is harmonic vibration, $\boldsymbol{q}$ in eq. (1) can be written as

$$\boldsymbol{q} = \{\boldsymbol{u}\}\cos(\omega t - \varphi), \tag{2}$$

where $\{\boldsymbol{u}\}$ is modal vector. By substituting eq. (2) into eq. (1), the eigenvalue problem of structural vibration is given as

$$\boldsymbol{K}\{\boldsymbol{u}\} = \lambda \boldsymbol{M}\{\boldsymbol{u}\}, \tag{3}$$

where $\lambda=\omega^2$ is the square of natural vibration frequency. The natural frequency of structural vibration and modal vectors will be obtained via solving eq. (3). To acquire a non-zero solution in the equation above, the sufficient and necessary condition should be

$$\det(\boldsymbol{K} - \lambda \boldsymbol{M}) = 0. \tag{4}$$

Eq. (4) is the structural eigenvalue equation. Let the structural DOF be N after discretization, thus both $\boldsymbol{K}$ and $\boldsymbol{M}$ are matrices with dimensions $N\times N$. If the structure involved has no rigid freedoms, $\{\boldsymbol{u}\}^{\mathrm{T}}\boldsymbol{K}\{\boldsymbol{u}\} > 0$ is true for any non-zero displacement $\{\boldsymbol{u}\}$. In other words, $\boldsymbol{K}$ is a positive definite matrix. If there exists rigid body displacement $\{\boldsymbol{u}_0\}$ in the structure, then $\{\boldsymbol{u}_0\}^{\mathrm{T}}\boldsymbol{K}\{\boldsymbol{u}_0\} = 0$, which means $\boldsymbol{K}$ is positive semi-definite. The corresponding eigenvalue is $\lambda=\omega_0^2=0$.

Concerning the engineering structure, apparently, structural vibration will generate the kinetic energy. Hence for any non-zero $\{\boldsymbol{u}\}$ we have $\{\boldsymbol{u}\}^{\mathrm{T}}\boldsymbol{M}\{\boldsymbol{u}\} > 0$, which implies $\boldsymbol{M}$ is a positive definite matrix. If mass matrix $\boldsymbol{M}$ is positive semi-definite, we can eliminate the pure static DOF by using the static-condensation method. In this way, mass matrix $\boldsymbol{M}$ will become positive definite matrix.

$\boldsymbol{M}$ and $\boldsymbol{K}$ in eq. (3) can be achieved by using the finite element method. Besides, $\boldsymbol{M}$ and $\boldsymbol{K}$ are real symmetric as well as positive definite matrix, or at least positive semi-definite. Therefore, the N eigenvalues $\lambda_i(i=1,2,\cdots,N)$ that are derived from eigenproblem can be arranged according to their magnitudes as $0\leqslant\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_N$; the corresponding eigenvectors are $\{\boldsymbol{u}_i\}(i=1,2,\cdots,N)$. Thus eq. (3) turns to

$$\boldsymbol{K}\{\boldsymbol{u}_i\} = \lambda_i \boldsymbol{M}\{\boldsymbol{u}_i\}, \tag{5}$$

considering the first $j(j\leqslant N)$ eigenvalues, we have

$$\boldsymbol{K}\{\boldsymbol{U}\} = \boldsymbol{M}\{\boldsymbol{U}\}\boldsymbol{\Gamma}, \tag{6}$$

where $\{\boldsymbol{u}\}$ denotes the modal matrix with dimensions $N\times j$, each column of matrix $\{\boldsymbol{u}\}$ is one of the structural j eigenvectors. $\boldsymbol{\Gamma}$ denotes diagonal matrix with dimensions $j\times j$, each diagonal element in $\boldsymbol{\Gamma}$ represents the corresponding eigenvalue. $\{\boldsymbol{u}\}$ and $\boldsymbol{\Gamma}$ in eq. (6) can be written as

$$\begin{cases} \{\boldsymbol{U}\} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_j], \\ \boldsymbol{\Gamma} = \mathrm{diag}(\lambda_i),\ i = 1, 2, \cdots, j. \end{cases} \tag{7}$$

Note that if $\{\boldsymbol{u}_i\}$ is an eigenvector, $a\{\boldsymbol{u}_i\}$ will also be an eigenvector that corresponds to the same eigenvalue λ_i. Therefore $\{\boldsymbol{u}_i\}$ is just a direction vector with dimensions $N\times 1$. By normalizing the eigenvectors, we make every eigenvector unique. Hereinafter, $\{\boldsymbol{u}_i\}$ is regarded as normalized eigenvector. Owing to the orthogonality of the eigenvector, we also have

$$\begin{cases} \{\boldsymbol{u}_i\}^{\mathrm{T}}\boldsymbol{M}\{\boldsymbol{u}_j\} = \delta_{ij}, \\ \{\boldsymbol{u}_i\}^{\mathrm{T}}\boldsymbol{K}\{\boldsymbol{u}_j\} = \lambda_i\delta_{ij}. \end{cases} \tag{8}$$

3 Matrix perturbation method in coping with the eigenproblem of structural vibration

Perturbation theory of distinct eigenvalue is the focus of this paper. Note that in eq. (3), the variations of structural parameters are reflected in the changes of mass matrix $\boldsymbol{M}$ and stiffness matrix $\boldsymbol{K}$. Hence, mass matrix $\boldsymbol{M}$ and stiffness matrix $\boldsymbol{K}$ under perturbation are represented as

$$\begin{cases} \boldsymbol{M} = \boldsymbol{M}_0 + \varepsilon\boldsymbol{M}_1, \\ \boldsymbol{K} = \boldsymbol{K}_0 + \varepsilon\boldsymbol{K}_1, \end{cases} \tag{9}$$

where ε is a minim, $\boldsymbol{K}_0$ denotes stiffness matrix of the baseline system and $\boldsymbol{M}_0$ denotes mass matrix of the baseline

system, respectively. $\varepsilon \boldsymbol{K}_1$ and $\varepsilon \boldsymbol{M}_1$ denote the perturbations of $\boldsymbol{K}_0$ and $\boldsymbol{M}_0$, respectively. In addition, we have

$$\begin{cases} \boldsymbol{K} \to \boldsymbol{K}_0, \\ \boldsymbol{M} \to \boldsymbol{M}_0, \end{cases} \text{s.t.} \begin{cases} \varepsilon \boldsymbol{K}_1 \to 0, \\ \varepsilon \boldsymbol{M}_1 \to 0. \end{cases} \tag{10}$$

According to the perturbation theory, we extend the eigenvector $\{\boldsymbol{u}\}$ and eigenvalue λ as power series

$$\{\boldsymbol{u}^i\} = \{\boldsymbol{u}_0^i\} + \varepsilon\{\boldsymbol{u}_1^i\} + \varepsilon^2\{\boldsymbol{u}_2^i\} + \cdots \quad (i = 1, 2, \cdots, n), \tag{11}$$

$$\lambda^i = \lambda_o^i + \varepsilon\lambda_1^i + \varepsilon^2\lambda_2^i + \cdots \quad (i = 1, 2, \cdots, n), \tag{12}$$

where λ_0^i is the ith eigenvalue in the baseline system, λ_1^i and λ_2^i are the first-order perturbation and the second-order perturbation of the ith eigenvalue, respectively. $\{\boldsymbol{u}_0^i\}$ is the ith eigenvector in the baseline system, $\{\boldsymbol{u}_1^i\}$ and $\{\boldsymbol{u}_2^i\}$ are the first-order perturbation and the second-order perturbation of the ith eigenvector, respectively.

Substituting eqs. (9), (11) and (12) into eq. (3) yields

$$\begin{aligned} &(\boldsymbol{K}_0 + \varepsilon\boldsymbol{K}_1)(\{\boldsymbol{u}_0^i\} + \varepsilon\{\boldsymbol{u}_1^i\} + \varepsilon^2\{\boldsymbol{u}_2^i\}) \\ &= (\lambda_o^i + \varepsilon\lambda_1^i + \varepsilon^2\lambda_2^i)(\boldsymbol{M}_0 + \varepsilon\boldsymbol{M}_1)(\{\boldsymbol{u}_0^i\} + \varepsilon\{\boldsymbol{u}_1^i\} + \varepsilon^2\{\boldsymbol{u}_2^i\}). \end{aligned} \tag{13}$$

Extending eq. (13) with the $O(\varepsilon^3)$ terms ignored, then comparing the coefficients of the same power terms of ε on each side of the equation, we get

$$\varepsilon^0: \quad \boldsymbol{K}_0\{\boldsymbol{u}_0^i\} = \lambda_0^i\boldsymbol{M}_0\{\boldsymbol{u}_0^i\}, \tag{14}$$

$$\varepsilon^1: \quad \boldsymbol{K}_0\{\boldsymbol{u}_1^i\} + \boldsymbol{K}_1\{\boldsymbol{u}_0^i\} = \lambda_0^i\boldsymbol{M}_0\{\boldsymbol{u}_1^i\} + \lambda_0^i\boldsymbol{M}_1\{\boldsymbol{u}_0^i\} + \lambda_1^i\boldsymbol{M}_0\{\boldsymbol{u}_0^i\}, \tag{15}$$

$$\begin{aligned} \varepsilon^2: \quad &\boldsymbol{K}_0\{\boldsymbol{u}_2^i\} + \boldsymbol{K}_1\{\boldsymbol{u}_1^i\} = \lambda_0^i\boldsymbol{M}_0\{\boldsymbol{u}_2^i\} + \lambda_0^i\boldsymbol{M}_1\{\boldsymbol{u}_1^i\} + \lambda_1^i\boldsymbol{M}_0\{\boldsymbol{u}_1^i\} \\ &+ \lambda_1^i\boldsymbol{M}_1\{\boldsymbol{u}_0^i\} + \lambda_2^i\boldsymbol{M}_0\{\boldsymbol{u}_0^i\}, \end{aligned} \tag{16}$$

here $\{\boldsymbol{u}_1^i\}$ and $\{\boldsymbol{u}_0^s\}$ are subject to

$$\{\boldsymbol{u}_1^i\} = \sum_{s=1}^{n} C_s^1\{\boldsymbol{u}_0^s\}. \tag{17}$$

Substituting eq. (17) into eq. (15) yields

$$\begin{aligned} &\boldsymbol{K}_0\sum_{s=1}^{n} C_s^1\{\boldsymbol{u}_0^s\} + \boldsymbol{K}_1\{\boldsymbol{u}_0^i\} = \lambda_0^i\boldsymbol{M}_0\sum_{s=1}^{n} C_s^1\{\boldsymbol{u}_0^s\} \\ &+ \lambda_0^i\boldsymbol{M}_1\{\boldsymbol{u}_0^i\} + \lambda_1^i\boldsymbol{M}_0\{\boldsymbol{u}_0^i\}. \end{aligned} \tag{18}$$

Premultiplying eq. (18) with $\{\boldsymbol{u}_o^s\}^{\mathrm{T}}$ gives

$$\begin{aligned} &\sum_{s=1}^{n} C_s^1\{\boldsymbol{u}_o^s\}^{\mathrm{T}}\boldsymbol{K}_0\{\boldsymbol{u}_0^s\} + \{\boldsymbol{u}_o^s\}^{\mathrm{T}}\boldsymbol{K}_1\{\boldsymbol{u}_0^i\} \\ &= \sum_{s=1}^{n} C_s^1\lambda_0^i\{\boldsymbol{u}_o^s\}^{\mathrm{T}}\boldsymbol{M}_0\{\boldsymbol{u}_0^s\} + \lambda_0^i\{\boldsymbol{u}_o^s\}^{\mathrm{T}}\boldsymbol{M}_1\{\boldsymbol{u}_0^i\} + \lambda_1^i\{\boldsymbol{u}_o^s\}^{\mathrm{T}}\boldsymbol{M}_0\{\boldsymbol{u}_0^i\}. \end{aligned} \tag{19}$$

Note that the orthogonality of the eigenvector in eqs. (8) and (19) can be rewritten as

$$C_s^1\lambda_0^s + \{\boldsymbol{u}_o^s\}^{\mathrm{T}}\boldsymbol{K}_1\{\boldsymbol{u}_0^i\} = C_S^1\lambda_0^i + \lambda_0^i\{\boldsymbol{u}_o^s\}^{\mathrm{T}}\boldsymbol{M}_1\{\boldsymbol{u}_0^i\} + \lambda_1^i\delta_{is}. \tag{20}$$

Rearranging the equation above, we have

$$C_S^1(\lambda_0^i - \lambda_0^s) + \lambda_1^i\delta_{is} = \{\boldsymbol{u}_o^s\}^{\mathrm{T}}\boldsymbol{K}_1\{\boldsymbol{u}_0^i\} - \lambda_0^i\{\boldsymbol{u}_o^s\}^{\mathrm{T}}\boldsymbol{M}_1\{\boldsymbol{u}_0^i\}. \tag{21}$$

It should be noted that if i=s, $\lambda_0^i = \lambda_0^s$, then eq. (21) will be expressed as

$$\lambda_1^i = \{\boldsymbol{u}_o^i\}^{\mathrm{T}}\boldsymbol{K}_1\{\boldsymbol{u}_0^i\} - \lambda_0^i\{\boldsymbol{u}_o^i\}^{\mathrm{T}}\boldsymbol{M}_1\{\boldsymbol{u}_0^i\}. \tag{22}$$

In the case of $i \neq s$, $\delta_{is} = 0$, it is derived from eq. (21) as

$$C_S^1 = (\{\boldsymbol{u}_o^s\}^{\mathrm{T}}\boldsymbol{K}_1\{\boldsymbol{u}_0^i\} - \lambda_0^i\{\boldsymbol{u}_o^s\}^{\mathrm{T}}\boldsymbol{M}_1\{\boldsymbol{u}_0^i\}) / (\lambda_0^i - \lambda_0^s). \tag{23}$$

Considering that $\{\boldsymbol{u}^i\}$ is subject to mass orthonormalization condition as

$$\{\boldsymbol{u}^i\}^{\mathrm{T}}\boldsymbol{M}\{\boldsymbol{u}^i\} = 1, \tag{24}$$

substituting eqs. (9) and (11) into eq. (24), we have

$$\begin{aligned} &(\{\boldsymbol{u}_0^i\} + \varepsilon\{\boldsymbol{u}_1^i\} + \varepsilon^2\{\boldsymbol{u}_2^i\})^{\mathrm{T}}(\boldsymbol{M}_0 + \varepsilon\boldsymbol{M}_1)(\{\boldsymbol{u}_0^i\} \\ &+ \varepsilon\{\boldsymbol{u}_1^i\} + \varepsilon^2\{\boldsymbol{u}_2^i\}) = 1. \end{aligned} \tag{25}$$

Expanding eq. (25) with the $O(\varepsilon^3)$ terms ignored, then comparing the coefficients of the same power terms of ε on each side of the equation, we have

$$\varepsilon^0: \quad \{\boldsymbol{u}_0^i\}^{\mathrm{T}}\boldsymbol{M}_0\{\boldsymbol{u}_0^i\} = 1, \tag{26}$$

$$\varepsilon^1: \quad \{\boldsymbol{u}_0^i\}^{\mathrm{T}}\boldsymbol{M}_0\{\boldsymbol{u}_1^i\} + \{\boldsymbol{u}_1^i\}^{\mathrm{T}}\boldsymbol{M}_0\{\boldsymbol{u}_0^i\} + \{\boldsymbol{u}_0^i\}^{\mathrm{T}}\boldsymbol{M}_1\{\boldsymbol{u}_0^i\} = 0. \tag{27}$$

Premultiplying eq. (17) with $\{\boldsymbol{u}_o^i\}^{\mathrm{T}}\boldsymbol{M}_0$ gives

$$\{\boldsymbol{u}_0^i\}^{\mathrm{T}}\boldsymbol{M}_0\{\boldsymbol{u}_1^i\} = \sum_{s=1}^{n} C_s^1\{\boldsymbol{u}_0^i\}^{\mathrm{T}}\boldsymbol{M}_0\{\boldsymbol{u}_0^s\}, \tag{28}$$

if i=s, it can be deduced from eq. (28) that

$$C_i^1 = \{\boldsymbol{u}_0^i\}^{\mathrm{T}}\boldsymbol{M}_0\{\boldsymbol{u}_1^i\}. \tag{29}$$

Note that C_i^1 is a scalar, by transposing eq. (29), we have

$$C_i^1 = \{\boldsymbol{u}_1^i\}^{\mathrm{T}}\boldsymbol{M}_0\{\boldsymbol{u}_0^i\}. \tag{30}$$

Substituting eqs. (29) and (30) into eq. (27) gives

$$C_i^1 = -\frac{1}{2}\{\boldsymbol{u}_0^i\}^{\mathrm{T}}\boldsymbol{M}_1\{\boldsymbol{u}_0^i\}. \tag{31}$$

Taking eqs. (31), (23) and (17) into consideration, the first-order perturbation of the structural eigenvector can be written as

$$\begin{aligned} \{\boldsymbol{u}_1^i\} &= \sum_{s=1}^{n} C_s^1\{\boldsymbol{u}_0^s\} \\ &= \sum_{\substack{s=1 \\ s\neq i}}^{n} \frac{\left[(\{\boldsymbol{u}_o^s\}^{\mathrm{T}}\boldsymbol{K}_1\{\boldsymbol{u}_0^i\} - \lambda_0^i\{\boldsymbol{u}_o^s\}^{\mathrm{T}}\boldsymbol{M}_1\{\boldsymbol{u}_0^i\})\{\boldsymbol{u}_0^s\}\right]}{(\lambda_0^i - \lambda_0^s)} \\ &\quad -\frac{1}{2}(\{\boldsymbol{u}_0^i\}^{\mathrm{T}}\boldsymbol{M}_1\{\boldsymbol{u}_0^i\})\{\boldsymbol{u}_0^i\}. \end{aligned} \tag{32}$$

4 Stochastic eigenvalue variance analysis of random parameter structure

Due to the uncertainties of material capability, structural parameters and loading environment, the structural mass matrix and stiffness matrix are always stochastic, which further leads to the stochastic properties of structural eigenvalue and eigenvector.

Concerning the structure with stochastic parameters, the stiffness matrix $\boldsymbol{K}$, mass matrix $\boldsymbol{M}$, eigenvalue λ and eigenvector $\{\boldsymbol{u}\}$ can be represented as

$$\boldsymbol{K} = \boldsymbol{K}_d + \varepsilon \boldsymbol{K}_r, \tag{33}$$

$$\boldsymbol{M} = \boldsymbol{M}_d + \varepsilon \boldsymbol{M}_r, \tag{34}$$

$$\lambda^i = \lambda_d^i + \varepsilon \lambda_r^i, \tag{35}$$

$$\{\boldsymbol{u}^i\} = \{\boldsymbol{u}_d^i\} + \varepsilon \{\boldsymbol{u}_r^i\}, \tag{36}$$

where ε is a minim parameter; $\boldsymbol{K}_d$, $\boldsymbol{M}_d$, $\boldsymbol{\lambda}_d^i$, $\{\boldsymbol{u}_d^i\}$ are deterministic parts of $\boldsymbol{K}$, $\boldsymbol{M}$, λ and $\{\boldsymbol{u}\}$, respectively; while $\boldsymbol{K}_r$, $\boldsymbol{M}_r$, $\boldsymbol{\lambda}_r^i$, $\{\boldsymbol{u}_r^i\}$ are random parts of $\boldsymbol{K}$, $\boldsymbol{M}$, λ and $\{\boldsymbol{u}\}$, respectively. Additionally, the mean values of $\boldsymbol{K}_r$, $\boldsymbol{M}_r$, $\boldsymbol{\lambda}_r^i$, $\{\boldsymbol{u}_r^i\}$ are all zero.

Taking expectation of eq. (33)–(36) yields

$$E[\boldsymbol{K}] = E[\boldsymbol{K}_d] + \varepsilon E[\boldsymbol{K}_r] = \boldsymbol{K}_d, \tag{37}$$

$$E[\boldsymbol{M}] = E[\boldsymbol{M}_d] + \varepsilon E[\boldsymbol{M}_r] = \boldsymbol{M}_d, \tag{38}$$

$$E[\lambda^i] = E[\lambda_d^i] + \varepsilon E[\lambda_r^i] = \lambda_d^i, \tag{39}$$

$$E[\{\boldsymbol{u}^i\}] = E[\{\boldsymbol{u}_d^i\}] + \varepsilon E[\{\boldsymbol{u}_r^i\}] = \{\boldsymbol{u}_d^i\}. \tag{40}$$

Squaring eq. (35) gives

$$(\lambda^i)^2 = (\lambda_d^i)^2 + 2\varepsilon \lambda_d^i \lambda_r^i + \varepsilon^2 (\lambda_r^i)^2. \tag{41}$$

Taking expectation of eq. (41) we have

$$E[(\lambda^i)^2] = E[(\lambda_d^i)^2] + \varepsilon^2 E[(\lambda_r^i)^2], \tag{42}$$

λ^i is subject to

$$\mathrm{Var}(\lambda^i) = E[(\lambda^i)^2] - (E[\lambda^i])^2. \tag{43}$$

Taking eqs. (43) and (42) into consideration we have

$$\mathrm{Var}(\lambda^i) = \varepsilon^2 E[(\lambda_r^i)^2]. \tag{44}$$

It should be noted that according to eq. (36),

$$\begin{aligned}\{\boldsymbol{u}^i\}\{\boldsymbol{u}^i\}^{\mathrm{T}} &= (\{\boldsymbol{u}_d^i\} + \varepsilon\{\boldsymbol{u}_r^i\})(\{\boldsymbol{u}_d^i\} + \varepsilon\{\boldsymbol{u}_r^i\})^{\mathrm{T}} \\ &= \{\boldsymbol{u}_d^i\}\{\boldsymbol{u}_d^i\}^{\mathrm{T}} + \varepsilon\{\boldsymbol{u}_d^i\}\{\boldsymbol{u}_r^i\}^{\mathrm{T}} + \varepsilon\{\boldsymbol{u}_r^i\}\{\boldsymbol{u}_d^i\}^{\mathrm{T}} + \varepsilon^2\{\boldsymbol{u}_r^i\}\{\boldsymbol{u}_r^i\}^{\mathrm{T}}.\end{aligned} \tag{45}$$

Taking expectation of eq. (45) yields

$$E[\{\boldsymbol{u}^i\}\{\boldsymbol{u}^i\}^{\mathrm{T}}] = E[\{\boldsymbol{u}_d^i\}\{\boldsymbol{u}_d^i\}^{\mathrm{T}}] + \varepsilon^2 E[\{\boldsymbol{u}_r^i\}\{\boldsymbol{u}_r^i\}^{\mathrm{T}}]. \tag{46}$$

The covariance matrix of the eigenvector can be written as

$$\begin{aligned}&\mathrm{Cov}[\{\boldsymbol{u}^i\},\{\boldsymbol{u}^i\}^{\mathrm{T}}] = E\left[(\{\boldsymbol{u}^i\} - E[\{\boldsymbol{u}^i\}]) \cdot (\{\boldsymbol{u}^i\}^{\mathrm{T}} - E[\{\boldsymbol{u}^i\}^{\mathrm{T}}])\right] \\ &= E\left[\{\boldsymbol{u}^i\}\{\boldsymbol{u}^i\}^{\mathrm{T}} - \{\boldsymbol{u}_d^i\}\{\boldsymbol{u}^i\}^{\mathrm{T}} - \{\boldsymbol{u}^i\}\{\boldsymbol{u}_d^i\}^{\mathrm{T}} + E[\{\boldsymbol{u}^i\}]E[\{\boldsymbol{u}^i\}^{\mathrm{T}}]\right] \\ &= E[\{\boldsymbol{u}^i\}\{\boldsymbol{u}^i\}^{\mathrm{T}}] - \{\boldsymbol{u}_d^i\}\{\boldsymbol{u}_d^i\}^{\mathrm{T}} \\ &= E[\{\boldsymbol{u}^i\}\{\boldsymbol{u}^i\}^{\mathrm{T}}] - E[\{\boldsymbol{u}^i\}]E[\{\boldsymbol{u}^i\}^{\mathrm{T}}].\end{aligned} \tag{47}$$

Considering eqs. (46) and (47), we have

$$\mathrm{Cov}[\{\boldsymbol{u}^i\},\{\boldsymbol{u}^i\}^{\mathrm{T}}] = \varepsilon^2 E[\{\boldsymbol{u}_r^i\}\{\boldsymbol{u}_r^i\}^{\mathrm{T}}]. \tag{48}$$

Note that as for the system with N-DOF, $\mathrm{Cov}[\{\boldsymbol{u}^i\},\{\boldsymbol{u}^i\}^{\mathrm{T}}]$ denotes a matrix with dimensions $N\times N$, in which the diagonal elements are the variance of each mode, while the off-diagonal elements are the covariance of each mode.

Substituting eqs. (33)–(36) into eq. (3), we get

$$\begin{aligned}&(\boldsymbol{K}_d + \varepsilon \boldsymbol{K}_r)(\{\boldsymbol{u}_d^i\} + \varepsilon\{\boldsymbol{u}_r^i\}) \\ &= (\lambda_d^i + \varepsilon\lambda_r^i)(\boldsymbol{M}_d + \varepsilon \boldsymbol{M}_r)(\{\boldsymbol{u}_d^i\} + \varepsilon\{\boldsymbol{u}_r^i\}).\end{aligned} \tag{49}$$

Expanding eq. (49) with the $O(\varepsilon^3)$ terms ignored, then comparing the coefficient of the same power terms of ε on each side of the equation, we yield

$$\varepsilon^0:\ \boldsymbol{K}_d\{\boldsymbol{u}_d^i\} = \lambda_d^i \boldsymbol{M}_d\{\boldsymbol{u}_d^i\}, \tag{50}$$

$$\varepsilon^1:\ \boldsymbol{K}_d\{\boldsymbol{u}_r^i\} - \lambda_d^i \boldsymbol{M}_d\{\boldsymbol{u}_r^i\} = \lambda_d^i \boldsymbol{M}_r\{\boldsymbol{u}_d^i\} + \lambda_r^i \boldsymbol{M}_d\{\boldsymbol{u}_d^i\} - \boldsymbol{K}_r\{\boldsymbol{u}_d^i\}. \tag{51}$$

Note that eqs. (50) and (51) are similar to eqs. (14) and (15) in form. Taking the distinct eigenvalue into consideration, and utilizing the method adopted in Section 3, λ_r^i and $\{\boldsymbol{u}_r^i\}$ are derived as

$$\lambda_r^i = \{\boldsymbol{u}_d^i\}^{\mathrm{T}}(\boldsymbol{K}_r - \lambda_d^i \boldsymbol{M}_r)\{\boldsymbol{u}_d^i\}, \tag{52}$$

$$\begin{aligned}\{\boldsymbol{u}_r^i\} &= \sum_{i=1}^{n} C_r(i,j)\{\boldsymbol{u}_d^i\} = \sum_{i=1,\ i\neq j}^{n} C_r(i,j)\{\boldsymbol{u}_d^i\} + C_r(i,i)\{\boldsymbol{u}_d^i\} \\ &= \sum_{i=1,\ i\neq j}^{n}\left[\frac{(\{\boldsymbol{u}_d^i\}^{\mathrm{T}}(\boldsymbol{K}_r - \lambda_d^i \boldsymbol{M}_r)\{\boldsymbol{u}_d^i\})}{(\lambda_d^j - \lambda_d^i)}\right]\{\boldsymbol{u}_d^i\} \\ &\quad - \left[\frac{\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{M}_r\{\boldsymbol{u}_d^i\}}{2}\right]\{\boldsymbol{u}_d^i\},\end{aligned} \tag{53}$$

where $\boldsymbol{K}_r$ and $\boldsymbol{M}_r$ in eqs. (52) and (53) are random matrices which are different from $\boldsymbol{K}_1$ and $\boldsymbol{M}_1$ in eq. (22).

Concerning the variance of λ^i, we have

$$\begin{aligned}\mathrm{Var}(\lambda^i) &= \mathrm{Var}(\lambda_d^i + \varepsilon\lambda_r^i) = \mathrm{Var}(\lambda_d^i) + \mathrm{Var}(\varepsilon\lambda_r^i) \\ &= 0 + \mathrm{Var}(\varepsilon\lambda_r^i) = \varepsilon^2 \mathrm{Var}(\lambda_r^i) \\ &= \varepsilon^2 E[(\lambda_r^i - E(\lambda_r^i))^2] = \varepsilon^2 E[(\lambda_r^i)^2].\end{aligned} \tag{54}$$

Eq. (54) is equivalent to

$$\mathrm{Var}(\lambda_r^i) = E[(\lambda_r^i)^2]. \tag{55}$$

Substituting eq. (52) into eq. (55) yields

$$\begin{aligned}\mathrm{Var}(\lambda_r^i) &= E[(\lambda_r^i)^2] = E[(\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{K}_r\{\boldsymbol{u}_d^i\} - \lambda_d^i\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{M}_r\{\boldsymbol{u}_d^i\})^2] \\ &= E[(\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{K}_r\{\boldsymbol{u}_d^i\})^2 - (\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{K}_r\{\boldsymbol{u}_d^i\}) \\ &\quad \cdot(\lambda_d^i\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{M}_r\{\boldsymbol{u}_d^i\}) - (\lambda_d^i\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{M}_r\{\boldsymbol{u}_d^i\}) \\ &\quad \cdot(\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{K}_r\{\boldsymbol{u}_d^i\}) + (\lambda_d^i\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{M}_r\{\boldsymbol{u}_d^i\})^2] \\ &= E[(\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{K}_r\{\boldsymbol{u}_d^i\})^2] + (\lambda_d^i)^2 E[(\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{M}_r\{\boldsymbol{u}_d^i\})^2].\end{aligned} \tag{56}$$

Then $(\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{K}_r\{\boldsymbol{u}_d^i\})^2$ and $(\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{M}_r\{\boldsymbol{u}_d^i\})^2$ in eq. (56) are expanded into quadratic forms as follows:

$$\begin{aligned}\left(\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{K}_r\{\boldsymbol{u}_d^i\}\right)^2 &= \left(\sum_{a_1,a_2=1}^{n} k_{a_1a_2}u_{a_1}u_{a_2}\right)\left(\sum_{a_3,a_4=1}^{n} k_{a_3a_4}u_{a_3}u_{a_4}\right) \\ &= \sum_{a_1,a_2,a_3,a_4=1}^{n} k_{a_1a_2}k_{a_3a_4}u_{a_1}u_{a_2}u_{a_3}u_{a_4},\end{aligned} \tag{57}$$

$$\begin{aligned}\left(\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{M}_r\{\boldsymbol{u}_d^i\}\right)^2 &= \left(\sum_{b_1,b_2=1}^{n} m_{b_1b_2}u_{b_1}u_{b_2}\right)\left(\sum_{b_3,b_4=1}^{n} m_{b_3b_4}u_{b_3}u_{b_4}\right) \\ &= \sum_{b_1,b_2,b_3,b_4=1}^{n} m_{b_1b_2}m_{b_3b_4}u_{b_1}u_{b_2}u_{b_3}u_{b_4},\end{aligned} \tag{58}$$

where $k_{a_1a_2}$ and $k_{a_3a_4}$ are elements in matrix $\boldsymbol{K}_r$, while $m_{b_1b_2}$ and $m_{b_3b_4}$ are elements in matrix $\boldsymbol{M}_r$.

Substituting eqs. (57) and (58) into eq. (56) gives

$$\begin{aligned}\mathrm{Var}(\lambda_r^i) &= E\left[(\lambda_r^i)^2\right] \\ &= E\left[(\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{K}_r\{\boldsymbol{u}_d^i\})^2\right] + (\lambda_d^i)^2 E\left[(\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{M}_r\{\boldsymbol{u}_d^i\})^2\right] \\ &= E\left[\sum_{a_1,a_2,a_3,a_4=1}^{n} k_{a_1a_2}k_{a_3a_4}u_{a_1}u_{a_2}u_{a_3}u_{a_4}\right] + (\lambda_d^i)^2 E\left[\sum_{b_1,b_2,b_3,b_4=1}^{n} m_{b_1b_2}m_{b_3b_4}u_{b_1}u_{b_2}u_{b_3}u_{b_4}\right] \\ &= E\left[\sum_{a_1,a_2,a_3,a_4=1}^{n} k_{a_1a_2}k_{a_3a_4}\right]u_{a_1}u_{a_2}u_{a_3}u_{a_4} + (\lambda_d^i)^2 E\left[\sum_{b_1,b_2,b_3,b_4=1}^{n} m_{b_1b_2}m_{b_3b_4}\right]u_{b_1}u_{b_2}u_{b_3}u_{b_4}.\end{aligned} \tag{59}$$

In terms of eq. (54), we have

$$\begin{aligned}\mathrm{Var}(\lambda^i) &= \varepsilon^2\mathrm{Var}(\lambda_r^i) \\ &= \varepsilon^2 E\left[\sum_{a_1,a_2,a_3,a_4=1}^{n} k_{a_1a_2}k_{a_3a_4}u_{a_1}u_{a_2}u_{a_3}u_{a_4}\right] + \varepsilon^2(\lambda_d^i)^2 E\left[\sum_{b_1,b_2,b_3,b_4=1}^{n} m_{b_1b_2}m_{b_3b_4}u_{b_1}u_{b_2}u_{b_3}u_{b_4}\right] \\ &= \varepsilon^2 E\left[\sum_{a_1,a_2,a_3,a_4=1}^{n} k_{a_1a_2}k_{a_3a_4}\right]u_{a_1}u_{a_2}u_{a_3}u_{a_4} + \varepsilon^2(\lambda_d^i)^2 E\left[\sum_{b_1,b_2,b_3,b_4=1}^{n} m_{b_1b_2}m_{b_3b_4}\right]u_{b_1}u_{b_2}u_{b_3}u_{b_4}.\end{aligned} \tag{60}$$

Furthermore, note that $E[\boldsymbol{K}_r] = 0$ and $E[\boldsymbol{M}_r] = 0$, hence the expectation of the elements in $\boldsymbol{K}_r$ and $\boldsymbol{M}_r$ is zero too, thus we have

$$\begin{cases} E[k_{ij}] = 0, \ (i = 1,2,\cdots,n;\ j = 1,2,\cdots,n), \\ E[m_{ij}] = 0, \ (i = 1,2,\cdots,n;\ j = 1,2,\cdots,n). \end{cases} \tag{61}$$

Assuming that each element in $\boldsymbol{K}_r$ and $\boldsymbol{M}_r$ is independent, then the expectation of each cross-element term in eqs. (59) and (60) is zero. Eventually, we have the simplified forms of eqs. (59) and (60) as

$$\begin{aligned}\mathrm{Var}(\lambda_r^i) &= E[(\lambda_r^i)^2] \\ &= E[(\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{K}_r\{\boldsymbol{u}_d^i\})^2] + (\lambda_d^i)^2 E[(\{\boldsymbol{u}_d^i\}^{\mathrm{T}}\boldsymbol{M}_r\{\boldsymbol{u}_d^i\})^2] \\ &= E\left[\sum_{a_1,a_2,a_3,a_4=1}^{n} k_{a_1a_2}k_{a_3a_4}u_{a_1}u_{a_2}u_{a_3}u_{a_4}\right] + (\lambda_d^i)^2 E\left[\sum_{b_1,b_2,b_3,b_4=1}^{n} m_{b_1b_2}m_{b_3b_4}u_{b_1}u_{b_2}u_{b_3}u_{b_4}\right] \\ &= E\left[\sum_{a_1,a_2,a_3,a_4=1}^{n} k_{a_1a_2}k_{a_3a_4}\right]u_{a_1}u_{a_2}u_{a_3}u_{a_4} + (\lambda_d^i)^2 E\left[\sum_{b_1,b_2,b_3,b_4=1}^{n} m_{b_1b_2}m_{b_3b_4}\right]u_{b_1}u_{b_2}u_{b_3}u_{b_4} \\ &= E\left[\sum_{a_1,a_2,=1}^{n} k_{a_1a_2}k_{a_1a_2}\right]u_{a_1}^2u_{a_2}^2 + (\lambda_d^i)^2 E\left[\sum_{b_1,b_2,=1}^{n} m_{b_1b_2}m_{b_1b_2}\right]u_{b_1}^2u_{b_2}^2,\end{aligned} \tag{62}$$

$$\begin{aligned}\mathrm{Var}(\lambda^i) &= \varepsilon^2 \mathrm{Var}(\lambda_r^i)\\ &= \varepsilon^2 E\left[\sum_{a_1,a_2,a_3,a_4=1}^{n} k_{a_1a_2}k_{a_3a_4}u_{a_1}u_{a_2}u_{a_3}u_{a_4}\right]\\ &\quad + \varepsilon^2(\lambda_d^i)^2 E\left[\sum_{b_1,b_2,b_3,b_4=1}^{n} m_{b_1b_2}m_{b_3b_4}u_{b_1}u_{b_2}u_{b_3}u_{b_4}\right]\\ &= \varepsilon^2 E\left[\sum_{a_1,a_2,a_3,a_4=1}^{n} k_{a_1a_2}k_{a_3a_4}\right]u_{a_1}u_{a_2}u_{a_3}u_{a_4}\\ &\quad + \varepsilon^2(\lambda_d^i)^2 E[\sum_{b_1,b_2,b_3,b_4=1}^{n} m_{b_1b_2}m_{b_3b_4}]u_{b_1}u_{b_2}u_{b_3}u_{b_4}\\ &= \varepsilon^2 E\left[\sum_{a_1,a_2=1}^{n} k_{a_1a_2}k_{a_1a_2}\right]u_{a_1}^2u_{a_2}^2\\ &\quad + \varepsilon^2(\lambda_d^i)^2 E\left[\sum_{b_1,b_2=1}^{n} m_{b_1b_2}m_{b_1b_2}\right]u_{b_1}^2u_{b_2}^2. \end{aligned} \tag{63}$$

5 Numerical study

5.1 Case Ⅰ: planar truss-system

By virtue of the numerical example in refs. [22,23], in this paper we take the planar truss-system in Figure 2 as a numerical example in order to verify the proposed method.

The system is composed of planar truss-structure, which is modeled with 4 nodes and 6 elements. The degrees of freedom in node 1 and node 4 are 0. The length of each member is shown in Figure 2, and length L takes a value of 1.0 m, the cross-sectional area of every element is assigned a value of A=1.6×10^{-3} m^2, the nominal values of Young's modulus and weight density in each member are E=2.1×10^{11} N/m^2 and ρ=7.3×10^3 kg/m^3, respectively.

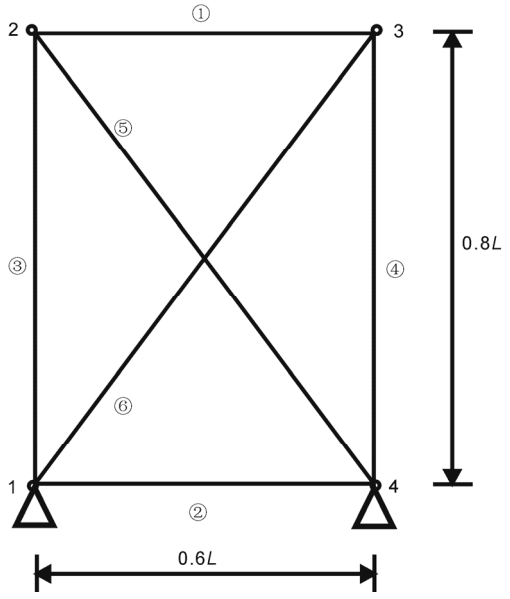

Figure 2 Planar truss-structure of numerical example.

Firstly, without taking the uncertain effects into account, we analyze the structural vibration properties by taking the nominal values of each structural parameter into calculation. In this way, 4 orders of vibration frequencies and their corresponding eigenvalues and eigenvectors (see Table 1) are obtained. Note that the system global stiffness matrix $\boldsymbol{K}$ can be written as

$$\boldsymbol{K} = \sum_{i=1}^{6} \frac{E_i A_i}{L_i} \boldsymbol{K}_i^e , \tag{64}$$

and the system lumped mass matrix $\boldsymbol{M}$ is represented as

$$\boldsymbol{M} = \frac{m_1 + m_3 + m_5}{2} \boldsymbol{I}_{4\times 4} , \tag{65}$$

here $\boldsymbol{K}_i^e$ is elementary stiffness matrix of each member under the global coordinate, m_1, m_3, m_5 denote the weight of members 1, 3, 5, respectively. The matrices $\boldsymbol{K}_i^e$ appearing in eq. (64) are as follows:

$$\boldsymbol{K}_1^e = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{K}_2^e = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{K}_3^e = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{K}_4^e = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{K}_5^e = \begin{bmatrix} 0.36 & -0.48 & 0 & 0 \\ -0.48 & 0.64 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{K}_6^e = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.36 & 0.48 \\ 0 & 0 & 0.48 & 0.64 \end{bmatrix}.$$

Let's take the errors caused by equipment in the process of manufacture and the differences in raw materials into consideration. Hereinafter, for the purpose of calculation, the Young's modulus E of each element is treated as Gaussian distribution $N(\mu_1, \sigma_1^2)$, here μ_1=2.1×10^{11} N/m^2

Table 1 Eigenvalues and Eigenvectors of the baseline-system

Order		1st	2nd	3rd	4th
Frequency (Hz)		367.10	1036.80	1109.70	1521.70
Eigenvalue		5319000	42436000	48619000	91411000
Eigenvector (normalized)	u_1	0.2296	–0.0579	0.0661	0.2318
	v_1	0.0661	–0.2318	–0.2296	–0.0579
	u_2	0.2296	0.0579	0.0661	–0.2318
	v_2	–0.0661	–0.2318	0.2296	–0.0579

and σ_1=0.1×10^{11} N/m^2. Additionally, the weight density ρ of each element is also assigned as Gaussian distribution $N(\mu_2,\ \sigma_2{}^2)$, here μ_2=7300 kg/m^3, σ_2=200 kg/m^3.

According to the proposed method for determining $\mathrm{Var}(\lambda_r^i)$ and $\mathrm{Var}(\lambda_i)$, variance of eigenvalues corresponding to 4 orders of natural frequencies is demonstrated in Table 2.

To validate the feasibility of the DVA method, the results obtained by the DVA method are compared with those of the Monte-Carlo method (see Figures 3–6).

As can be seen in Figures 3–6, the results deduced with the Monte-Carlo method are often bigger in value than those deduced with the DVA method. With the increment of sampling number, the curves of outcomes that are determined by the Monte-Carlo method tend to be smooth. There are two dominant reasons that account for the deviation in results between these two methods.

The deducing process of the DVA method is based on the first-order matrix perturbation theory. Therefore it could lead to certain errors from the accurate value when the variance of structural eigenvalue is calculated. It is safe to say, by adopting the second-order matrix perturbation theory, the deviation will be diminished notably.

In the DVA method, we assume that all the matrix elements in the perturbation parts of structural stiffness matrix and structural mass matrix are independent, which could be expressed as

$$\begin{cases} E(k_{ij}\cdot k_{pq}) = E(k_{ij})\cdot E(k_{pq}) = 0, \\ M(k_{ij}\cdot k_{pq}) = M(k_{ij})\cdot M(k_{pq}) = 0. \end{cases} \tag{66}$$

Table 2 Variance of eigenvalues determined by the proposed method

Order	1st	2nd	3rd	4th
Frequency (Hz)	367.1	1036.8	1109.7	1521.7
Eigenvalue	5319000	42436000	48619000	91411000
Variance	3.8234×10^{10}	3.4646×10^{12}	4.9275×10^{12}	6.257×10^{12}

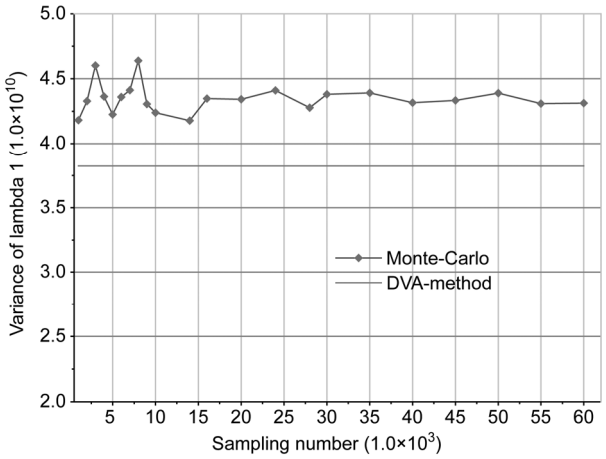

Figure 3 Comparison of the variance of λ_1 obtained by the Monte-Carlo and DVA methods.

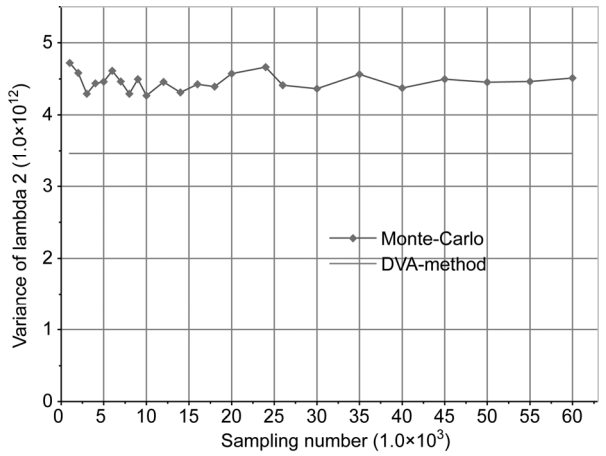

Figure 4 Comparison of the variance of λ_2 obtained by the Monte-Carlo and DVA methods.

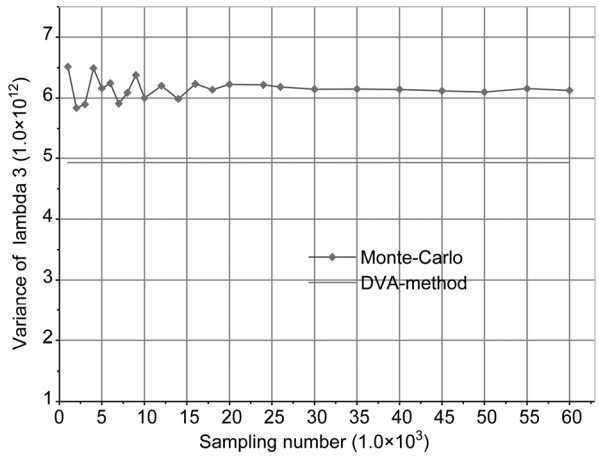

Figure 5 Comparison of the variance of λ_3 obtained by the Monte-Carlo and DVA methods.

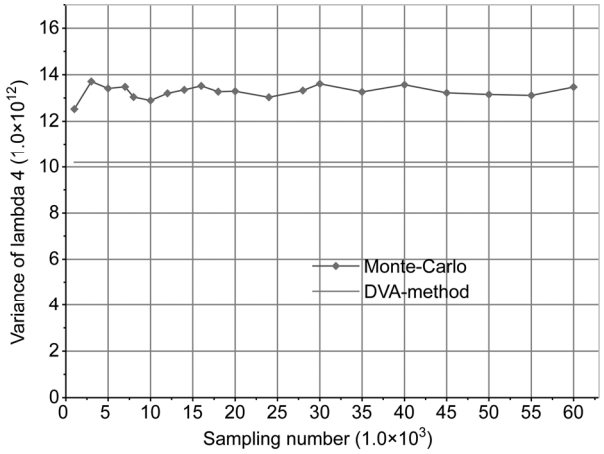

Figure 6 Comparison of the variance of λ_4 obtained by the Monte-Carlo and DVA methods.

The relativity between any two matrix elements is ignored, therefore the cross-terms of matrix elements are absent when we calculate the variance of structural eigenvalue.

With respect to the calculative time consumption, it costs merely about 1.3 s in calculating the variance of eigenvalues with the DVA method. The comparison of calculative time consumption between these two methods is demonstrated in Table 3. Both the DVA and Monte-Carlo methods are executed with an Intel Core i7-2600@3.40GHz computer.

As shown in Table 3, the computational time consumption of the Monte-Carlo method increases conspicuously when the sampling number increases. Therefore, in the case of coping with some large and complex structures such as aircraft and vehicle, the computational cost will be implausible. Quite on the contrary, the DVA method is low-cost in computational effort. In another word, the DVA method could be adopted in eigenvalue re-analysis of large and complex structure potentially.

5.2 Case II: wing structure of MA700 commercial aircraft

Aiming at verifying the proposed method with a large and complex engineering structure, we employ the wing structure of MA700 commercial aircraft as the numerical study object, as can be seen in Figure 7. Initially, the modal analysis of the wing structure is accomplished, and the first two orders free vibration mode fringe is illustrated in Figures 8 and 9. Secondly, in order to extract the global stiffness matrix and global mass matrix in f06 file created by Nastran, the bdf file of the wing structure, which is generated by Patran, is modified. Thirdly, the global stiffness matrix and global mass matrix are read via Matlab with a reader program.

Additionally, by adopting the DVA method (as presented in eq. (63)), the variance of the first-order eigenvalue of MA700 wing structure is achieved, which is illustrated in Figure 10. Simultaneously, the outcomes of the proposed method are compared with those of the Monte-Carlo method in Figure 10. It should be identified complementarily that the sampling number of the Monte-Carlo method is 60000. Both the DVA and Monte-Carlo methods are executed with an Intel Core i7-2600@3.40GHz computer.

Table 3 Time consumption for two types of computation

Method		Computational time consumption (s)			
DVA		λ_1	λ_2	λ_3	λ_4
		1.33	1.35	1.32	1.36
Monte-Carlo sampling number	7.0×10^3	745	222	362	1207
	8.0×10^3	869	261	319	1380
	1.0×10^4	1331	315	408	1681
	2.0×10^4	1741	1374	1577	3317
	4.0×10^4	2287	2443	2816	6480
	6.0×10^4	4830	4862	5862	6331

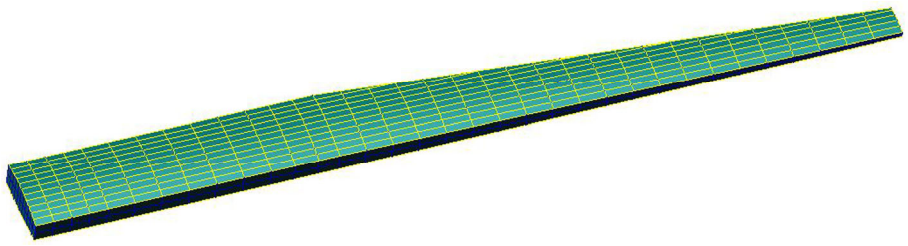

Figure 7 (Color online) Wing structure of MA700 commercial aircraft.

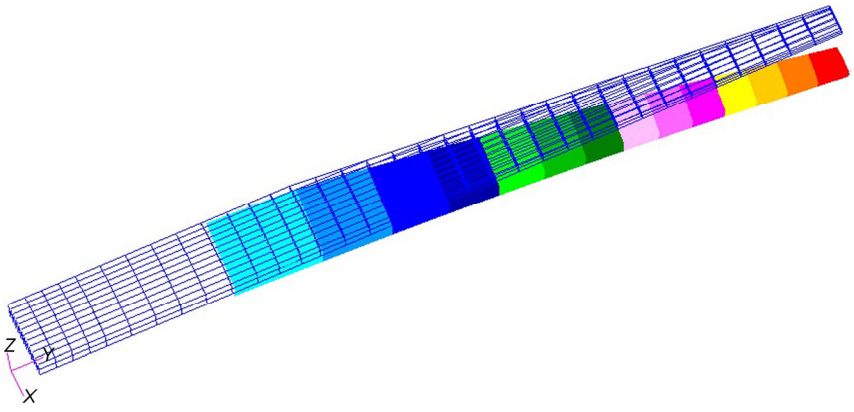

Figure 8 (Color online) First order free vibration mode.

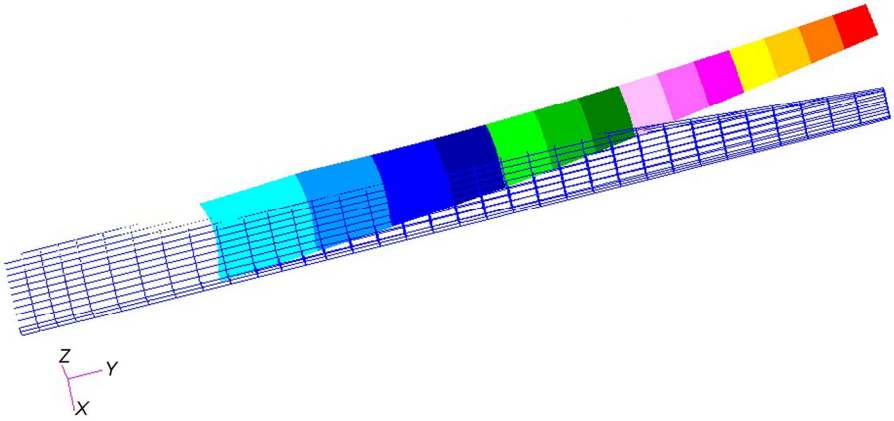

Figure 9 (Color online) Second order free vibration mode.

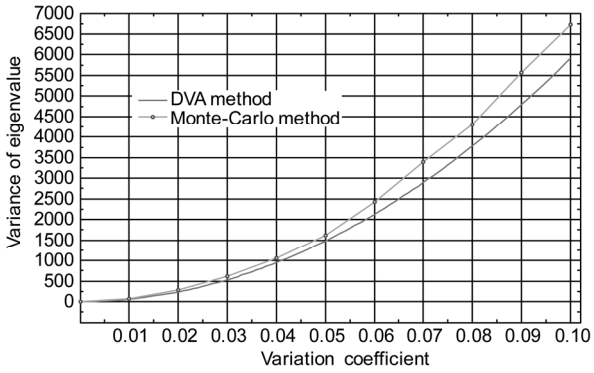

Figure 10 Outcomes of DVA and Monte-Carlo methods.

According to the outcomes of Case II, the following perspectives could be summarized.

Variance of eigenvalue which is determined by the DVA method increases alongside the increment of variation coefficient.

The result of the DVA method is relatively smaller than that of the Monte-Carlo method, which is similar to the situation in Case I. The reasons have already been systematically analyzed in Case I.

It is observed that when the value of variation coefficient is small, the result of the DVA method is consistent with that of the Monte-Carlo method; with regard to the fact that the value of variation coefficient is approximate to 0.1, the gap between these two curves in Figure 10 enhances gradually. Particularly, when the value of variation coefficient is zero, which implies that all the structural parameters are deterministic without uncertainty impacts, the outcomes of these two methods both converge at zero value.

With respect to the calculative time consumption, it costs hours of computational time in each variation coefficient value when the Monte-Carlo method is utilized; while the DVA method only takes a few seconds to fulfill the whole calculative procedure.

Due to the calculation process of case II, another remarkable advantage of the proposed method is that the application of the DVA method is easy to fulfill via the secondary development on the basis of commercial FEM/CAE software.

6 Conclusion

Based upon the matrix perturbation theory, the present research has developed a DVA method which is oriented towards generalized random eigenvalue and has deduced in detail the expressions of structural eigenvalue's variance after perturbation. According to the expressions deduced in the present research, the variance of structural eigenvalue after perturbation can be calculated with the perturbation of stiffness matrix, the perturbation of mass matrix and the eigenvector of baseline structure directly.

To verify the proposed method, two numerical examples are utilized. One is planar truss-system and the other is wing structure of MA700 commercial aircraft. By virtue of the results in these two examples, the feasibility and validity of our research have been demonstrated clearly. The drawback of the proposed method, if there exists any, is that it lacks calculating accuracy to some extent when compared with the Monte-Carlo method and the reasons are analyzed in detail in Section 5. However, the DVA method enjoys a significant advantage in calculating efficiency and is easy to fulfill by means of secondary development of FEM/CAE software.

By all accounts, the method proposed in the present research could meet the requirement of calculating the variation range of structural eigenvalue quickly and efficiently when structural parameters have changed. It is an efficient method to do structural re-analysis and structural dynamic sensitivity analysis. Simultaneously, it is highly applicable to the fields of engineering practice. This method will become a powerful tool in dealing with the random eigenproblem in large complex structures.

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