SCIENCE CHINA
Technological Sciences

May 2013 Vol.56 No.5: 1273–1281
doi: 10.1007/s11431-013-5200-y

A random medium model for simulation of concrete failure

LIANG ShiXue[1], REN XiaoDan[1] & LI Jie[1, 2*]

[1] *School of Civil Engineering, Tongji University, Shanghai 200092, China;*
[2] *The State Key Laboratory on Disaster Reduction in Civil Engineering, Tongji University, Shanghai 200092, China*

Received December 3, 2012; accepted March 11, 2013; published online March 29, 2013

A random medium model is developed to describe damage and failure of concrete. In the first place, to simulate the evolving cracks in a mesoscale, the concrete is randomly discretized as irregular finite elements. Moreover, the cohesive elements are inserted into the adjacency of finite elements as the possible cracking paths. The spatial variation of the material properties is considered using a 2-D random field, and the stochastic harmonic function method is adopted to simulate the sample of the fracture energy random field in the analysis. Then, the simulations of concrete specimens are given to describe the different failure modes of concrete under tension. Finally, based on the simulating results, the probability density distributions of the stress-strain curves are solved by the probability density evolution methods. Thus, the accuracy and efficiency of the proposed model are verified in both the sample level and collection level.

failure simulation, cohesive elements, random field, probability density evolution

Citation: Liang S X, Ren X D, Li J. A random medium model for simulation of concrete failure. Sci China Tech Sci, 2013, 56: 1273–1281, doi: 10.1007/s11431-013-5200-y

1 Introduction

Concrete, as the most widely used construction material for infrastructures, is featured by its stochastic nonlinearities due to the random distribution of the multiple phases and defects [1]. After years of investigations, two groups of models, i.e., the continuum models based on the continuum damage models and the fracture simulation based discontinuous models, have been proposed to characterize the nonlinear behaviors of concrete.

Within the framework of the continuum damage mechanics, the continuum models [2–5] incline to consider the degeneration of mechanical behaviors by using the continuum damage variable. Based on the definition of the generalized damage variable (avoiding the explicit simulation of cracks and voids), the continuum damage mechanics has its intrinsic advantage to drive the physical mechanisms into the irreversible thermodynamics principles to model a wide range of softening materials, especially the concrete. However, the irreversible thermodynamics only provides a framework for the damage evolution. That is to say, the thermo dynamical inequalities could not define the specified forms of the damage evolution functions, which also play a very important role in continuum damage mechanics. Hence, several empirical expressions of damage evolution were proposed based on the curve fitting of experimental results. In this case, the forms of damage evolution and the corresponding parameters are often lack of physical meanings. What's more, the generalized damage variable cannot describe the detailed crack propagation and interaction, which are critical to the strength and the failure modes of concrete.

Meanwhile, some researchers turned to the discontinuous models, which tend to simulate the propagation and coalescence of micro-voids and cracks in a direct way. As for a single crack, lying within the infinite homogenous solid, the closed-form solution [6] can be derived based on the classical

*Corresponding author (email: lijie@tongji.edu.cn.)

linear elastic fracture mechanics (LEFM) for several stress conditions. Referring to the interaction of the cracks, many analytical techniques were also developed, such as differential method, Mori-Tanaka method and so on [7]. However, most of these methods are restricted to the weak interaction of cracks. When it comes to the complex coalescence, bifurcation even the intersection of cracks, it is still extraordinarily challenging to obtain the analytical solution nowadays. Some other researchers tried the numerical approaches that explicitly model the cracking process, e.g., the extended finite element method [8] (X-FEM) and the interface finite element method. In the X-FEM, the additional discontinuous enrichments are introduced to model the presence of cracks, voids or heterogeneities within the finite element, which is the geometric domain. In the interface finite element method, the variational equations are built with the terms representing the matrix and the interfaces separately. Then, a solid matrix is modeled as elastic media and the interface is described by the cohesive law. The cohesive law was firstly proposed by Dugdale [9] to describe the crack growth in metal. Later, Barenblatt [10] gave the mathematical explanation of the cohesive models. Hillerborg [11] investigated the applied formulations of several materials, such as the metal and concrete. Xu and Needleman [12] attained the convincible simulations of the fracture in solid, which was based on the combination of the finite element and cohesive element method.

When taking a close observation at the concrete, it is naturally to find out that its mechanical properties, ranging from the strength to the Young's modulus, are endowed with stochastic features due to the random distribution of the aggregates and the micro-defects (cracks and voids). Thereby, to deem the concrete as a kind of random medium will be more reasonable, as compared to the homogeneous description [1]. The random nature within the domain of concrete can be defined by the random medium. As for the simulation of concrete fracture and failure process, the random medium is modeled by a 2-D random field in this paper.

In the present paper, the randomly distributed finite elements and cohesive elements are developed in the first step to introduce the possible cracking paths. Then, the concrete is considered as a random medium. In this case, we use the 2-D random field, which is generated by the newly developed stochastic harmonic function method to model the spatial variation of fracture energy. It turns out that the numerical results of the tensile tests not only agree well with the experimental results in the mean and standard deviation curves, but also offer the probability density distributions of the stress-strain curves against the histograms of the experimental data.

2 Cohesive model based simulation

2.1 Cohesive crack method

Consider a solid Ω, which contains a series of zero-thickness cracks or shear bands. And let S be the interface within the solid (Figure 1(a)). As it depicts, S cuts Ω into two parts. Herein, we denote the two sides of interface S by S^+ and S^-, and the two parts of solid Ω by Ω^+ and Ω^-.

To establish the weak form equation of the solid in both parts, we have

$$\int_{\Omega^+} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \mathrm{d}\Omega = \int_{\partial\Omega^+} \boldsymbol{t} \cdot \boldsymbol{v} \mathrm{d}\Gamma + \int_{S^+} \boldsymbol{T}^+ \cdot \boldsymbol{v}^+ \mathrm{d}\Gamma , \tag{1}$$

$$\int_{\Omega^-} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \mathrm{d}\Omega = \int_{\partial\Omega^-} \boldsymbol{t} \cdot \boldsymbol{v} \mathrm{d}\Gamma + \int_{S^+} \boldsymbol{T}^- \cdot \boldsymbol{v}^- \mathrm{d}\Gamma , \tag{2}$$

where T^+, T^- indicate the stress at the interfaces of the solid and $T=T^+=-T^-$. Combining eq. (1) and eq. (2), we have

$$\int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \mathrm{d}\Omega + \int_{S} \boldsymbol{T} \cdot (\boldsymbol{v}^- - \boldsymbol{v}^+) \mathrm{d}\Gamma = \int_{\partial\Omega} \boldsymbol{t} \cdot \boldsymbol{v} \mathrm{d}\Gamma . \tag{3}$$

To solve the interfacial stress $\boldsymbol{T}$ in eq. (3), a nonlinear zone is introduced between the real crack and the non-cracking elastic area. Thanks to the nonlinear zone, the stress at the tip of the crack is reasonable, rather than the analytical infinity by the LEFM. Generally, the stress at the crack-tip is named as cohesive stress. Hillerborg proposed that the crack would propagate when the cohesive stress reached the strength of concrete, and the cohesive stress reduced along the opening of the crack. To apply the cohesive stress as the external tractions applying on the crack surfaces, the cohesive crack model is established (Figure 2) [10, 11].

As for the tensile fracture (Model- Ⅰ fracture), the cohesive

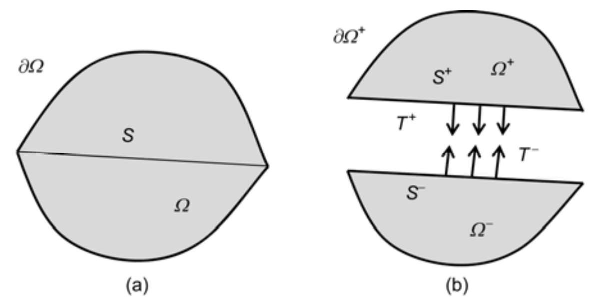

Figure 1 Solid analysis. (a) Solid with an interface; (b) split the solid along the interface.

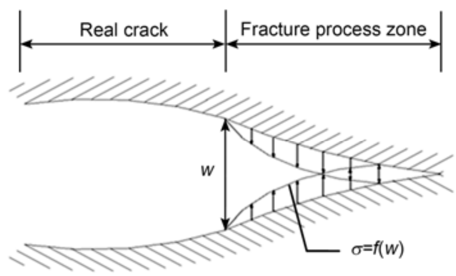

Figure 2 Cohesive crack model.

stress can be deemed as the function of COD (crack opening displacement) as follows

$$\boldsymbol{T} = \boldsymbol{T}[\boldsymbol{w}(\boldsymbol{u})]. \tag{4}$$

In this paper, the following stress-COD relationship distribution (Figure 3(a)) of the cohesive model is adopted, among various expressions [11],

$$f = f_t - kw, \tag{5}$$

Where $f=\boldsymbol{T}\cdot\boldsymbol{n}$ denotes the normal cohesive stress, $w=\boldsymbol{w}\cdot\boldsymbol{n}$ denotes the crack width, $\boldsymbol{n}$ is the normal unit vector, and k is the strength intensity factor.

Introduce the fracture energy as an intrinsic property of concrete, and specify its formulation as

$$G_c = \int_0^{w_1} f\,\mathrm{d}w, \tag{6}$$

where w_1 is the maximum width of cracks.

Apparently, for the $f-w$ diagram shown in Figure 3 (a), G_c can be expressed as

$$\int_0^{w_1} f\,\mathrm{d}w = f_t \cdot w_1/2. \tag{7}$$

Hillerborg suggests that the stress-strain relationship of the cohesive element should be linear before the stress reaches the tensile strength. The ascending line is also linear which is determined by G_c after the tensile peak stress (Figure 3 (b)).

As for the shear fracture (Model-II fracture), the cohesive stress can be expressed as a function of CSD (crack shear displacement). Then the shear fracture energy G_s is introduced for Model-II fracture and the cohesive stress can be obtained as the Model-I fracture [13]. Although in this paper, the tensile fracture is critical to the failure of the concrete, the shear fracture is considered for the integrity of the model.

Substituting eq. (4) into eq. (3), we obtain the functional equation as

$$\int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\mathrm{d}\Omega + \int_S \boldsymbol{T}[\boldsymbol{w}(\boldsymbol{u})] \cdot \boldsymbol{w}(\boldsymbol{v})\mathrm{d}\Gamma = \int_{\partial\Omega} \boldsymbol{t}\cdot\boldsymbol{v}\mathrm{d}\Gamma. \tag{8}$$

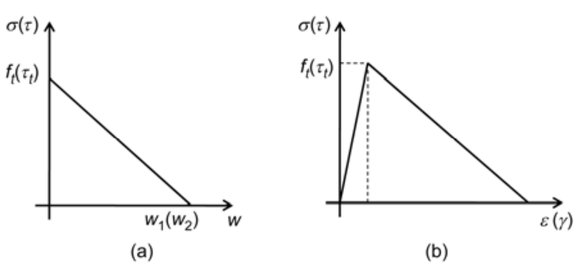

Figure 3 The stress diagram of cohesive model. (a) Stress-crack width relationship; (b) stress-strain relationship of cohesive model.

It could be noted in eq. (8) that the presence of a cohesive surface results in the addition of a new term to the functional equation of the finite element [12]. Thus, the finite elements and the cohesive elements are compatible in a unified model.

2.2 Random cohesive model

As discussed in section 2.1, the cohesive elements may give the potential crack paths when connecting the finite elements. As we know, the cracks in concrete are highly irregular and randomly oriented. Herein, the irregular elements are developed to model the random initial micro cracks in the concrete. The irregular cohesive model [14] is established as follows.

1) Define the boundary of the 2D domain.

2) Generate the set of random points in the domain and boundary.

3) Decompose the domain by the Delaunay triangulation scheme.

4) Contract the triangles to gain the cohesive elements that connect the adjacent finite elements.

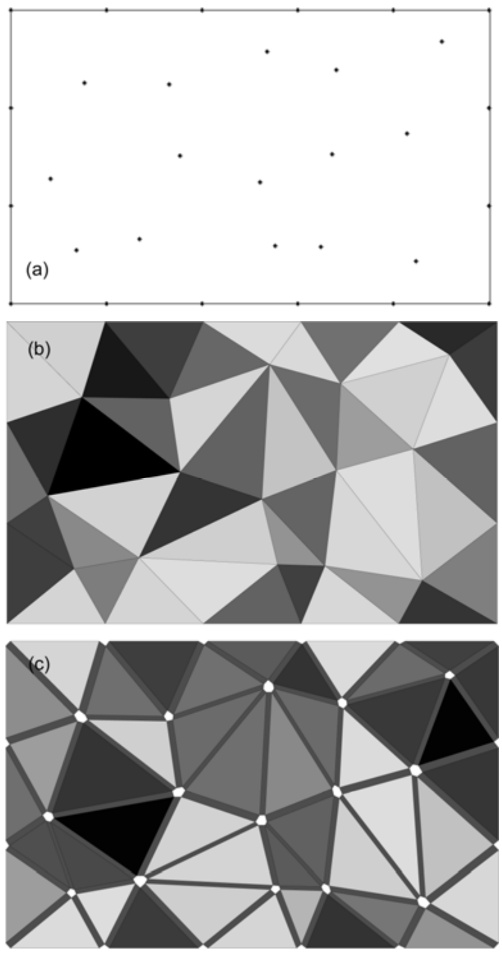

Figure 4 Generation of the irregular cohesive elements. (a) Random point set; (b) delauney triangulation; (c) finite elements and cohesive elements.

3 Stochastic harmonic function and random field

When it comes to the classic fracture mechanics, the fracture energy is considered as the intrinsic property of the material. Admittedly, the deterministic description, which relies on the mean value of the fracture energy, simplifies the model as well as the simulation. However, the heterogeneity and the randomness that are induced by the randomly distributed micro-cracks cannot be taken into account in the deterministic simulation. What's more, the various cracking branches and the unpredictable failure modes cannot be well tackled either. Therefore, to define the concrete as a kind of random media would represent the essential randomness of the concrete [1]. In order to capture the random propagation of cracks and the failure modes, the fracture energy is described as a 2-D random field in the present work.

Chen and Li [15] originally developed the stochastic harmonic function for the simulation of random process, such as the time histories of strong ground motion and the wind. Based on the development of the stochastic harmonic function for spectrum representation, Liang et al. [16] extended it into the multi-dimensional random fields. The stochastic harmonic function method is introduced to characterizing the random field of fracture energy in the present paper. Without loss of generality, the 2-D, homogeneous Gaussian random field $f_0(x,y)$ with μ=0 and σ=1 can be represented as follows

$$f(x,y)=\sqrt{2}\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\Big[A_{n_1n_2}\cos\left(K_{1n_1}x+K_{2n_2}y+\Phi_{n_1n_2}^{(1)}\right)+\tilde{A}_{n_1n_2}\cos\left(K_{1n_1}x-K_{2n_2}y+\Phi_{n_1n_2}^{(2)}\right)\Big],\quad(9)$$

where $f(x,y)$ is the random field generated by the stochastic harmonic function; $A_{n_1n_2}$ and $\tilde{A}_{n_1n_2}$ denote the amplitudes of the n_1-th and n_2-th harmonic components; K_{1n_1} and K_{2n_2} indicate respectively the wave numbers. $\Phi_{n_1n_2}^{(1)}$, $\Phi_{n_1n_2}^{(2)}$, n_1= 1, 2,$\cdots$ N_1, and n_2=1, 2,$\cdots$ N_2 are the independent random phases which uniformly distribute in the range [0, 2π]. K_{1n_1} and K_{2n_2} are the independent random wave numbers , which obey the uniform distribution in the ranges $\left(K_{1n_1-1}^p, K_{1n_1}^p\right]$ and $\left(K_{2n_2-1}^p, K_{2n_2}^p\right]$, respectively, i.e.,

$$p_{K_{jnj}}\left(K_j\right)=\begin{cases}\dfrac{1}{K_{jn_j}^p-K_{jn_j-1}^p}=\dfrac{1}{\Delta K_{jn_j}}, K\in\left(K_{jn_j-1}^p, K_{jn_j}^p\right], j=1,2,\\ 0, \qquad\qquad \text{other},\end{cases}\quad(10)$$

$A_{n_1n_2}$ and $\tilde{A}_{n_1n_2}$ are determined by the following equations

$$A_{n_1n_2}=\sqrt{2S_{f_0f_0}\left(K_{1n_1},K_{2n_2}\right)\Delta K_{1n_1}\Delta K_{2n_2}},\quad(11)$$

$$\tilde{A}_{n_1n_2}=\sqrt{2S_{f_0f_0}\left(K_{1n_1},-K_{2n_2}\right)\Delta K_{1n_1}\Delta K_{2n_2}},\quad(12)$$

where $S_{f_0f_0}$ is the target spectrum density function. From eqs. (11) and (12), it can be clearly observed that the random field could be fully determined by the target spectrum density function. It can be also proved that the power spectral density function of the stochastic harmonic function based random filed is equal to the target one. What's more, it has been proved that the stochastic harmonic random field asymptotically approaches the normal distribution.

According to eqs. (5) and (8), the fracture energy will not only affect the tensile strength of the concrete but also influence the descending branch of the stress-strain relationship of the concrete. Therefore, the mean value, fluctuation and the correlation of fracture energy should be determined.

The correlation function [17] is chosen as

$$R(\tau_1,\tau_2)=\exp\left(-\left(\frac{\xi_1}{b_1}\right)^2-\left(\frac{\xi_2}{b_2}\right)^2\right),\quad(13)$$

where $\xi_1=x_2-x_1$ indicates the distance of x direction in cartesian coordinates; $\xi_2=y_2-y_1$ indicates the distance of y direction; b_1 and b_2 are the correlation length of x and y directions, respectively.

Using the Fourier transform, the corresponding power spectrum density function can be obtained as follows:

$$S_{f_0f_0}(K_1,K_2)=\frac{b_1b_2}{4\pi}\exp\left[-\left(\frac{b_1K_1}{2}\right)^2-\left(\frac{b_2K_2}{2}\right)^2\right],\quad(14)$$
$$-K_{1u}\leqslant K_1\leqslant K_{1u}, -K_{2u}\leqslant K_2\leqslant K_{2u}.$$

Substituting eq. (14) into eqs. (11) and (12), the random field of the fracture energy can be established by eq. (9).

The random field $f_1(x,y)$ with μ_{G_c} and σ_{G_c} could be expressed as follows

$$f_1(x,y)=\mu_{G_c}+\sigma_{G_c}^2 f(x,y).\quad(15)$$

4 Tensile failure simulation of concrete

4.1 Numerical model

The simulations for the tensile failure of concrete were carried out based on the proposed methods in the former sections. The geometric dimensions of the numerical specimens were b=150 mm and h=150 mm. According to the test results obtained by Ren et al. [18], the material parameters were taken to be E=37559 MPa and v=0.2. The mean value of the tensile strength of the concrete applied to both the finite elements and the cohesive elements was f_t=3.28 MPa.

The random field of fracture energy was introduced based on the methods proposed in section 3. Due to the strong nonlinearities induced by the cracking process, we chose the explicit solution to get the temporal integration of the crack process. The numerical specimen was developed by the finite element package ABAQUS Explicit. Based on the previous irregular discretization method, more than 20000 finite elements and 30000 cohesive elements were generated. The numerical specimen and its boundary conditions are given in Figure 5.

4.2 Random field model

Bazant and Pfeiffer [19] compared the notched and non-notched beams to localize the heterogeneities of concrete. With the test results, the correlation length could be considered as the fracture influenced length governed by the heterogeneous of material. Thus in the present work, the correlation lengths in eq. (13) were specified as b_1=b_2=3$d_{\max}$= 24 mm, where $d_{\max}$ is the maximum aggregate size.

Carpinteri and Chiaia [20, 21] introduced the multifractal scaling law for fracture energy based on a series of compact tests and three-point bending tests. As for the cohesive crack model, the fracture energy is

$$G_c = G_c^{\infty}\left(1+\frac{l_{ch}}{b}\right)^{-1/2}, \tag{16}$$

$$l_{ch} = \alpha d_{\max}, \tag{17}$$

where G_c^{∞} is the nominal asymptotic fracture energy valid within the limit of infinite structure size ($b \to \infty$), b is the size of the structure and is 150 mm in the simulation, l_{ch} is the characteristic length in determining the size dependent G_c along with the direction of the fracture process, and α is the non-dimensional parameter. The material parameters adopted in the present paper are $G_c^{\infty} = 160\ \text{N/m}$ and α=30.

Based on eqs. (16) and (17) as well as the parameters, the mean value of the fracture energy was calculated as μ_{G_c} = 100 N/m, and the standard deviation of the fracture energy was $\sigma_{G_c} = 0.1\mu_{G_c} = 10\ \text{N/m}$. Thus, the fracture energy random field could be modeled by eq. (15).

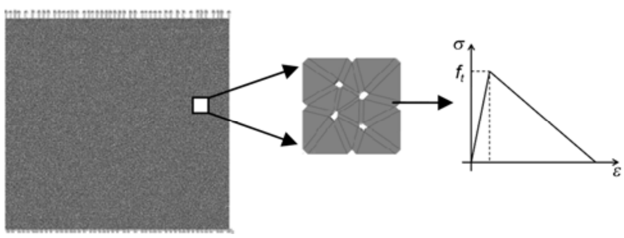

Figure 5 The numerical model and the stress-strain relationship of cohesive elements.

The target power spectrum density function and the power spectrum density function based on the stochastic harmonic function are plotted in Figures 6 and 7, respectively. In addition, the comparison between these two PSDs is depicted in Figure 8 at given wave numbers. The samples of the random field are shown in Figure 9.

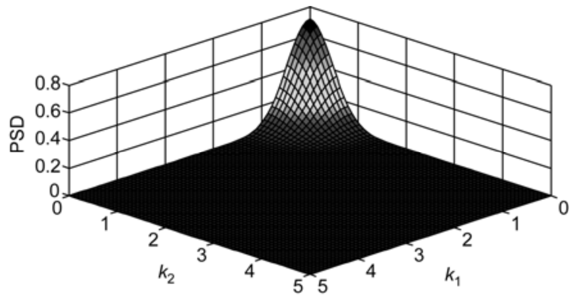

Figure 6 Target power spectral density function.

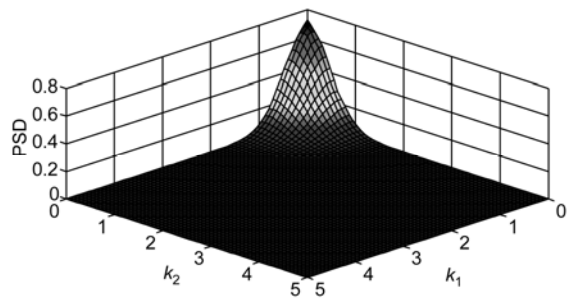

Figure 7 Power spectral density function by stochastic harmonic function.

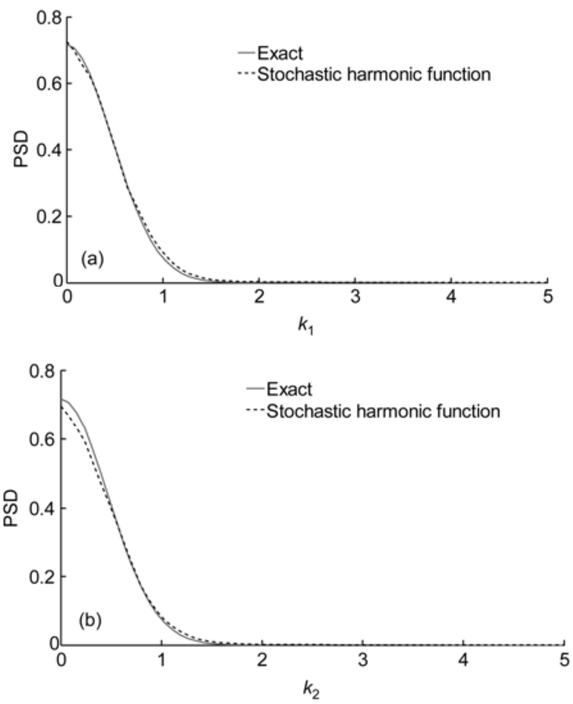

Figure 8 Power spectral density function. (a) K_2=0; (b) K_1=0.

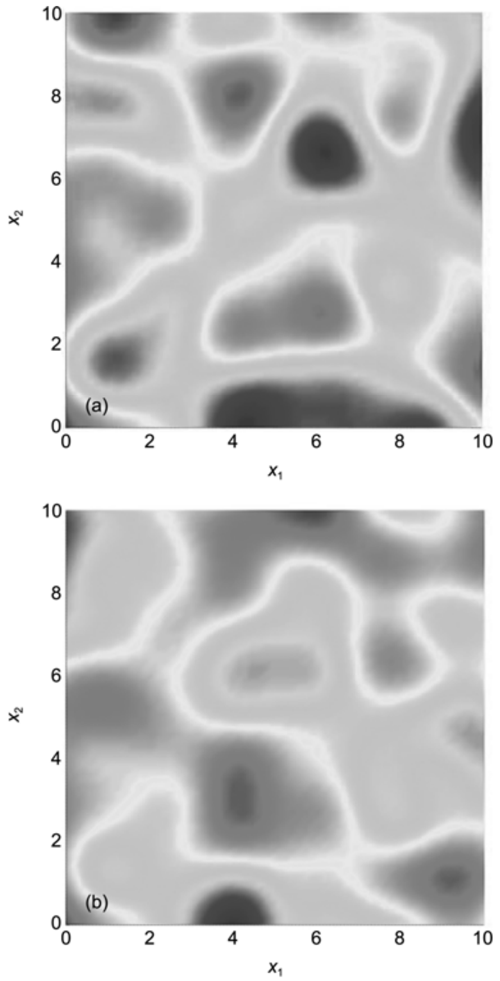

Figure 9 Samples by stochastic harmonic function for fracture energy. (a) Sample 1; (b) sample 2.

4.3 Failure simulation

As is shown in Figures 10 and 11, the overall procedure of the nonlinearity and the cracks can be observed as follows: At the very beginning, the stress and strain both increase linearly; when the loading increases, there are micro-cracks uniformly distributed in the whole specimen; then, with further increasing of the loading, the micro cracks concentrate on a certain area and the stress concentration happens on the tips of the cracks; at the final stage, a severe strain concentration occurs on the cracking zones and a main crack which is perpendicular to the tensile loading cuts through the specimen of the concrete. Figure 10 indicates the comparison between the numerical experiment and the test result of the end-crack failure of the concrete under tensile test, and Figure 11 indicates the non-end-crack failure as well.

5 Probability density analysis of the stress-strain relationship

When taking a close look at the concrete based on the random medium model, the most accurate way to describe the random stress-strain relationship is the probability density function. Thus, the probability density evolution method [22] can be applied. Without loss of generality, the monotonic stress-strain relationship is governed by the following equation:

$$\sigma = f(\Theta, \varepsilon), \tag{18}$$

where $\Theta = (\theta_1, \theta_2, \cdots, \theta_n)$ is the random parameter vector, whose joint probability density function is $p_\Theta(\theta)$. When it comes to the fracture energy random field by eq. (9), Θ is composed of the random wave numbers K_{1n_1} and K_{2n_2} and the random phase angles $\Phi_{n_1n_2}^{(1)}$ and $\Phi_{n_1n_2}^{(2)}$.

To replace the generalized time parameter t by ε, the probability density evolution equation [23] is

$$\frac{\partial p_{\sigma\Theta}(\sigma, \theta, \varepsilon)}{\partial \varepsilon} + \dot{\sigma}(\theta, \varepsilon)\frac{\partial p_{\sigma\Theta}(\sigma, \theta, \varepsilon)}{\partial \sigma} = 0, \tag{19}$$

Where $\dot{\sigma}$ denotes the first-order derivative of the stress.

Eq. (19) can be solved by combining the initial conditions, the simulations and the finite-difference method. The solving procedures are as follows.

1) Select the representative points of the random wave numbers K_{1n_1}, K_{2n_2} and the random phase angles $\Phi_{n_1n_2}^{(1)}$, $\Phi_{n_1n_2}^{(2)}$. The selected points are $\theta_q (q = 1, 2, \cdots, N_{sel})$, where N_{sel} is the total number of the selected points.

2) Simulation based on the proposed cohesive model is done for each representative point.

3) The finite-difference computation method [24] is used to get the joint probability density function.

4) Take numerical integration with respect to θ_q to acquire the numerical PDF.

In this paper, the quasi-symmetric method [25] was chosen in selecting the representative points and the number of random variables was 32. The total number of the representative points was 300. As is illustrated in Figure 12, the comparison between the simulating results and the test results suggests the validation of the model. Figure 13 depicts the probability density function of the stress-strain relationship of the concrete. Figure 14 illustrates the comparison between the experimental results and the theoretical results of the stress probability density function at certain strain.

It can be noted from eq. (19) that the probability density evolution of the stress will transfer in the whole loading process and will be affected by the random parameters and the complexity of the model. To the best knowledge of the authors, the probability flow of the stress in the numerical model will lead to the further random cracks and the failure modes. As for the simulation in this paper, there were 187 end-crack failures and 113 non-end-crack failures of the numerical samples. It can be observed in Figures 13 and 14

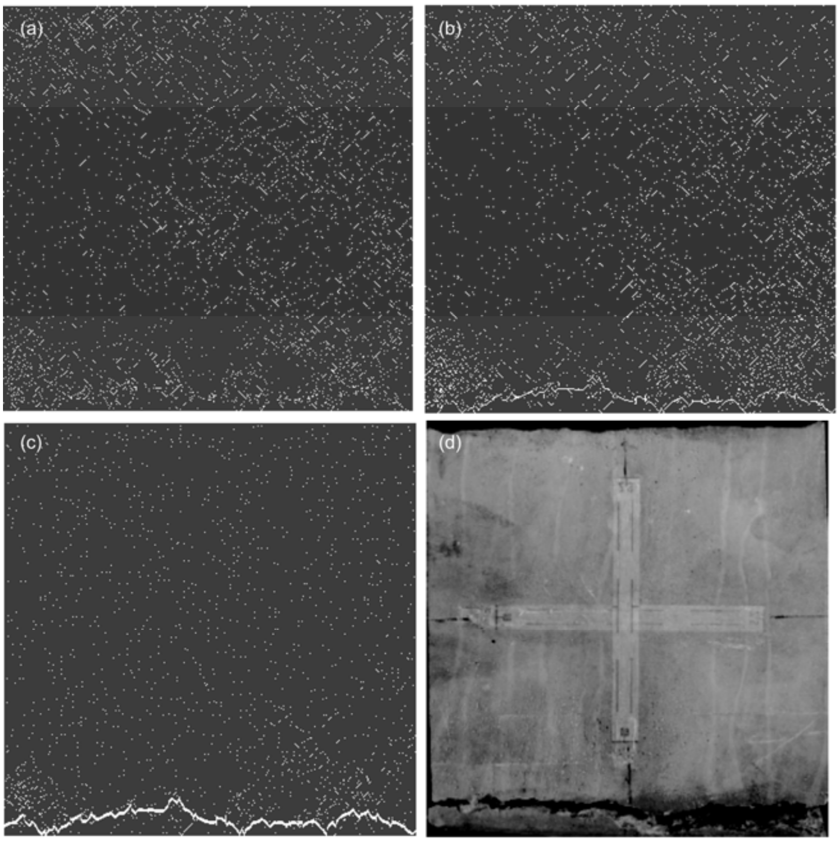

Figure 10 Simulation and test result of end-crack failure (ε_u ultimate tensile strain). (a) ε=0.2ε_u; (b) ε=0.7ε_u; (c) ε=ε_u; (d) test result.

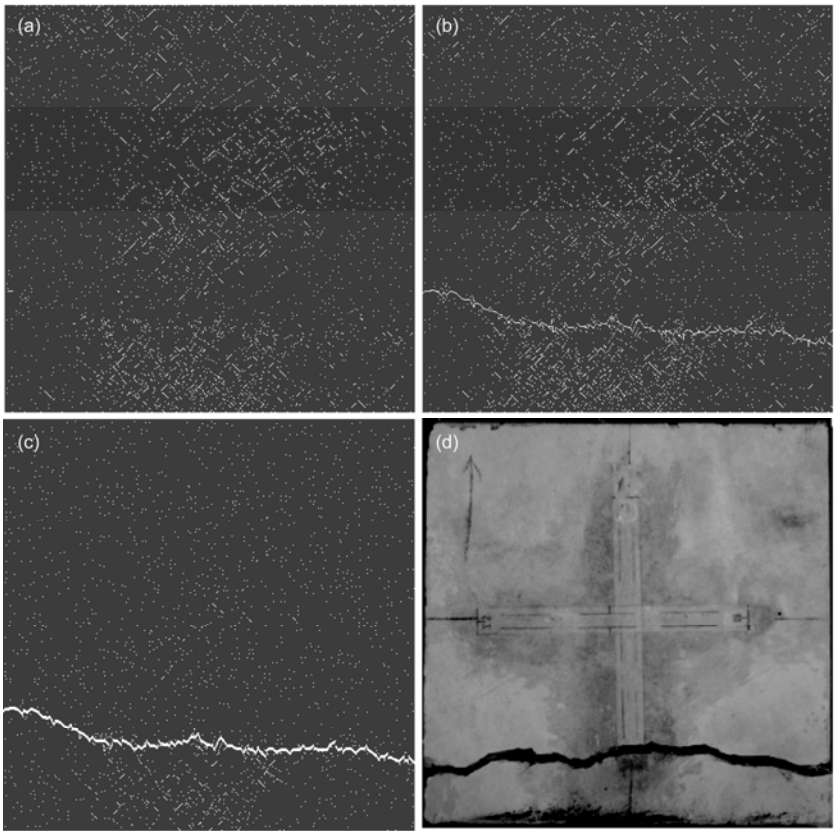

Figure 11 Simulation and test result of non-end-crack failure (ε_u ultimate tensile strain). (a) ε=0.2ε_u; (b) ε=0.7ε_u; (c) ε=ε_u; (d) test result.

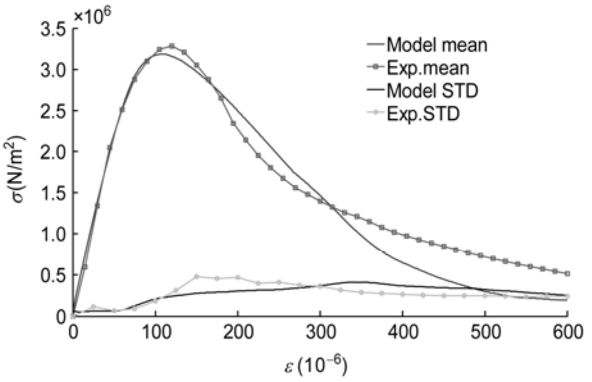

Figure 12 Mean and standard deviation curves for stress.

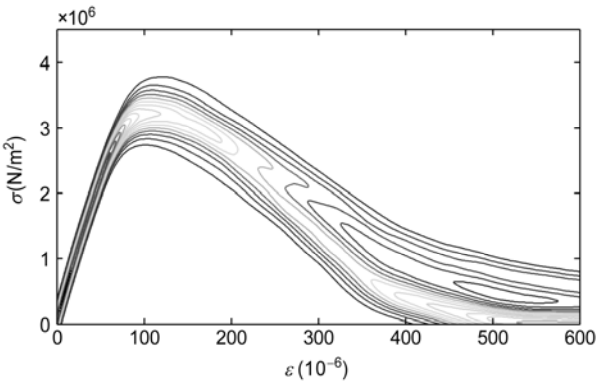

Figure13 Probability density functions of stress-strain curve

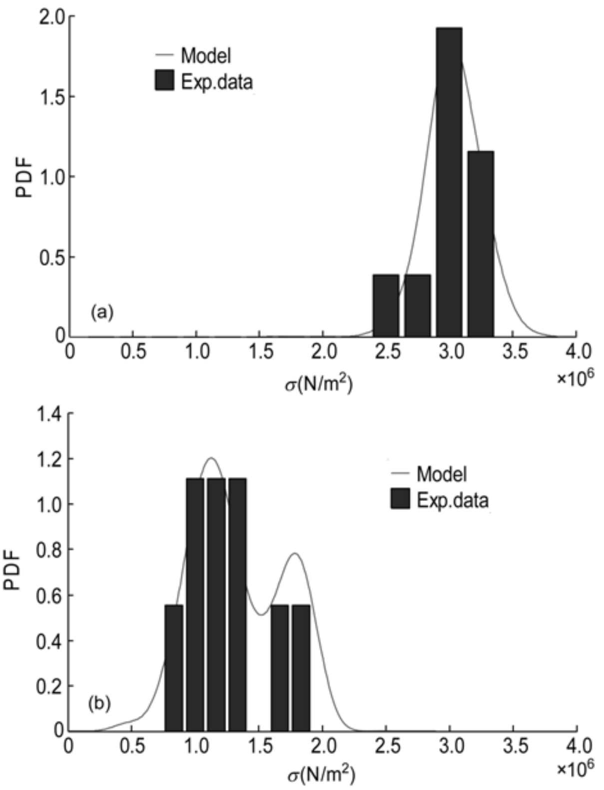

Figure14 Comparison of the probability density functions for the experimental and theoretical results. (a) ε=0.0001; (b) ε=0.0003.

that the probability density of the stress changed from one peak to two peaks with the increasing of strain due to the random failure modes. Thus, the probability density of the stress-strain relationship can represent the stochastic features of concrete and reveal the essential transfer of probability density.

6 Conclusion

In this paper, the failure simulation of the concrete based on the random medium model is presented. The randomness of concrete is considered in two aspects. On the one hand, the random micro cracks are introduced by the irregular finite elements and cohesive elements. On the other hand, the stochastic harmonic function is adopted to model the 2-D random field of fracture energy. Then, simulations of the cracking process and the failure modes are given to testify the proposed model in the sample level. In addition, the probability distribution evolution method is introduced to clarify the probability density distributions of the stress-strain relationship in the collection level. With the efforts of the present work, a new random medium based model for the accurate simulation of concrete is developed.

This work was supported by the National Natural Science Foundation of China (Grant Nos. 90715033, 51261120374, 51208374).

1 Li J. Research on the stochastic damage mechanics for concrete material and structures. J Tongji Univ (Nat Sci), 2004, 32(10): 1270–1277

2 Mazars J. A description of micro-and macroscale damage of concrete structures. Eng Fract Mech, 1986, 25(5): 729–737

3 Simo J C, Ju J W. Strain-and stress-based continuum damage models—I. Formulation. Int J Solids Struct, 1987, 23(7): 821–840

4 Lubliner J, Oliver J, Oller S, et al. A plastic-damage model for concrete. Int J Solids Struct, 1989, 25(3): 299–326

5 Wu J Y, Li J, Faria R. An energy release rate-based plastic-damage model for concrete. Int J Solids Struct, 2006, 43(3): 583–612

6 Nemat-Nasser S, Hori M. Micromechanics: Overall Properties of Heterogeneous Materials. North-Holland: Elsevier Amsterdam, 1999

7 Gross D, Seelig T. Fracture mechanics: with an introduction to micromechanics. Heidelberg: Springer, 2011

8 Moës N, Belytschko T. Extended finite element method for cohesive crack growth. Eng Fract Mech, 2002, 69(7): 813–833

9 Dugdale D S. Yielding of steel sheets containing slits. J Mech Phys Solids, 1960, 8(2): 100–104

10 Barenblatt G I. The mathematical theory of equilibrium cracks in brittle fracture. Adv Appl Mech, 1962, 7: 55–129

11 Hillerborg A, Modeer M, Petersson P E. Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements. Cement Concr Res, 1976, 6(6): 773–781

12 Xu X P, Needleman A. Numerical simulations of fast crack growth in brittle solids. J Mech Phys Solids, 1994, 42(9): 1397–1434

13 Rots J G, de Borst R. Analysis of mixed-mode fracture in concrete. J Eng Mech, 1987, 113(11): 1739–1758

14 Ren X, Li J. Dynamic fracture in irregularly structured systems. Phys Rev E, 2012, 85(5): 55102

15 Chen J B, Li J. Stochastic harmonic function and spectrum representations. Chinese J Theo Appl Mech, 2011, 43(3): 505–513

16 Liang S X, Sun W L, Li J. Simulation of multi-dimensional random

fields by stochastic harmonic functions. J Tongji Univ (Nat Sci), 2012, 40(7): 965–970
17 Xu X F, Graham-Brady L. A stochastic computational method for evaluation of global and local behavior of random elastic media. Comput Method Appl M, 2005, 194(42): 4362–4385
18 Ren X D, Yang W Z, Zhou Y, et al. Behavior of high-performance concrete under uniaxial and biaxial loading. Aci Mater J, 2009, 105(6): 548–557
19 Bazant Z P, Pfeiffer P A. Determination of fracture energy from size effect and brittleness number. Aci Mater J, 1987, 84(6): 755–767
20 Carpinteri A, Chiaia B. Size effects on concrete fracture energy: Dimensional transition from order to disorder. Mater Struct, 1996, 29(5): 259–266
21 Carpinteri A, Chiaia B, Ferro G. Size effects on nominal tensile strength of concrete structures: Multifractality of material ligaments and dimensional transition from order to disorder. Mater Struct, 1995, 28(6): 311–317
22 Li J, Chen J B. Probability density evolution method for analysis of stochastic structural dynamic response. Chinese J Theo Appl Mech, 2003, 35(4): 437–442
23 Zeng S J, Li J. Analysis on constitutive law of plain concrete subjected to uniaxial compressive stress based on generalized probability density evolution method. J Tongji Univ (Nat Sci), 2010, 38(6): 798–804
24 Chen J B, Li J, Difference method for probability density evolution equation of stochastic structural response. Chinese Quart Mech, 2005, 25(1): 22–28
25 Li J, Xu J, Chen J B. The use of quasi-symmetric point method in probability density evolution theory. J Wuhan Tech Univ, 2010, 32(9): 1–5