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A perturbation-incremental scheme for studying Hopf bifurcation in delayed differential systems

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A new method, called perturbation-incremental scheme (PIS), is presented to investigate the periodic solution derived from Hopf bifurcation due to time delay in a system of first-order delayed differential equations. The method is summarized as three steps, namely linear analysis at critical value, perturbation and increment for continuation. The PIS can bypass and avoid the tedious calculation of the center manifold reduction (CMR) and normal form. Meanwhile, the PIS not only inherits the advantages of the method of multiple scales (MMS) but also overcomes the disadvantages of the incremental harmonic balance (IHB) method. Three delayed systems are used as illustrative examples to demonstrate the validity of the present method. The periodic solution derived from the delay-induced Hopf bifurcation is obtained in a closed form by the PIS procedure. The validity of the results is shown by their consistency with the numerical simulation. Furthermore, an approximate solution can be calculated in any required accuracy.

delayed differential equation, perturbation-incremental scheme, Hopf bifurcation, synchronization, center manifold

1 Introduction

Dynamics of systems with time delay is of interest since time delay is ubiquitous in nature, science and engineering. A general mathematical model can be written as

$$\dot{\boldsymbol{Z}}(t)=\boldsymbol{C}\boldsymbol{Z}(t)+\boldsymbol{D}\boldsymbol{Z}(t-\tau)+\varepsilon\boldsymbol{F}(\boldsymbol{Z}(t),\boldsymbol{Z}(t-\tau)), \quad (1)$$

where $\boldsymbol{Z}(t)\in\mathbf{R}^n$, $\boldsymbol{C}$ and $\boldsymbol{D}$ are $n\times n$ real constant matrixes, $\boldsymbol{F}(\cdot)$ is a nonlinear function of its variables with $\boldsymbol{F}(0,0)=0$, ε is a parameter representing the couple degree between nonlinearities, τ is the time delay, and n is a positive integer. Eq. (1) may model many real systems, such as neural[1,2], ecological[3], biological[4], mechanical[5−8], controlling[9], secure communication via chaotic synchronization[10,11] and other natural systems suject to finite propagation speeds of signals, finite reaction times and finite processing times[12]. It has been shown that the time delay in various systems has not only quantitative but also qualitative effects on dynamics even for small time delays[13,14].

As a result, various qualitative and quantitative theories for delayed differential equations (DDEs) are developed and extended in recent years. Eq. (1) has been used as a mathematical model for the investigation of stability of systems with time delay. The delay-induced Hopf bifurcation may be the most simple but basic bifurcation on stability analysis.

In the qualitative treatment of Hopf bifurcation, many authors[1,2,15−17] suggested a center manifold reduction (CMR) then a normal form procedure to classify stability of the periodic solution derived from the Hopf bifurcation. The main steps of the analysis are schemed as follows: (i) To consider an equilibrium at a critical parameter; (ii) to solve the eigenvalue problem for this equilibrium to find its linear stability; (iii) to localize the

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critical point in parameter space where bifurcation occurs; (iv) to calculate the eigenvalues and eigenfunctions at the bifurcation point and reduce DDEs on the center manifold; (v) to compute the appropriate normal form coefficients.

In the quantitative treatment of Hopf bifurcation, interesting attentions are focused on the analytical expression of the periodic solutions derived from the Hopf bifurcation. Periodic solutions can be obtained through Hopf bifurcation of a steady state solution by means of the center manifold theory. However, it is in general tedious to reduce a given DDE to a finite dimensional system. Instead of the center manifold reduction, Das and Chatterjee[18] employed the method of multiple scales (MMS) to obtain analytical solutions of bifurcation parameters close to Hopf bifurcation point for DDEs. It should be noted that Das' version is somewhat different from that proposed by Nayfeh et al.[5] The MMS can bypass the explicit center manifold and normal form calculation. For those values closed to the Hopf bifurcation point, analytical approximations from Das' version are in good agreement with the numerical solutions. However, the method is invalid for values far away from the bifurcation point. For this case, the IHB method can be employed to express periodic or even doubling periodic solutions analytically[7]. The key technique for IHB is to find an initial value for iteration, which is difficult in general. Moiola et al.[19] have presented a so-called domain approach to analyze graphically the structure of degenerate Hopf bifurcation and to recovery the continuation of the bifurcated periodic paths for the single-input single-output (SISO) feedback systems with odd nonlinearities and time delays. The method is valid for bifurcation parameter close to the Hopf bifurcation point, but is invalid for those far away.

Motivated by the above problems, we will consider eq. (1) as a mathematical model to propose a new method called the perturbation-incremental scheme (PIS) and extend it to investigate delay-induced Hopf bifurcation and periodic solution in the present paper. The paper is organized as follows. Firstly, the PIS is described in section 2. In sections 3－5, three systems are considered as illustrative examples to demonstrate the validity of the PIS by comparing with results from MMS, CMR and numerical simulation.

It will be seen that the PIS avoids tedious computations which one quite often comes across in center manifold reduction. The obtained results from the perturbation step can be used as an initial guess for incremental iteration. Through continuation step, periodic solutions at bifurcation parameter far away from the bifurcation point can be determined in any desired accuracy. Therefore, the PIS not only inherits the advantages of MMS and Moiola's method, but also efficiently overcomes the disadvantages of MMS and IHB.

2 Perturbation-incremental scheme

The perturbation-incremental scheme (PIS) has been used to study nonlinear oscillations without time delay[20,21]. In this section, we will outline a procedure of the PIS for eq. (1). Without loss of generality, we assume that the trivial equilibrium of eq. (1) without time delay is not saddle-type. Then, the procedure of the PIS is divided into three steps.

2.1 Linear analysis ($\varepsilon = 0$)

To determine the stability of the trivial solution for $\tau \neq 0$, one linearizes eq. (1) around $\boldsymbol{Z}$=0 and obtains the characteristic equation

$$\det(\lambda \boldsymbol{I} - \boldsymbol{C} - \boldsymbol{D}\,\mathrm{e}^{-\lambda\tau}) = 0, \tag{2}$$

where $\boldsymbol{I}$ is the identity matrix and $\tau > 0$.

The roots of the characteristic eq. (2) are commonly called the eigenvalues of the equilibrium point of eq. (1). The stability of the trivial equilibrium is changed when the eigenvalues of eq. (2) have zero or imaginary pairs. The former may lead to a static bifurcation of equilibrium points such that the number of equilibrium points changes when the bifurcation parameters vary. The latter deals with a Hopf bifurcation such that dynamical behaviors of the system change from a static stable state to a periodic motion or vice versa. Especially, if there is only a pair of purely imaginary eigenvalues given by $\pm \mathrm{i}\omega$ at $\tau = \tau_c$ and $\lambda = 0$ is not a root of eq. (2), then a Hopf bifurcation may occur in eq. (1). We will concentrate on this case in the subsequent discussion.

For $n \leqslant 2$, the explicit expression of the critical stability boundaries for the trivial equilibrium is easily determined in one-dimensional parameter space, such as τ, by solving eq. (2) and using the Hopf bifurcation theorem. For $n > 2$, it is very difficult to find the explicit expression since there are infinite roots to be examined in eq. (2). Fortunately, Olgac and Sipahi[22] have proposed a novel treatment yielding a practical and structured methodology to obtain the critical boundaries. To proceed to the next step, we assume that the critical values of a Hopf bifurcation in τ are found.

2.2 Perturbation method at a critical value (small $\varepsilon\tau_\varepsilon$)

The second step deals with the problem of calculating an approximation of a small-amplitude periodic solution near a Hopf bifurcation point. We first consider the value of time delay close to a Hopf bifurcation point. A perturbation to one of the critical values, $\tau = \tau_c + \varepsilon\tau_\varepsilon$, of eq. (1) yields

$$\begin{aligned}\dot{\boldsymbol{Z}}(t) &= \boldsymbol{CZ}(t) + \boldsymbol{DZ}(t-\tau_c)\\ &\quad + \widetilde{\boldsymbol{F}}(\boldsymbol{Z}(t),\ \boldsymbol{Z}(t-\tau_c),\ \boldsymbol{Z}(t-\tau_c-\varepsilon\tau_\varepsilon),\varepsilon), \end{aligned} \tag{3}$$

where

$$\begin{aligned}&\widetilde{\boldsymbol{F}}(\boldsymbol{Z}(t),\boldsymbol{Z}(t-\tau_c),\boldsymbol{Z}(t-\tau_c-\varepsilon\tau_\varepsilon),\varepsilon)\\ &= \boldsymbol{D}\left[\boldsymbol{Z}(t-\tau_c-\varepsilon\tau_\varepsilon) - \boldsymbol{Z}(t-\tau_c)\right]\\ &\quad + \varepsilon F(\boldsymbol{Z}(t),\boldsymbol{Z}(t-\tau_c-\varepsilon\tau_\varepsilon)),\end{aligned} \tag{4}$$

and τ_c is a critical value or a Hopf bifurcation point. It follows from section 2.1 that eq. (3) undergoes a Hopf bifurcation when $\varepsilon = 0$. Consider the equation given by

$$\dot{\boldsymbol{W}}(t) = -\boldsymbol{C}^{\mathrm{T}}\boldsymbol{W}(t) - \boldsymbol{D}^{\mathrm{T}}\boldsymbol{W}(t+\tau_c). \tag{5}$$

Suppose that the periodic solution of eq. (5) with period $2\pi/\omega$ is expressed as

$$\boldsymbol{W}(t) = \boldsymbol{p}\cos(\phi) + \boldsymbol{q}\sin(\phi), \tag{6}$$

which results in

$$\boldsymbol{W}(t-\tau) = \boldsymbol{p}\cos(\phi-\omega\tau_c) + \boldsymbol{q}\sin(\phi-\omega\tau_c), \tag{7}$$

where $\boldsymbol{p}=(p_1,p_2,\ldots,p_n)^{\mathrm{T}}$, $\boldsymbol{q}=(q_1,q_2,\ldots,q_n)^{\mathrm{T}}\in\mathbf{R}^n$, $\phi=\omega t$, τ_c and ω are determined by eq. (2). Substituting eqs. (6) and (7) into eq. (5) and using the harmonic balance, one may obtain

$$\boldsymbol{M}^{\mathrm{T}}\boldsymbol{p} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{q}, \tag{8}$$

and

$$\boldsymbol{M}^{\mathrm{T}}\boldsymbol{q} = -\boldsymbol{N}^{\mathrm{T}}\boldsymbol{p}, \tag{9}$$

where $\boldsymbol{M} = \omega\boldsymbol{I} + \boldsymbol{D}\sin(\omega\tau_c)$ and $\boldsymbol{N} = \boldsymbol{C} + \boldsymbol{D}\cos(\omega\tau_c)$. Obviously, there are only two independent constants in the periodic solution eq. (6) arising from the Hopf bifurcation. If p_1 and q_1 are chosen to be independent, then p_i and q_i ($i = 2,\ldots,n$) can be determined by eqs. (8) and (9) in terms of p_1 and q_1. Similarly, for $\varepsilon = 0$, the periodic solution of eq. (3) with period $2\pi/\omega$ can be expressed as

$$\boldsymbol{Z}(t) = \boldsymbol{a}\cos(\phi) + \boldsymbol{b}\sin(\phi), \tag{10}$$

where $\boldsymbol{a} = (a_1,a_2,\ldots,a_n)^{\mathrm{T}}$, $\boldsymbol{b} = (b_1,b_2,\ldots,b_n)^{\mathrm{T}} \in \mathbf{R}^n$, a_1 and b_1 are independent, a_i and b_i ($i = 2,\ldots,n$) are functions of a_1 and b_1, given by

$$\boldsymbol{Mb} = \boldsymbol{Na}, \tag{11}$$

and

$$-\boldsymbol{Ma} = \boldsymbol{Nb}. \tag{12}$$

Eq. (10) in a polar coordinate system can be given by

$$\boldsymbol{Z}(t) = \boldsymbol{r}\cos(\phi+\theta), \tag{13}$$

where $\boldsymbol{r} = (r_1,r_2,\ldots,r_n)^{\mathrm{T}}$ and r_i ($i = 2,\ldots,n$) are linearly represented in r_1. Therefore, for a small ε, the solution of eq. (3) may be considered as a perturbation of eq. (13), given by

$$\boldsymbol{Z}(t) = \boldsymbol{r}(\varepsilon)\cos((\omega+\sigma(\varepsilon))t), \tag{14}$$

where $r(0) = r$, $\sigma(0) = 0$, and $r_i(\varepsilon) = r_i(r_1(\varepsilon))$ ($i = 2,..,n$). To obtain $r_1(\varepsilon)$ and $\sigma(\varepsilon)$, multiplying both sides of eq. (3) by $[\boldsymbol{W}(t)]^{\mathrm{T}}$ from eq. (6) and integrating with respect to t from zero to $2\pi/\omega$, we obtain

$$\begin{aligned}&\int_0^{2\pi/\omega}[\boldsymbol{W}(t)]^{\mathrm{T}}\dot{\boldsymbol{Z}}(t)\mathrm{d}t\\ &= \int_0^{2\pi/\omega}[\boldsymbol{W}(t)]^{\mathrm{T}}[\boldsymbol{CZ}(t) + \boldsymbol{DZ}(t-\tau_c)\\ &\quad + \widetilde{\boldsymbol{F}}(\boldsymbol{Z}(t),\boldsymbol{Z}(t-\tau_c),\boldsymbol{Z}(t-\tau_c-\varepsilon\tau_\varepsilon),\varepsilon)]\mathrm{d}t.\end{aligned} \tag{15}$$

Eq. (15) yields that

$$\begin{aligned}&\int_0^{2\pi/\omega}\left[\dot{\boldsymbol{W}}(t) + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{W}(t) + \boldsymbol{D}^{\mathrm{T}}\boldsymbol{W}(t+\tau_c)\right]^{\mathrm{T}}\boldsymbol{Z}(t)\mathrm{d}t\\ &+\int_{-\tau_c}^0[\boldsymbol{D}^{\mathrm{T}}\boldsymbol{W}(t+\tau_c)]^{\mathrm{T}}\left[\boldsymbol{Z}(t) - \boldsymbol{Z}(t+2\pi/\omega)\right]\mathrm{d}t\\ &-[\boldsymbol{W}(0)]^T[\boldsymbol{Z}(2\pi/\omega) - \boldsymbol{Z}(0)]\\ &+\int_0^{2\pi/\omega}[\boldsymbol{W}(t)]^T\widetilde{\boldsymbol{F}}(\boldsymbol{Z}(t),\boldsymbol{Z}(t-\tau_c),\\ &\boldsymbol{Z}(t-\tau_c-\varepsilon\tau_\varepsilon),\varepsilon)\mathrm{d}t = 0.\end{aligned} \tag{16}$$

Since $\boldsymbol{W}(t)$ is a periodic solution of eq. (5) and $\boldsymbol{W}(t) = \boldsymbol{W}\left(t + \dfrac{2\pi}{\omega}\right)$, eq. (16) becomes

$$\begin{aligned}&\int_{-\tau_c}^0[\boldsymbol{D}^{\mathrm{T}}\boldsymbol{W}(t+\tau_c)]^{\mathrm{T}}[\boldsymbol{Z}(t) - \boldsymbol{Z}(t+2\pi/\omega)]\mathrm{d}t\\ &\quad -[\boldsymbol{W}(0)]^{\mathrm{T}}[\boldsymbol{Z}(2\pi/\omega) - \boldsymbol{Z}(0)]\\ &\quad +\int_0^{2\pi/\omega}[\boldsymbol{W}(t)]^{\mathrm{T}}\widetilde{\boldsymbol{F}}(\boldsymbol{Z}(t),\boldsymbol{Z}(t-\tau_c),\\ &\quad \boldsymbol{Z}(t-\tau_c-\varepsilon\tau_\varepsilon),\varepsilon)\mathrm{d}t=0.\end{aligned} \tag{17}$$

Eq. (17) is a sufficient and necessary condition for eq. (14) to be the periodic solution of eq. (3), which is induced from the trivial equilibrium by the Hopf bifurcation at τ_c. Substituting eqs. (6) and (13) into eq. (17) and noting the independence of p_1 and q_1 yields a set of algebraic equations for the determination of $r_1(\varepsilon)$ and $\sigma(\varepsilon)$.

It should be noted that eq. (17) is a set of explicit algebraic equations in $r_1(\varepsilon)$ and $\sigma(\varepsilon)$. The roots of eq. (17) cannot be expressed in a closed form since the equations contain transcendent functions. To obtain an analytical form in $r_1(\varepsilon)$ and $\sigma(\varepsilon)$, the transcendent functions in eq. (17) are expanded in Taylor's series in ε and then high-order terms in power ε are neglected. Thus, one may only obtain an approximate formula in $r_1(\varepsilon)$ and $\sigma(\varepsilon)$. Correspondingly, an analytical expression of the periodic solution derived from a Hopf bifurcation is approximately given by eq. (14). The accuracy of the analytical expression strongly depends on the magnitude of ε. When the value of the time delay is very close to the Hopf bifurcation point (i.e., $\varepsilon\tau_\varepsilon$ is very small), the approximation eq. (14) is accurate enough to describe the periodic solutions of eq. (3). However, when the value of the time delay is far away from the Hopf bifurcation point, such approximation is quantitatively invalid. However, the approximate expression can be modified to approach the exact solution by the incremental step as discussed in the following subsection.

2.3 Parameter incremental method (large $\varepsilon\tau_\varepsilon$)

A time transformation is first introduced as

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \Phi(\varphi),\ \ \Phi(\varphi + 2\pi) = \Phi(\varphi), \tag{18}$$

where φ is the new time, which is different from $\phi = \omega t$. In the φ domain, eq. (1) is rewritten as

$$\Phi \boldsymbol{Z} = \boldsymbol{CZ} + \boldsymbol{DZ}_\tau + \varepsilon \boldsymbol{F}(\boldsymbol{Z}, \boldsymbol{Z}_\tau), \tag{19}$$

where prime denotes differentiation with respect to φ and $\tau = \tau_c + \varepsilon\tau_\varepsilon$.

If φ_1 is the new time corresponding to $t-\tau$, it follows from eq. (18) that

$$\mathrm{d}t = \frac{\mathrm{d}\varphi}{\Phi(\varphi)} = \frac{\mathrm{d}\varphi_1}{\Phi(\varphi_1)} \Rightarrow \Phi(\varphi)\frac{\mathrm{d}\varphi_1}{\mathrm{d}\varphi} = \Phi(\varphi_1), \tag{20}$$

which yields that $\varphi_1 - \varphi$ is a periodic function in φ with period 2π.

If eq. (19) possesses a periodic solution at $\tau = \tau_0 = \tau_c + \varepsilon\tau_\varepsilon$ by the second step and the expression in the form of eq. (14) provides a sufficiently accurate representation for a small $\varepsilon\tau_\varepsilon$, then one can assume that the expression given by

$$\boldsymbol{Z}(t) = \sum_{j=0}^{m} (\boldsymbol{a}_j \cos j\varphi + \boldsymbol{b}_j \sin j\varphi) \tag{21}$$

is a periodic solution at $\tau = \tau_0 + \Delta\tau$, where $a_j, b_j \in \mathbf{R}^n$.

Initially, for $\Delta\tau = 0$, one can easily obtain that

$$\boldsymbol{a}_j = \begin{cases} \boldsymbol{r}(\varepsilon), & j = 1, \\ 0, & j \neq 1, \end{cases} \qquad \boldsymbol{b}_j = 0 \ \text{ for any } j. \tag{22}$$

Correspondingly, one has

$$\Phi(\varphi) = \sum_{j=0}^{m} (p_j \cos j\varphi + q_j \sin j\varphi), \tag{23}$$

where

$$p_0 = \omega + \sigma(\varepsilon),\ p_j = 0,\ q_j = 0 \text{ for all } j > 0. \tag{24}$$

To consider the continuation with the delay τ as the bifurcation parameter, an increment of τ from τ_0 to $\tau_0 + \Delta\tau$ corresponds to changes of the following quantities:

$$\begin{aligned} &\boldsymbol{Z} \to \boldsymbol{Z} + \Delta\boldsymbol{Z},\ \boldsymbol{Z}_\tau \to \boldsymbol{Z}_\tau + \Delta\boldsymbol{Z}_\tau, \\ &\Phi \to \Phi + \Delta\Phi \ \text{ and } \ \varphi_1 \to \varphi_1 + \Delta\varphi_1. \end{aligned} \tag{25}$$

Substituting eq. (25) into eqs. (19) and (20), and expanding in Taylor's series about an initial guess or solution (e.g. for $\Delta\tau = 0$), one can obtain linearized incremental equations by ignoring all the non-linear terms of small increments as below:

$$\begin{aligned} &\boldsymbol{Z}'\Delta\Phi(\varphi) + \Phi(\varphi)\Delta\boldsymbol{Z}' - \boldsymbol{C}\Delta\boldsymbol{Z} - \boldsymbol{D}\Delta\boldsymbol{Z}_\tau \\ &\quad - \varepsilon\left(\left.\frac{\partial \boldsymbol{F}(\boldsymbol{Z},\boldsymbol{Z}_\tau)}{\partial \boldsymbol{Z}}\right|_0 \Delta\boldsymbol{Z} - \left.\frac{\partial \boldsymbol{F}(\boldsymbol{Z},\boldsymbol{Z}_\tau)}{\partial \boldsymbol{Z}_\tau}\right|_0 \Delta\boldsymbol{Z}_\tau \right) \\ &= \boldsymbol{CZ} + \boldsymbol{DZ}_\tau + \varepsilon \boldsymbol{F}(\boldsymbol{Z}, \boldsymbol{Z}_\tau) - \Phi(\varphi)\boldsymbol{Z}, \end{aligned} \tag{26}$$

$$\begin{aligned} &\varphi_1 \Delta\Phi(\varphi) + \Phi(\varphi)\Delta\varphi_1 - \Delta\Phi(\varphi_1) - \Phi'(\varphi_1)\Delta\varphi_1 \\ &= \Phi(\varphi_1) - \Phi(\varphi)\varphi_1, \end{aligned} \tag{27}$$

where the subscript 0 represents the evaluation of the relevant quantities corresponding to the initial solution. From eq. (21), the terms $\Delta\boldsymbol{Z}$ and $\Delta\boldsymbol{Z}'$ are expressed respectively as

$$\begin{aligned} \Delta\boldsymbol{Z} &= \sum_{j=0}^{m} (\Delta\boldsymbol{a}_j \cos j\varphi + \Delta\boldsymbol{b}_j \sin j\varphi), \\ \Delta\boldsymbol{Z}' &= \sum_{j=1}^{m} j(\Delta\boldsymbol{b}_j \cos j\varphi - \Delta\boldsymbol{a}_j \sin j\varphi). \end{aligned} \tag{28}$$

Since Φ and $\varphi_1 - \varphi$ are both periodic functions in φ with period 2π, we write

$$\Phi(\varphi) = \sum_{j=0}^{m} (p_j \cos j\varphi + q_j \sin j\varphi),$$

$$\Delta\Phi(\varphi) = \sum_{j=0}^{m} (\Delta p_j \cos j\varphi + \Delta q_j \sin j\varphi), \tag{29}$$

$$\Delta\Phi'(\varphi) = \sum_{j=1}^{m} j(\Delta q_j \cos j\varphi - \Delta p_j \sin j\varphi),$$

and

$$\varphi_1 = \varphi + \sum_{j=0}^{m} (r_j \cos j\varphi + s_j \sin j\varphi),$$

$$\Delta\varphi_1 = \sum_{j=0}^{m} (\Delta r_j \cos j\varphi + \Delta s_j \sin j\varphi), \tag{30}$$

$$\Delta\varphi_1 = \sum_{j=1}^{m} j(\Delta s_j \cos j\varphi - \Delta r_j \sin j\varphi).$$

Similarly, for $\Delta\tau = 0$, the initial guess of φ_1 can be chosen as

$$r_0 = -(\omega + \sigma(\varepsilon))\tau_c,\ r_j = 0\ (j \neq 0),\ s_j = 0 \text{ for any } j. \tag{31}$$

For the delay term $\boldsymbol{Z}_\tau$, we have

$$\boldsymbol{Z}_\tau = \sum_{j=0}^{m} (\boldsymbol{a}_j \cos j\varphi_1 + \boldsymbol{b}_j \sin j\varphi_1),$$
$$\Delta\boldsymbol{Z}_\tau = \sum_{j=0}^{m} (\Delta\boldsymbol{a}_j \cos j\varphi_1 + \Delta\boldsymbol{b}_j \sin j\varphi_1) + \frac{\partial \boldsymbol{Z}_\tau}{\partial\varphi_1}\Delta\varphi_1. \tag{32}$$

The integration constant of eq. (20) provides information about the delay τ. Since φ_1 is the new time corresponding to $t-\tau$, it follows from eq. (20) that

$$\int_{t-\tau}^{t} \mathrm{d}t_1 = \int_{\varphi_1}^{\varphi} \frac{\mathrm{d}\theta}{\Phi(\theta)} \Rightarrow \tau = \int_{\varphi_1}^{\varphi} \frac{\mathrm{d}\theta}{\Phi(\theta)}. \tag{33}$$

For a small increment of τ to $\tau + \Delta\tau$, the linearized incremental eq. (33) is given by

$$\int_{\varphi_1}^{\varphi} \frac{\Delta\Phi(\theta)}{\Phi^2(\theta)}\mathrm{d}\theta + \frac{\Delta\varphi_1}{\Phi(\varphi_1)} = \int_{\varphi_1}^{\varphi} \frac{\mathrm{d}\theta}{\Phi(\theta)} - \tau - \Delta\tau, \tag{34}$$

which implies, for $\varphi = 0$,

$$\int_{\xi}^{0} \frac{\Delta\Phi(\theta)}{\Phi^2(\theta)}\mathrm{d}\theta + \frac{\Delta\varphi_1(0)}{\Phi(\alpha)} = \int_{\xi}^{0} \frac{\mathrm{d}\theta}{\Phi(\theta)} - \tau - \Delta\tau. \tag{35}$$

where $\xi = \varphi_1(0)$. The harmonic balance method is applied to eqs. (26), (27) and (35). Rewriting the linearized eq. (26) in terms of the increments Δa_j, Δb_j, Δp_j, Δq_j, Δr_j and Δs_j, we have

$$\sum_{j=0}^{m} [\boldsymbol{\Psi}_{1,j}\Delta\boldsymbol{a}_j + \boldsymbol{\Psi}_{2,j}\Delta\boldsymbol{b}_j + \boldsymbol{\Psi}_{3,j}\Delta p_j + \boldsymbol{\Psi}_{4,j}\Delta q_j + \boldsymbol{\Psi}_{5,j}\Delta r_j + \boldsymbol{\Psi}_{6,j}\Delta s_j] = \boldsymbol{\Lambda}_1, \tag{36}$$

where

$$\boldsymbol{\Psi}_{1,j} = -j\Phi(\varphi)\sin j\varphi\boldsymbol{I} - \boldsymbol{C}\cos j\varphi - \boldsymbol{D}\cos j\varphi_1 - \varepsilon\left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{Z}}\Big|_0 \cos j\varphi + \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{Z}_\tau}\Big|_0 \cos j\varphi_1\right),$$

$$\boldsymbol{\Psi}_{2,j} = j\Phi(\varphi)\cos j\varphi\boldsymbol{I} - \boldsymbol{C}\sin j\varphi - \boldsymbol{D}\sin j\varphi_1 - \varepsilon\left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{Z}}\Big|_0 \sin j\varphi + \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{Z}_\tau}\Big|_0 \sin j\varphi_1\right), \tag{37}$$

$$\boldsymbol{\Psi}_{3,j} = \boldsymbol{Z}\cos j\varphi,\ \boldsymbol{\Psi}_{4,j} = \boldsymbol{Z}\sin j\varphi,$$

$$\boldsymbol{\Psi}_{5,j} = -\boldsymbol{D}\frac{\partial \boldsymbol{Z}_\tau}{\partial\varphi_1}\cos j\varphi - \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{Z}_\tau}\Big|_0 \frac{\partial \boldsymbol{Z}_\tau}{\partial\varphi_1}\cos j\varphi,$$

$$\boldsymbol{\Psi}_{6,j} = -\boldsymbol{D}\frac{\partial \boldsymbol{Z}_\tau}{\partial\varphi_1}\sin j\varphi - \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{Z}_\tau}\Big|_0 \frac{\partial \boldsymbol{Z}_\tau}{\partial\varphi_1}\sin j\varphi,$$

$$\boldsymbol{\Lambda}_1 = \boldsymbol{C}\boldsymbol{Z} + \boldsymbol{D}\boldsymbol{Z}_\tau + \varepsilon\boldsymbol{F}(\boldsymbol{Z}, \boldsymbol{Z}_\tau) - \Phi(\varphi)\boldsymbol{Z},$$

Similarly, from eqs. (27) and (35), we obtain, respectively,

$$\sum_{j=0}^{m} [\Psi_{7,j}\Delta p_j + \Psi_{8,j}\Delta q_j + \Psi_{9,j}\Delta r_j + \Psi_{10,j}\Delta s_j] = \Lambda_2, \tag{38}$$

and

$$\sum_{j=0}^{m} [\Psi_{11,j}\Delta p_j + \Psi_{12,j}\Delta q_j + \Psi_{13,j}\Delta r_j] = \Lambda_3, \tag{39}$$

where

$$\Psi_{7,j} = \varphi_1\cos j\varphi - \cos j\varphi_1, \Psi_{8,j} = \varphi_1\sin j\varphi - \sin j\varphi_1,$$

$$\Psi_{9,j} = -j\Phi(\varphi)\sin j\varphi - \Phi'(\varphi_1)\cos j\varphi,$$

$$\Psi_{10,j} = j\Phi(\varphi)\cos j\varphi - \Phi'(\varphi_1)\sin j\varphi, \tag{40}$$

$$\Psi_{11,j} = \int_{\xi}^{0}\frac{\cos j\theta}{\Phi^2(\theta)}\mathrm{d}\theta,\ \Psi_{12,j} = \int_{\xi}^{0}\frac{\sin j\theta}{\Phi^2(\theta)}\mathrm{d}\theta,\ \Psi_{13,j} = \frac{1}{\Phi(\xi)},$$

$$\Lambda_2 = \Phi(\varphi_1) - \Phi(\varphi)\varphi_1,\ \Lambda_3 = \int_{\xi}^{0}\frac{\mathrm{d}\theta}{\Phi(\theta)} - \tau - \Delta\tau.$$

Since $\Psi_{i,j}$ $(1 \leqslant i \leqslant 13, 1 \leqslant j \leqslant m)$ and Λ_k $(1 \leqslant k \leqslant 3)$ are periodic functions in φ, they can be expressed in Fourier series in which coefficients can easily be obtained by the method of fast Fourier transform (FFT). Let a_{ij}, $b_{ij} \in R$ $(1 \leqslant i \leqslant n, 0 \leqslant j \leqslant m)$ be the i-th elements in $\boldsymbol{a}_j$ and $\boldsymbol{b}_j$, respectively. By comparing the coefficients of harmonic terms of eqs. (36), (38) and (39), a system of linear equations is thus obtained with unknowns Δa_{ij}, Δb_{ij}, Δp_j, Δq_j, Δr_j and Δs_j in the form

$$\sum_{i=1}^{n}\sum_{j=0}^{m}(A_{k,ij}\Delta a_{ij}+B_{k,ij}\Delta b_{ij})$$
$$+\sum_{j=0}^{m}(P_{k,j}\Delta p_j+Q_{k,j}\Delta q_j+R_{k,j}\Delta r_j+S_{k,j}\Delta s_j)=T_k, \quad (41)$$

where T_k are residue terms. The values of $\boldsymbol{a}_j$, $\boldsymbol{b}_j$, p_j, q_j, r_j and s_j are updated by adding the original values and the corresponding incremental values. The iteration process continues until $T_k \to 0$ for all k (in practice, $|T_k|$ is less than a desired degree of accuracy). The entire incremental process proceeds by adding the $\Delta\tau$ increment to the converged value of τ, using the previous solution as the initial approximation until a new converged solution is obtained.

The stability of a periodic solution can be determined by the Floquet method[23,24]. Let $\boldsymbol{\zeta}\in\mathbf{R}^n$ be a small perturbation from a periodic solution of eq. (1). Then,

$$\frac{\mathrm{d}\boldsymbol{\zeta}}{\mathrm{d}\varphi}=\frac{1}{\Phi}[\boldsymbol{A}(\varphi,\varphi_1)\boldsymbol{\zeta}+\boldsymbol{B}(\varphi,\varphi_1)\boldsymbol{\zeta}_\tau]+O(\boldsymbol{\zeta}^2,\boldsymbol{\zeta}_\tau^2), \quad (42)$$

where $\boldsymbol{A}(\varphi,\varphi_1)=\boldsymbol{C}+\varepsilon\dfrac{\partial\Phi(\boldsymbol{Z},\boldsymbol{Z}_\tau)}{\partial\boldsymbol{Z}}$ and $\boldsymbol{B}(\varphi,\varphi_1)=\boldsymbol{D}+\varepsilon\dfrac{\partial\boldsymbol{F}(\boldsymbol{Z},\boldsymbol{Z}_\tau)}{\partial\boldsymbol{Z}_\tau}$. The entities of $\boldsymbol{A}$ and $\boldsymbol{B}$ are all periodic functions of φ with period 2π, which can be determined by using the incremental procedure. The time delay interval $I_1=[-\tau,0]$ corresponds to $I_2=[\alpha,0]$ in the φ domain. Discrete points in I_2 are selected for the computation of Floquet multipliers.

From the incremental procedure, the Fourier coefficients of φ_1 in eq. (30) are obtained. Assume that $\varphi=\beta$ when $\varphi_1=0$ and let $I_3=[0,\beta]$. For each $\varphi\in I_3$, there is a unique $\varphi_1\in I_2$. We choose a mesh size $h=\dfrac{\beta}{N-1}$ and discrete points $\varphi^{(i)}=ih\ (0\leqslant i\leqslant N-1)$ in I_3, which correspond to $\varphi_1^{(i)}=\varphi_1(\varphi^{(i)})$ in I_2. Let $\zeta(\varphi_1^{(i)})$ be the $(i+1)$-th unit vector in $\mathbf{R}^n$. By applying numerical integration to eq. (42), we obtain the monodromy matrix $\boldsymbol{M}$ as

$$\boldsymbol{M}=[\zeta(\varphi_1^{(0)}+2\pi),\zeta(\varphi_1^{(1)}+2\pi),\zeta(\varphi_1^{(N-1)}+2\pi)]. \quad (43)$$

The eigenvalues of $\boldsymbol{M}$ are used to determine the stability of the periodic solution. One of the eigenvalues or Floquet multipliers of $\boldsymbol{M}$ must be unity which provides a check for the accuracy of the calculation. If all the other eigenvalues are inside the unit circle, the periodic solution under consideration is stable; otherwise, it is unstable.

It should be noted that the procedure of the PIS discussed in this section is proposed only for the space of τ. For other bifurcation parameter varying in one-dimensional space, the present method can be formulated in a similar way. In the next three sections, we will study three examples and demonstrate the validity and advantage of the PIS by comparing the obtained results with those from MMS, CMR and numerical simulation.

3 First-order delayed differential equation with a limit cycle

As the first example, we consider an autonomous equation with a limit cycle given by

$$\dot{x}=-\alpha\, x(t-\tau)-\varepsilon\, x^3(t), \quad (44)$$

where τ is the time delay and $\alpha>0$. Das and Chatterjee[18] showed that eq. (44) undergoes a Hopf bifurcation when $\varepsilon=0$, $\tau=\pi/2$ and $\alpha=\alpha_c=1$. At the bifurcation point, there is a pair of purely imaginary roots, $\lambda=\pm i\omega$. Substituting $\alpha=1+\varepsilon$ and $\tau=\pi/2$ into eq. (44) yields

$$\dot{x}=-x\left(t-\frac{\pi}{2}\right)-\varepsilon\left[x\left(t-\frac{\pi}{2}\right)+x^3(t)\right]. \quad (45)$$

For a small ε, eq. (45) was investigated by Das and Chatterjee[18] with the method of multiple scales. However, the method is invalid for a large ε. We use the PIS proposed in the last section to obtain the approximate solution in a closed form for a large ε.

It is easily seen that $\boldsymbol{C}$=0 and $\boldsymbol{D}$= −1 in eq. (5) and $W(t)$ is determined by

$$\dot{W}(t)=W\left(t+\frac{\pi}{2}\right). \quad (46)$$

$\omega=1$ and $\tau=\pi/2$ yield $\boldsymbol{M}$=$\boldsymbol{N}$=0 in eqs. (8) and (9). It implies that

$$W(t)=p\cos(\phi)+q\sin(\phi), \quad (47)$$

where $\phi=t$. It follows from the above steps of the PIS that the periodic solution of eq. (45) is expressed as

$$x(t)=r(\varepsilon)\cos((1+\sigma(\varepsilon))t+\theta), \tag{48}$$

where θ is an initial phase. From eq. (17), one has

$$\begin{aligned}&-\int_{-\pi/2}^{0} W\left(t+\frac{\pi}{2}\right)\left[x(t)-x(t+2\pi/\omega)\right]\mathrm{d}t\\ &-W(0)[x(2\pi)-x(0)]\\ &-\varepsilon\int_{0}^{2\pi} W(t)\left[x\left(t-\frac{\pi}{2}\right)+x^3(t)\right]\mathrm{d}t=0.\end{aligned} \tag{49}$$

If one seeks an approximation at $O(\varepsilon^2)$ for eq. (45), then $r(\varepsilon)$ and $\sigma(\varepsilon)$ can be expanded in powers of ε, i.e.,

$$r(\varepsilon)=r_0+r_1\,\varepsilon,\ \ \sigma(\varepsilon)=\sigma_1\,\varepsilon+\sigma_2\,\varepsilon^2. \tag{50}$$

Substituting eqs. (47), (48) and (50) into eq. (49), noting the independence of p and q and using the symbolic algebra package MATHEMATICA, one has a set of algebraic equations in r_0, r_1, σ_1 and σ_2 at $O(\varepsilon^2)$ as follows:

$$\begin{aligned}&-8\varepsilon r_1(-1+\sigma_0)-6\pi\varepsilon r_0{}^3\sigma_0+r_0(8-4(2+\varepsilon)\sigma_0\\ &+(4+3\pi^2)\varepsilon\sigma_0{}^2-8\varepsilon\sigma_1)=0,\\ &-18\varepsilon r_0^2 r_1+4\pi\varepsilon r_1\sigma_0+3r_0^3(-2+\varepsilon\sigma_0)\\ &+2\pi r_0(-2(-1+\varepsilon)\sigma_0+3\varepsilon\sigma_0^2+2\varepsilon\sigma_1)=0,\\ &-8\varepsilon r_1(-1+\sigma_0)-6\pi\varepsilon r_0{}^3\sigma_0+r_0(8+4(-2+\varepsilon)\sigma_0\\ &+(-4+3\pi^2)\varepsilon\sigma_0{}^2-8\varepsilon\sigma_1)=0,\\ &-18\varepsilon r_0{}^2 r_1+4\pi\varepsilon r_1\sigma_0-3r_0{}^3(2+\varepsilon\sigma_0)\\ &+2\pi r_0(-2(-1+\varepsilon)\sigma_0+5\varepsilon\sigma_0^2+2\varepsilon\sigma_1)=0.\end{aligned} \tag{51}$$

Eq. (51) yield that

$$r(\varepsilon)=\sqrt{\frac{2\pi}{3}}+\left(\sqrt{\frac{\pi}{6}}-\frac{\pi^{\frac{5}{2}}}{8\sqrt{6}}\right)\varepsilon,\ \ \sigma(\varepsilon)=\varepsilon\left(1-\frac{\pi^2\varepsilon}{8}\right). \tag{52}$$

The approximation represented in eq. (52) is completely the same as that from MMS[18] at $O(1)$, but is distinct at $O(\varepsilon)$, as shown in Figures 1 and 2, where the thin solid line represents the approximate solution eq. (48) with eq. (52), dot-dashed line represents that obtained in ref. [18] by means of the MMS, and the crossing symbol is the result from numerical simulation.

Figure 1(a) shows that the three solutions from eq. (48), the MMS[18] and numerical simulation are almost the same in the phase plane for a small ε, say ε=0.1. However, they are separated for a large ε, say ε=2, as

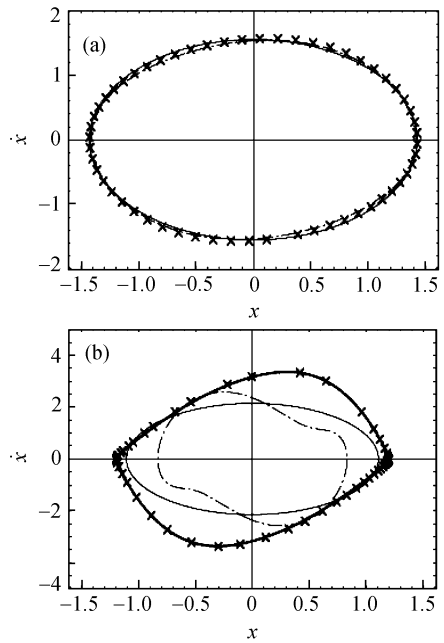

Figure 1 Comparison between the approximate solution eq. (48) (thin solid), MMS solution[18] (dot-dashing), PIS solution (thick solid) and the numerical simulation (crossing symbol) in $\dot{x}$ vs. x for the periodic solution of eq. (45) when ε is chosen as (a) ε=1.0, (b) ε=2.0, respectively.

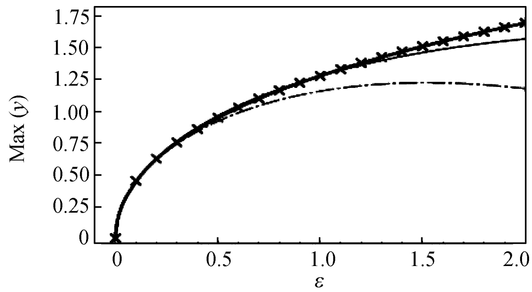

Figure 2 Comparison of bifurcation curves between the approximate solution eq. (48) (thin solid), MMS solution [18] (dot-dashed), PIS solution (thick solid) and the numerical simulation (crossing symbol) in Max(y) vs. ε for the periodic solution of eq. (45), where $y=\sqrt{\varepsilon}x$.

shown in Figure 1(b). Their differences are more apparent in Figure 2 with $\mathrm{Max}(y)$ $(y=\sqrt{\varepsilon}x)$ vs. ε. Figures 1 and 2 suggest that both the perturbation method and the MMS are valid for small ε but invalid for large ε. Next, we implement the third step (see section 2.3) to update the approximate solution given by eq. (48) with eq. (52) for a large ε. The approximate solution is considered as an initial guess of the incremental method as it is closer to the numerical solution than that from MMS. If the periodic solution of eq. (45) is assumed as

$$x(\varphi)=\sum_{j=0}^{m}(a_j\cos j\varphi+b_j\sin j\varphi),$$

$$x_\tau(\varphi)=\sum_{j=0}^{m}(a_j\cos j\varphi_1+b_j\sin j\varphi_1), \qquad (53)$$

corresponding to $\varepsilon=0.1+\Delta\varepsilon$, then the initial guess (or solution) for the increment of ε is easily given by

$$a_1=r(0.1)=\sqrt{\frac{2\pi}{3}}+0.1\left(\sqrt{\frac{\pi}{6}}-\frac{\pi^{\frac{5}{2}}}{8\sqrt{6}}\right), \qquad (54)$$
$$a_j=0\ (j\neq 1),\ \ b_j=0 \text{ for any } j.$$

From the incremental step, we obtain an expression of the periodic solution for ε=2, which is shown up to ten harmonic terms and is represented in thick solid line in Figure 1(b). Figure 2 shows the continuation of ε from 0.1 to 2.0 with $Max(y)$ vs. ε. All three approximate solutions are plotted to compare with that from numerical simulation in Figures 1 and 2. It is seen that the PIS solution is in a good agreement with the numerical solution. This suggests that the PIS is valid for eq. (45). A perturbation solution can be iteratively updated to reach any required accuracy by means of the incremental step. Such a conclusion is also seen in the following two examples.

4 Second-order delayed differential equation related to machining dynamics

A dynamical model of machining with degrees of freedom is considered as our second example, as given by ref. [5]

$$\ddot{x}+2\xi\dot{x}+\gamma^2(x+\beta_2x^2+\beta_3x^3)$$
$$=-\gamma^2 w[x-x_\tau+\alpha_2(x-x_\tau)^2+\alpha_3(x-x_\tau)^3], \qquad (55)$$

where $x_\tau=x(t-\tau)$. Nayfeh et al.[5] used eq. (55) to investigate the impact of time delay and nonlinearity on cutting in the machine tool. It is known from ref. [5] that the trivial solution of eq. (55) undergoes a Hopf bifurcation at $\tau=\tau_c$ for any $w>2\xi\omega/\gamma^2$ with τ varying, where τ_c is determined by

$$\begin{aligned}&\gamma^2-\omega^2+\gamma^2w(1-\cos\omega\tau_c)=0,\\&2\xi\omega+\gamma^2w\sin\omega\tau_c=0,\end{aligned} \qquad (56)$$

where ω represents the chatter frequency and $\omega\neq 0$.

We will present the procedure of the PIS and the continuation of the periodic paths bifurcated from the trivial equilibrium by delay-induced Hopf bifurcation for eq. (55). To this end, rescale $x\to\varepsilon x$ and perturb τ_c, say $\tau=\tau_c+\varepsilon^2\tau_\varepsilon$ transform eq. (55) into the form of eq. (3), where

$$\boldsymbol{Z}(t)=\begin{Bmatrix}x(t)\\y(t)\end{Bmatrix}\in\mathbf{R}^2,$$
$$\boldsymbol{C}=\begin{bmatrix}0&1\\-\gamma^2(1+w)&-2\xi\end{bmatrix},$$
$$\boldsymbol{D}=\begin{bmatrix}0&0\\\gamma^2w&0\end{bmatrix}, \qquad (57)$$
$$\tilde{\boldsymbol{F}}(\boldsymbol{Z},\boldsymbol{Z}_{\tau_c},\boldsymbol{Z}_{\tau_c+\varepsilon^2\tau_\varepsilon})=\boldsymbol{D}\left[\boldsymbol{Z}_{\tau_c+\varepsilon^2\tau_\varepsilon}-\boldsymbol{Z}_{\tau_c}\right]+\varepsilon\boldsymbol{F}(\boldsymbol{Z},\boldsymbol{Z}_{\tau_c+\varepsilon^2\tau_\varepsilon}),$$
$$\boldsymbol{F}(\boldsymbol{Z},\boldsymbol{Z}_{\tau_c+\varepsilon^2\tau_\varepsilon})=\begin{Bmatrix}0\\-\gamma^2(\beta_2x^2+\varepsilon\beta_3x^3)\\-\gamma^2w(\alpha_2(x-x_{\tau_c+\varepsilon^2\tau_\varepsilon})^2\\+\varepsilon\alpha_3(x-x_{\tau_c+\varepsilon^2\tau_\varepsilon})^3)\end{Bmatrix},$$

and $\gamma=1088.56$ rad/sec, $\xi=24792/\omega$, $\varepsilon\beta_2=479.3$ 1/in, $\varepsilon^2\beta_3$=264500 1/in^2, $\varepsilon\alpha_2$=5.668 1/in, and $\varepsilon^2\alpha_3$=−3715.2 1/in^2[5]. Since $\boldsymbol{M}$ and $\boldsymbol{N}$ are non-singular matrixes in eqs. (8) and (9), then $\boldsymbol{W}(t)$ in eq. (6) can be further simplified as

$$\boldsymbol{W}(t)=\begin{Bmatrix}(2p\xi-q\omega^2)\cos(\phi)+(p+2q\xi)\omega\sin(\phi)\\p\cos(\phi)+q\omega\sin(\phi)\end{Bmatrix}, \qquad (58)$$

where $\phi=\omega t$, and p and q are independent. It is easily verified that $\boldsymbol{W}(t)$ given in eq. (58) is a periodic solution of eq. (5). Based on eq. (58), the periodic solution of eq. (3) with eq. (57) can be expressed as

$$\boldsymbol{Z}(t)=((\boldsymbol{N}^{-1})^{\mathrm{T}}\det(\boldsymbol{N})\cos(\omega t+\sigma(\varepsilon)t)$$
$$+(\boldsymbol{M}^{-1})^{\mathrm{T}}\det(\boldsymbol{M})\sin(\omega t+\sigma(\varepsilon)t))\begin{Bmatrix}a(\varepsilon)\\b(\varepsilon)\end{Bmatrix} \qquad (59)$$

for small values of ε. Substituting M and N into eq. (59) and letting $a(\varepsilon)=-r(\varepsilon)\omega\sin\theta$ and $b(\varepsilon)=-2\xi a(\varepsilon)-r(\varepsilon)\cos\theta$, one has

$$\boldsymbol{Z}(t)=\begin{Bmatrix}r(\varepsilon)\cos(\omega t+\sigma(\varepsilon)t+\theta)\\-r(\varepsilon)\omega\sin(\omega t+\sigma(\varepsilon)t+\theta)\end{Bmatrix}. \qquad (60)$$

Letting $r(\varepsilon)=r_0+O(\varepsilon)$ and $\sigma(\varepsilon)=\varepsilon^2\sigma_2+O(\varepsilon^3)$ and substituting eqs. (58) and (60) into eq. (17), one can obtain an approximation at $O(\varepsilon^2)$ as

$$\frac{-3r^3\varepsilon^2\omega^3\alpha_3}{4w\gamma^2}+\frac{3\gamma^2\varepsilon^2r_0^{\ 3}\alpha_3}{2\omega}-\frac{3\gamma^2\varepsilon^2r_0^{\ 3}\alpha_3}{4w\omega}$$
$$-\frac{3\varepsilon^2\omega r_0^{\ 3}\alpha_3}{2}+\frac{3\varepsilon^2\omega r_0^{\ 3}\alpha_3}{2w}$$

$$+\frac{3\pi\varepsilon^2\xi^2\omega r_0{}^3\alpha_3}{w\gamma^2}-\frac{3\gamma^2\varepsilon^2 r_0{}^3\beta_3}{4\omega}+2\varepsilon^2 r_0\sigma_2$$

$$+2\varepsilon^2\xi r_0\sigma_2\tau_c+2\varepsilon^2\xi\omega r_0\tau_\varepsilon=0,$$

$$\frac{3\varepsilon^2\xi r_0{}^3\alpha_3}{w}-\frac{3\varepsilon^2\xi\omega^2 r_0{}^3\alpha_3}{w\gamma^2}+\frac{2\varepsilon^2\xi r_0\sigma_2}{\omega}$$

$$+\frac{\gamma^2\varepsilon^2 r_0\sigma_2\tau_c}{\omega}+\frac{w\gamma^2\varepsilon^2 r_0\sigma_2\tau_c}{\omega}$$

$$-\varepsilon^2\omega r_0\sigma_2\tau_c+\gamma^2\varepsilon^2 r_0\tau_\varepsilon$$

$$+w\gamma^2\varepsilon^2 r_0\tau_\varepsilon-\varepsilon^2\omega^2 r_0\tau_\varepsilon=0. \quad (61)$$

Thus, the periodic solution of eq. (55) at $w=0.1$ can approximately be given by eq. (60), where $r(\varepsilon)$ and $\sigma(\varepsilon)$ are solved from eq. (61) as

$$r(\varepsilon)=r_0,\quad \sigma(\varepsilon)=\varepsilon^2\sigma_2, \quad (62)$$

and ω is determined from eq. (56) such that $\omega_s=1093.5427$ (corresponding to the stable branches) and $\omega_u=1187.88996$ (corresponding to the unstable branches). Figure 3 shows an approximate prediction of the Hopf bifurcation from the trivial solution by using eq. (62) with τ varying, where thin solid line denotes stable periodic solution and thin dashed line denotes unstable solution, given by eq. (60) with eq. (62). Comparing with the numerical solution (symbol "×") in Figure 3, one can see that the approximate solution eq. (60) with eq. (62) is not in a good agreement. This is more apparent in Figure 4(a) and (b), in which the consistency with the numerical solution fails even for values of τ close to τ_c. This is because eq. (55) includes quadratic stiffness. In fact, the quadratic stiffness results in the unsymmetrical nature of the system such that the periodic solutions should include the constant and second harmonic terms. However, the approximate solution in the form of eqs. (60) and (62) contains only the first harmonic term. Fortunately, it can be modified by the third step or incremental method of the PIS. Now, the solution represented in eqs. (60) and (62) is regarded as the initial guess. The PIS solution is assumed to be

$$\boldsymbol{Z}(\varphi)=\sum_{j=0}^{m}(\boldsymbol{a}_j\cos j\varphi+\boldsymbol{b}_j\sin j\varphi),$$
$$\boldsymbol{Z}_\tau(\varphi)=\sum_{j=0}^{m}(\boldsymbol{a}_j\cos j\varphi_1+\boldsymbol{b}_j\sin j\varphi_1), \quad (63)$$

where $\boldsymbol{Z}=\begin{Bmatrix}x\\y\end{Bmatrix}$, $\boldsymbol{a}_j=\begin{Bmatrix}a_{j,1}\\a_{j,2}\end{Bmatrix}$, $\boldsymbol{b}_j=\begin{Bmatrix}b_{j,1}\\b_{j,2}\end{Bmatrix}$ and $\Phi(\varphi)$ and φ_1 are expressed in eqs. (29) and (30), respectively. One can compute the solution by the third step of the PIS starting from τ_c as shown in the previous section, as shown in Figures 3, 4 and 5.

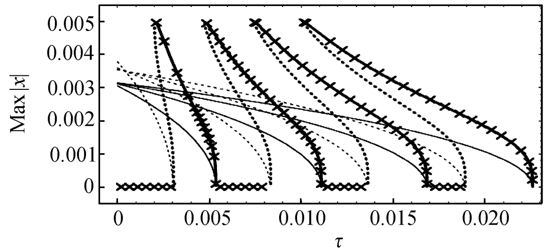

Figure 3 Comparison between the approximate solution eq. (60) with eq. (62) (thin), PIS solution (thick) and the numerical simulation (crossing symbol) in Max$|x|$ vs. τ for the periodic solution of eq. (55), where w=0.1, solid line denotes stable periodic solution and dashed line denotes unstable periodic solution.

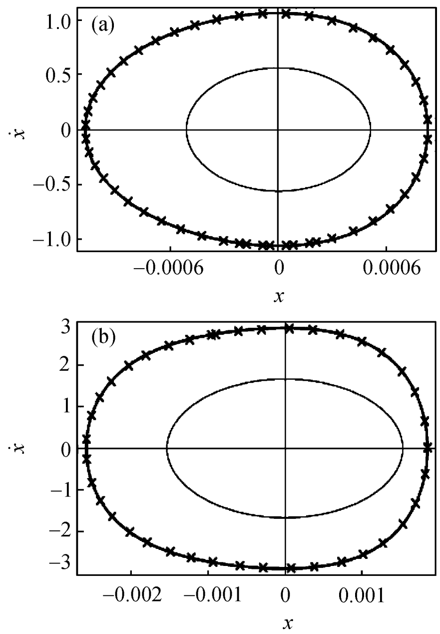

Figure 4 Comparison between the approximate solution eq. (60) with eq. (62) (thin), PIS solution (thick) and the numerical simulation (crossing symbol) in phase plane for the periodic solution of eq. (55), where w=0.1, τ_c=0.0053509, ω=1093.5428. (a) τ=0.0052; (b) τ=0.004.

5 Synchronization solution in a network of three identical neurons

For our final example, we consider a network of three identical electronic (artificial) neurons interconnected through nearest neighborhoods. Dynamics of the system under consideration is governed by the Hopfield's model with delay, given by[25]

$$\dot{x}_1=-x_1+\alpha f(x_{1,\tau})+\beta\left[f(x_{2,\tau})+f(x_{3,\tau})\right],$$

$$\dot{x}_2 = -x_2 + \alpha f(x_{2,\tau}) + \beta\left[f(x_{1,\tau}) + f(x_{3,\tau})\right],$$
$$\dot{x}_3 = -x_3 + \alpha f(x_{3,\tau}) + \beta\left[f(x_{1,\tau}) + f(x_{2,\tau})\right], \quad (64)$$

where $x_{i,\tau} = x_i(t-\tau)$ ($i = 1,2,3$), $f(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, α and β measure respectively the coupled strengths of self-connection and neighborhood-interaction, and τ is time delay due to the finite switching speed of amplifiers. Using the CMR, Wu et al.[25] approximately obtained a synchronization solution of eq. (64), which bifurcates from the trivial solution by a Hopf bifurcation at $\tau = \tau_c = \frac{\pi - \cos^{-1}(1/|\alpha + 2\beta|)}{\sqrt{(\alpha + 2\alpha)^2 - 1}}$ when $(\alpha, \beta) \in D$ and $\beta < 0$, where $D = \{(\alpha,\beta) : \alpha - \beta < -1, \alpha + 2\beta < -1\}$. Following the PIS in section 2, we can obtain the synchronization solution of eq. (64) and compare it with that from the CMR and the numerical simulation, as shown in Figure 5.

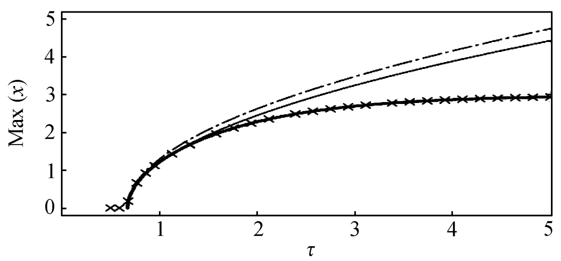

Figure 5 Synchronization solution derived from Hopf bifurcation in Max(x) vs. τ for eq. (64) when α=–2 and β=–0.5, where thin solid line denotes perturbation solution, dot-dashed line CMR solution from ref. [25], thick solid line the PIS solution and crossing symbol the numerical simulation.

6 Conclusions

A semi-analytical/numerical method, called perturbation-incremental scheme (PIS), has been developed to investigate the periodic solution derived from Hopf bifurcation due to time delay in a system of first-order delayed differential equations by three steps. The main attention is focused on representing the continuation of the bifurcated periodic solutions in a closed form with the quantitatively high accuracy. Three delayed systems are introduced as illustrative examples. The validity of the results is shown by their consistency with the numerical simulation.

The results obtained in this paper suggest that the PIS can be considered as an effective approach to investigate delayed differential equations (DDEs) when the time delay is the bifurcation parameter. Firstly, the periodic solution obtained from the perturbation step of the PIS has higher accuracy than that from both the method of multiple scales and the center manifold reduction for values of the parameter close to the Hopf bifurcation point. Therefore, the perturbation step can provide an appropriate initial guess to accelerate the convergence to the solution when the PIS is applied. Secondly, the PIS not only inherits the advantages of the method of multiple scales (MMS), but also overcomes the disadvantage of the IHB method. It has a very clear procedure such that some symbolic algebraic packages, such as MATHEMATICA, can easily be programmed to compute the solution. Thirdly, the delayed differential equations can also be reduced on center manifolds by using the PIS rather than the CMR. Using the PIS can avoid the tedious computation often encountered in the CMR. Thus, the PIS can easily be extended to the study of high-codimension DDEs[26]. Finally, the periodic solution arising from a Hopf bifurcation due to the time delay can be calculated in any desired accuracy even for values of the parameter far away from the Hopf bifurcation point by using the PIS.

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