DOI: 10.1007/s11431-006-0332-y

Improvement of tree-like network constructal method for heat conduction optimization

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Received June 5, 2005; accepted March 20, 2006

Abstract The analysis of the "tree-like network" construct method has been repeated. The high effective conduction channel distribution has been optimized again, without the premise that the new order assembly construct must be assembled by the optimized last order construct. It is proved that the "tree-like network" construct method is faultiness. A more optimal construct is obtained, and when the thermal conductivity and the proportion of the two heat conduction materials are constant, the limit of the minimum heat resistance is derived. All these conclusions can be used to guide the engineering application.

Keywords: constructal theory, "tree-like network" construct method, high effective conduction channel distribution, generalized thermodynamic optimization.

1 Introduction

As the rapid development of the technology, more and more components have been integrated in the circuit, while the size of components becomes smaller and smaller, so the problem of cooling it becomes more important. One fundamental problem of it is the volume-to-point flow problem, which is about how to conduct the heat generated in a fixed volume to a heat sink located on its boundary effectively. One solution of it is arranging high conductivity material in it to reduce thermal resistance. For this problem, Bejan put forward the "tree-like network" construct method based on the constructal theory[1−9], which derived the optimal high effective conduction channel distribution with the premise that the heat generation is uniform and the ratio of conductivity of high conductive material to conductivity of electronic material is high. Cheng *et al*.[10,11] constructed the high effective conduction channel in different conditions by bionic optimization method.

The "tree-like network" construct method started the optimization with a rectangular area, which is very small and nearly the same as the area can be manufactured in industry. Optimizing this area, the optimal shape is deduced. Then, assembling the first order assembly construct with the optimal area and optimizing this construct with respect to the number of rectangular area and the distribution of high conductivity material. In this way, higher order assembly construct can also be assembled. It ends when the area covers the given area. However, there are some doubts for the "tree-like network" construct method. This method assembles the new construct with the optimized last construct. However, whether this optimization process is the best has not been proved. Having compared the thermal resistance of each order construct, Ghodoossi[12] found that the thermal resistance does not decrease with the increase of the construct complex, which is against the claim of the construct theory. In this paper, the high effective conduction channel distribution is optimized again without the premise that the new order construct is assembled by the optimized last one. It is proved that the "tree-like network" construct method is faultiness. A more optimal construct is obtained, and when the thermal conductivity and the proportion of the two heat conduction materials are constant, the limit of the minimum heat resistance is derived. A different conclusion is obtained. All these conclusions can be used to guide the engineering application.

2 Rectangular elemental area

As shown in Fig. 1, rectangular area ($H_0 \times L_0 \times 1$) generates heat at a constant rate q volumetrically. The heat generation rate per unit volume is constant [$q''' = q/(H_0 \times L_0)$]. The area size A_0 is constant but the aspect ratio H_0 / L_0 is free to vary. The heat generated in the rectangular area is first directed to a relatively high conductive link of width D_0, which is located on the longer axes of the rectangular elemental area. Then it is channeled to a heat sink located at point M_0 by the D_0 link. The boundary of the rectangular elemental area is adiabatic except for the heat sink point M_0. It is assumed that the thermal conductivity of a high conductive link (k_p) is much higher than the thermal conductivity of electronic material (k_0) and the area occupied by high conductive material is much smaller than the area of electronic material. It is also assumed that the rec-

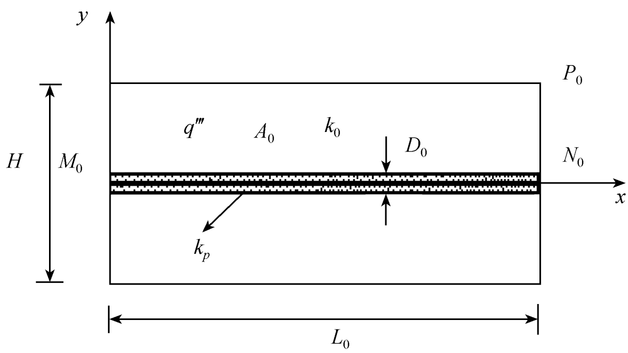

Fig. 1. Rectangular elemental area[9].

tangular elemental area is slender enough to have one-dimensional (y-direction) heat conduction on the heat generating area.

The temperature difference (ΔT_0) between the point P_0, which involves the highest temperature, and the heat sink point M_0, can be divided into two parts. One is the temperature difference between the point P_0 and N_0, and the other is the temperature difference between N_0 and M_0. It was derived in ref. [9]:

$$\Delta T_0 = \Delta T_{N_0 P_0} + \Delta T_{M_0 N_0} = \frac{q'''}{2k_0}\left(\frac{H_0}{2}\right)^2 + \frac{q''' H_0}{k_p D_0}\frac{L_0^2}{2}. \tag{1}$$

Nondimensionalizing eq. (1) gives

$$\frac{\Delta T_0}{q''' H_0 L_0 / k_0} = \frac{1}{8}\times\frac{H_0}{L_0} + \frac{k_0}{2\phi_0 k_p}\times\frac{L_0}{H_0}. \tag{2}$$

Optimizing eq. (2) with respect to H_0 / L_0 gives the optimized geometrically results[9]

$$\left(\frac{H_0}{L_0}\right)_{\text{opt}} = 2\left(\frac{1}{\hat{k}\phi_0}\right)^{1/2}, \tag{3}$$

$$\frac{\Delta T_{0,\min}}{q''' A_0 / k_0} = \frac{1}{2}\left(\frac{1}{\hat{k}\phi_0}\right)^{1/2}, \tag{4}$$

where $\hat{k} = k_p / k_0$, $\phi_0 = D_0 / H_0$. ϕ_0 is called "porosity" of the electronic material, which is the ratio of volume of high conductive material (k_p) to the volume of electronic material (k_0). $\hat{k}$ is the ratio of conductivity of high conductive material to conductivity of electronic material.

3 First order assembly construct

A large number of elemental areas are assembled in the way shown in Fig. 2. The elemental areas are lined up to both sides of the new high-conductivity path of width D_1. The heat currents of each elemental area are first conducted to the new path. And with the new path, the heat currents are then directed to the heat sink point M_1. The outer boundary of this area is adiabatic except for the heat sink point M_1. The area size A_1 is constant but the area size of A_0 and the aspect ratio H_1 / L_1 is free to vary. It can also be considered that the area size of A_0 is constant, while the area size of A_1 and the aspect ratio H_1 / L_1 is free to vary. These two methods are called inward design method and forward design method, which all optimize the first construct by taking the thermal resistance as the objective and derive consistent conclusions. Here, the inward design method is adopted.

The original "tree-like network" construct method assembled the first order assembly construct with the optimal elemental area. However, the optimal first construct is not ob-

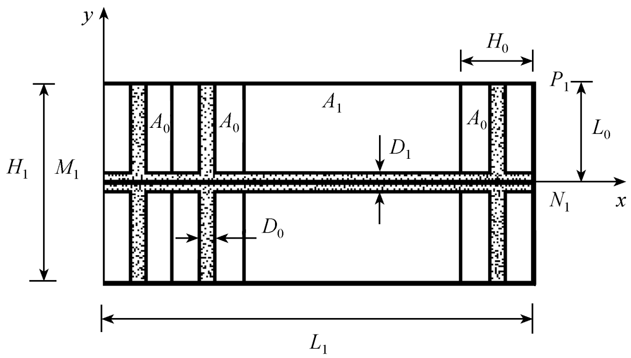

Fig. 2. The first order assembly construct[9].

tained by assembling it with the optimal elemental area. So, one takes the elemental area's geometric aspect ratio (H_0/L_0) as a new degree of freedom, and optimizes the first construct again.

For simplicity, one can assume that the heat generated in the elements conducting toward the D_1 link uniformly all the way from N_1 to M_1. It is reasonable when the number of constituent (n_1) is sufficiently large. So the temperature gradient distribution along D_1 is linear, which is just the same as the elemental area. One can obtain the temperature difference along the new high- conductivity path by replacing H_0, L_0 and D_0 in $\Delta T_{M_0N_0}$ with H_1, L_1 and D_1 respectively, then

$$\Delta T_{M_1N_1} = \frac{q'''H_1}{k_pD_1}\frac{L_1^2}{2}. \tag{5}$$

When n_1 is sufficiently large, the temperature difference between N_1 and P_1 ($\Delta T_{N_1P_1}$) is nearly the same as the maximum temperature difference of the elemental area (ΔT_0), so

$$\Delta T_1 = \Delta T_{P_1N_1} + \Delta T_{M_1N_1} = \frac{q'''H_0^2}{8k_0} + \frac{q'''H_0}{k_pD_0}\frac{L_0^2}{2} + \frac{q'''H_1}{k_pD_1}\frac{L_1^2}{2}. \tag{6}$$

The maximum temperature difference of the first order assembly construct is nondimensionalized. Note that q''' and k_0 are constants and $A_1 = n_1A_0$.

$$\frac{\Delta T_1}{q'''A_1/k_0} = \left(\frac{H_0/L_0}{8} + \frac{L_0/H_0}{2\hat{k}\phi_0}\right)\frac{1}{n_1} + \frac{H_0/L_0}{8\hat{k}\tilde{D}_1}n_1, \tag{7}$$

where $\tilde{D}_i = D_i/H_i\,(i \neq 1)$. Eq. (7) shows that the nondimensional maximum temperature difference of the construct can be minimized with respect to the number of constituents (n_1). The minimization result is

$$n_{1,\text{opt}} = \left(\tilde{D}_1\hat{k} + \frac{4\tilde{D}_1}{\phi_0}\frac{1}{a^2}\right)^{1/2}, \tag{8}$$

where $a = H_0 / L_0$. Combining eq. (8) with eq. (7) gives

$$\frac{\Delta T_{1,\min}}{q''' A_1 / k_0} = \frac{1}{4}\left(\frac{a^2}{\tilde{D}_1 \hat{k}} + \frac{4}{\tilde{D}_1 \hat{k}^2 \phi_0}\right)^{1/2}. \tag{9}$$

Eq. (9) shows that the thermal resistance reaches the minimum when $a = 0$,

$$\frac{\Delta T_{1,\min,\min}}{q''' A_1 / k_0} = \frac{1}{2}\left(\frac{1}{\tilde{D}_1 \hat{k}^2 \phi_0}\right)^{1/2}. \tag{10}$$

However, $a = 0$ means $H_0 = 0$ or $L_0 = \infty$, and it is unpractical. It is the most important that eq. (9) shows that the thermal resistance decreases monotonically as a decreases, the minimum thermal resistance limit can be approached by decreasing a.

Eq. (10) also shows that the minimum thermal resistance can be optimized with respect to high conductive material allocation in the construct. The total area of high conductive material (A_{P_1}) in the construct is constant, it is

$$A_{P_1} = D_1 L_1 + n_{1,\text{opt}} D_0 L_0 = \text{const.} \tag{11}$$

Because the construct area A_1 is also constant, the porosity of the construct is constant. The porosity of the construct is

$$\phi_1 = A_{P_1} / A_1. \tag{12}$$

Note that $A_1 = n_{1,\text{opt}} A_0$, $\phi_0 = D_0 / H_0$, one has

$$\phi_1 = \tilde{D}_1 + \phi_0. \tag{13}$$

Combining eq. (13) with eq. (9) and $a \to 0$ gives

$$\frac{\Delta T_{1,\min,\min}}{q''' A_1 / k_0} \to \frac{1}{2}\left(\frac{1}{\hat{k}^2 \phi_0 (\phi_1 - \phi_0)}\right)^{1/2}. \tag{14}$$

Optimizing eq. (14) with respect to ϕ_0, one reads

$$\phi_{0,\text{opt}} = \frac{1}{2}\phi_1. \tag{15}$$

Therefore, when $a \to 0$,

$$\frac{\Delta T_{1,\min,\min,\min}}{q''' A_1 / k_0} \to \frac{1}{\hat{k}\phi_1}. \tag{16}$$

Substituting $a \to 0$ into eq. (8) yields $n_{1,\text{opt}} \to \infty$. The number of constituent (n_1) is sufficiently large, and it proves that the simplification made in the beginning of this section is reasonable. When $a \to 0$, the optimal aspect ratio (H_1 / L_1) is

$$\left(\frac{H_1}{L_1}\right)_{\text{opt}} = \frac{2L_0}{n_{1,\text{opt}} H_0 / 2} = \frac{4}{n_{1,\text{opt}} a} \to 2. \tag{17}$$

The results above show that the nondimensional thermal resistance decreases as the elemental area's geometric aspect ratio (a) decreases, and its limit is $1/\hat{k}\phi_1$. The original "tree-like network" construct method assembled the first order assembly construct with the optimal elemental area. Substituting $a = 2(1/\hat{k}\phi_1)^{1/2}$ into eq. (9), one obtains

$$\frac{\Delta T_{1,\min}}{q'''A_1/k_0} = \frac{1}{4}\left(\frac{4}{\tilde{D}_1\hat{k}^2\phi_1} + \frac{4}{\tilde{D}_1\hat{k}^2\phi_0}\right)^{1/2}. \tag{18}$$

Optimizing it with respect to high conductive material allocation, we read

$$\frac{\Delta T_{1,\min\min}}{q'''A_1/k_0} = \sqrt{2}\frac{1}{\hat{k}\phi_1}. \tag{19}$$

Contrasting it with the result of this paper, the thermal resistance reduces about 30%.

4 Second order assembly construct

The new second order assembly construct is shown in Fig. 3. It is assembled just as the first order assembly constructs. The first order assembly constructs are lined up to both sides of the new high-conductivity path of width D_2. The heat currents of the first order assembly constructs are first conducted to the new path. And with the new path, the heat currents are then directed to the heat sink point M_2. The boundary of the construct is adiabatic except for the heat sink point located at M_2. The area of A_2 is fixed, but the aspect ratio of the new construct (H_2/L_2) is free to vary. Contrary to the "tree-like network" construct method, the number of first order assembly construct (n_2), the porosity of first order assembly construct (ϕ_1), the porosity (ϕ_0) and aspect ratio of element (a) are all new degrees of freedom.

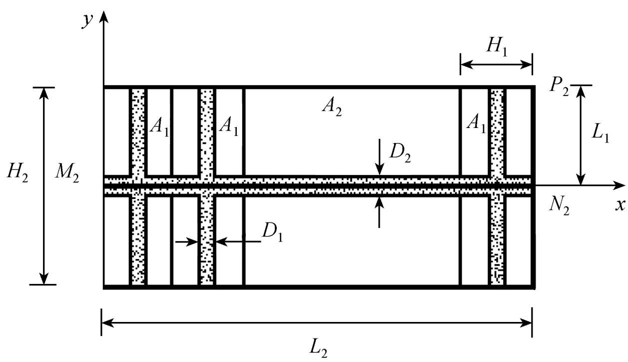

Fig. 3. The second order assembly construct[9].

Similarly, the maximum temperature difference of the second order assembly construct is composed of two parts: the temperature difference ($\Delta T_{N_2P_2}$) between N_2 and P_2, and the temperature difference ($\Delta T_{M_2N_2}$) along the high conductivity path M_2N_2. When the number of constituent (n_2) is sufficiently large, it is reasonable to assume that the heat generated in the first order assembly construct conducting toward the D_2 link uniformly all the way from N_2 to M_2. So the temperature gradient distribution along D_1 is linear, which is just the same as the elemental area. One can obtain the temperature difference along the new high-conductivity path by replacing H_0, L_0 and D_0 in $\Delta T_{M_0N_0}$ with H_2, L_2 and D_2, respectively, then

$$\Delta T_{M_2N_2} = \frac{q''' H_2}{k_p D_2} \frac{L_2^2}{2}. \tag{20}$$

When n_2 is sufficiently large, $\Delta T_{N_2P_2}$ is approximate to ΔT_1, then

$$\Delta T_2 = \Delta T_{P_2N_2} + \Delta T_{M_2N_2} = \frac{q''' H_0^2}{8k_0} + \frac{q''' L_0^2}{2k_p \phi_0} + \frac{q''' H_1}{k_p D_1} \frac{L_1^2}{2} + \frac{q''' H_2}{k_p D_2} \frac{L_2^2}{2}. \tag{21}$$

The maximum temperature difference of the second order assembly construct is nondimensionalized. Note that q''' and k_0 are constants and $A_2 = n_2 A_1$, one has

$$\frac{\Delta T_2}{q''' A_2 / k_0} = \left(\frac{a}{8n_1} + \frac{1}{2\hat{k}\phi_0 a n_1} + \frac{a n_1}{8\hat{k}D_1} \right) \frac{1}{n_2} + \frac{1}{2\hat{k}\tilde{D}_2 n_1 a} n_2 . \tag{22}$$

Eq. (22) shows that the nondimensional maximum temperature difference of the construct can be minimized with respect to the number of constituents (n_2). The minimization result is

$$n_{2,\mathrm{opt}} = \left(\frac{\hat{k}\tilde{D}_2 a^2}{4} + \frac{\tilde{D}_2}{\phi_0} + \frac{\tilde{D}_2 n_1^2 a^2}{4\tilde{D}_1} \right)^{1/2} . \tag{23}$$

Substituting eq. (23) into eq. (22) yields

$$\frac{\Delta T_{2,\min}}{q''' A_2 / k_0} = \left(\frac{1}{4\hat{k}\tilde{D}_2 n_1^2} + \frac{1}{n_1^2 a^2 \hat{k}^2 \phi_0 \tilde{D}_2} + \frac{1}{4\tilde{D}_1 \tilde{D}_2 \hat{k}^2} \right)^{1/2} . \tag{24}$$

Eq. (24) shows that when $n_1 \to \infty$ and $n_1 a \to \infty$, the thermal resistance approaches its minimum. Therefore, when $n_1 \to \infty$ and $n_1 a \to \infty$,

$$\frac{\Delta T_{2,\min,\min}}{q''' A_2 / k_0} \to \left(\frac{1}{4\tilde{D}_1 \tilde{D}_2 \hat{k}^2} \right)^{1/2} . \tag{25}$$

Substituting $n_1 \to \infty$ and $n_1 a \to \infty$ into eq. (23) yields

$$n_{2,\mathrm{opt}} \to \infty >> 1 . \tag{26}$$

So the simplification made in this section is reasonable. Note that

$$\frac{H_1}{L_1} = \frac{2L_0}{n_1 H_0 / 2} = \frac{4}{n_1 a} . \tag{27}$$

When $n_1 a \to \infty$, $H_1 / L_1 \to 0$. It shows that when the aspect ratio of first order assembly construct approaches to 0, the thermal resistance of second order assembly approaches to its minimum.

Eq. (23) also shows that the minimum thermal resistance can be optimized with respect to high conductive material allocation in the construct. The total area of high conductive material (A_{P_2}) in the construct is constant, it is

$$A_{P_2} = D_2 L_2 + n_{2,\mathrm{opt}} A_{p_1} = \mathrm{const.} \tag{28}$$

Because the construct area A_2 is also constant, the porosity of the construct is constant. The porosity of the construct is

$$\phi_2 = A_{P_2} / A_2 . \tag{29}$$

Note that $A_2 = n_{2,\mathrm{opt}} A_1$ and $\phi_1 = A_{P_1} / A_1$, one has

$$\phi_2 = \tilde{D}_2 + \phi_1 . \tag{30}$$

Substituting eqs. (13), (30), $n_1 \to \infty$ and $n_1 a \to \infty$ into eq. (25) yields

$$\frac{\Delta T_{2,\min,\min}}{q''' A_2 / k_0} \to \left(\frac{1}{4\hat{k}^2 (\phi_2 - \phi_1)(\phi_1 - \phi_0)} \right)^{1/2} . \tag{31}$$

Optimizing eq. (31) gives

$$\phi_{1,\mathrm{opt}} = \frac{1}{2}\phi_2 , \quad \phi_{0,\mathrm{opt}} = 0 . \tag{32}$$

Substituting eq. (32), $n_1 \to \infty$ and $n_1 a \to \infty$ into eq. (31) yields

$$\frac{\Delta T_{2,\min,\min,\min}}{q''' A_2 / k_0} \to \frac{1}{\hat{k}\phi_2} . \tag{33}$$

In addition,

$$\left(\frac{H_2}{L_2} \right)_{\mathrm{opt}} = \frac{2L_1}{n_{2,\mathrm{opt}} H_1 / 2} = \frac{4}{n_{2,\mathrm{opt}} H_1 / L_1} \to 2 . \tag{34}$$

Note that $\phi_{0,\mathrm{opt}} = 0$ means the second order assembly construct becomes the first order assembly construct. So the second order assembly construct will be converted to first order assembly construct when it is optimal. Optimizing the third and forth construct as doing for the second construct will deduce the same result.

The original "tree-like network" construct method assembled the second assembly construct with the optimized first assembly construct which is also composed of the optimized elemental area. The optimization results of the second assembly construct derived with the "tree-like network" construct method have been shown in ref. [9]

$$n_{2,\mathrm{opt}} = 2 , \tag{35}$$

$$\left(\frac{H_2}{L_2} \right)_{\mathrm{opt}} = \sqrt{2} , \tag{36}$$

$$\frac{\Delta T_{2,\min,\min}}{q'''A_2/k_0}=\frac{9\sqrt{2}}{8}\frac{1}{\hat{k}\phi_2}. \tag{37}$$

Contrasting these with the results of this section, the thermal resistance derived in this paper reduced about 37%. Table 1 lists the results of the two methods, which also shows that the results of this paper improve a lot.

Table 1 The minimal thermal resistance of the "tree-like network" construct method and the method of this paper[a)]

	Elemental area	1st order	2nd order	3rd order	4th order
"Tree-like network" construct method	$\frac{1}{2}\left(1/(\hat{k}\phi)\right)^{1/2}$	$\sqrt{2}/(\hat{k}\phi)$	$(8\sqrt{2}/9)/(\hat{k}\phi)$	$(1.7223)/(\hat{k}\phi)$	$(1.8182)/(\hat{k}\phi)$
The method of this paper	$\frac{1}{2}\left(1/(\hat{k}\phi)\right)^{1/2}$	$1/(\hat{k}\phi)$	$1/(\hat{k}\phi)$	$1/(\hat{k}\phi)$	$1/(\hat{k}\phi)$

a) $\hat{k}$ is the ratio of conductivity of high conductive material to conductivity of electronic material. ϕ is the ratio of volume of high conductive material (k_p) to the volume of electronic material (k_0).

5 Conclusion

The original "tree-like network" construct method assembled the new construct with the optimized last one. But whether this optimization is true has not been proved. This paper optimized the high effective conduction channel again without the premise that the constituents must be the optimized ones. The results prove that the original "tree-like network" construct method is unreasonable. The optimal construct should not be composed of optimal parts.

There are also several other conclusions, which are showed bellow: First, the minimum thermal resistance decreases as the elemental area's aspect ratio (H_0/L_0) decreases, and when the elemental area's aspect ratio approaches zero and the number of constituents is infinite, the first order assembly approaches to its optimum. Obviously, it is not practical. So the minimum thermal resistance can just be close to it. Second, when the thermal conductivity and the proportion of the two heat conduction materials are constant, the limit of the minimum nondimensional thermal resistance is $1/\hat{k}\phi$. Third, the higher order assembly construct are converted to first order construct when it approaches to its optimum. Besides, the result also confirms the expectation of Constructal Theory that the more complex the construct is, the less the thermal resistance is.

The conclusion can be put to engineering application. The engineer should choose the right structure with respect to the project demand, the cost and the craft lever.

Acknowledgements This work was supported by the Program for New Century Excellent Talents in University of P. R. China (Grant No. NCET-04-1006) and the Foundation for the Author of National Excellent Doctoral Dissertation of P. R. China (Grant No. 200136).

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