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Three-dimensional decomposition method of global atmospheric circulation

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By adopting the idea of three-dimensional Walker, Hadley and Rossby stream functions, the global atmospheric circulation can be considered as the sum of three stream functions from a global perspective. Therefore, a mathematical model of three-dimensional decomposition of global atmospheric circulation is proposed and the existence and uniqueness of the model are proved. Besides, the model includes a numerical method leading to no truncation error in the discrete three-dimensional grid points. Results also show that the three-dimensional stream functions exist and are unique for a given velocity field. The mathematical model shows the generalized form of three-dimensional stream functions equal to the velocity field in representing the features of atmospheric motion. Besides, the vertical velocity calculated through the model can represent the main characteristics of the vertical motion. In sum, the three-dimensional decomposition of atmospheric circulation is convenient for the further investigation of the features of global atmospheric motions.

three-dimensional decomposition of the global atmospheric circulation, generalized Walker, Hadley and Rossby stream functions, existence and uniqueness, vertical motion

Many key progresses have been achieved in climate research in recent years, for instance, the formation of the idea of climate system, the discovery of multiple equilibrium climate systems, the abrupt change of climate and the activity of human beings being the external forcing of climate change[1]. Improving the accuracy of short-term climate prediction is a main objective of climate research. Nowadays, the current climate prediction methods can be classified into two broad categories, namely, the statistical methods and dynamical numerical models. Scientists can use either of them to forecast the future climate[2]. It is almost impossible to predict the climate successfully by using statistical methods because of the inherent limits[2]. However, using dynamical numerical model has become the most popular and robust methods in climate prediction in recent years[3]. Nowadays, land-air-sea coupled climate models have been developed from simple atmospheric model and the fully coupled models including all the subsystem of climate are being developed[4,5]. Meanwhile, the performance of dynamical numerical models is constantly improved owing to much work in improving the model resolution and refining the parameterization of physical processes[2,3,6]. However, the climate models still have some defects. Doubts remains in both simulations and

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climate prediction[7] and the progress of the accuracy of climate prediction can not well satisfy the demand of operational service[2]. The main reasons may be the lack of awareness in developing fundamental theory and the defects in research method[2].

By recalling the successful history of numerical weather forecasting, we may find that it is the achievement of fundamental theory in dynamical meteorology that leads to the encouraging success of short-range and medium-range weather forecasting[8]. Nowadays, the fundamental theory of climate prediction is still immature, thus there will not be any reliable and accurate climate prediction before the great advance in the research of basic theory[8]. However, scholars have achieved many outstanding progresses in the research of fundamental theory. Gu[9] as early as in 1958 pointed out that evolutive data at that time could be incorporated into numerical models after changing numerical prediction from initial-value problem to evolving problem. Chou and Guo[10,11] proposed that an air-sea model could be regarded as an initial problem the same as the problem that an air-sea model can be thought as a historical evolution of temperature and pressure fields. Cao[12] gained self-memory equations of atmospheric motion based on the perspective that atmospheric movement is an irreversible process. Chou[13] initially proposed the qualitative theory of the dynamical equations of external forced, dissipative and nonlinear atmospheric system. Based on what Chou had found, Wang, Huang and Li et al.[14−16] further extended the qualitative theory to infinite Hilbert space, large-scale ocean dynamical equations, large-scale sea-air coupling systems and non-stationary external forced systems. Nevertheless, there is much work remaining to be further explored[2].

Lorenz pointed out that in rotating annulus experiments, both Hadley circulation pattern and Rossby circulation pattern would occur under certain rotating speed conditions[17]. Because of the asymmetry distribution of solar radiation on the earth, the uneven distribution of the land and sea and the rotation of the earth, it is well known that the atmospheric circulation in medium and high latitude areas displays mainly quasi-horizontal and quasi-geostrophic Rossby circulation. However, the atmospheric circulation in tropic regions appears mainly Hadley circulation and Walker circulation. Atmospheric circulations in different regions show distinct movement features. Many investigations have proved that atmospheric circulations in different regions do not exist alone and that they have strong interactions. It must be necessary to consider the impacts of atmospheric circulations of remote areas while predicting[18−26]. Ye analyzed the relationship and features of vertical circulations on various vertical meridional and zonal planes[18,19]. Qian et al.[20] showed that the interannual change of the rotation of the solid earth could cause the anomalies of the zonal winds of atmosphere. Fu et al.[21] detected that the variations of sea surface temperature in equatorial Pacific regions could influence significantly the patterns and the intensity of both the meridional circulations over Pacific regions and the averaged circulations along the equator. Wu and Cubasch[22] found that continual anomaly of eastern Pacific Ocean along the equator could cause the strengthening of Hadley circulation. Ji and Cao[23] showed that when the eastern tropical sea surface temperature is higher than the western tropical sea surface temperature, the meridional circulations in the atmosphere would be strengthened and zonal circulations would be weakened. Many results pointed out that restraint interactions between Rossby circulations, Walker circulations and Hadley circulations exist[24−26]. However, research results have not been reported yet, being an object to further discover the internal dynamical mechanism of the interaction between the circulations. One main reason may be that proper method is not available for the moment.

We know that the quasi-geostrophic theory is the basis of successful numerical weather forecasting models and that vertical motion is important in tropic regions. However, the robust vertical motion in tropic areas is omitted as a negligible variable in the quasi-geostrophic theory. Thus, it is not proper to apply quasi-geostrophic theory in low latitude regions. Although the importance of vertical circulations is well known, the characteristics of vertical circulations have not been thoroughly understood yet because of the lack of atmospheric sounding data and the suitable research model.

We know that climate system is external forcing, dissipative, nonlinear and long-term evolutive. We also know that geostrophic dynamics theory is an adiabatic, and nondissipative theory only suitable for short-range weather in medium or high latitudes regions. Therefore, Fundamental theory suitable for the study of dynamical process of global atmospheric circulation should not be geostrophic dynamics theory. To solve this problem, we try to set up a new set of equations that represent straightly the physical features of the global atmospheric circulation. Thus, we can think Hadley circulations, Rossby circulations and Rossby circulations exist all over the globe in the global point of view. Based on this idea, we may define the generalized three-dimensional Walker circulation on the horizontal plane, in so doing, we can distinguish it from traditional two-dimensional Rossby circulation. Similarly, we may define the generalized three-dimensional Hadley circulations and the generalized three-dimensional Walker circulations. The global atmospheric circulation can be considered as the sum of the Walker circulations, Hadley circulations and Rossby circulations. We call this idea the three-dimensional decomposition method of global atmospheric circulation.

1 Three-dimensional decomposition model of global atmospheric circulation

We think the atmospheric circulation can be expressed in the form of three-dimensional stream functions from a global perspective based on the current findings. To make the issue more clearly expressed, we discuss the generalized definition of the three-dimensional stream functions first.

1.1 Definition of generalized three-dimensional Rossby, Walker and Hadley circulations

According to the definition of two-dimensional circulations, generalized form of three-dimensional Rossby circulations in pressure Cartesian coordinate system refers to the three-dimensional velocity field which satisfies the following equation

$$\boldsymbol{V}_{\mathrm{R}}(x,y,p) \equiv \boldsymbol{i}u_{\mathrm{R}}(x,y,p) + \boldsymbol{j}v_{\mathrm{R}}(x,y,p) + \boldsymbol{k}\omega_{\mathrm{R}}(x,y,p), \tag{1}$$

and the vertical condition

$$\omega_{\mathrm{R}}(x,y,p) \equiv 0, \tag{2}$$

and also the underlying condition

$$\frac{\partial u_{\mathrm{R}}(x,y,p)}{\partial x} + \frac{\partial v_{\mathrm{R}}(x,y,p)}{\partial y} = 0. \tag{3}$$

For generalized three-dimensional Rossby circulations, if there exists function $R(x, y, p)$ in the pressure Cartesian coordinate system, which satisfies

$$\begin{cases} u_{\mathrm{R}}(x,y,p) = -\dfrac{\partial R}{\partial y}, \\ v_{\mathrm{R}}(x,y,p) = \dfrac{\partial R}{\partial x}, \end{cases} \tag{4}$$

then we call $R(x, y, p)$ the three-dimensional Rossby stream function. If we can decide the three-dimensional stream function $R(x, y, p)$ by using an already known velocity field $\boldsymbol{V}_{\mathrm{R}}(x, y, p)$, we can obtain $\boldsymbol{V}_{\mathrm{R}}(x,y,p)$ from $R(x, y, p)$ in turn. Therefore, the stream function $R(x, y, p)$ can represent the features of the three-dimensional velocity field of Rossby circulations.

Similarly, the generalized three-dimensional Walker circulations refer to the three-dimensional velocity field

$$\boldsymbol{V}_{\mathrm{W}}(x,y,p) \equiv \boldsymbol{i}u_{\mathrm{W}}(x,y,p) + \boldsymbol{j}v_{\mathrm{W}}(x,y,p) + \boldsymbol{k}\omega_{\mathrm{W}}(x,y,p), \tag{5}$$

which satisfies the meridional condition

$$v_{\mathrm{W}}(x,y,p) \equiv 0, \tag{6}$$

and the condition

$$\frac{\partial u_{\mathrm{W}}(x,y,p)}{\partial x} + \frac{\partial \omega_{\mathrm{W}}(x,y,p)}{\partial p} = 0. \tag{7}$$

Generalized three-dimensional Walker stream function of $W(x, y, p)$ refers to that satisfying the following equation

$$\begin{cases} u_{\mathrm{W}}(x,y,p) = \dfrac{\partial W}{\partial p}, \\ \omega_{\mathrm{W}}(x,y,p) = -\dfrac{\partial W}{\partial x}. \end{cases} \tag{8}$$

Generalized three-dimensional Hadley stream function refers to the three-dimensional velocity field which satisfies the following equation

$$\boldsymbol{V}_{\mathrm{H}}(x,y,p) \equiv \boldsymbol{i}u_{\mathrm{H}}(x,y,p) + \boldsymbol{j}v_{\mathrm{H}}(x,y,p) + \boldsymbol{k}\omega_{\mathrm{H}}(x,y,p), \tag{9}$$

and the latitudinal component condition

$$u_{\mathrm{H}}(x,y,p) \equiv 0, \tag{10}$$

and in addition, satisfies the underlying condition

$$\frac{\partial v_{\mathrm{H}}(x,y,p)}{\partial y} + \frac{\partial \omega_{\mathrm{H}}(x,y,p)}{\partial p} = 0. \tag{11}$$

Generalized three-dimensional Hadley stream function of $H(x, y, p)$ denotes the functions as follows:

$$\begin{cases} v_{\mathrm{H}}(x,y,p) = -\dfrac{\partial H}{\partial p}, \\ \omega_{\mathrm{H}}(x,y,p) = \dfrac{\partial H}{\partial y}. \end{cases} \tag{12}$$

Similarly, we can give the definitions of generalized three-dimensional stream functions in pressure spherical coordinates, but we shall not discuss them here.

1.2 Three-dimensional decomposition of velocity field

If $\boldsymbol{V}$ denotes the three-dimensional velocity field in pressure Cartesian coordinates, then velocity field $\boldsymbol{V}$ can be expressed as the sum of the generalized three-dimensional circulations as follows:

$$\boldsymbol{V}=\boldsymbol{V}_{\mathrm{R}}+\boldsymbol{V}_{\mathrm{W}}+\boldsymbol{V}_{\mathrm{H}}. \tag{13}$$

By using the definitions of the three-dimensional circulations, the components of (13) can be denoted in the Cartesian coordinate system in the following:

$$\begin{cases} u = u_{\mathrm{W}} + u_{\mathrm{R}} = \dfrac{\partial W(x,y,p)}{\partial p} - \dfrac{\partial R(x,y,p)}{\partial y}, \\ v = v_{\mathrm{R}} + v_{\mathrm{H}} = \dfrac{\partial R(x,y,p)}{\partial x} - \dfrac{\partial H(x,y,p)}{\partial p}, \\ \omega = \omega_{\mathrm{H}} + \omega_{\mathrm{W}} = \dfrac{\partial H(x,y,p)}{\partial y} - \dfrac{\partial W(x,y,p)}{\partial x}. \end{cases} \tag{14}$$

The three-dimensional velocity field $\boldsymbol{V}(x, y, p)$ satisfies equation (14) and the following condition

$$\frac{\partial u(x,y,p)}{\partial x} + \frac{\partial v(x,y,p)}{\partial y} + \frac{\partial \omega(x,y,p)}{\partial p} = 0. \tag{15}$$

Therefore, eq. (14) can be thought to be the three-dimensional decomposition of velocity in pressure Cartesian coordinate system.

We can draw the conclusion that three-dimensional stream functions H, W and R can be determined by using observational velocity field of u and v through the three-dimensional decomposition model while considering the generalized definitions of the three-dimensional circulations and eq. (14). We can also conclude that the velocity field u^*, v^* ω^* can be calculated through already known three-dimensional stream functions of H, W and R.

While the familiar form of two-dimensional stream functions can be determined through the total differential equation, the three-dimensional stream functions cannot be expressed in total differential form. An arbitrary function must be considered in the differential equation. The problem becomes much more complicated since the increase of one degree of freedom. The generalized three-dimensional stream functions discussed in this study come from the movement of atmosphere and are essentially different from that in the total differential equation discussed in ref. [27].

2 Partial differential equations of three-dimensional stream functions

If not considering water vapor condensation and evaporation, neglecting the air temperature change in Cartesian coordinates system of pressure, we obtain

$$\nabla \cdot \boldsymbol{V} \equiv 0. \tag{16}$$

Then velocity $\boldsymbol{V}$ can be expressed in the form of curl of vector $\boldsymbol{A}$ as follows:

$$\boldsymbol{V} = -\nabla \times \boldsymbol{A}, \tag{17}$$

where A is the vector potential of vector velocity field $\boldsymbol{V}$. If we denote vector field $\boldsymbol{A}$ as $\boldsymbol{A}=iA_x+jA_y+kA_p$, then eq. (17) in Cartesian coordinate system changes to

$$\begin{cases} u(x,y,p) = \dfrac{\partial A_y(x,y,p)}{\partial p} - \dfrac{\partial A_p(x,y,p)}{\partial y}, \\ v(x,y,p) = \dfrac{\partial A_p(x,y,p)}{\partial x} - \dfrac{\partial A_x(x,y,p)}{\partial p}, \\ \omega(x,y,p) = \dfrac{\partial A_x(x,y,p)}{\partial y} - \dfrac{\partial A_y(x,y,p)}{\partial x}. \end{cases} \tag{18}$$

We can find that the three components A_x, A_y, A_p of $\boldsymbol{A}$ equal to H, W, R by comparing eq. (14) with eq. (18). From eq. (17) we know that for the given velocity field, there will be many infinite different expressions of vector potential. Therefore, certain conditions are added, so $\boldsymbol{A}$ is determined uniquely and the three components of $\boldsymbol{A}$ are just the previous definitions of generalized three-dimensional stream functions according to the needs of the vector potential under certain conditions. Therefore, we can denote the potential of velocity field as $\boldsymbol{A} = \boldsymbol{i}H + \boldsymbol{j}W + \boldsymbol{k}R.$

According to eq. (17), the extra condition of $\boldsymbol{A}$ is

$$\nabla \cdot \boldsymbol{A} = 0. \tag{19}$$

Taking the curl of vector operation on both sides of eq. (17) and substituting it into eq. (19) in Cartesian coordinate system, we have

$$\begin{cases} \dfrac{\partial^2 H}{\partial x^2} + \dfrac{\partial^2 H}{\partial y^2} + \dfrac{\partial^2 H}{\partial p^2} = \dfrac{\partial \omega}{\partial y} - \dfrac{\partial v}{\partial p}, \\ \dfrac{\partial^2 W}{\partial x^2} + \dfrac{\partial^2 W}{\partial y^2} + \dfrac{\partial^2 W}{\partial p^2} = \dfrac{\partial u}{\partial p} - \dfrac{\partial \omega}{\partial x}, \\ \dfrac{\partial^2 R}{\partial x^2} + \dfrac{\partial^2 R}{\partial y^2} + \dfrac{\partial^2 R}{\partial p^2} = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}. \end{cases} \tag{20}$$

The velocity field $\boldsymbol{V}$ satisfies equation (16) and the third component ω of $\boldsymbol{V}$ satisfies the condition $\omega \to 0$(as $p \to 0$) in the pressure coordinate system. So in theory, ω

can be determined by the horizontal part of velocity field. The expression of ω in Cartesian coordinate pressure system is

$$\omega(x,y,p) = -\int_0^p \left(\frac{\partial u(x,y,t)}{\partial x} + \frac{\partial v(x,y,t)}{\partial y} \right) \mathrm{d}p. \quad (21)$$

Therefore, substituting eq. (18) into the right side of eq. (20), we have

$$\begin{cases} \dfrac{\partial^2 H}{\partial x^2} + \dfrac{\partial^2 H}{\partial p^2} + \dfrac{\partial^2 W}{\partial x \partial y} = -\dfrac{\partial v}{\partial p}, \\ \dfrac{\partial^2 W}{\partial y^2} + \dfrac{\partial^2 W}{\partial p^2} + \dfrac{\partial^2 H}{\partial x \partial y} = \dfrac{\partial u}{\partial p}, \\ \dfrac{\partial^2 R}{\partial x^2} + \dfrac{\partial^2 R}{\partial y^2} + \dfrac{\partial^2 R}{\partial p^2} = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}. \end{cases} \quad (22)$$

If we can prove that the solution of H, W, R of eq. (22) exists and is unique under certain boundary conditions and that H, W, R also satisfy eq. (14), then we can gain the three-dimensional decomposition model. We already know that velocity field $\boldsymbol{V}$ can be equivalently expressed in the form of stream functions through eqs. (17) and (19). We also note that the velocity field of $\boldsymbol{V}$ can be replaced by $\boldsymbol{A}$ for the large-scale movement. This replacement will be of some convenience for the future research.

We can regard eq. (22) as the partial differential equation that three-dimensional stream functions H, W, R are satisfied in pressure Cartesian coordinate system. Nevertheless, the pressure spherical coordinate system will be the best choice for the research of large-scale motion. From the above discussions, we can see the three-dimensional decomposition equations are also valid in the pressure spherical coordinate system. Similarly, from the eqs. (13), (16), (17) and (19), we know that three-dimensional stream functions in pressure spherical coordinates satisfy the partial differential equations as follows:

$$\frac{1}{a^2 \sin^2\theta}\frac{\partial^2 H}{\partial \lambda^2} + \frac{H}{a^2 \sin^2\theta} + \beta^2 \frac{\partial^2 H}{\partial p^2} + \frac{1}{a^2 \sin\theta}\frac{\partial^2 W}{\partial \lambda \partial \theta} - \frac{\cos\theta}{a^2 \sin^2\theta}\frac{\partial W}{\partial \lambda} = -\beta \frac{\partial v_\theta}{\partial p}, \quad (23)$$

$$\frac{1}{a^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{\cos\theta}{a^2 \sin\theta}\frac{\partial W}{\partial \theta} + \beta^2 \frac{\partial^2 W}{\partial p^2} + \frac{1}{a^2 \sin\theta}\frac{\partial^2 H}{\partial \lambda \partial \theta} + \frac{\cos\theta}{a^2 \sin^2\theta}\frac{\partial H}{\partial \lambda} = \beta \frac{\partial v_\lambda}{\partial p}, \quad (24)$$

$$\frac{1}{a^2 \sin^2\theta}\frac{\partial^2 R}{\partial \lambda^2} + \frac{\cos\theta}{a^2 \sin\theta}\frac{\partial R}{\partial \theta} + \frac{1}{a^2}\frac{\partial^2 R}{\partial \theta^2} + \beta^2 \frac{\partial^2 R}{\partial p^2} = \frac{1}{a\sin\theta}\frac{\partial v_\theta}{\partial \lambda} - \frac{1}{a\sin\theta}\frac{\partial (v_\lambda \sin\theta)}{\partial \theta}. \quad (25)$$

In eqs. (23)—(25), a denotes the radius of the earth, and β satisfies

$$\beta = -\bar{\rho} g,$$

where $\bar{\rho}$ is the average density of the atmosphere and g represents the gravitational acceleration of the earth.

There are infinite many three-dimensional stream functions of H, W, R satisfying partial differential eqs. (22)—(24). According to the atmospheric motion, reasonable boundary conditions deciding the stream functions can be given. So the existence and uniqueness of the problem for determining solution can be proved in theory, and the three-dimensional model can be achieved.

3 Problem for determining solutions of partial differential equations and the proof of the existence and uniqueness of solutions

We think that local problem is unrelated to its state in the infinite distant horizontal direction in Cartesian coordinate system under normal conditions. We also think the global problem satisfies the periodic conditions in spherical coordinates system. Furthermore, we think the stream functions and their derivatives of first order are approaching to zero with respect to p at the top of the atmosphere. Therefore, for partial differential eqs. (22)—(24), we can give the following boundary conditions:

(1) The infinitely distant condition in Cartesian coordinate system is

$$|\boldsymbol{A}| \to 0 \text{ and } |\nabla \boldsymbol{A}| \to 0, \text{ as } x^2 + y^2 \to +\infty. \quad (26)$$

(2) The periodic condition in spherical coordinates system is

$$\begin{cases} \boldsymbol{A}(0,\theta,p) = \boldsymbol{A}(2\pi,\theta,p), \\ \left.\dfrac{\partial \boldsymbol{A}(\lambda,\theta,p)}{\partial \lambda}\right|_{\theta=0} = \left.\dfrac{\partial \boldsymbol{A}(\lambda,\theta,p)}{\partial \lambda}\right|_{\theta=\pi} = 0. \end{cases} \quad (27)$$

(3) The condition as $p \to 0$ is

$$|\boldsymbol{A}| \to 0 \text{ and } \left|\frac{\partial \boldsymbol{A}}{\partial p}\right| \to 0. \quad (28)$$

If we define the set as $\Phi = \{\boldsymbol{A} =$

$\boldsymbol{i}H + \boldsymbol{j}W + \boldsymbol{k}R \mid \nabla \cdot \boldsymbol{A} = 0$, where H, W, R are continuously differentiable functions of second order, then Φ represents the set of vector functions that are continuously differentiable functions of second order and have the divergence equaling zero. Thus, we obtain the following two problems for determining solutions.

Problem for determining solution 1: if $\boldsymbol{A} = \boldsymbol{i}H + \boldsymbol{j}W + \boldsymbol{k}R \in \Phi$, then $\boldsymbol{A}$ satisfies partial differential eq. (22) and boundary conditions (26) and (28).

Problem for determining solution 2: if $\boldsymbol{A} = \boldsymbol{i}H + \boldsymbol{j}W + \boldsymbol{k}R \in \Phi$, then $\boldsymbol{A}$ satisfies partial differential eqs. (23) and (24) and boundary conditions (27) and (28).

So, we discuss the existence and uniqueness of the two problems of determining solutions. In fact, we can prove the two problems are equivalent to the similar equations of first order.

Theorem 1: if $\boldsymbol{A} = \boldsymbol{i}H + \boldsymbol{j}W + \boldsymbol{k}R \in \Phi$, then Problem 1 and 2 equal the equations of first order described as follows:

$$\begin{cases} \dfrac{\partial \varphi}{\partial p} + A\dfrac{\partial \varphi}{\partial x} + B\dfrac{\partial \varphi}{\partial y} + C\varphi = f, \\ \varphi\,|_{p=0} \to 0. \end{cases} \tag{29}$$

For Problem 1, the vector functions φ and f are defined as follows:

$$\varphi = (E_1, E_2, E_3, F_1, F_2, G_1, G_2, H, W, R)^T,$$

$$f = \left(-\frac{\partial v}{\partial p}, \frac{\partial u}{\partial p}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}, 0,0,0,0,0,0,0 \right)^T, \tag{30}$$

where $\boldsymbol{A} = (a_{ij})$, $\boldsymbol{B} = (b_{ij})$, $\boldsymbol{C} = (c_{ij})$ denote sparse constant matrix of 10th order. For matrix $\boldsymbol{A}$, there exist $a_{14} = a_{16} = a_{35} = 1$, $a_{41} = a_{53} = -1$, $a_{41} = a_{53} = -1$, and other elements of $\boldsymbol{A}$ equate to zero. For matrix $\boldsymbol{B}$, we obtain $b_{24} = b_{26} = b_{37} = 1$, $b_{24} = b_{26} = b_{37} = 1$ and $b_{62} = b_{73} = -1$, $b_{62} = b_{73} = -1$, and other elements of $\boldsymbol{B}$ equate to zero. For matrix $\boldsymbol{C}$, we get $c_{81} = c_{92} = c_{10,3} = -1$, $c_{81} = c_{92} = c_{10,3} = -1$, and other elements of $\boldsymbol{C}$ equate to zero. For Problem 2, the independent variables satisfy $x = \lambda, y = \theta$. The vector functions φ and f can be expressed as follows:

$$\varphi = (E_1, E_2, E_3, F_1, F_2, F_3, G_1, G_2, H, W, R)^T, \tag{31}$$

$$f = \left(-\frac{1}{\beta}\frac{\partial v_\theta}{\partial p}, \frac{1}{\beta}\frac{\partial v_\lambda}{\partial p}, -\frac{1}{a\beta^2 \sin\theta}\frac{\partial v_\theta}{\partial \lambda} - \frac{1}{a\beta^2 \sin\theta}\frac{\partial (v_\lambda \sin\theta)}{\partial \theta}, 0,0,0,0,0,0,0,0 \right)^T, \tag{32}$$

where $\boldsymbol{A} = (a_{ij})$, $\boldsymbol{B} = (b_{ij})$, $\boldsymbol{C} = (c_{ij})$ denote sparse functional matrices of 11th order, and matrix $\boldsymbol{A}$ satisfies $a_{14} = a_{36} = \dfrac{1}{a^2\beta^2 \sin^2\theta}$, and $a_{41} = a_{52} = a_{63} = -1$, and other elements of $\boldsymbol{A}$ equate to zero. For matrix $\boldsymbol{B}$, there exist $b_{15} = b_{24} = \dfrac{1}{a^2\beta^2 \sin\theta}$, $b_{27} = b_{38} = \dfrac{1}{a^2\beta^2}$ and $b_{72} = b_{83} = -1$, other elements of $\boldsymbol{B}$ are 0. For matrix $\boldsymbol{C}$, there exist $c_{15} = -\dfrac{\cos\theta}{a^2\beta^2 \sin^2\theta}$, $c_{19} = -\dfrac{1}{a^2\beta^2 \sin^2\theta}$, $c_{24} = \dfrac{\cos\theta}{a^2\beta^2 \sin^2\theta}$, $c_{27} = c_{38} = \dfrac{\cos\theta}{a^2\beta^2 \sin\theta}$ and $c_{91}= c_{10,2}= c_{11,3}=-1$, and other elements of $\boldsymbol{C}$ are zero. Detailed proof of Theorem 1 is given in appendix 1. Therefore, proofs of the existence and uniqueness of Problem 1 and Problem 2 are the same as what discussed in partial differential eq. (29) of first order. For convenience, eq. (29) can be denoted in the form of its component as follows:

$$\begin{cases} \dfrac{\partial \varphi_i}{\partial p} = \displaystyle\sum_{j=1}^{N}\sum_{k=1}^{2} a_{ij}^{(k)} \frac{\partial \varphi_j}{\partial x_k} + \sum_{j=1}^{N} b_{ij}\varphi_j + f_i, \\ \varphi_i\big|_{p=0} = 0, \qquad i = 1, 2, \cdots, N \end{cases} \tag{33}$$

where φ_i and f_i represent the ith component of vector functions φ and f, respectively. We know $a_{ij}^{(k)}$ and b_{ij} can be determined by functional matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$. For Problem 1, we obtain $x_1 = x, x_2 = y$, $N = 10$. For Problem 2, we have $x_1 = \lambda, x_2 = \theta$, and $N = 11$. According to Cauchy-Dunayevskaya theorem, which is an important theorem in the theory of partial differential equations, eq. (33) has a unique analytic solution.

Theorem 2 (Cauchy-Dunayevskaya theorem): in eq. (33), if the points $(x_0, y_0, 0)$ of $a_{ij}^{(k)}$, b_{ij} and f_i ($i, j = 1, 2, \cdots, N, k = 1, 2$) are analytic in their certain neighborhood in space R^3, then eq. (33) has a unique analytic solution $\varphi_i (i = 1, 2, \cdots, N)$ in the neighborhood of $(x_0, y_0, 0)$ in space R^3.

Detailed proof of Theorem 2 can be found in ref. [28]. From Theorem 1 and 2, we know that if the u, v field of velocity field $\boldsymbol{V}$ is already known, the three-dimensional stream functions can be determined uniquely through Problem 1 and 2, thus the three-dimensional decomposition is theoretically achieved.

4 New method of calculating the large-scale vertical velocity

Vertical movement occupies a very important position and is the key factor to affect the weather. Usually vertical velocity is not gained through meteorological observation; it is often calculated by other variables through diagnostic equations. The vertical velocity in NCEP reanalysis project is thus obtained[29,30]. Theorem 1 and Theorem 2 are theoretical guarantees for the existence and uniqueness of the velocity field through the three-dimensional decomposition model. We also note that the three-dimensional stream functions$H$, W, R, determined by the velocity field $\boldsymbol{V}$, can be inversely used to decide the original velocity field $\boldsymbol{V}$, especially the vertical velocity. In pressure Cartesian coordinates system, we obtain

$$\omega = \omega_{\mathrm{H}} + \omega_{\mathrm{W}} = \frac{\partial H(x,y,p)}{\partial y} - \frac{\partial W(x,y,p)}{\partial x}. \tag{34}$$

In pressure spherical coordinates, we obtain

$$\omega = \omega_{\mathrm{H}} + \omega_{\mathrm{W}} = \frac{1}{a\sin\theta}\frac{\partial(H\sin\theta)}{\partial\theta} - \frac{1}{a\sin\theta}\frac{\partial W}{\partial\lambda}. \tag{35}$$

Eqs. (34) and (35) can be thought to be a new method of calculating vertical movement of the atmosphere.

From eqs. (34) and (35), we know the large-scale vertical velocity cab be regarded as the sum of the impacts of Hadley stream functions and Walker stream functions. Therefore, one of the advantages of calculating the vertical velocity through the three-dimensional decomposition model is that vertical velocity ω can be decomposed into two parts, namely ω_{H} and ω_{W}. This will provide convenience to the study of the interaction of global three-dimensional circulations. Furthermore, it is helpful to the research of the respective contribution that Hadley circulations and Walker circulations have to the vertical movement.

5 Algorithm of three-dimensional decomposition

For global issues, using spherical coordinates system is the best choice. However, we find that partial differential eqs. (23)—(25) have complex forms in spherical coordinates system that will cause substantial difficulties in calculation. Therefore, we will simplify eqs. (23) —(25), so we can put forward a proper simplified model while preserving the nature of the problem unchanged. By simplifying the model, we try to reveal the main features of three-dimensional circulations.

5.1 Simplified model in spherical coordinate system

In the derivation of equations (23)—(25), if we neglect the gradient effect of spherical Earth on the operator ∇, we find that in low-latitude areas, the partial differential equation is essentially eq. (22) in rectangular coordinates system. Therefore, we call eq. (22) the simplified model in spherical coordinates. For this simplified model, the boundary conditions are eq. (28) and the periodic condition in spherical coordinates system. However, the simplified model can only be applied to low-latitude areas. To promote the discussion, we consider theoretically areas between 60ºS and 60ºN, and we calculate the average of velocity field along 60ºS and 60ºN. Therefore, we let u, v satisfy the periodic conditions as follows:

$$\begin{cases} \boldsymbol{A}(0,y,p) = \boldsymbol{A}(2\pi,y,p), \\ \boldsymbol{A}\left(x,\dfrac{\pi}{6},p\right) = \boldsymbol{A}\left(x,\dfrac{5\pi}{6},p\right). \end{cases} \tag{36}$$

The above handling skills led the following numerical method to have no truncation error in theory. To remove the artificial impact added to the periodic condition (36), we only consider the results in areas between 30ºS and 30ºN. Therefore, we obtain an important theorem for the simplified model (22) in spherical coordinate.

Theorem 3: If $\boldsymbol{A} = \boldsymbol{i}H + \boldsymbol{j}W + \boldsymbol{k}R \in \Phi$, the vector $\boldsymbol{A}$ exists and is unique, which satisfies partial differential equations of (22) and the boundary conditions (28) and (36).

The detailed proof of Theorem 1 is given in appendix 2. From Theorem 1, we note that we can uniquely decide the stream functions of H, W, R in spherical coordinates system by using the simplified eq. (22) and boundary conditions (28) and (36). We also note that the stream functions are just the above-generalized three-dimensional stream functions. Thus, we obtain the three-dimensional decomposition model. In fact, if stream functions of H, W, R satisfy Theorem 3, then we have

$$\frac{\partial H}{\partial x}+\frac{\partial W}{\partial y}+\frac{\partial R}{\partial p}=0. \quad (37)$$

Substituting eq. (37) into the first expression of partial differential eq. (22), if we can exchange the order of partial derivative of stream functions of H, W, R about independent variable x, y, p, we obtain

$$\frac{\partial^2 H}{\partial p^2}-\frac{\partial^2 R}{\partial p \partial x}=-\frac{\partial v}{\partial p}. \quad (38)$$

We conduct definite integral operation on both sides of the above equation from p_s to p, and we assume that on the earth surface, we have $\left.\frac{\partial W}{\partial p}-\frac{\partial R}{\partial y}\right|_{p=p_s}$ = $u_s(x, y)$, so we obtain

$$\frac{\partial H}{\partial p}-\frac{\partial R}{\partial x}=-v, \quad \forall p \in (0, p_s). \quad (39)$$

Similarly, we can prove the equation as follows:

$$\frac{\partial W}{\partial p}-\frac{\partial R}{\partial y}=u, \quad \forall p \in (0, p_s). \quad (40)$$

If $\left.\frac{\partial H}{\partial y}-\frac{\partial W}{\partial x}\right|_{p=0}=0$, $\left.\frac{\partial H}{\partial y}-\frac{\partial W}{\partial x}\right|_{p=0}=0$,

then we substitute equations (39) and (40) into equation of continuity and performing integral operation, we obtain

$$\frac{\partial H}{\partial y}-\frac{\partial W}{\partial x}=\omega, \quad \forall p \in (0, p_s). \quad (41)$$

From eqs. (39)—(41), we know the simplified model can reflect the main features of three-dimensional decomposition of velocity field.

5.2 Numerical scheme of the simplified model

We know the simplified model (22) satisfies the periodic boundary condition of eq. (28), and the periodic function can be expanded in Fourier series. If we conduct proper prolongation operation on all functions of the simplified model with respect to the independent variable p to enable them to satisfy periodic conditions, we can expand the periodic functions into new equations in the form of Fourier series with respect to the arguments of x, y, p. According to uniqueness of the expansion of the Fourier series, we will be able to obtain stream functions.

The most important advantage of Fourier series solution is that there is no truncation error in calculating at the mesh point. Stream functions H, W, R can be calculated accurately through horizontal components of velocity. Before introducing the numerical methods, we need perform the zero dimension process about the equations. However, we know the problem discussed is a simple form of linear problems and the boundary conditions are linear. We also know the dimensionless parameters are some constants. Therefore, to simplify the description, we still use the original equations in the following discussions (actually, they are already dimensionless equations.). In the actual calculation, these dimensionless parameters can be substituted into the simplified model.

To simplify the discussion, we may let $p_s=1$. Based on the simplified model of (22) and boundary conditions (28) and (36), we may conduct prolongation operations on stream functions of H, W and R and the three components of the given $\boldsymbol{V}$ in the forms of the following:

(1) Carrying on continuation operations along the x-axis to let functions have a period of 2π;

(2) keeping continuation operations along the y-axis to let functions have a period of $2\pi/3$;

(3) for u, v, R, conducting even function continuation operations along the p-axis as follows:

$$\varphi(x,y,p)=\begin{cases}\varphi(x,y,p), & 0<p<1\\ 0, & p=0\\ \varphi(x,y,-p), & -1<p<0\end{cases}$$

and for ω, H, W, performing the odd function prolongation operation in the following

$$\varphi(x,y,p)=\begin{cases}\varphi(x,y,p), & 0<p<1\\ 0, & p=0\\ -\varphi(x,y,-p), & -1<p<0.\end{cases}$$

Finally, we prolong the functions in the entire domain of p-axis with a period of 2.

Thus, the prolonged functions are periodic functions with respect to x, y, p in the entire set of real numbers. Then u, v, R, can be expressed in the forms of Fourier series as follows:

$$\begin{aligned}\varphi(x,y,p)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}[&\varphi_{mnk}^{(1)}\cos mx\cos 3ny\cos k\pi p\\ &+\varphi_{mnk}^{(2)}\cos mx\sin 3ny\cos k\pi p\\ &+\varphi_{mnk}^{(3)}\sin mx\cos 3ny\cos k\pi p\\ &+\varphi_{mnk}^{(4)}\sin mx\sin 3ny\cos k\pi p]. \quad (42)\end{aligned}$$

Then ω, H, W can be denoted as follows:

$$\varphi(x,y,p)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}[\varphi_{mnk}^{(1)}\cos mx\cos 3ny\sin k\pi p$$
$$+\varphi_{mnk}^{(2)}\cos mx\sin 3ny\sin k\pi p$$
$$+\varphi_{mnk}^{(3)}\sin mx\cos 3ny\sin k\pi p$$
$$+\varphi_{mnk}^{(4)}\sin mx\sin 3ny\sin k\pi p]. \quad (43)$$

The above two expressions show that as $p\to 0$, H, W, R and u, v, ω satisfy the following boundary conditions

$$\begin{cases}\dfrac{\partial u}{\partial p}=\dfrac{\partial v}{\partial p}=\omega=0,\\ \dfrac{\partial R}{\partial p}=H=W=0.\end{cases} \quad (44)$$

Substituting H, W, R and u,v into the simplified model (22) in the forms of eqs. (42) and (43), we can decide the coefficients of Fourier series of stream functions of H, W, R by making use of the orthogonality of trigonometric functions and boundary condition (44). Thus, we obtain their expressions in forms of Fourier series and the simplified numerical model in spherical coordinates. For the solution of the stream functions in the forms of Fourier series, we derive an important conclusion as follows:

Corollary: The stream functions of H, W, R determined by (42) and (43) satisfy the following three equations

$$\begin{cases}\dfrac{\partial W}{\partial p}-\dfrac{\partial R}{\partial y}=u,\\ \dfrac{\partial R}{\partial x}-\dfrac{\partial H}{\partial p}=v,\\ \dfrac{\partial H}{\partial x}+\dfrac{\partial W}{\partial y}+\dfrac{\partial R}{\partial p}=0.\end{cases} \quad (45)$$

Corollary 1 shows that for a given velocity field of u,v, expressed in the forms of Fourier series, we can obtain the velocity field of u^*, v^* by using the stream functions of H, W, R. We also find that the velocity field of u^*, v^* is identical with the original velocity field of u, v. This result is consistent with the result of theoretical analysis of (39)—(41) (analysis of calculated results is omitted here). We note that by using the numerical method, we can not only solve the numerical solution of the simplified model, but also consider the characteristics of the three-dimensional velocity field decomposition put forward in the beginning. The detailed derivation of the simplified model in the forms of Fourier series and the detailed proof of the Corollary is given in appendix 3.

In sum, if the field of u, v is expressed in form of eq. (42), then the stream functions of H, W, R can be calculated through the above expansion of Fourier series. Besides, the method has no numerical truncation error. The numerical error only exists in the Fourier series representation of u,v. However, through the discrete Fourier transform, we know discrete u,v can be expressed in the form of limited Fourier series and the series are precise at discrete grid points. This shows that the solving process of Fourier series of the simplified model has no numerical truncation error. Meanwhile the stream functions in forms of truncated Fourier series are convenient for the study of the main characteristics of atmospheric circulation. Furthermore, if we substitute equations (42) and (43) into equation (34), we obtain a method of calculating large-scale vertical motion.

6 Analysis of results

The above discussion shows that the three-dimensional decomposition model is feasible. By using the NCEP/NCAR reanalysis data[29,30], vertical velocities of ω^*, ω_H and ω_W are calculated through three-dimensional decomposition model, and then we can do a preliminary analysis of vertical movement of the global atmosphere.

Usually, the meridional mean of a field can reflect its main characteristics, therefore we first analyze the characteristics of the meridional mean distribution of vertical velocities ω^* and ω_H and their variance.

Figure 1 shows the meridional average distribution of vertical velocity ω^* bears the same basic characteristics as those of vertical velocity ω_W derived from Walker circulations. Results also display that the high centers of the mean circulations and its variance are both located at 500 hPa and we know that high values of variance reveal dramatic changes of vertical movement. These mean characteristics are consistent with those of NCEP data[31]. Another point to note is that almost all the zonal averages of the vertical velocity ω_W are smaller than or equal to 10^{-6}, and zonal averaged field is almost a zero constant field. The above results prove that ω_W represents the main features of the Walker circulations, which are obtained through the three-dimensional decomposition model.

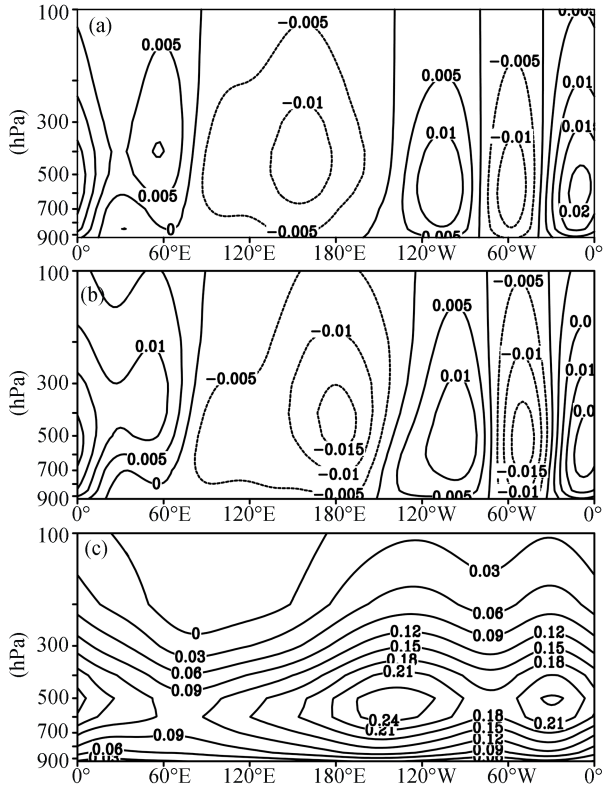

Figure 1 Meridional mean distribution of ω^*, ω_H and W (1981－2000). (a) Vertical velocity ω^*; (b) vertical velocity of Walker stream function ω_W;(c) Walker stream function W. Unit of ω^* and ω_W: Pa/s; unit of stream function W: 10^6 m·Pa/s; meridional mean refers to the average from latitude 30ºS to 30ºN.

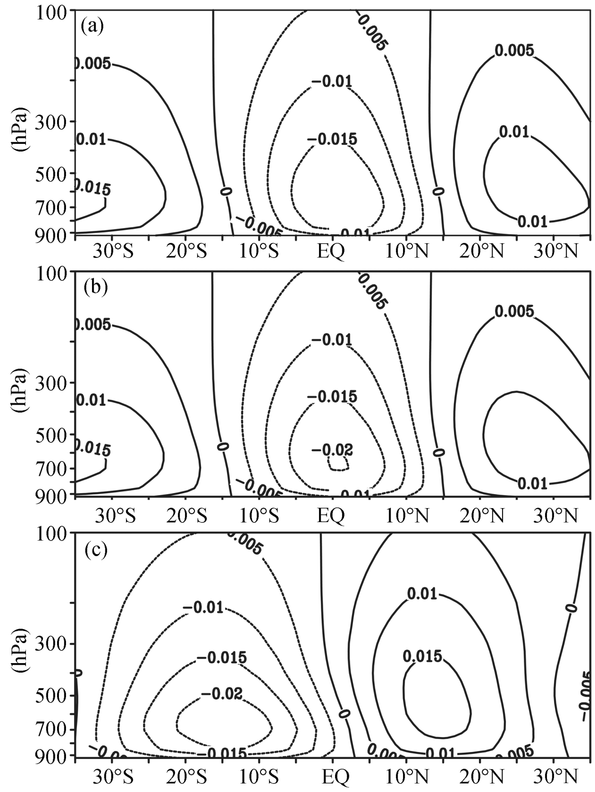

Figure 2 Zonal mean distribution of ω^*, ω_H and H (1981－2000). (a) Vertical velocity ω^*; (b) vertical velocity of Walker stream function ω_H; (c) stream function H. Unit of ω^* and ω_H: Pa/s; unit of stream function H: 10^6 m·Pa/s; zonal mean refers to the average from longitude 0º to 360º.

Furthermore, we will analyze the zonal distribution of the vertical velocity of Hadley circulations and its variance.

Figure 2 shows that the zonal average distribution of vertical velocity ω_H bears the same basic characteristics as those of vertical velocity ω^*. The air rises in the equatorial regions and sinks in mid-latitude regions. Results also show that the maximum value of ω_H is located at 700 hPa while the maximum value of its variances is located at 500 hPa. These mean characteristics are consistent with those of NCEP data[31]. Another point to note is that almost all the meridional averages of the vertical velocity ω_H are smaller than or equal to 10^{-5}, the meridional averaged field of ω_H is almost a zero constant field. The above results show that ω_H represents the main features of the Hadley circulations, which are obtained through the three-dimensional decomposition model.

From Figures 1 and 2, we clearly see the main features of global vertical movement. The three-dimensional decomposition of circulations has led to the characteristics of vertical movement and ω^* can be expressed in the form of the sum of zonal vertical velocity ω^* and meridional vertical velocity ω_H, and this is helpful to further understanding the global atmospheric circulation.

7 Summary

This study shows that the global atmospheric circulation can be expressed in the form of three-dimensional stream functions. The realization of the three-dimensional decomposition has a clear physical meaning. Furthermore, we obtain new scheme of calculating vertical movement. We get conclusions as follows:

(1) The three-dimensional decomposition model of global atmospheric circulation is put forward by using the idea of generalized three-dimensional stream functions. The existence and uniqueness of the model is proved mathematically.

(2) A simplified model of three-dimensional decom-

position is put forward and it has the main features of the three-dimensional decomposition model. The existence and uniqueness of the simplified model are also proved. Furthermore, a new numerical scheme of Fourier series is given to calculate the three-dimensional stream functions and velocity field. Through the three-dimensional decomposition model, it can be proved that the velocity field is equivalent to the stream functions in representing the main features of atmospheric motion.

(3) Practical calculation shows that the NCEP velocity field of u, v, ω is identical with the velocity field of u^*, v^*, ω^*. The above conclusion is consistent with that of theoretical analysis. A new method of calculating large-scale vertical motion is put forward. Results show that the main characteristics of vertical velocity ω^* are very close to those of ω and that ω^* can reflect the features of real atmospheric vertical motion.

(4) Results show that $\omega_{\rm H}$ and $\omega_{\rm W}$ can represent the main dynamic characteristics of vertical movements near tropic regions.

Through the three-dimensional decomposition model, we have resolved the problem of the unified description of the global circulation. The three-dimensional decomposition model is expected to become a new method for climate research. From the relevant results, we see that it can be applied to many sorts of problems and has broad prospects for application and theory study. In short, although we obtain some valuable results, we still have much work to do.

Appendix 1: Detailed proof of Theorem 1

For Problem 1 of determining solution, if $\boldsymbol{A} = \boldsymbol{i}H + \boldsymbol{j}W + \boldsymbol{k}R \in \Phi$, and A satisfies eq. (22), we obtain

$$\begin{cases} \dfrac{\partial H}{\partial p} = E_1, \ \dfrac{\partial W}{\partial p} = E_2, \ \dfrac{\partial R}{\partial p} = E_3, \\ \dfrac{\partial H}{\partial x} = F_1, \ \dfrac{\partial R}{\partial x} = F_2, \\ \dfrac{\partial W}{\partial y} = G_1, \ \dfrac{\partial R}{\partial y} = G_2. \end{cases} \tag{a1.1}$$

Substituting (a1.1)into equation (22), we obtain

$$\frac{\partial E_1}{\partial p} + \frac{\partial F_1}{\partial x} + \frac{\partial G_1}{\partial x} = -\frac{\partial v}{\partial p},$$

$$\frac{\partial E_2}{\partial p} + \frac{\partial G_1}{\partial y} + \frac{\partial F_1}{\partial y} = \frac{\partial u}{\partial p},$$

$$\frac{\partial E_2}{\partial p} + \frac{\partial G_1}{\partial y} + \frac{\partial F_1}{\partial y} = \frac{\partial u}{\partial p},$$

$$\frac{\partial E_3}{\partial p} + \frac{\partial F_2}{\partial x} + \frac{\partial G_2}{\partial y} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

From (a1.1) we obtain

$$\frac{\partial F_1}{\partial p} = \frac{\partial E_1}{\partial x}, \ \frac{\partial F_2}{\partial p} = \frac{\partial E_3}{\partial x}, \ \frac{\partial G_1}{\partial p} = \frac{\partial E_2}{\partial y}, \ \frac{\partial G_2}{\partial p} = \frac{\partial E_3}{\partial y}.$$

From the boundary condition (28), we know that as $p \to 0$

$$E_1 |_{p=0} \to 0, \ E_2 |_{p=0} \to 0, \ E_3 |_{p=0} \to 0,$$

$$F_1\big|_{p=0} = \frac{\partial H}{\partial x}\bigg|_{p=0} = \frac{\partial}{\partial x}(H|_{p=0}) \to 0, \ F_2\big|_{p=0} \to 0,$$

$$G_1\big|_{p=0} = \frac{\partial W}{\partial y}\bigg|_{p=0} = \frac{\partial}{\partial y}(W|_{p=0}) \to 0, \ G_2\big|_{p=0} \to 0.$$

Thus, Problem 1 will be expressed as the following problem of first order

$$\begin{cases} \dfrac{\partial \varphi}{\partial p} + A\dfrac{\partial \varphi}{\partial x} + B\dfrac{\partial \varphi}{\partial y} + C\varphi = f, \\ \varphi|_{p=0} \to 0, \end{cases} \tag{a1.2}$$

where φ and f are vector functions. We have

$$\varphi = (E_1, E_2, E_3, F_1, F_2, G_1, G_2, H, W, R)^T,$$

$$f = \left(-\frac{\partial v}{\partial p}, \frac{\partial u}{\partial p}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}, 0,0,0,0,0,0,0\right)^T,$$

where $\boldsymbol{A} = (a_{ij})$, $\boldsymbol{B} = (b_{ij})$, $\boldsymbol{C} = (c_{ij})$ denote sparse matrices of 10th order. For matrix $\boldsymbol{A}$, we obtain $a_{14} = a_{16} = a_{35} = 1$ and $a_{41} = a_{53} = -1$, and other elements of $\boldsymbol{A}$ equate to 0. For matrix $\boldsymbol{B}$, we obtain $b_{24} = b_{26} = b_{37} = 1$ and $b_{62} = b_{73} = -1$, and other elements of B equate to zero. For matrix $\boldsymbol{C}$, we obtain $c_{81} = c_{92} = c_{10,3} = -1$ and other elements of $\boldsymbol{C}$ equate to zero.

For Problem 2, since $\boldsymbol{A} = \boldsymbol{i}H + \boldsymbol{j}W + \boldsymbol{k}R \in \Phi$ and $\boldsymbol{A}$ satisfy eqs. (23) and (24), we have

$$\begin{cases} \dfrac{\partial H}{\partial p} = E_1, \ \dfrac{\partial W}{\partial p} = E_2, \ \dfrac{\partial R}{\partial p} = E_3, \\ \dfrac{\partial H}{\partial \lambda} = F_1, \ \dfrac{\partial W}{\partial \lambda} = F_2, \ \dfrac{\partial R}{\partial \lambda} = F_3, \\ \dfrac{\partial W}{\partial \theta} = G_1, \ \dfrac{\partial R}{\partial \theta} = G_2. \end{cases} \tag{a1.3}$$

Since $\beta \neq 0$ in eqs. (23) and (24), dividing both sides of eqs. (23) and (24) by β^2, and substituting (a1.3) into eqs. (23) and (24), we obtain

$$\frac{\partial E_1}{\partial p}+\frac{1}{a^2\beta^2\sin^2\theta}\frac{\partial F_1}{\partial \lambda}+\frac{H}{a^2\beta^2\sin^2\theta}+\frac{1}{a^2\beta^2\sin\theta}\frac{\partial F_2}{\partial\theta}-\frac{\cos\theta}{a^2\beta^2\sin^2\theta}F_2=-\frac{1}{\beta}\frac{\partial v_\theta}{\partial p},$$

$$\frac{\partial E_2}{\partial p}+\frac{1}{a^2\beta^2\sin\theta}\frac{\partial F_1}{\partial\theta}+\frac{\cos\theta}{a^2\beta^2\sin^2\theta}F_1+\frac{1}{a^2\beta^2}\frac{\partial G_1}{\partial\theta}+\frac{\cos\theta}{a^2\beta^2\sin\theta}G_1=\frac{1}{\beta}\frac{\partial v_\lambda}{\partial p},$$

$$\frac{\partial E_3}{\partial p}+\frac{1}{a^2\beta^2\sin^2\theta}\frac{\partial F_3}{\partial\lambda}+\frac{\cos\theta}{a^2\beta^2\sin\theta}G_2+\frac{1}{a^2\beta^2}\frac{\partial G_2}{\partial\theta}=\frac{1}{a\sin\theta}\frac{\partial v_\theta}{\partial\lambda}-\frac{1}{a\sin\theta}\frac{\partial(v_\lambda\sin\theta)}{\partial\theta}.$$

From (a1.3), we obtain

$$\frac{\partial F_1}{\partial p}=\frac{\partial E_1}{\partial\lambda},\quad \frac{\partial F_2}{\partial p}=\frac{\partial E_2}{\partial\lambda},$$

$$\frac{\partial F_3}{\partial p}=\frac{\partial E_3}{\partial\lambda},\quad \frac{\partial G_1}{\partial p}=\frac{\partial E_2}{\partial\theta},\quad \frac{\partial G_2}{\partial p}=\frac{\partial E_3}{\partial\theta}.$$

According to the boundary condition (28), we know that as $p\to 0$ we have

$$E_1|_{p=0}\to 0, E_2|_{p=0}\to 0, E_3|_{p=0}\to 0;$$

$$F_1|_{p=0}=\left.\frac{\partial H}{\partial\lambda}\right|_{p=0}=\frac{\partial}{\partial\lambda}(H|_{p=0})\to 0;$$

$$F_2|_{p=0}\to 0, F_3|_{p=0}\to 0;$$

$$G_1|_{p=0}=\left.\frac{\partial W}{\partial\theta}\right|_{p=0}=\frac{\partial}{\partial\theta}(W|_{p=0})\to 0\ \ G_2|_{p=0}\to 0.$$

Thus, Problem 1 and 2 can be denoted as the following problem of first order

$$\begin{cases}\dfrac{\partial\varphi}{\partial p}+A\dfrac{\partial\varphi}{\partial x}+B\dfrac{\partial\varphi}{\partial y}+C\varphi=f,\\ \varphi|_{p=0}\to 0,\end{cases}\qquad\text{(a1.4)}$$

where φ and f are vector functions. We have

$$\varphi=(E_1,E_2,E_3,F_1,F_2,F_3,G_1,G_2,H,W,R)^T,$$

$$f=\Big(-\frac{1}{\beta}\frac{\partial v_\theta}{\partial p},\frac{1}{\beta}\frac{\partial v_\lambda}{\partial p},-\frac{1}{a\beta^2\sin\theta}\frac{\partial v_\theta}{\partial\lambda}-\frac{1}{a\beta^2\sin\theta}\frac{\partial(v_\lambda\sin\theta)}{\partial\theta}\ 0,0,0,0,0,0,0,0\Big)^T,$$

where $\boldsymbol{A}=(a_{ij})$, $\boldsymbol{B}=(b_{ij})$, $\boldsymbol{C}=(c_{ij})$ denote sparse functional matrices of 11th order. And matrix $\boldsymbol{A}$ satisfies $a_{14}=a_{36}=\dfrac{1}{a^2\beta^2\sin^2\theta}$. For $\boldsymbol{A}$ we obtain $a_{41}=a_{52}=a_{63}=-1$ and other elements of $\boldsymbol{A}$ equate to 0. For matrix $\boldsymbol{B}$, we obtain $b_{15}=b_{24}=\dfrac{1}{a^2\beta^2\sin\theta}$, $b_{27}=b_{38}=\dfrac{1}{a^2\beta^2}$ and $b_{72}=b_{83}=-1$, other elements of $\boldsymbol{B}$ are 0. For matrix $\boldsymbol{C}$ we obtain $c_{15}=-\dfrac{\cos\theta}{a^2\beta^2\sin^2\theta}$, $c_{19}=-\dfrac{1}{a^2\beta^2\sin^2\theta}$, $c_{24}=\dfrac{\cos\theta}{a^2\beta^2\sin^2\theta}$, $c_{27}=c_{38}=\dfrac{\cos\theta}{a^2\beta^2\sin\theta}$ and $c_{91}=c_{10,2}=c_{11,3}=-1$ and other elements of $\boldsymbol{C}$ are 0.

Appendix 2: Detailed proof of Theorem 3

We give a solution by use of the method of separation of variables, and then we prove the existence and uniqueness of solutions. First, assuming that

$$\begin{cases}H(x,y,p)=X^H(x)\cdot Y^H(y)\cdot Z^H(p),\\ W(x,y,p)=X^W(x)\cdot Y^W(y)\cdot Z^W(p),\\ R(x,y,p)=X^R(x)\cdot Y^R(y)\cdot Z^R(p),\end{cases}\qquad\text{(a2.1)}$$

substituting (a2.1) into the third homogeneous equation of (22), we obtain

$$\begin{cases}\ddot{X}^R(x)+\alpha X^R(x)=0,\\ \ddot{Y}^R(y)+\gamma Y^R(y)=0.\end{cases}\qquad\text{(a2.2)}$$

Furthermore, substituting the third equation of eq. (a2.1) into periodic condition (36), we obtain

$$\begin{cases}X^R(0)=X^R(2\pi),\\ Y^R(\frac{\pi}{6})=Y^R(\frac{5\pi}{6}).\end{cases}\qquad\text{(a2.3)}$$

Jointing eq. (a2.2) with condition (a2.3), we obtain the following two-eigenvalue problems

$$\begin{cases}\ddot{X}^R(x)+\alpha X^R(x)=0,\\ X^R(0)=X^R(2\pi),\end{cases}\qquad\text{(a2.4)}$$

$$\begin{cases}\ddot{Y}^R(y)+\gamma Y^R(y)=0,\\ Y^R(\frac{\pi}{3})=Y^R(\frac{2\pi}{3}).\end{cases}\qquad\text{(a2.5)}$$

From eqs. (a2.4) and (a2.5), we obtain eigenvalue

$$\alpha = m^2,\ \gamma = 9n^2,\ (m, n = 0,\ 1,\ 2,\ \cdots)$$

and eigenfunction

$$X_m^R(x) = A_m^R \cos mx + B_m^R \sin mx\,,$$

$$Y_n^R(y) = A_n^R \cos 3ny + B_n^R \sin 3ny\,.$$

Therefore, we obtain the expression of stream function $R(x, y, p)$ as follows:

$$R = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[A_{mn}^R(p)\cos mx\cos 3ny + B_{mn}^R(p)\cos mx\sin 3ny + C_{mn}^R(p)\sin mx\cos 3ny + D_{mn}^R(p)\sin mx\sin 3ny]. \quad \text{(a2.6)}$$

Now we will decide $A_{mn}^R(p)$, $B_{mn}^R(p)$ $C_{mn}^R(p)$ and $D_{mn}^R(p)$ in the above expression. Substituting (a2.6) into the third equation of (22), by using the orthogonality of trigonometric functions, we easily obtain

$$\ddot{A}_{mn}^R(p) - (m^2 + 9n^2)A_{mn}^R(p) = \frac{3}{\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} h\cos mx\cos 3ny\,\mathrm{d}x\mathrm{d}y \triangleq h^A,\ \ m, n = 1, 2\cdots$$

$$\ddot{B}_{mn}^R(p) - (m^2 + 9n^2)B_{mn}^R(p) = \frac{3}{\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} h\cos mx\sin 3ny\,\mathrm{d}x\mathrm{d}y \triangleq h^B,\ \ m, n = 1, 2\cdots$$

$$\ddot{C}_{mn}^R(p) - (m^2 + 9n^2)C_{mn}^R(p) = \frac{3}{\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} h\sin mx\cos 3ny\,\mathrm{d}x\mathrm{d}y \triangleq h^C,\ \ m, n = 1, 2\cdots$$

$$\ddot{D}_{mn}^R(p) - (m^2 + 9n^2)D_{mn}^R(p) = \frac{3}{\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} h\sin mx\sin 3ny\,\mathrm{d}x\mathrm{d}y \triangleq h^D,\ \ m, n = 1, 2\cdots$$

Besides, $A_{mn}^R(p)$, $B_{mn}^R(p)$, $C_{mn}^R(p)$ and $D_{mn}^R(p)$ satisfy the second-order differential equation of the same type as follows:

$$\ddot{y} - a^2 y = f(p)\,. \quad \text{(a2.7)}$$

Before solving this equation, we must first consider the boundary conditions. By using the expression (a2.6), we obtain

$$A_{mn}^R(p) = \frac{3}{\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} R\cos mx\sin 3ny\,\mathrm{d}x\mathrm{d}y,\quad m, n = 1, 2, \cdots \quad \text{(a2.8)}$$

$$A_{0n}^R(p) = \frac{3}{2\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} R\sin 3ny\,\mathrm{d}x\mathrm{d}y,\quad n = 1, 2, \cdots \quad \text{(a2.9)}$$

$$B_{mn}^R(p) = \frac{3}{\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{5}} R\sin mx\sin 3ny\,\mathrm{d}x\mathrm{d}y,\quad m, n = 1, 2, \cdots \quad \text{(a2.10)}$$

For the above three equations, we conduct derivative operation of first order with respect to p, and we get

$$\dot{A}_{mn}^R(p) = \frac{3}{\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{\partial R}{\partial p}\cos mx\sin 3ny\,\mathrm{d}x\mathrm{d}y,\quad m, n = 1, 2, \cdots \quad \text{(a2.11)}$$

$$\dot{A}_{0n}^R(p) = \frac{3}{2\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{\partial R}{\partial p}\sin 3ny\,\mathrm{d}x\mathrm{d}y,\quad n = 1, 2, \cdots \quad \text{(a2.12)}$$

$$\dot{B}_{mn}^R(p) = \frac{3}{\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{\partial R}{\partial p}\sin mx\sin 3ny\,\mathrm{d}x\mathrm{d}y,\quad m, n = 1, 2, \cdots \quad \text{(a2.13)}$$

We conduct limit operation as $p \to 0$ on both sides of (a2.8), and assume that we can change the order of integral operation of (a2.8) with that of the limit operation of (a2.8), and substituting the boundary condition (28) into (a2.8), we obtain

$$\lim_{p\to 0} A_{mn}^R(p) = \lim_{p\to 0}\frac{2}{\pi^2}\int_0^{2\pi}\int_0^{\pi} R\cos mx\sin ny\,\mathrm{d}x\mathrm{d}y = \frac{2}{\pi^2}\int_0^{2\pi}\int_0^{\pi}(\lim_{p\to 0} R)\cos mx\sin ny\,\mathrm{d}x\mathrm{d}y = 0\,.$$

Perform the same operation for (a2.9)—(a2.13), we get

$$\lim_{p\to 0} A_{0n}^R(p) = \lim_{p\to 0} B_{mn}^R(p) = \lim_{p\to 0} \dot{A}_{mn}^R(p) = \lim_{p\to 0} \dot{A}_{0n}^R(p) = \lim_{p\to 0} \dot{B}_{mn}^R(p) = 0.$$

Thus, we can add the boundary condition to second-order differential equation (a2.7) as follows:

$$y(0) = \dot{y}(0) = 0. \quad \text{(a2.14)}$$

So we obtain general solution of the non-homogeneous eq. (a2.7) which satisfies condition (a2.14) as follows:

$$y(p) = \frac{\mathrm{e}^{ap}}{2a}\int_0^p \mathrm{e}^{-at} f(t)\mathrm{d}t - \frac{\mathrm{e}^{-ap}}{2a}\int_0^p \mathrm{e}^{at} f(t)\mathrm{d}t. \quad \text{(a2.15)}$$

Thus, for $A_{mn}^R(p)$, $B_{mn}^R(p)$, $C_{mn}^R(p)$ and $D_{mn}^R(p)$, by use of (a2.15), we obtain

$$A_{mn}^R(p)=\frac{e^{p\sqrt{m^2+9n^2}}}{2\sqrt{m^2+9n^2}}\int_0^p e^{-t\sqrt{m^2+9n^2}}h^A(t)dt-$$
$$\frac{e^{-p\sqrt{m^2+9n^2}}}{2\sqrt{m^2+9n^2}}\int_0^p e^{t\sqrt{m^2+9n^2}}h^A(t)dt,$$
$$B_{mn}^R(p)=\frac{e^{p\sqrt{m^2+9n^2}}}{2\sqrt{m^2+9n^2}}\int_0^p e^{-t\sqrt{m^2+9n^2}}h^B(t)dt-$$
$$\frac{e^{-p\sqrt{m^2+9n^2}}}{2\sqrt{m^2+9n^2}}\int_0^p e^{t\sqrt{m^2+9n^2}}h^B(t)dt,$$
$$C_{mn}^R(p)=\frac{e^{p\sqrt{m^2+9n^2}}}{2\sqrt{m^2+9n^2}}\int_0^p e^{-t\sqrt{m^2+9n^2}}h^C(t)dt-$$
$$\frac{e^{-p\sqrt{m^2+9n^2}}}{2\sqrt{m^2+9n^2}}\int_0^p e^{t\sqrt{m^2+9n^2}}h^C(t)dt,$$
$$D_{mn}^R(p)=\frac{e^{p\sqrt{m^2+9n^2}}}{2\sqrt{m^2+9n^2}}\int_0^p e^{-t\sqrt{m^2+9n^2}}h^D(t)dt-$$
$$\frac{e^{-p\sqrt{m^2+9n^2}}}{2\sqrt{m^2+9n^2}}\int_0^p e^{t\sqrt{m^2+9n^2}}h^D(t)dt.$$

Substituting the above four equations into (25), we obtain expression of stream function $R(x,y,p)$. Afterward, we manage to obtain expressions of stream function $H(x,y,p)$ and $W(x,y,p)$. Similarly, we substitute (a2.1) into the homogeneity of the first and the second equations of (22). By use of periodic condition (36), we solve problems of four eigenvalues, so we get the expressions of H and W as follows

$$H=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[A_{mn}^H(p)\cos mx\cos 3ny+B_{mn}^H(p)\cos mx\sin 3ny$$
$$+C_{mn}^H(p)\sin mx\cos 3ny+D_{mn}^H(p)\sin mx\sin 3ny], \quad \text{(a2.16)}$$
$$W=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[A_{mn}^W(p)\cos mx\cos 3ny+B_{mn}^W(p)\cos mx\sin 3ny$$
$$+C_{mn}^W(p)\sin mx\cos 3ny+D_{mn}^W(p)\sin mx\sin 3ny]. \quad \text{(a2.17)}$$

Now we determine $A_{mn}^H(p)$, $B_{mn}^H(p)$, $C_{mn}^H(p)$, $D_{mn}^H(p)$, $A_{mn}^W(p)$, $B_{mn}^W(p)$, $C_{mn}^W(p)$ and $D_{mn}^W(p)$ in the above equations. Similarly, substituting (a2.16) and (a2.17) into the first and the second equations of (22), and by using the orthogonality of trigonometric functions, we can easily obtain

$$\ddot{A}_{mn}^H(p)-m^2A_{mn}^H(p)+3mnD_{mn}^W(p)$$
$$=\frac{3}{\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} l\cos mx\cos 3ny\,dx\,dy\triangleq l^A,$$
$$\ddot{B}_{mn}^H(p)-m^2B_{mn}^H(p)-3mnC_{mn}^W(p)$$
$$=\frac{3}{\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} l\cos mx\sin 3ny\,dx\,dy\triangleq l^B,$$
$$\ddot{C}_{mn}^H(p)-m^2C_{mn}^H(p)-3mnB_{mn}^W(p)$$
$$=\frac{3}{\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} l\sin mx\cos 3ny\,dx\,dy\triangleq l^C,$$
$$\ddot{D}_{mn}^H(p)-m^2D_{mn}^H(p)+3mnA_{mn}^W(p)$$
$$=\frac{3}{\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} l\sin mx\sin 3ny\,dx\,dy\triangleq l^D,$$
$$\ddot{A}_{mn}^W(p)-9n^2A_{mn}^W(p)+3mnD_{mn}^H(p)$$
$$=\frac{3}{\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} g\cos mx\cos 3ny\,dx\,dy\triangleq g^A,$$
$$\ddot{B}_{mn}^W(p)-9n^2B_{mn}^W(p)-3mnC_{mn}^H(p)$$
$$=\frac{3}{\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} g\cos mx\sin 3ny\,dx\,dy\triangleq g^B,$$
$$\ddot{C}_{mn}^W(p)-9n^2C_{mn}^W(p)-3mnB_{mn}^H(p)$$
$$=\frac{3}{\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} g\sin mx\cos 3ny\,dx\,dy\triangleq g^C,$$
$$\ddot{D}_{mn}^W(p)-9n^2D_{mn}^W(p)+3mnA_{mn}^H(p)$$
$$=\frac{3}{\pi^2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} g\sin mx\sin 3ny\,dx\,dy\triangleq g^D.$$

If $\boldsymbol{A}=\boldsymbol{i}H+\boldsymbol{j}W+\boldsymbol{k}R\in\Phi$, then we obtain

$$\frac{\partial H}{\partial x}+\frac{\partial W}{\partial y}+\frac{\partial R}{\partial p}=0. \quad \text{(a2.18)}$$

Afterwards, substituting eqs. (a2.16) and (a2.17) which are expressions of H and W into (a2.18), by using the orthogonality of trigonometric functions we can easily obtain

$$-mA_{mn}^H(p)+3nD_{mn}^W(p)+\dot{C}_{mn}^R(p)=0,$$
$$-mB_{mn}^H(p)-3nC_{mn}^W(p)+\dot{D}_{mn}^R(p)=0,$$
$$mC_{mn}^H(p)+3nB_{mn}^W(p)+\dot{A}_{mn}^R(p)=0,$$
$$mD_{mn}^H(p)-3nA_{mn}^W(p)+\dot{B}_{mn}^R(p)=0.$$

Substituting the derivatives of first order of the above obtained $A_{mn}^R(p)$, $B_{mn}^R(p)$, $C_{mn}^R(p)$ and $D_{mn}^R(p)$ into the above equations, so we obtain

$$\ddot{A}_{mn}^H(p)=m\dot{C}_{mn}^R(p)+l^A,\quad \ddot{B}_{mn}^H(p)=m\dot{D}_{mn}^R(p)+l^B,$$

$$\ddot{C}^H_{mn}(p) = -m\dot{A}^R_{mn}(p) + l^C, \quad \ddot{D}^H_{mn}(p) = -m\dot{B}^R_{mn}(p) + l^D,$$

$$\ddot{A}^W_{mn}(p) = 3n\dot{B}^R_{mn}(p) + g^A, \quad \ddot{B}^W_{mn}(p) = -3n\dot{A}^R_{mn}(p) + g^B,$$

$$\ddot{C}^H_{mn}(p) = 3n\dot{D}^R_{mn}(p) + g^C, \ \ddot{D}^H_{mn}(p) = -3n\dot{C}^R_{mn}(p) + g^D.$$

Similarly, by using the boundary conditions (28) as $p \to 0$ we easily get

$$A^H_{mn}(p) = \int_0^p [\int_0^t [m\dot{C}^R_{mn}(s) + l^A(s)]\mathrm{d}s]\mathrm{d}t,$$

$$B^H_{mn}(p) = \int_0^p [\int_0^t [m\dot{D}^R_{mn}(s) + l^B(s)]\mathrm{d}s]\mathrm{d}t,$$

$$C^H_{mn}(p) = \int_0^p [\int_0^t [-m\dot{A}^R_{mn}(s) + l^C(s)]\mathrm{d}s]\mathrm{d}t,$$

$$D^H_{mn}(p) = \int_0^p [\int_0^t [-m\dot{B}^R_{mn}(s) + l^D(s)]\mathrm{d}s]\mathrm{d}t,$$

$$A^W_{mn}(p) = \int_0^p [\int_0^t [3n\dot{B}^R_{mn}(s) + g^A(s)]\mathrm{d}s]\mathrm{d}t,$$

$$B^W_{mn}(p) = \int_0^p [\int_0^t [-3n\dot{A}^R_{mn}(s) + g^B(s)]\mathrm{d}s]\mathrm{d}t,$$

$$C^W_{mn}(p) = \int_0^p [\int_0^t [3n\dot{D}^R_{mn}(s) + g^C(s)]\mathrm{d}s]\mathrm{d}t,$$

$$D^W_{mn}(p) = \int_0^p [\int_0^t [-3n\dot{C}^R_{mn}(s) + g^D(s)]\mathrm{d}s]\mathrm{d}t.$$

Finally, substituting the above eight equations into (a2.16)—(a2.17), which are expressions of H and W, we obtain expressions of the stream functions of $H(x,y,p)$ and $W(x,y,p)$.

Now we obtain the analytical expressions of stream functions, thus by use of the analytical expressions of stream functions, we prove the existence and uniqueness of solutions. In fact, if we assume there exist H', W', R', which are another group of solutions of Theorem 3, from the linear simplified model, we know there exist functions as follows

$H^* = H - H'$, $W^* = W - W'$, $R^* = R - R'$. These functions satisfy the homogeneous form of Theorem 3. Stream functions H', W', R' satisfy the condition when the right side of equation (22) equates to zero. Thus, from the above expressions of H, W, R we obtain $H^* = W^* = R^* \equiv 0$. This equation proves that Theorem 3 has a unique solution.

Now we discuss the existence of solutions. According to the above discussion, we have found a group of solutions in the form of a series sequence of Theorem 3 by using the method of separation of variables. Clearly, if these solutions are convergence within the definition domain ,then they will satisfy conditions of Theorem 3. Therefore, proof of existence of the solutions is equal to the proof of convergence of eqs. (a2.6), (a2.16) and (a2.17)which are expressed in the form of a series sequence. Moreover, the proof of the convergence of (a2.6), (a2.16) and (a2.17) is clear. Because they actually are the Fourier series of stream functions of H, W, R, as we know that continuous functions in form of the Fourier series are always convergent, we can prove the theorem.

Appendix 3: Derivation of the Fourier series solution of the simplified model and the detailed proof of Corollary

Firstly we denote coefficient of the expansion (42) of u, v, R as follows:

$$u_i = u^{(i)}_{mnk}, \quad v_i = v^{(i)}_{mnk}, \quad R_i = R^{(i)}_{mnk}, \quad (i = 1,2,3,4).$$

Denote coefficient of expansion (43) of H, W as follows:

$$H_i = H^{(i)}_{mnk}, \quad W_i = W^{(i)}_{mnk}, \quad (i = 1,2,3,4).$$

We can let

$$a = m, \quad b = 3n, \quad c = k\pi.$$

To ensure the uniqueness of the solution, we request Fourier series of (42) and (43) not to contain the constant term. So m,n,k in (42) and (43) equate to zero asynchronously. We let the sign $\sum_{mnk}$ denote the sum with respect to m,n,k which asynchronously equate to zero. If we calculate coefficients Fourier series, then we can obtain the expressions of H, W, R in form of Fourier series. Substituting v, H *and* W in the form of eqs. (42) and (43) into the simplified model (22), and by use of the uniqueness of Fourier expansion and the orthogonality of the trigonometry functions, we obtain the equations as follows:

$$-(a^2 + c^2)H_1 + abW_4 = cv_1,$$

$$-(a^2 + c^2)H_2 - abW_3 = cv_2,$$

$$-(a^2 + c^2)H_3 - abW_2 = cv_3,$$

$$-(a^2 + c^2)H_4 + abW_1 = cv_4,$$

$$-(b^2 + c^2)W_1 + abH_4 = -cu_1,$$

$$-(b^2 + c^2)W_2 - abH_3 = -cu_2,$$

$$-(b^2 + c^2)W_3 - abH_2 = -cu_3,$$

$$-(b^2 + c^2)W_4 + abH_1 = -cu_4,$$

$$-(a^2 + b^2 + c^2)R_1 = av_3 - bu_2,$$

$$-(a^2+b^2+c^2)R_2 = av_4 + bu_1,$$
$$-(a^2+b^2+c^2)R_1 = -av_1 - bu_4,$$
$$-(a^2+b^2+c^2)R_1 = -av_2 + bu_3.$$

From the above simultaneous equations, we obtain solutions:

(a) when $c \neq 0$ (i.e. $k \neq 0$)

$$\begin{cases} H_1 = \dfrac{1}{\Delta}[-v_1(b^2+c^2) + u_4 ab], \\ H_2 = \dfrac{1}{\Delta}[-v_2(b^2+c^2) - u_3 ab], \\ H_3 = \dfrac{1}{\Delta}[-v_3(b^2+c^2) - u_2 ab], \\ H_4 = \dfrac{1}{\Delta}[-v_4(b^2+c^2) + u_1 ab], \end{cases} \quad \text{(a3.1)}$$

$$\begin{cases} W_1 = \dfrac{1}{\Delta}[u_1(a^2+c^2) - v_4 ab], \\ W_2 = \dfrac{1}{\Delta}[u_2(a^2+c^2) + v_3 ab], \\ W_3 = \dfrac{1}{\Delta}[u_3(a^2+c^2) + v_2 ab], \\ W_4 = \dfrac{1}{\Delta}[u_4(a^2+c^2) - v_1 ab], \end{cases} \quad \text{(a3.2)}$$

$$\begin{cases} R_1 = \dfrac{c^2}{\Delta}(bu_2 - av_3), \\ R_2 = \dfrac{c^2}{\Delta}(-bu_1 - av_4), \\ R_3 = \dfrac{c^2}{\Delta}(bu_4 + av_1), \\ R_4 = \dfrac{c^2}{\Delta}(-bu_3 + av_2), \end{cases} \quad \text{(a3.3)}$$

where $\Delta = c^2(a^2+b^2+c^2)$, and because m, n, k asynchronously equate to 0, we obtain $a^2+b^2+c^2 \neq 0$.

(b) when $c = 0$ (i.e. $k = 0$)

$$\begin{cases} H_1 = H_2 = H_3 = H_4 = 0, \\ W_1 = W_2 = W_3 = W_4 = 0, \end{cases} \quad \text{(a3.4)}$$

for R_i $(i = 1,2,3,4)$, and when $a = 0$ (m=0) and $b \neq 0$ ($n \neq 0$), we obtain

$$\begin{cases} R_1 = \dfrac{1}{b}u_2,\ R_2 = -\dfrac{1}{b}u_1, \\ R_3 = \dfrac{1}{b}u_4,\ R_4 = -\dfrac{1}{b}u_3, \\ v_1 = v_2 = v_3 = v_4 = 0, \end{cases} \quad \text{(a3.5)}$$

when $a \neq 0$ ($m \neq 0$) and $b = 0$ ($n = 0$), we get

$$\begin{cases} R_1 = -\dfrac{1}{a}v_3,\ R_2 = -\dfrac{1}{a}v_4, \\ R_3 = \dfrac{1}{a}v_1,\ R_4 = \dfrac{1}{a}v_2, \\ u_1 = u_2 = u_3 = u_4 = 0, \end{cases} \quad \text{(a3.6)}$$

when $a \neq 0$ ($m \neq 0$) and $b \neq 0$ ($n \neq 0$), we obtain

$$\begin{cases} R_1 = -\dfrac{1}{a}v_3 = \dfrac{1}{b}u_2, \\ R_2 = -\dfrac{1}{a}v_4 = -\dfrac{1}{b}u_1, \\ R_3 = \dfrac{1}{a}v_1 = \dfrac{1}{b}u_4, \\ R_4 = \dfrac{1}{a}v_2 = -\dfrac{1}{b}u_3. \end{cases} \quad \text{(a3.7)}$$

Thus, we obtain the coefficients H_i, W_i, R_i (i= 1, 2, 3, 4). Finally, we obtain analytic expression of stream functions in form of Fourier series by substituting (a3.1) and (a3.7) into (42) and (43), respectively. Now we prove Corollary 1. Substituting the expressions of W, R and u into the first part of (44), we obtain

$$\sum_{mnk}[(cW_1 - bR_2)\cos mx \cos 3ny \cos k\pi p +$$
$$(cW_2 + bR_1)\cos mx \sin 3ny \cos k\pi p +$$
$$(cW_3 - bR_4)\sin mx \cos 3ny \cos k\pi p +$$
$$(cW_4 + bR_3)\sin mx \sin 3ny \cos k\pi p]$$
$$= \sum_{mnk}[u_1 \cos mx \cos 3ny \cos k\pi p$$
$$+u_2 \cos mx \sin 3ny \cos k\pi p +$$
$$u_3 \sin mx \cos 3ny \cos k\pi p + u_4 \sin mx \sin 3ny \cos k\pi p].$$

Therefore, the first part of (44) is valid only when the following four equations are valid:

$$\begin{cases} cW_1 - bR_2 = u_1, \quad cW_2 + bR_1 = u_2, \\ cW_3 - bR_4 = u_3, \quad cW_4 + bR_3 = u_4. \end{cases} \quad \text{(a3.8)}$$

When $c \neq 0$ ($k \neq 0$), substituting (a3.5) and (a3.6) into (a3.8), we obtain

$$cW_1 - bR_2 = \frac{c}{\Delta}[u_1(a^2+c^2) - v_4 ab] - b\frac{c^2}{\Delta}[-bu_1 - av_4] = u_1,$$

$$cW_2 + bR_1 = \frac{c}{\Delta}[u_2(a^2+c^2) + v_3 ab] + b\frac{c^2}{\Delta}[bu_2 - av_3] = u_2,$$

$$cW_3 - bR_4 = \frac{c}{\Delta}[u_3(a^2+c^2)+v_2ab] - b\frac{c^2}{\Delta}(-bu_3+av_2) = u_3,$$

$$cW_4 + bR_3 = \frac{c}{\Delta}[u_4(a^2+c^2)-v_1ab] + b\frac{c^2}{\Delta}(bu_4+av_1) = u_4,$$

where $c=0$ ($k=0$). From (a3.4) and (a3.7), we know (a3.8) is valid. Similarly, we can prove that the second and third equations of (45) are valid.

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