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Statistical distribution of nonlinear random wave height

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Abstract A statistical model of random wave is developed using Stokes wave theory of water wave dynamics. A new nonlinear probability distribution function of wave height is presented. The results indicate that wave steepness not only could be a parameter of the distribution function of wave height but also could reflect the degree of wave height distribution deviation from the Rayleigh distribution. The new wave height distribution overcomes the problem of Rayleigh distribution that the prediction of big wave is overestimated and the general wave is underestimated. The prediction of small probability wave height value of new distribution is also smaller than that of Rayleigh distribution. Wave height data taken from East China Normal University are used to verify the new distribution. The results indicate that the new distribution fits the measurements much better than the Rayleigh distribution.

Keywords: Random ocean wave, nonlinearity, wave height distribution.

1 Introduction

The probability distribution of ocean wave element is one of the results of application of random process theory to the ocean wave study. A great amount of outcome has obtained in this field[1,2]. In fact, a great deal of research on linear ocean wave has been done and great benefit has been acquired in the application. However, linear ocean wave only is approximate to the ocean wave process and its resolution could not meet the requirement of application such as remote sensing. In fact, the ocean wave is more like a nonlinear process. Hence, a great amount of work has been carried out to develop proper expressions for the ocean wave[3−6]. However, these results could not easily apply to the ocean wave probability theoretical research and statistical analysis because some of them could not reflect the physical process using their complex expressions, or some of them could not calculate their important parameter. Therefore, it is still significant work to seek simple probability distribution formula to reflect the nonlinear character of ocean wave.

The author has developed a random Stokes wave model[7] to reflect the nonlinear character of ocean wave-the asymmetry of wave crest and wave trough and obtained a non-Gaussian distribution. The formula has explicit formula and the nonlinear control parameter is the wave steepness which is physically significant. Applying the model to the statistical analysis of TOPEX altimeter data, the result is satisfying[8]. The wave height distribution of nonlinear random ocean wave is obtained in this paper based on the model.

2 Formulation

Assuming that the water wave surface elevation $\zeta(t)$ is expressed as a sum of countless independent random variables

$$\zeta(t)=\sum_{n=1}^{\infty}\zeta_n(t)=\sum_{n=1}^{\infty}a_n(\omega_n t+\varepsilon_n), \tag{1}$$

where t is the time, a is the amplitude, ω is the angular frequency and ε is the random phase uniformly distributed over the interval $(0,\ 2\pi)$[1]. From the central limit theorem, the random wave surface obeys the normal distribution.

Another assumption is that the water wave spectrum is the narrowband spectrum. Therefore, the wave surface elevation $\zeta(t)$ can be expressed as

$$\zeta(t)=\mathrm{Re}\,a(t)e^{\mathrm{i}(\omega t+\varepsilon(t))}. \tag{2}$$

Assuming $\zeta(t)$ obeys the normal distribution, we could obtain that the amplitude and the phase obey the Rayleigh distribution and the equality distribution, respectively[9].

On the other hand, the wave motion obeys the dynamic principles, and is governed by the dynamic equations and initial conditions[2]. After separating variables, the evolution equation of the linear gravity wave is

$$\frac{d^2\zeta(t)}{dt^2}+\omega^2\zeta(t)=0. \tag{3}$$

Using the approximation that both amplitude and phase vary slowly with time, eq. (3) also could be derived from eq. (2). Therefore, given the proper random initial conditions, such as

$$\begin{aligned}&\zeta(0)=A\cos\varphi,\\&\zeta'(0)=-A\sin\varphi,\end{aligned} \tag{4}$$

where A and φ denote random quantity which obey the Rayleigh distribution and the equality distribution, respectively. Thus from the probability theory $\zeta(t)$ must be normal process.

The above analyses indicate that treating the water waves as linear waves, the wave surface elevation model can be derived from differential equations with random initial conditions. This also implies that using hydrodynamic method to find the solution of linear gravity waves, and treating the wave field parameters as random variables, the linear statistical model of the water waves can be derived from the dynamic method. Using this method, the dynamical results of nonlinear gravity waves can be introduced to the random wave statistical distribution.

It is well known that the Stokes wave theory is a classic research result in nonlinear gravity wave field[10]. The theory reflects the asymmetry of wave crest and wave trough: the wave trough is slightly flat and the wave crest is slightly steep. It is more obvious when the nonlinearity increasing. This is an important character of ocean wave.

Similar to the analysis of linear ocean wave and based on the Stokes wave theory the author[7] pointed out that the variable wave surface at a fixed point can be written as

$$\zeta(t)=ae^{\delta\zeta(t)}\cos(\omega t+\varepsilon), \tag{5}$$

where $\zeta(t)$ and a are the wave surface and amplitude, respectively which has been non-dimensionalised by the wave surface covariance σ. ε is the random phase and $\delta=\sigma k$. k is proportional to the reciprocal of wavelength, the relation between k and ω is decided by the dispersive relation. In statistics ω could be looked as some mean frequency which is corresponding to the frequency of crossing-zero period or is computed by spectral moment[2]. Thus k is inverse proportional to mean wavelength. σ is proportional to the characteristic wave height. Therefore, the physical meaning of δ is the characteristic wave steepness.

Eq. (5) is developed from Eq. (2) in the nonlinear wave field. Nonlinear wave surface probability distribution could be derived from Eq. (5)[7]. Because the Stokes wave is a result of water wave dynamics, and δ is a parameter of nonlinear water wave dynamics, Eq. (5) is a product combining the wave dynamics and statistics.

We adopt zero-moment approximation of δ in Eq. (5), let $\omega t=\tau$ and combine the initial condition Eq. (4), the random initial equation of linear wave could be written as

$$L:\delta=0$$
$$\begin{cases}\zeta(t)=a\cos(\tau+\varepsilon),\\ \zeta(0)=A\cos\varphi,\\ \zeta'(0)=-A\sin\varphi.\end{cases} \tag{6}$$

It is easy to obtain that a=A and ε=φ: the amplitude obeys Rayleigh distribution in linear wave and the phrase obeys equality distribution. Thus eq. (5) includes the result of traditional wave model.

The important significance of the analysis above is that when $\delta \neq 0$ the random initial problem of nonlinear wave could be written as

$$N:\delta \neq 0$$

$$\begin{cases} \zeta = a\cos(\tau+\varepsilon) + \frac{1}{2}\delta a^2 + \frac{1}{2}\delta a^2\cos 2(\tau+\varepsilon) \\ \quad + \frac{3}{8}\delta^2 a^3 \cos 3(\tau+\varepsilon), \\ \zeta(0) = A\cos\varphi, \\ \zeta'(0) = -A\sin\varphi. \end{cases} \tag{7}$$

In order to compute conveniently the first formula in eq. (7) has been approximate to δ^2 order that is three order Stokes wave.

The amplitude or wave height distribution of nonlinear wave could be derived from eq. (7).

3 Statistical distribution of nonlinear random wave height

To develop the distribution of nonlinear wave amplitude a the function which includes δ as a parameter should be built using eq. (7):

$$\begin{cases} A = A(a,\varepsilon,\delta) \\ \varphi = \varphi(a,\varepsilon,\delta) \end{cases} \text{or} \begin{cases} a = a(A,\varphi,\delta) \\ \varepsilon = \varepsilon(A,\varphi,\delta). \end{cases} \tag{8}$$

In fact, we could get

$$\begin{cases} A = a + \frac{\delta a^2}{4}(5\cos\varepsilon - \cos 3\varepsilon) \\ \quad + \frac{\delta^2 a^3}{32}(2 + 25\cos 2\varepsilon - 14\cos 4\varepsilon - \cos 6\varepsilon), \\ \varphi = \varepsilon + \frac{\delta a}{4}(\sin\varepsilon + \sin 3\varepsilon) \\ \quad + \frac{\delta^2 a^2}{64}(5\sin 2\varepsilon + 8\sin 4\varepsilon + 4\sin 6\varepsilon), \end{cases} \tag{9}$$

and

$$\begin{cases} a = A - \frac{\delta A^2}{4}(5\cos\varphi - \cos 3\varphi) \\ \quad - \frac{\delta^2 A^2}{32}(2 + 25\cos 2\varphi - 14\cos 4\varphi - \cos 6\varphi), \\ \varepsilon = \varphi - \frac{\delta A}{4}(\sin\varphi + \sin 3\varphi) \\ \quad - \frac{\delta^2 A^2}{64}(5\sin 2\varphi + 8\sin 4\varphi + 4\sin 6\varphi). \end{cases} \tag{10}$$

Because only the distribution of a is discussed in this paper, the equality distribution φ could be averaged in [0, 2π] in eqs. (9) and (10), then get

$$\begin{cases} A = a\left[1 + \lambda a^2\right], \\ a = A\left[1 - \lambda A^2\right], \end{cases} \tag{11}$$

where $\lambda = \frac{1}{8}\delta^2$, because A obeys Rayleigh distribution: $f(A) = Ae^{-\frac{A^2}{2}}$, thus the distribution of a could be written as

$$\begin{aligned} f(a) &= f(A)\left|\frac{dA}{da}\right| \\ &= a\left[1 + \lambda a^2\right]\left[1 + 3\lambda a^2\right]e^{-\frac{1}{2}a^2\left(1+\lambda a^2\right)^2}. \end{aligned} \tag{12}$$

Eq. (12) is the amplitude distribution of nonlinear wave developed in this paper. Fig. 1 shows the amplitude distribution change with wave steepness. The results just overcome the problem of Rayleigh distribution that the prediction of big wave is overestimated and the general wave is underestimated. The problem of Rayleigh distribution is the shortcoming of the probability prediction of wave height. This has been put great attention and managed to improve. In this paper the nonlinear control parameter is introduced and enriches the research of this field, furthermore, it is promising in the future application.

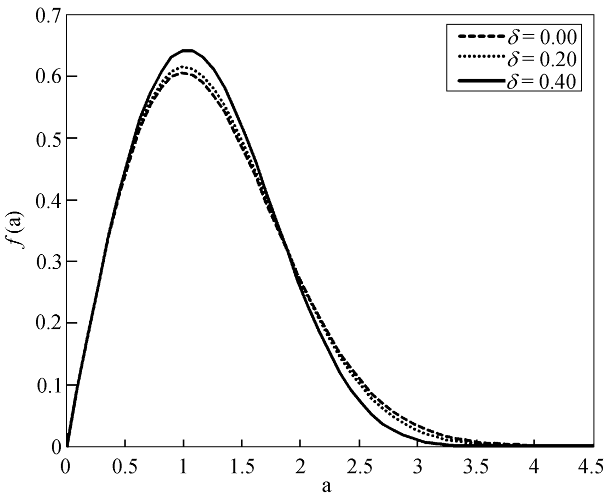

Fig. 1. The amplitude distribution change with wave steepness (δ =0 is Rayleigh distribution).

Finally, the dimensional equation is written as

$$f\left(a\right)=\frac{a}{\sigma^2}\left[1+\frac{\lambda}{\sigma^2}a^2\right]\left[1+\frac{3\lambda}{\sigma^2}a^2\right]e^{-\frac{a^2}{2\sigma^2}\left(1+\frac{\lambda}{\sigma^2}a^2\right)^2}, \quad (13)$$

where σ^2 is the zero order moment of spectrum that is the total energy of ocean wave[2] is a pivotal parameter of wave probability prediction. σ could be written as the function of mean amplitude and root mean square (rms) of amplitude just as the equations above.

As the analysis of nonlinear wave, twice of amplitude (a) is wave height (H), the distribution is obtained from eq. (13):

$$f\left(H\right)=\frac{H}{4\sigma^2}\left[1+\frac{\lambda}{4\sigma^2}H^2\right] \cdot \left[1+\frac{3\lambda}{4\sigma^2}H^2\right]e^{-\frac{H^2}{8\sigma^2}\left(1+\frac{\lambda}{4\sigma^2}H^2\right)^2}. \quad (14)$$

$f(H)$ is integrated from H to ∞ and gets the function of exceeding probability:

$$F\left(H\right)=\int_H^\infty f\left(H\right)=e^{-\frac{H^2}{8\sigma^2}\left(1+\frac{\lambda}{4\sigma^2}H^2\right)^2}. \quad (15)$$

Fig. 2 shows the exceeding probability ($f(H)$) change with wave steepness. It is obvious that the result of new distribution is better than that of Rayleigh. The wave height value of small probability such as the swell wave height is put more attention in application. Because the Rayleigh distribution has the shortcoming that the wave height value of small probability is overestimated many authors try to use Weibull distribution[11]. However, the Weibull distribution is a truely empirical function and the pivotal parameter of Weibull is obtained by fitting the data[11]. The new wave height distribution in this paper is obtained by combination of the theory of statistics and dynamics. The influence of wave nonlinear is reflected by the parameter of wave steepness. Thus the new distribution is significant both in theory and in practice.

4 Verification of nonlinear wave height distribution

The data used to verified the new wave height distribution (NWHD) and the new exceeding probability distribution (NEPD) is provided by State Key Laboratory of Estuarine and Coastal research, East China Normal University[12]. The verification result is expressed as rms error and relative error. The rms error and relative error are $R=\sqrt{\sum_{i=1}^{n}(p_i-s_i)^2/n}$ and $X=\sum_{i=1,}^{n}\left(\frac{\left|p_i-s_i\right|}{s_i}\right)$, respectively, where R is the rms error, X is relative error, s_i is the measurement data and n is the number of measurement.

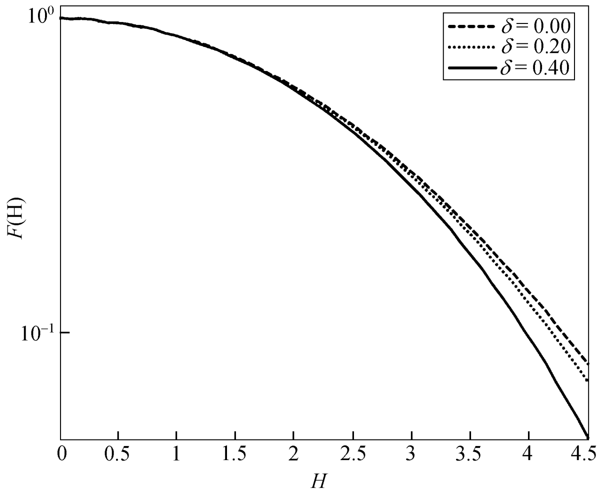

Fig. 2. The exceeding probability change with wave steepness (δ=0 is Rayleigh distribution).

The nonlinear control parameter δ is decided by the element of characteristic wave. Two kinds of element of characteristic wave are chosen to compute δ. One is δ_1 which calculated from mean period of cross-zero period ($\overline{T}$) and mean height; another is δ_2 which is calculated from mean period $T_{0,2}$ by zero-order spectrum and second-order spectrum and mean height H_0 by zero-order spectrum.

The measurement data of East China Normal University collected from September 21, 1999 to September 26, 1999 during 9912 Typhoon and located at Niupi Reef of Changjiang estuary (31°08′ North and 122°15′ East). The raw data were collected as 2,048 18-min records of surface elevation measurement every three hours. A representative measurement results were selected and Fig. 3 shows the results of statistical analysis. The solid line is NWHD with δ_1=0.36, the dashdotted line is NWHD with δ_2=0.38, the dotted line is Rayleigh distribution, the dashed line is Weibull distribution (α =2.03, β=7.95), and the star is the statistical results of measurement every 0.2 meter. The rms error and relative error between every kind of distribution and measurement are listed in Table 1. The rms error of NWHD δ_1 reduces 23.7% as against of Rayleigh distribution and reduces 14.6% as against that of Weibull distribution. The rms error of NWHD δ_2 reduces 23.5% as against that of Rayleigh distribution and reduces 14.3% as against that of Weibull distribution. The relative error of NWHD δ_1 reduces 17.8% as against that of Rayleigh distribution and reduces 9.4% as against that of Weibull distribution. The relative error of NWHD δ_2 reduces 17.4% as against that of Rayleigh distribution and reduces 8.9% as against that of Weibull distribution. The NWHD δ_1 fits the measurement best.

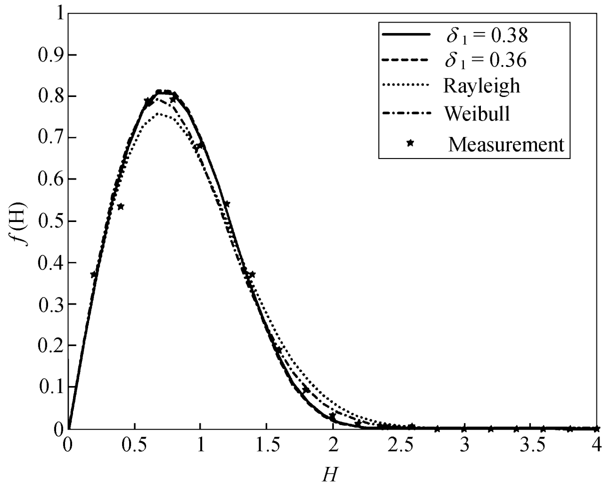

Fig. 3. Comparison between the distribution of nonlinear wave height and measurement.

Fig. 4 shows the comparison between each exceeding probability of nonlinear wave height distribution and measurement. The solid line is new exceeding probability distribution with δ_1=0.36, the dashdotted line is new exceeding probability distribution with δ_2=0.38, the dotted line is the exceeding probability of Rayleigh distribution, the dashed line is the exceeding probability Weibull distribution. The star is the exceeding probability of measurement. Table 1 also shows the rms error and relative error of exceeding probability between each distribution and measurement. The rms error of NEPD δ_1 reduces 52.1% as against that of Rayleigh distribution and reduces 17.9% as against that of Weibull distribution. The rms-error of NEPD δ_2 reduces 51.0% as against that of Rayleigh distribution and reduces 16% as against that of Weibull distribution. The relative error of NEPD δ_1 reduces 52.5% as against that of Rayleigh distribution and reduces 36.3% as against that of Weibull distribution. The relative error of NEPD δ_2 reduces 49.7% as against that of Rayleigh distribution and reduces 32.6% as against that of Weibull distribution. The NEPD δ_1 fits the measurement best.

Table 1 The rms error and relative error between every kind of distribution and measurement

	Wave height distribution		Exceeding	
	rms error	relative error	rms error	relative error
Rayleigh	2.989×10^{-2}	1.789	9.013×10^{-3}	6.539
Weibull	2.670×10^{-2}	1.623	5.254×10^{-3}	4.875
NWHD δ_1	2.281×10^{-2}	1.471	4.316×10^{-3}	3.107
NWHD δ_2	2.287×10^{-2}	1.478	4.415×10^{-3}	3.288

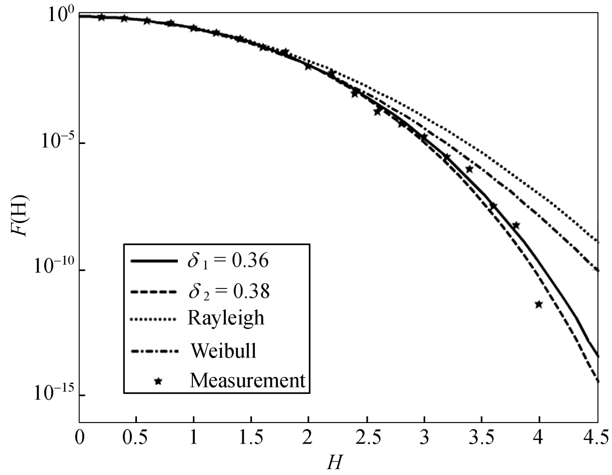

Fig. 4. Comparison between exceeding probability of nonlinear wave height and measurement.

From the verification above it is obvious that the NWHD and NEPD are better than Rayleigh distribution and Weibull distribution. Furthermore, the result of δ_1 is a little better than that of δ_2.

5 Conclusions

Applying the Stokes wave theory of hydrodynamic results to random water wave study, a new wave height distribution is derived in this paper. From the results, one can see that the wave steepness is an important parameter not only in dynamics but also in statistics. The value of wave steepness reflects the degree of wave height distribution skew from the Rayleigh distribution, and also describes variety of distribution function. Comparisons between results derived from the new distribution and the measurement data show two advantages: one is that the new wave height distribution solves the problem of Rayleigh distribution that the prediction of big wave is overestimated and the general wave is underestimated. Another is that the wave height value of small probability is lower than that of Rayleigh distribution. These results will work well in application.

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