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Non-weight modules over the algebra $\mathcal{SW}(b)$

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Abstract In this paper, we denote the semi-direct product of the Witt algebra and the loop Schrödinger algebra by $\mathcal{SW}(b)$, where b belongs to $\mathbb{C}$. Our primary focus is on classifying $U(\mathbb{C}d_0 \oplus \mathbb{C}h_0)$-free modules of rank 1 over $\mathcal{SW}(b)$. We characterize both the irreducibility and isomorphism classes of these modules. Furthermore, we construct new non-weight modules over $\mathcal{SW}(0)$ by taking the tensor product of $U(\mathbb{C}d_0 \oplus \mathbb{C}h_0)$-free modules with irreducible highest weight modules. We also consider the irreducibility and isomorphism classes for the tensor product modules. Finally, we reformulate some tensor product modules over $\mathcal{SW}(0)$ as induced modules derived from modules over certain subalgebras.

Keywords Witt algebra, non-weight modules, $U(\mathfrak{h})$-free modules, tensor product, Schrödinger algebra

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1 Introduction

Throughout this paper, $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{Z}$, and $\mathbb{Z}^*$ denote the sets of complex numbers, non-zero complex numbers, integers, and non-zero integers, respectively. Let $\mathbb{Z}_{\geqslant k}$ be the set of all the integers i satisfying $i \geqslant k$. For a given Lie algebra $\mathfrak{g}$, its universal enveloping algebra is denoted by $U(\mathfrak{g})$. It should be noted that all the algebras, vector spaces, and modules discussed in this paper are assumed to be over $\mathbb{C}$.

Let $A = \mathbb{C}[t, t^{-1}]$. The Witt algebra $\mathcal{W} = \mathrm{Der}\mathbb{C}[t, t^{-1}]$ is an infinite-dimensional Lie algebra with a basis $\{d_n = t^{n+1}\frac{d}{dt} \mid n \in \mathbb{Z}\}$ and satisfies the commutation relation

$$[d_m, d_n] = (n-m)d_{m+n}, \quad \forall\, m, n \in \mathbb{Z}. \tag{1.1}$$

$\mathcal{W}$ serves as a classical research object, finding extensive applications across various physics domains and mathematical branches (see [2, 12, 15, 20, 25] and the references therein).

Let $\mathcal{S}$ denote the Schrödinger algebra with the basis $\{f, q, h, z, p, e\}$ and non-trivial commutation relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h,$$
$$[h, p] = p, \quad [h, q] = -q, \quad [p, q] = z,$$

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$$[e,q]=p,\quad [p,f]=-q.$$

From [24], $\mathcal{S}$ admits a diagonalizable derivation d, i.e.,

$$d(h)=d(e)=d(f)=0,\quad d(z)=2z,\quad d(p)=p,\quad d(q)=q.$$

Recall that the loop Schrödinger algebra $\mathcal{S}\otimes A$ is defined by the following commutation relation:

$$[x\otimes t^i, y\otimes t^j]=[x,y]\otimes t^{i+j},\quad \forall\, i,j\in\mathbb{Z},\quad x,y\in\mathcal{S}. \tag{1.2}$$

In this paper, we briefly write $x_i=x\otimes t^i$ for all $i\in\mathbb{Z}$ and $x\in\mathcal{S}$.

For any $b\in\mathbb{C}$, the Lie algebra $\mathcal{SW}(b)=\mathcal{W}\ltimes(\mathcal{S}\otimes A)$, associated with $\mathcal{S}$, is defined by the relations (1.1)–(1.2) and

$$\begin{aligned}&[d_i,h_j]=jh_{i+j},\quad [d_i,e_j]=je_{i+j},\quad [d_i,f_j]=jf_{i+j},\\ &[d_i,z_j]=(j+2bi)z_{i+j},\quad [d_i,p_j]=(j+bi)p_{i+j},\quad [d_i,q_j]=(j+bi)q_{i+j},\end{aligned}$$

where $i,j\in\mathbb{Z}$. This algebra has been previously discussed in [3, 21] and includes several notable subalgebras. For example,

$$\mathcal{H}=\operatorname{span}_{\mathbb{C}}\{d_i,h_i\mid i\in\mathbb{Z}\}\cong\operatorname{span}_{\mathbb{C}}\{d_i,f_i\mid i\in\mathbb{Z}\}\cong\operatorname{span}_{\mathbb{C}}\{d_i,e_i\mid i\in\mathbb{Z}\}$$

is the Heisenberg-Virasoro algebra with the one-dimensional center (see [5, 7]),

$$\mathcal{L}_b=\operatorname{span}_{\mathbb{C}}\{d_i,z_i\mid i\in\mathbb{Z}\}\cong\operatorname{span}_{\mathbb{C}}\{d_i,q_i\mid i\in\mathbb{Z}\}\cong\operatorname{span}_{\mathbb{C}}\{d_i,p_i\mid i\in\mathbb{Z}\}$$

is the centerless Ovsienko-Roger algebra (see [19]), and $\mathcal{A}=\operatorname{span}_{\mathbb{C}}\{d_i,f_i,h_i,e_i\mid i\in\mathbb{Z}\}$ is the centerless affine-Virasoro algebra of type A_1 (see [14]). The weight module theory over $\mathcal{SW}(b)$ is a significant topic in its representation theory, as it was explored in [3]. This study classified all the irreducible Harish-Chandra modules over this algebra.

In recent years, the exploration of non-weight modules has attracted increasing attention from mathematicians. A class of non-weight modules on which the Cartan subalgebra $\mathfrak{h}$ acts freely has been constructed and studied. These modules are called $U(\mathfrak{h})$-free modules. This concept was initially introduced by Nilsson [22] for the simple Lie algebra $\mathfrak{sl}_{n+1}$. Since then, numerous researchers have constructed $U(\mathfrak{h})$-free modules for various Lie algebras (see [4, 6–8, 13, 17]). Furthermore, considering the tensor product of $U(\mathfrak{h})$-free modules with known irreducible modules is an effective approach to constructing new non-weight modules and studying original modules (see [5, 13, 16, 23]). Additionally, $U(\mathbb{C}h_0)$-free modules of rank 1 over the Schrödinger algebra have been considered in [8]. $U(\mathbb{C}d_0\oplus\mathbb{C}h_0)$-free modules of rank 1 and the modules obtained by taking the tensor product of $U(\mathbb{C}d_0\oplus\mathbb{C}h_0)$-free modules with irreducible highest weight modules over $\mathcal{A}$ were explored in [9–11]. These findings inspire our investigation into non-weight modules over $\mathcal{SW}(b)$, including $U(\mathbb{C}d_0\oplus\mathbb{C}h_0)$-free modules of rank 1 and the tensor product modules in this paper.

The rest of this paper is organized as follows. In Section 2, we recall several established results and subsequently construct a class of non-weight modules over $\mathcal{SW}(b)$. In Section 3, we classify all the $\mathcal{SW}(b)$-module structures whose restriction to $U(\mathbb{C}d_0\oplus\mathbb{C}h_0)$ is free of rank 1. We further investigate the properties of these modules, including the irreducibility and isomorphism classes. In Section 4, we construct modules over $\mathcal{SW}(0)$ by taking the tensor product of modules constructed in Section 2 with irreducible highest weight modules. Concurrently, we also consider the irreducibility and isomorphism classes for these tensor product modules. Additionally, in Section 5, we reformulate some certain tensor product modules as induced modules from modules of certain subalgebras over $\mathcal{SW}(0)$.

2 Preliminaries

In this section, we first recall some known results which will be used later. Then, we construct a class of non-weight modules over $\mathcal{SW}(b)$ inspired by [8, 9].

It is clear that $\mathcal{SW}(b)$ has a triangular decomposition

$$\mathcal{SW}(b) = \mathcal{SW}(b)_- \oplus \mathcal{SW}(b)_0 \oplus \mathcal{SW}(b)_+,$$

where

$$\begin{aligned}\mathcal{SW}(b)_- &= \mathrm{span}_{\mathbb{C}}\{f_{-i}, q_{-i}, h_{-i}, z_{-i}, p_{-i}, e_{-i}, d_{-i}, q_0, f_0 \mid i \in \mathbb{Z}_{\geqslant 1}\},\\ \mathcal{SW}(b)_0 &= \mathrm{span}_{\mathbb{C}}\{h_0, z_0, d_0\},\end{aligned}$$

and

$$\mathcal{SW}(b)_+ = \mathrm{span}_{\mathbb{C}}\{f_i, q_i, h_i, z_i, p_i, e_i, d_i, p_0, e_0 \mid i \in \mathbb{Z}_{\geqslant 1}\}.$$

Note that $\mathfrak{h} = \mathbb{C}d_0 \oplus \mathbb{C}h_0$ is the Cartan subalgebra. Furthermore, the center of this algebra is

$$Z(\mathcal{SW}(b)) = \begin{cases}\mathbb{C}z_0, & \text{if } b = 0,\\ 0, & \text{otherwise.}\end{cases}$$

In [9], a class of non-weight modules over $\mathcal{A} = \mathrm{span}_{\mathbb{C}}\{d_i, f_i, h_i, e_i \mid i \in \mathbb{Z}\}$ was constructed. We recall such modules and their properties in the subsequent theorem and proposition, respectively.

Theorem 2.1 (See [9]). *For $\lambda, \alpha \in \mathbb{C}^*$, $\beta, \gamma \in \mathbb{C}$, and $i \in \mathbb{Z}$, any $U(\mathcal{A})$-module such that its restriction to $U(\mathbb{C}d_0 \oplus \mathbb{C}h_0)$ is free of rank 1 is isomorphic to one of the following modules*:

$$\Omega(\lambda, \alpha, \beta, \gamma), \quad \Delta(\lambda, \alpha, \beta, \gamma), \quad \Theta(\lambda, \alpha, \beta, \gamma),$$

whose module structures are given as follows:

$$\begin{aligned}
\Omega(\lambda, \alpha, \beta, \gamma):\ & f_i \cdot g(d_0, h_0) = -\frac{\lambda^i}{\alpha}\left(\frac{h_0}{2} - \beta\right)\left(\frac{h_0}{2} + \beta + 1\right)g(d_0 - i, h_0 + 2),\\
& h_i \cdot g(d_0, h_0) = \lambda^i h_0 g(d_0 - i, h_0), \quad e_i \cdot g(d_0, h_0) = \lambda^i \alpha g(d_0 - i, h_0 - 2),\\
& d_i \cdot g(d_0, h_0) = \lambda^i (d_0 + i\gamma) g(d_0 - i, h_0),\\
\Delta(\lambda, \alpha, \beta, \gamma):\ & f_i \cdot g(d_0, h_0) = \lambda^i \alpha g(d_0 - i, h_0 + 2), \quad h_i \cdot g(d_0, h_0) = \lambda^i h_0 g(d_0 - i, h_0),\\
& e_i \cdot g(d_0, h_0) = -\frac{\lambda^i}{\alpha}\left(\frac{h_0}{2} + \beta\right)\left(\frac{h_0}{2} - \beta - 1\right)g(d_0 - i, h_0 - 2),\\
& d_i \cdot g(d_0, h_0) = \lambda^i (d_0 + i\gamma) g(d_0 - i, h_0),\\
\Theta(\lambda, \alpha, \beta, \gamma):\ & f_i \cdot g(d_0, h_0) = -\frac{\lambda^i}{\alpha}\left(\frac{h_0}{2} - \beta\right)g(d_0 - i, h_0 + 2),\\
& h_i \cdot g(d_0, h_0) = \lambda^i h_0 g(d_0 - i, h_0),\\
& e_i \cdot g(d_0, h_0) = \lambda^i \alpha\left(\frac{h_0}{2} + \beta\right)g(d_0 - i, h_0 - 2),\\
& d_i \cdot g(d_0, h_0) = \lambda^i (d_0 + i\gamma) g(d_0 - i, h_0).
\end{aligned}$$

Proposition 2.2 (See [9]). *Let $\lambda, \lambda_1, \alpha, \alpha_1 \in \mathbb{C}^*$ and $\beta, \beta_1, \gamma, \gamma_1 \in \mathbb{C}$. Then, as $\mathcal{A}$-modules,*

(1) *$\Omega(\lambda, \alpha, \beta, \gamma)$ and $\Delta(\lambda, \alpha, \beta, \gamma)$ are irreducible and $\Theta(\lambda, \alpha, \beta, \gamma)$ is irreducible if and only if $2\beta \notin \mathbb{Z}_{\geqslant 0}$*;

(2) *$\Omega(\lambda, \alpha, \beta, \gamma)$, $\Delta(\lambda, \alpha, \beta, \gamma)$, and $\Theta(\lambda, \alpha, \beta, \gamma)$ are pairwise non-isomorphic. Moreover,*

$$\begin{aligned}
&\Omega(\lambda, \alpha, \beta, \gamma) \cong \Omega(\lambda_1, \alpha_1, \beta_1, \gamma_1) \Leftrightarrow (\lambda, \alpha, \beta, \gamma) = (\lambda_1, \alpha_1, \beta_1, \gamma_1) \ \textit{or}\ (\lambda, \alpha, \beta, \gamma) = (\lambda_1, \alpha_1, -\beta_1 - 1, \gamma_1),\\
&\Delta(\lambda, \alpha, \beta, \gamma) \cong \Delta(\lambda_1, \alpha_1, \beta_1, \gamma_1) \Leftrightarrow (\lambda, \alpha, \beta, \gamma) = (\lambda_1, \alpha_1, \beta_1, \gamma_1) \ \textit{or}\ (\lambda, \alpha, \beta, \gamma) = (\lambda_1, \alpha_1, -\beta_1 - 1, \gamma_1),\\
&\Theta(\lambda, \alpha, \beta, \gamma) \cong \Theta(\lambda_1, \alpha_1, \beta_1, \gamma_1) \Leftrightarrow (\lambda, \alpha, \beta, \gamma) = (\lambda_1, \alpha_1, \beta_1, \gamma_1).
\end{aligned}$$

Let $\mathcal{D}$ be the subalgebra of $\mathcal{SW}(b)$ spanned by $\{q_i, z_i, p_i \mid i \in \mathbb{Z}\}$. It follows naturally that the aforementioned $\mathcal{A}$-modules can be extended to $\mathcal{SW}(b)$-modules.

Definition 2.3. Let $\mathbb{C}[s,t]$ denote the polynomial algebra in variables s and t. For any $g(s,t)\in\mathbb{C}[s,t]$ and $\lambda,\alpha,\mu\in\mathbb{C}^*$, $\beta,\gamma\in\mathbb{C}$, and $i\in\mathbb{Z}$, define the action of $\mathcal{SW}(b)$ on $\mathbb{C}[s,t]$ as follows:

$$\begin{aligned}
\Omega(\lambda,\alpha,\beta,\gamma,0): f_i\cdot g(s,t) &= -\frac{\lambda^i}{\alpha}\left(\frac{t}{2}-\beta\right)\left(\frac{t}{2}+\beta+1\right)g(s-i,t+2),\\
h_i\cdot g(s,t) &= \lambda^i t g(s-i,t),\quad e_i\cdot g(s,t)=\lambda^i\alpha g(s-i,t-2),\\
d_i\cdot g(s,t) &= \lambda^i(s+i\gamma)g(s-i,t),\quad \mathcal{D}\cdot g(s,t)=0,\\
\Delta(\lambda,\alpha,\beta,\gamma,0): f_i\cdot g(s,t) &= \lambda^i\alpha g(s-i,t+2),\quad h_i\cdot g(s,t)=\lambda^i t g(s-i,t),\\
e_i\cdot g(s,t) &= -\frac{\lambda^i}{\alpha}\left(\frac{t}{2}+\beta\right)\left(\frac{t}{2}-\beta-1\right)g(s-i,t-2),\\
d_i\cdot g(s,t) &= \lambda^i(s+i\gamma)g(s-i,t),\quad \mathcal{D}\cdot g(s,t)=0,\\
\Theta(\lambda,\alpha,\beta,\gamma,0): f_i\cdot g(s,t) &= -\frac{\lambda^i}{\alpha}\left(\frac{t}{2}-\beta\right)g(s-i,t+2),\quad h_i\cdot g(s,t)=\lambda^i t g(s-i,t),\\
e_i\cdot g(s,t) &= \lambda^i\alpha\left(\frac{t}{2}+\beta\right)g(s-i,t-2),\quad d_i\cdot g(s,t)=\lambda^i(s+i\gamma)g(s-i,t),\\
\mathcal{D}\cdot g(s,t) &= 0,\\
\overline{\Omega}\left(\lambda,\alpha,-\frac{3}{4},\gamma,\mu\right): f_i\cdot g(s,t) &= -\frac{\lambda^i}{\alpha}\left(\frac{t}{2}+\frac{3}{4}\right)\left(\frac{t}{2}+\frac{1}{4}\right)g(s-i,t+2),\\
h_i\cdot g(s,t) &= \lambda^i t g(s-i,t),\quad e_i\cdot g(s,t)=\lambda^i\alpha g(s-i,t-2),\\
d_i\cdot g(s,t) &= \lambda^i(s+i\gamma)g(s-i,t),\quad q_i\cdot g(s,t)=-\frac{\lambda^i\mu}{2\alpha}\left(t+\frac{1}{2}\right)g(s-i,t+1),\\
z_i\cdot g(s,t) &= \frac{\lambda^i\mu^2}{2\alpha}g(s-i,t),\quad p_i\cdot g(s,t)=\lambda^i\mu g(s-i,t-1),\\
\overline{\Delta}\left(\lambda,\alpha,-\frac{3}{4},\gamma,\mu\right): f_i\cdot g(s,t) &= \lambda^i\alpha g(s-i,t+2),\quad h_i\cdot g(s,t)=\lambda^i t g(s-i,t),\\
e_i\cdot g(s,t) &= -\frac{\lambda^i}{\alpha}\left(\frac{t}{2}-\frac{3}{4}\right)\left(\frac{t}{2}-\frac{1}{4}\right)g(s-i,t-2),\\
d_i\cdot g(s,t) &= \lambda^i(s+i\gamma)g(s-i,t),\quad q_i\cdot g(s,t)=\lambda^i\mu g(s-i,t+1),\\
z_i\cdot g(s,t) &= -\frac{\lambda^i\mu^2}{2\alpha}g(s-i,t),\quad p_i\cdot g(s,t)=\frac{\lambda^i\mu}{2\alpha}\left(t-\frac{1}{2}\right)g(s-i,t-1).
\end{aligned}$$

Remark 2.4. Let the notations remain as previously defined.

(1) The subalgebra $\mathcal{H}=\mathrm{span}_{\mathbb{C}}\{h_i,d_i\mid i\in\mathbb{Z}\}$ in each case has the same module structure on $\mathbb{C}[s,t]$. Further results on this module over $\mathcal{H}$ can be found in [7] or [17].

(2) In this paper, for convenience, $\Omega(\lambda,\alpha,\beta,\gamma,0)$, $\Delta(\lambda,\alpha,\beta,\gamma,0)$, $\Theta(\lambda,\alpha,\beta,\gamma,0)$, $\overline{\Omega}(\lambda,\alpha,-\frac{3}{4},\gamma,a)$, and $\overline{\Delta}(\lambda,\alpha,-\frac{3}{4},\gamma,a)$ are denoted by $\Omega,\Delta,\Theta,\overline{\Omega}$, and $\overline{\Delta}$, respectively.

Proposition 2.5. *Under the action of* $\mathcal{SW}(b)$ *on* $\mathbb{C}[s,t]$*, given in Definition* 2.3*, then*

(1) Ω, Δ, *and* Θ *are* $\mathcal{SW}(b)$*-modules*;

(2) $\overline{\Omega}$ *and* $\overline{\Delta}$ *are* $\mathcal{SW}(0)$*-modules.*

Proof. (1) Observing Definition 2.3, it becomes evident that the action of $\mathcal{D}$ on Ω, Δ or Θ is trivial. This observation, coupled with Theorem 2.1, implies that Ω, Δ, and Θ are $\mathcal{SW}(b)$-modules.

(2) We only show that $\overline{\Omega}$ is an $\mathcal{SW}(0)$-module as an example with the other case being similar. According to Theorem 2.1, $\overline{\Omega}$ is an $\mathcal{A}$-modules. Consequently, it is only necessary to verify the following relations.

For any $\lambda,\alpha,\mu\in\mathbb{C}^*$, $\beta,\gamma\in\mathbb{C}$ and $g(s,t)\in\mathbb{C}[s,t]$, $i,j\in\mathbb{Z}$, we obtain

$$\begin{aligned}
&p_i\cdot q_j\cdot g(s,t)-q_j\cdot p_i\cdot g(s,t)\\
&\quad = p_i\cdot\left(-\frac{\lambda^j\mu}{2\alpha}\left(t+\frac{1}{2}\right)g(s-j,t+1)\right)-q_j\cdot(\lambda^i\mu g(s-i,t-1))
\end{aligned}$$

$$= -\frac{\lambda^{i+j}\mu^2}{2\alpha}\left(t-\frac{1}{2}\right)g(s-i-j,t) + \frac{\lambda^{i+j}\mu^2}{2\alpha}\left(t+\frac{1}{2}\right)g(s-i-j,t)$$
$$= \frac{\lambda^{i+j}\mu^2}{2\alpha}g(s-i-j,t) = z_{i+j}\cdot g(s,t) = [p_i,q_j]\cdot g(s,t),$$
$$h_i\cdot p_j\cdot g(s,t) - p_j\cdot h_i\cdot g(s,t)$$
$$= h_i\cdot(\lambda^j\mu g(s-j,t-1)) - p_j\cdot(\lambda^i t g(s-i,t))$$
$$= \lambda^{i+j}\mu t g(s-i-j,t-1) - \lambda^{i+j}\mu(t-1)g(s-i-j,t-1)$$
$$= \lambda^{i+j}\mu g(s-i-j,t-1) = p_{i+j}\cdot g(s,t) = [h_i,p_j]\cdot g(s,t),$$
$$h_i\cdot q_j\cdot g(s,t) - q_j\cdot h_i\cdot g(s,t)$$
$$= h_i\cdot\left(-\frac{\lambda^j\mu}{2\alpha}\left(t+\frac{1}{2}\right)g(s-j,t+1)\right) - q_j\cdot(\lambda^i t g(s-i,t))$$
$$= -\frac{\lambda^{i+j}\mu}{2\alpha}t\left(t+\frac{1}{2}\right)g(s-i-j,t+1) + \frac{\lambda^{i+j}\mu}{2\alpha}\left(t+\frac{1}{2}\right)(t+1)g(s-i-j,t+1)$$
$$= \frac{\lambda^{i+j}\mu}{2\alpha}\left(t+\frac{1}{2}\right)g(s-i-j,t+1) = -q_{i+j}\cdot g(s,t) = [h_i,q_j]\cdot g(s,t),$$
$$e_i\cdot q_j\cdot g(s,t) - q_j\cdot e_i\cdot g(s,t)$$
$$= e_i\cdot\left(-\frac{\lambda^j\mu}{2\alpha}\left(t+\frac{1}{2}\right)g(s-j,t+1)\right) - q_j\cdot(\lambda^i\alpha g(s-i,t-2))$$
$$= -\frac{\lambda^{i+j}\mu}{2}\left(t-\frac{3}{2}\right)g(s-i-j,t-1) + \frac{\lambda^{i+j}\mu}{2}\left(t+\frac{1}{2}\right)g(s-i-j,t-1)$$
$$= \lambda^{i+j}\mu g(s-i-j,t-1) = p_{i+j}\cdot g(s,t) = [e_i,q_j]\cdot g(s,t),$$
$$p_i\cdot f_j\cdot g(s,t) - f_j\cdot p_i\cdot g(s,t)$$
$$= p_i\cdot\left(-\frac{\lambda^j}{\alpha}\left(\frac{t}{2}+\frac{3}{4}\right)\left(\frac{t}{2}+\frac{1}{4}\right)g(s-j,t+2)\right) - f_j\cdot(\lambda^i\mu g(s-i,t-1))$$
$$= -\frac{\lambda^{i+j}\mu}{\alpha}\left(\frac{t}{2}+\frac{1}{4}\right)\left(\frac{t}{2}-\frac{1}{4}\right)g(s-i-j,t+1) + \frac{\lambda^{i+j}\mu}{\alpha}\left(\frac{t}{2}+\frac{3}{4}\right)\left(\frac{t}{2}+\frac{1}{4}\right)g(s-i-j,t+1)$$
$$= \frac{\lambda^{i+j}\mu}{2\alpha}\left(t+\frac{1}{2}\right)g(s-i-j,t+1) = -q_{i+j}\cdot g(s,t) = [p_i,f_j]\cdot g(s,t),$$
$$e_i\cdot p_j\cdot g(s,t) - p_j\cdot e_i\cdot g(s,t)$$
$$= e_i\cdot(\lambda^j\mu g(s-j,t-1)) - p_j\cdot(\lambda^i\alpha g(s-i,t-2))$$
$$= \lambda^{i+j}\alpha\mu g(s-i-j,t-3) - \lambda^{i+j}\mu\alpha g(s-i-j,t-3)$$
$$= 0 = [e_i,p_j]\cdot g(s,t),$$
$$f_i\cdot q_j\cdot g(s,t) - q_j\cdot f_i\cdot g(s,t)$$
$$= f_i\cdot\left(-\frac{\lambda^j\mu}{2\alpha}\left(t+\frac{1}{2}\right)g(s-j,t+1)\right) - q_j\cdot\left(-\frac{\lambda^i}{\alpha}\left(\frac{t}{2}+\frac{3}{4}\right)\left(\frac{t}{2}+\frac{1}{4}\right)g(s-i,t+2)\right)$$
$$= \frac{\lambda^{i+j}\mu}{2\alpha^2}\left(\frac{t}{2}+\frac{3}{4}\right)\left(\frac{t}{2}+\frac{1}{4}\right)\left(t+\frac{5}{2}\right)g(s-i-j,t+3)$$
$$-\frac{\lambda^{i+j}\mu}{2\alpha^2}\left(t+\frac{1}{2}\right)\left(\frac{t}{2}+\frac{5}{4}\right)\left(\frac{t}{2}+\frac{3}{4}\right)g(s-i-j,t+3) = 0 = [f_i,q_j]\cdot g(s,t),$$
$$d_i\cdot p_j\cdot g(s,t) - p_j\cdot d_i\cdot g(s,t)$$
$$= d_i\cdot(\lambda^j\mu g(s-j,t-1)) - p_j\cdot(\lambda^i(s+i\gamma)g(s-i,t))$$
$$= \lambda^{i+j}\mu(s+i\gamma)g(s-i-j,t-1) - \lambda^{i+j}\mu(s-j+i\gamma)g(s-i-j,t-1)$$
$$= j\lambda^{i+j}\mu g(s-i-j,t-1) = jp_{i+j}\cdot g(s,t) = [d_i,p_j]\cdot g(s,t),$$
$$d_i\cdot q_j\cdot g(s,t) - q_j\cdot d_i\cdot g(s,t)$$

$$
\begin{aligned}
&= d_i \cdot \left(-\frac{\lambda^j \mu}{2\alpha}\left(t+\frac{1}{2}\right) g(s-j,t+1) \right) - q_j \cdot (\lambda^i (s+i\gamma) g(s-i,t)) \\
&= -\frac{\lambda^{i+j}\mu}{2\alpha}(s+i\gamma)\left(t+\frac{1}{2}\right) g(s-i-j,t+1) + \frac{\lambda^{i+j}\mu}{2\alpha}\left(t+\frac{1}{2}\right)(s-j+i\gamma) g(s-i-j,t+1) \\
&= -j\frac{\lambda^{i+j}\mu}{2\alpha}\left(t+\frac{1}{2}\right) g(s-i-j,t+1) = jq_{i+j}\cdot(s,t) = [d_i,q_j]\cdot g(s,t),
\end{aligned}
$$

and

$$
\begin{aligned}
&d_i \cdot z_j \cdot g(s,t) - z_j \cdot d_i \cdot g(s,t) \\
&= d_i \cdot \left(\frac{\lambda^j \mu^2}{2\alpha} g(s-j,t) \right) - z_j \cdot (\lambda^i (s+i\gamma) g(s-i,t)) \\
&= \frac{\lambda^{i+j}\mu^2}{2\alpha}(s+i\gamma) g(s-i-j,t) - \frac{\lambda^{i+j}\mu^2}{2\alpha}(s-j+i\gamma) g(s-i-j,t) \\
&= j\frac{\lambda^{i+j}\mu^2}{2\alpha} g(s-i-j,t) = jz_{i+j}\cdot g(s,t) = [d_i,z_j]\cdot g(s,t).
\end{aligned}
$$

Furthermore, it is straightforward to verify that

$$
\begin{aligned}
p_i \cdot p_j \cdot g(s,t) - p_j \cdot p_i \cdot g(s,t) = 0 = [p_i,p_j]\cdot g(s,t), \\
q_i \cdot q_j \cdot g(s,t) - q_j \cdot q_i \cdot g(s,t) = 0 = [q_i,q_j]\cdot g(s,t), \\
z_i \cdot x_j \cdot g(s,t) - x_j \cdot z_i \cdot g(s,t) = 0 = [z_i,x_j]\cdot g(s,t),
\end{aligned}
$$

where $x_j \in \{f_j, q_j, h_j, z_j, p_j, e_j \mid j \in \mathbb{Z}\}$. The proof is now conclusive. □

3 $U(\mathbb{C}d_0 \oplus \mathbb{C}h_0)$-free modules over $\mathcal{SW}(b)$

This section is devoted to classifying the modules over $\mathcal{SW}(b)$ whose restriction to $U(\mathbb{C}d_0 \oplus \mathbb{C}h_0)$ is free of rank 1. Additionally, we study the properties of these modules, including the irreducibility and isomorphism classes.

Theorem 3.1. *Let $\mathcal{M}$ be an $\mathcal{SW}(b)$-module whose restriction to $U(\mathbb{C}d_0 \oplus \mathbb{C}h_0)$ is free of rank 1. Then,*
(1) *if $b \neq 0$, $\mathcal{M}$ is isomorphic to one of the modules Ω, Δ, and Θ;*
(2) *if $b = 0$, $\mathcal{M}$ is isomorphic to one of the modules Ω, Δ, Θ, $\overline{\Omega}$, and $\overline{\Delta}$.*

We initially present several lemmas that will be used to prove the aforementioned theorem. Assume that $\mathcal{M} = U(\mathbb{C}d_0 \oplus \mathbb{C}h_0)$. Note that $\mathcal{M}$ can be viewed as an $\mathcal{H}$-module. Following [17, Theorem 3.1], we have

$$
\begin{aligned}
d_i \cdot g(d_0,h_0) = \lambda^i (d_0 + g_i(h_0)) g(d_0 - i, h_0), \\
h_i \cdot g(d_0,h_0) = \lambda^i h_0 g(d_0 - i, h_0),
\end{aligned}
$$

where $g(d_0,h_0) \in \mathcal{M}, \lambda \in \mathbb{C}^*, i \in \mathbb{Z}$, and $g_i(h_0) \in \{g_i(h_0) \mid g_i(h_0) = \sum_{l=0}^{\infty} g^{(l)} i h_0^l \in \mathbb{C}[h_0], g^{(l)} \in \mathbb{C}\}$. For any $i \in \mathbb{Z}$, let

$$
F_i(d_0,h_0) = f_i \cdot 1, \quad Q_i(d_0,h_0) = q_i \cdot 1, \quad Z_i(d_0,h_0) = z_i \cdot 1, \quad P_i(d_0,h_0) = p_i \cdot 1, \quad E_i(d_0,h_0) = e_i \cdot 1.
$$

Lemma 3.2. *The actions of f_i, q_i, z_i, p_i, and e_i on $\mathcal{M}$ are completely determined by $F_i(d_0,h_0)$, $Q_i(d_0,h_0)$, $Z_i(d_0,h_0)$, $P_i(d_0,h_0)$, and $E_i(d_0,h_0)$, respectively.*

Proof. It is easy to show that the following equations hold through an induction on $m \in \mathbb{Z}_{\geqslant 0}$:

$$
\begin{aligned}
x_i d_0^m = (d_0 - i)^m x_i, \quad f_i h_0^m = (h_0 + 2)^m f_i, \\
q_i h_0^m = (h_0 + 1)^m q_i, \quad z_i h_0^m = h_0^m z_i,
\end{aligned}
$$

$$p_i h_0^m = (h_0 - 1)^m p_i, \quad e_i h_0^m = (h_0 - 2)^m e_i,$$

where $x_i \in \{f_i, q_i, z_i, p_i, e_i \mid i \in \mathbb{Z}\}$.

Take any polynomial

$$g(d_0, h_0) = \sum_{j,k \in \mathbb{Z}_{\geqslant 0}} g_{j,k} d_0^j h_0^k \in \mathcal{M},$$

where $g_{j,k} \in \mathbb{C}$. Then, we obtain

$$\begin{aligned} f_i \cdot g(d_0, h_0) &= f_i \cdot \sum_{j,k \in \mathbb{Z}_{\geqslant 0}} g_{j,k} d_0^j h_0^k = \sum_{j,k \in \mathbb{Z}_{\geqslant 0}} g_{j,k} (d_0 - i)^j f_i \cdot h_0^k \\ &= \sum_{j,k \in \mathbb{Z}_{\geqslant 0}} g_{j,k} (d_0 - i)^j (h_0 + 2)^k F_i(d_0, h_0) \\ &= g(d_0 - i, h_0 + 2) F_i(d_0, h_0). \end{aligned}$$

Similarly, we have

$$\begin{aligned} q_i \cdot g(d_0, h_0) &= \sum_{j,k \in \mathbb{Z}_{\geqslant 0}} g_{j,k} (d_0 - i)^j (h_0 + 1)^k Q_i(d_0, h_0) = g(d_0 - i, h_0 + 1) Q_i(d_0, h_0), \\ z_i \cdot g(d_0, h_0) &= \sum_{j,k \in \mathbb{Z}_{\geqslant 0}} g_{j,k} (d_0 - i)^j h_0^k Z_i(d_0, h_0) = g(d_0 - i, h_0) Z_i(d_0, h_0), \\ p_i \cdot g(d_0, h_0) &= \sum_{j,k \in \mathbb{Z}_{\geqslant 0}} g_{j,k} (d_0 - i)^j (h_0 - 1)^k P_i(d_0, h_0) = g(d_0 - i, h_0 - 1) P_i(d_0, h_0), \\ e_i \cdot g(d_0, h_0) &= \sum_{j,k \in \mathbb{Z}_{\geqslant 0}} g_{j,k} (d_0 - i)^j (h_0 - 2)^k E_i(d_0, h_0) = g(d_0 - i, h_0 - 2) E_i(d_0, h_0). \end{aligned}$$

We complete the proof. □

Lemma 3.3. *Keeping the notations as above, we have $Z_0(d_0, h_0) \in \mathbb{C}$. Moreover,*

$$\begin{cases} P_0(d_0, h_0) = 0 \ \text{or} \ Q_0(d_0, h_0) = 0, & \text{if } Z_0(d_0, h_0) = 0, \\ P_0(d_0, h_0) \in \mathbb{C}^* \ \text{or} \ Q_0(d_0, h_0) \in \mathbb{C}^*, & \text{if } Z_0(d_0, h_0) \in \mathbb{C}^*. \end{cases}$$

Proof. Since $[e_0, f_0] \cdot 1 = h_0 \cdot 1$, we obtain

$$E_0(d_0, h_0) F_0(d_0, h_0 - 2) - E_0(d_0, h_0 + 2) F_0(d_0, h_0) = h_0, \tag{3.1}$$

which follows that $E_0(d_0, h_0) \neq 0$ and $F_0(d_0, h_0) \neq 0$. Therefore, based on

$$0 = [z_0, e_0] \cdot 1 = Z_0(d_0, h_0) E_0(d_0, h_0) - Z_0(d_0, h_0 - 2) E_0(d_0, h_0)$$

and

$$0 = [h_1, z_0] \cdot 1 = \lambda h_0 Z_0(d_0 - 1, h_0) - \lambda h_0 Z_0(d_0, h_0),$$

we conclude that $Z_0(d_0, h_0) \in \mathbb{C}[d_0] \cap \mathbb{C}[h_0] = \mathbb{C}$.

Let $Z_0(d_0, h_0) = a \in \mathbb{C}$. The equation $[p_0, q_0] \cdot 1 = z_0 \cdot 1$ implies that

$$P_0(d_0, h_0) Q_0(d_0, h_0 - 1) - P_0(d_0, h_0 + 1) Q_0(d_0, h_0) = a. \tag{3.2}$$

We may assume that

$$P_0(d_0, h_0) = \sum_{j=0}^{m} a_j(d_0) h_0^j$$

and

$$Q_0(d_0, h_0) = \sum_{j=0}^{n} b_j(d_0) h_0^j,$$

where $a_j(d_0), b_j(d_0) \in \mathbb{C}[d_0]$ and $a_m(d_0)b_n(d_0) \neq 0$. Substituting these expressions into (3.2), we obtain

$$\sum_{j=0}^{m} a_j(d_0)h_0^j \sum_{l=0}^{n} b_l(d_0)(h_0-1)^l - \sum_{j=0}^{m} a_j(d_0)(h_0+1)^j \sum_{l=0}^{n} b_l(d_0)h_0^l = a. \tag{3.3}$$

Then, the highest degree term on the left-hand side of (3.3) with respect to h_0 is written as

$$-(m+n)a_m(d_0)b_n(d_0)h_0^{m+n-1}. \tag{3.4}$$

If $a = 0$, (3.4) means that $m+n=0$, i.e., $m=n=0$. Hence, $P_0(d_0,h_0) = a_0(d_0)$ and $Q_0(d_0,h_0) = b_0(d_0)$. The following equations

$$0 = [e_0, p_0] \cdot 1 = a_0(d_0)E_0(d_0,h_0) - a_0(d_0)E_0(d_0,h_0-1)$$

and

$$0 = [f_0, q_0] \cdot 1 = b_0(d_0)F_0(d_0,h_0) - b_0(d_0)F_0(d_0,h_0+1)$$

yield that $E_0(d_0,h_0), F_0(d_0,h_0) \in \mathbb{C}[d_0]$, which contradicts (3.1). Therefore, we conclude that $P_0(d_0,h_0) = 0$ or $Q_0(d_0,h_0) = 0$.

If $a \neq 0$, (3.3) indicates that $m+n > 0$. Furthermore, from (3.3)–(3.4), we easily get $m+n=1$ and $a_m(d_0)b_n(d_0) = -a$. Conversely, if $m+n>1$, we have $a_m(d_0)b_n(d_0) = 0$, which results in a contradiction with $a_m(d_0) \neq 0$ and $b_n(d_0) \neq 0$. Thus, $a_m(d_0), b_n(d_0) \in \mathbb{C}^*$. This suggests that $P_0(d_0,h_0) \in \mathbb{C}^*$ or $Q_0(d_0,h_0) \in \mathbb{C}^*$. We now complete the proof. □

Lemma 3.4. *For $b \neq 0$, we have $Z_0(d_0,h_0) = 0$.*

Proof. From Lemma 3.3, we may assume that $Z_0(d_0,h_0) = a \in \mathbb{C}$. Considering $\mathcal{M}$ as an $\mathcal{A}$-module, we see from the discussions presented in [9, Theorem 3.2] that

$$d_i \cdot g(d_0,h_0) = \lambda^i(d_0+i\gamma)g(d_0-i,h_0)$$

for some $\gamma \in \mathbb{C}$ and all $i \in \mathbb{Z}$.

Case 1. $b \neq \frac{1}{2}$.

For any $i \in \mathbb{Z}^*$, the equation

$$0 = \lambda^i(d_0+i\gamma)a - a\lambda^i(d_0+i\gamma) = [d_i, z_0] \cdot 1 = 2biz_i \cdot 1$$

yields that $2biZ_i(d_0,h_0) = 0$. Then,

$$Z_i(d_0,h_0) = 0, \quad \forall\, i \in \mathbb{Z}^*.$$

This, in conjunction with

$$0 = [d_i, z_{-i}] \cdot 1 = (2b-1)iz_0 \cdot 1, \quad \forall\, i \in \mathbb{Z}^*,$$

demonstrates that $a = 0$.

Case 2. $b = \frac{1}{2}$.

Suppose that $a \neq 0$. It follows from Lemma 3.3 that $P_0(d_0,h_0) \in \mathbb{C}^*$ or $Q_0(d_0,h_0) \in \mathbb{C}^*$. If $P_0(d_0,h_0) \in \mathbb{C}^*$, we may assume that $P_0(d_0,h_0) = \mu$. For all $i \in \mathbb{Z}^*$, we have

$$0 = \lambda^i(d_0+i\gamma)\mu - \mu\lambda^i(d_0+i\gamma) = [d_i, p_0] \cdot 1 = \frac{1}{2}ip_i \cdot 1.$$

Then,

$$P_i(d_0,h_0) = 0, \quad \forall\, i \in \mathbb{Z}^*.$$

According to

$$0 = [d_i, p_{-i}] \cdot 1 = -\frac{1}{2}ip_0 \cdot 1, \quad \forall\, i \in \mathbb{Z}^*,$$

we obtain $P_0(d_0,h_0) = 0$, which leads to a contradiction. Therefore, $a = 0$.

Similarly, if $Q_0(d_0,h_0) \in \mathbb{C}^*$, we also have $a = 0$. Thus, the proof is conclusively complete. □

Lemma 3.5. *If $Z_0(d_0,h_0)=0$, then $Q_i(d_0,h_0)=Z_i(d_0,h_0)=P_i(d_0,h_0)=0$ for any $i\in\mathbb{Z}$, i.e., $\mathcal{D}=\mathrm{span}_{\mathbb{C}}\{q_i,z_i,p_i\mid i\in\mathbb{Z}\}$ vanishes on $\mathcal{M}$.*

Proof. From Lemma 3.3, we have $P_0(d_0,h_0)=0$ or $Q_0(d_0,h_0)=0$. If $P_0(d_0,h_0)=0$, the equation $[p_0,f_0]\cdot 1=-q_0\cdot 1$ leads to that $Q_0(d_0,h_0)=0$. Then, for all $i\in\mathbb{Z}$, we have

$$P_i(d_0,h_0)=p_i\cdot 1=[h_i,p_0]\cdot 1=0,$$
$$Q_i(d_0,h_0)=q_i\cdot 1=[q_0,h_i]\cdot 1=0,$$
$$Z_i(d_0,h_0)=z_i\cdot 1=[p_0,q_i]\cdot 1=0.$$

If $Q_0(d_0,h_0)=0$, the same conclusion is evidently valid. The proof is thus complete. □

Proof of Theorem 3.1. (1) Based on Lemmas 3.4 and 3.5, the subalgebra $\mathcal{D}$ vanishes on $\mathcal{M}$. Consequently, according to Theorem 2.1, $\mathcal{M}$ is isomorphic to Ω, Δ or Θ.

(2) From Lemma 3.3, let $Z_0(d_0,h_0)=a\in\mathbb{C}$. If $a=0$, then Theorem 2.1, in conjunction with Lemmas 3.4 and 3.5, also suggests that $\mathcal{M}$ is isomorphic to Ω, Δ or Θ.

If $a\neq 0$, we may assume that

$$P_0(d_0,h_0)=\sum_{j=0}^{m}a_j(d_0)h_0^j,\quad Q_0(d_0,h_0)=\sum_{j=0}^{n}b_j(d_0)h_0^j,$$

where $a_j(d_0),b_j(d_0)\in\mathbb{C}[d_0]$ and $a_m(d_0)b_n(d_0)\neq 0$. The proof of Lemma 3.3 yields that both $m+n=1$ and $a_m(d_0)b_n(d_0)=-a$ hold. Hence, we can divide the discussions into the following two cases.

Case 1. $m=0$ and $n=1$.

Let $a_0(d_0)=\mu\in\mathbb{C}^*$ and $b_1(d_0)=-\frac{a}{\mu}\in\mathbb{C}^*$. Then, we have

$$P_0(d_0,h_0)=\mu,\quad Q_0(d_0,h_0)=-\frac{a}{\mu}h_0+b_0(d_0).$$

By the equation

$$0=[d_i,p_0]\cdot 1=\lambda^i\mu(d_0+g_i(h_0))-\lambda^i\mu(d_0+g_i(h_0-1)),$$

one can deduce that $g_i(h_0)\in\mathbb{C}$, i.e., $g_i(h_0)=i\gamma$ for some $\gamma\in\mathbb{C}$ and all $i\in\mathbb{Z}$. Hence,

$$d_i\cdot g(d_0,h_0)=\lambda^i(d_0+i\gamma)g(d_0-i,h_0).$$

Furthermore, according to $[d_1,q_0]\cdot 1=0$, we obtain

$$\lambda(d_0+\gamma)\Big(-\frac{a}{\mu}h_0+b_0(d_0-1)\Big)=\lambda(d_0+\gamma)\Big(-\frac{a}{\mu}h_0+b_0(d_0)\Big).$$

Then, $b_0(d_0)=b_0(d_0-1)$, i.e., $b_0(d_0)\in\mathbb{C}$. We may assume that $b_0(d_0)=\frac{a'}{\mu}$, where $a'\in\mathbb{C}$. Thus, $Q_0(d_0,h_0)=\frac{1}{\mu}(-ah_0+a')$.

Based on the equation

$$0=[e_0,p_0]\cdot 1=E_0(d_0,h_0)P_0(d_0,h_0-2)-E_0(d_0,h_0-1)P_0(d_0,h_0),$$

and $P_0(d_0,h_0)=\mu$, we obtain $E_0(d_0,h_0)\in\mathbb{C}[d_0]$. Since $[e_0,q_0]\cdot 1=p_0\cdot 1$, the equations

$$E_0(d_0,h_0)Q_0(d_0,h_0-2)-E_0(d_0,h_0+1)Q_0(d_0,h_0)=\mu$$

and

$$Q_0(d_0,h_0)=\frac{1}{\mu}(-ah_0+a')$$

yield that $E_0(d_0,h_0)=\frac{\mu^2}{2a}\in\mathbb{C}^*$. Let $E_0(d_0,h_0)=\alpha\in\mathbb{C}^*$. Then $a=\frac{\mu^2}{2\alpha}$. Assume that

$$F_0(d_0,h_0)=\sum_{j=0}^{m}u_j(d_0)h_0^j,\quad u_j(d_0)\in\mathbb{C}[d_0].$$

By (3.1), we can readily obtain

$$F_0(d_0,h_0)=-\frac{1}{4\alpha}h_0^2-\frac{1}{2\alpha}h_0+u_0(d_0). \tag{3.5}$$

The equation $[h_1,f_0]\cdot 1=-2f_1\cdot 1$ demonstrates that

$$F_1(d_0,h_0)=\lambda\bigg(-\frac{1}{4\alpha}h_0^2-\frac{1}{2\alpha}h_0+\frac{1}{2}u_0(d_0)h_0-\frac{1}{2}u_0(d_0-1)h_0+u_0(d_0)\bigg). \tag{3.6}$$

Moreover, the equation $[f_0,f_1]\cdot 1=0$ is equivalent to

$$F_0(d_0,h_0)F_1(d_0,h_0+2)=F_0(d_0-1,h_0+2)F_1(d_0,h_0). \tag{3.7}$$

Upon substituting (3.5)–(3.6) into (3.7), we see that $u_0(d_0)\in\mathbb{C}$. In fact, if $u_0(d_0)\notin\mathbb{C}$, a comparison of the terms independent of h_0 on both sides yields that $(u_0(d_0))^2=u_0(d_0)u_0(d_0-1)$, which results in a contradiction. Therefore, we conclude that $F_0(d_0,h_0)=-\frac{1}{4\alpha}h_0^2-\frac{1}{2\alpha}h_0+\beta'$, where $\beta'\in\mathbb{C}$.

Again from the equation

$$0=[f_0,q_0]\cdot 1=F_0(d_0,h_0)Q_0(d_0,h_0+2)-F_0(d_0,h_0+1)Q_0(d_0,h_0)$$

and the expressions of $Q_0(d_0,h_0)$ and $F_0(d_0,h_0)$ above, it follows that $a=-2a'$ and $\beta'=-\frac{3}{16\alpha}$. Since $a=\frac{\mu^2}{2\alpha}$, we get

$$Q_0(d_0,h_0)=-\frac{\mu}{2\alpha}\left(h_0+\frac{1}{2}\right)$$

and

$$F_0(d_0,h_0)=-\frac{1}{\alpha}\left(\frac{h_0}{2}+\frac{3}{4}\right)\left(\frac{h_0}{2}+\frac{1}{4}\right).$$

As a result, for all $i\in\mathbb{Z}$, we have

$$\begin{aligned}
F_i(d_0,h_0)&=f_i\cdot 1=-\frac{1}{2}[h_i,f_0]\cdot 1=-\frac{\lambda^i}{\alpha}\left(\frac{h_0}{2}+\frac{3}{4}\right)\left(\frac{h_0}{2}+\frac{1}{4}\right),\\
Q_i(d_0,h_0)&=q_i\cdot 1=-[h_i,q_0]\cdot 1=-\frac{\lambda^i\mu}{2\alpha}\left(h_0+\frac{1}{2}\right),\\
Z_i(d_0,h_0)&=z_i\cdot 1=[p_0,q_i]\cdot 1=\frac{\lambda^i\mu^2}{2\alpha},\\
P_i(d_0,h_0)&=p_i\cdot 1=[h_i,p_0]\cdot 1=\lambda^i\mu,\\
E_i(d_0,h_0)&=e_i\cdot 1=\frac{1}{2}[h_i,e_0]\cdot 1=\lambda^i\alpha.
\end{aligned}$$

Thus, from Lemma 3.2, we infer that $\mathcal{M}\cong\overline{\Omega}$.

Case 2. $m=1$ and $n=0$.

We interchange $P_0(d_0,h_0)$ and $Q_0(d_0,h_0)$. As in Case 1, we see that $\mathcal{M}\cong\overline{\Delta}$. The proof is complete. □

Remark 3.6. For $b\neq 0$, Theorem 3.1 suggests that considering modules over $\mathcal{SW}(b)$ whose restriction to $U(\mathbb{C}d_0\oplus\mathbb{C}h_0)$ is free of rank 1 is equivalent to considering such modules over $\mathcal{A}$.

Hence, we immediately deduce the following propositions from Proposition 2.2.

Proposition 3.7. *For $b\neq 0$, as $\mathcal{SW}(b)$-modules, Ω and Δ are irreducible and Θ is irreducible if and only if $2\beta\notin\mathbb{Z}_{\geqslant 0}$.*

Proposition 3.8. *Let $\lambda,\lambda_1,\alpha,\alpha_1\in\mathbb{C}^*$ and $\beta,\beta_1,\gamma,\gamma_1\in\mathbb{C}$. For $b\neq 0$, as $\mathcal{SW}(b)$-modules, Ω, Δ, and Θ are pairwise non-isomorphic. Moreover, we have*

$$\begin{aligned}
&\Omega(\lambda,\alpha,\beta,\gamma,0)\cong\Omega(\lambda_1,\alpha_1,\beta_1,\gamma_1,0)\Leftrightarrow(\lambda,\alpha,\beta,\gamma)=(\lambda_1,\alpha_1,\beta_1,\gamma_1)\\
&\quad \textit{or } (\lambda,\alpha,\beta,\gamma)=(\lambda_1,\alpha_1,-\beta_1-1,\gamma_1),
\end{aligned}$$

$$\begin{aligned}
&\Delta(\lambda,\alpha,\beta,\gamma,0)\cong\Delta(\lambda_1,\alpha_1,\beta_1,\gamma_1,0)\Leftrightarrow(\lambda,\alpha,\beta,\gamma)=(\lambda_1,\alpha_1,\beta_1,\gamma_1)\\
&\quad \text{or } (\lambda,\alpha,\beta,\gamma)=(\lambda_1,\alpha_1,-\beta_1-1,\gamma_1),\\
&\Theta(\lambda,\alpha,\beta,\gamma,0)\cong\Theta(\lambda_1,\alpha_1,\beta_1,\gamma_1,0)\Leftrightarrow(\lambda,\alpha,\beta,\gamma)=(\lambda_1,\alpha_1,\beta_1,\gamma_1).
\end{aligned}$$

Next, we focus on the case $b=0$.

Proposition 3.9. *As $\mathcal{SW}(0)$-modules, Ω, Δ, $\overline{\Omega}$, and $\overline{\Delta}$ are irreducible and Θ is irreducible if and only if $2\beta\notin\mathbb{Z}_{\geqslant0}$.*

Proof. This irreducibility is a direct consequence of Proposition 2.2. In addition, if $2\beta\in\mathbb{Z}_{\geqslant0}$, it is easy to verify that the vector space

$$V=\mathbb{C}[s,t]\prod_{n=0}^{2\beta}\left(\frac{t}{2}+\beta-n\right)$$

is a proper $\mathcal{SW}(0)$-submodule of Θ. □

Proposition 3.10. *Let $\lambda,\lambda_1,\alpha,\alpha_1,\mu,\mu_1\in\mathbb{C}^*$ and $\beta,\beta_1,\gamma,\gamma_1\in\mathbb{C}$. As $\mathcal{SW}(0)$-modules, Ω, Δ, Θ, $\overline{\Omega}$, and $\overline{\Delta}$ are pairwise non-isomorphic. Moreover, we have*

$$\begin{aligned}
&\Omega(\lambda,\alpha,\beta,\gamma,0)\cong\Omega(\lambda_1,\alpha_1,\beta_1,\gamma_1,0)\Leftrightarrow(\lambda,\alpha,\beta,\gamma)=(\lambda_1,\alpha_1,\beta_1,\gamma_1)\\
&\quad \textit{or } (\lambda,\alpha,\beta,\gamma)=(\lambda_1,\alpha_1,-\beta_1-1,\gamma_1),\\
&\Delta(\lambda,\alpha,\beta,\gamma,0)\cong\Delta(\lambda_1,\alpha_1,\beta_1,\gamma_1,0)\Leftrightarrow(\lambda,\alpha,\beta,\gamma)=(\lambda_1,\alpha_1,\beta_1,\gamma_1)\\
&\quad \textit{or } (\lambda,\alpha,\beta,\gamma)=(\lambda_1,\alpha_1,-\beta_1-1,\gamma_1),\\
&\Theta(\lambda,\alpha,\beta,\gamma,0)\cong\Theta(\lambda_1,\alpha_1,\beta_1,\gamma_1,0)\Leftrightarrow(\lambda,\alpha,\beta,\gamma)=(\lambda_1,\alpha_1,\beta_1,\gamma_1),\\
&\overline{\Omega}\left(\lambda,\alpha,-\frac{3}{4},\gamma,\mu\right)\cong\overline{\Omega}\left(\lambda_1,\alpha_1,-\frac{3}{4},\gamma_1,\mu_1\right)\Leftrightarrow(\lambda,\alpha,\gamma,\mu)=(\lambda_1,\alpha_1,\gamma_1,\mu_1),\\
&\overline{\Delta}\left(\lambda,\alpha,-\frac{3}{4},\gamma,\mu\right)\cong\overline{\Delta}\left(\lambda_1,\alpha_1,-\frac{3}{4},\gamma_1,\mu_1\right)\Leftrightarrow(\lambda,\alpha,\gamma,\mu)=(\lambda_1,\alpha_1,\gamma_1,\mu_1).
\end{aligned}$$

Proof. Suppose that Y and Y_1 are $\mathcal{SW}(0)$-modules given in Definition 2.3. Let $\varphi:Y\to Y_1$ be an isomorphism of $\mathcal{SW}(0)$-modules. From $\varphi(d_0^jh_0^k\cdot1)=d_0^jh_0^k\cdot\varphi(1)$ for any $j,k\in\mathbb{Z}_{\geqslant0}$, we know that $\varphi(s^jt^k)=s^jt^k\varphi(1)$. Then, for any $g(s,t)\in\mathbb{C}[s,t]$, we have $\varphi(g(s,t))=g(s,t)\varphi(1)$. From this, it is easy to see that $1=\varphi(\varphi^{-1}(1))=\varphi^{-1}(1)\varphi(1)$. Thus, $\varphi(1)\in\mathbb{C}^*$.

Let $c=e_0\cdot1$ and $c'=p_0\cdot1$ when $1\in Y$, and $c_1=e_0\cdot1$ and $c_1'=p_0\cdot1$ when $1\in Y_1$. Then, it is clear that

$$\begin{aligned}
&\varphi(1)c=\varphi(c)=\varphi(e_0\cdot1)=e_0\cdot\varphi(1)=\varphi(1)c_1,\\
&\varphi(1)c'=\varphi(c')=\varphi(p_0\cdot1)=p_0\cdot\varphi(1)=\varphi(1)c_1',
\end{aligned}\tag{3.8}$$

which mean that $c=c_1$ and $c'=c_1'$. According to Definition 2.3, Ω, Δ, Θ, $\overline{\Omega}$, and $\overline{\Delta}$ are pairwise non-isomorphic.

From Proposition 2.2, it is evident that

$$\begin{aligned}
&\Omega(\lambda,\alpha,\beta,\gamma,0)\cong\Omega(\lambda_1,\alpha_1,\beta_1,\gamma_1,0)\Leftrightarrow(\lambda,\alpha,\beta,\gamma)=(\lambda_1,\alpha_1,\beta_1,\gamma_1)\\
&\quad \text{or } (\lambda,\alpha,\beta,\gamma)=(\lambda_1,\alpha_1,-\beta_1-1,\gamma_1),\\
&\Delta(\lambda,\alpha,\beta,\gamma,0)\cong\Delta(\lambda_1,\alpha_1,\beta_1,\gamma_1,0)\Leftrightarrow(\lambda,\alpha,\beta,\gamma)=(\lambda_1,\alpha_1,\beta_1,\gamma_1)\\
&\quad \text{or } (\lambda,\alpha,\beta,\gamma)=(\lambda_1,\alpha_1,-\beta_1-1,\gamma_1),\\
&\Theta(\lambda,\alpha,\beta,\gamma,0)\cong\Theta(\lambda_1,\alpha_1,\beta_1,\gamma_1,0)\Leftrightarrow(\lambda,\alpha,\beta,\gamma)=(\lambda_1,\alpha_1,\beta_1,\gamma_1).
\end{aligned}$$

Let

$$\varphi_\Omega:\overline{\Omega}\left(\lambda,\alpha,-\frac{3}{4},\gamma,\mu\right)\to\overline{\Omega}\left(\lambda_1,\alpha_1,-\frac{3}{4},\gamma_1,\mu_1\right)$$

be an isomorphism of $\mathcal{SW}(0)$-modules. Since $\overline{\Omega}(\lambda,\alpha,-\frac{3}{4},\gamma,\mu)$ and $\overline{\Omega}(\lambda_1,\alpha_1,-\frac{3}{4},\gamma_1,\mu_1)$ can be regarded as $\mathcal{A}$-modules, it follows from Proposition 2.2 that $(\lambda,\alpha,\gamma)=(\lambda_1,\alpha_1,\gamma_1)$. In fact, $\mu=p_0\cdot 1$ when $1\in\overline{\Omega}(\lambda,\alpha,-\frac{3}{4},\gamma,\mu)$ and $\mu_1=p_0\cdot 1$ when $1\in\overline{\Omega}(\lambda_1,\alpha_1,-\frac{3}{4},\gamma_1,\mu_1)$. Consequently, (3.8) suggests that $\mu=\mu_1$.

Let

$$\varphi_\Delta:\overline{\Delta}\Big(\lambda,\alpha,-\frac{3}{4},\gamma,\mu\Big)\to\overline{\Delta}\Big(\lambda_1,\alpha_1,-\frac{3}{4},\gamma_1,\mu_1\Big)$$

be an isomorphism of $\mathcal{SW}(0)$-modules. Likewise, we have $(\lambda,\alpha,\gamma)=(\lambda_1,\alpha_1,\gamma_1)$. Additionally, $\frac{\mu}{2\alpha}(t-\frac{1}{2})=p_0\cdot 1$ when $1\in\overline{\Delta}(\lambda,\alpha,-\frac{3}{4},\gamma,\mu)$ and $\frac{\mu_1}{2\alpha_1}(t-\frac{1}{2})=p_0\cdot 1$ when $1\in\overline{\Delta}(\lambda_1,\alpha_1,-\frac{3}{4},\gamma_1,\mu_1)$. (3.8) along with $\alpha=\alpha_1$ gives us that $\mu=\mu_1$. This finishes the proof. □

In the subsequent sections, we consider the modules over $\mathcal{SW}(b)$ by taking the tensor product of $U(\mathbb{C}d_0\oplus\mathbb{C}h_0)$-free modules with irreducible highest weight modules (or Verma modules). The structures of these highest weight modules (or Verma modules) indicate that the properties of the tensor product $\mathcal{SW}(b)$-modules are primarily dependent on $U(\mathbb{C}d_0\oplus\mathbb{C}h_0)$-free modules. From Theorem 3.1, it is evident that $U(\mathbb{C}d_0\oplus\mathbb{C}h_0)$-free $\mathcal{SW}(b)$-modules is equivalent to such modules over $\mathcal{A}$ if $b\neq 0$. Consequently, the findings regarding the tensor product modules over $\mathcal{SW}(b)$ are derived from similar discussions to those in [11] when $b\neq 0$. Next, we only focus on the case $b=0$.

4 Tensor product modules over $\mathcal{SW}(0)$

In this section, we construct the irreducible highest weight module $V(\epsilon,\xi,\eta)$ over $\mathcal{SW}(0)$ and obtain the tensor product $\mathcal{SW}(0)$-modules $M(\lambda,\alpha,\beta,\gamma,\mu)\otimes V(\epsilon,\xi,\eta)$, where $M(\lambda,\alpha,\beta,\gamma,\mu)=\Omega,\Delta,\Theta,\overline{\Omega}$ or $\overline{\Delta}$ is defined in Definition 2.3. We then investigate the irreducibility and isomorphism classes for $M(\lambda,\alpha,\beta,\gamma,\mu)\otimes V(\epsilon,\xi,\eta)$, where $2\beta\notin\mathbb{Z}_{\geqslant 0}$ when $M(\lambda,\alpha,\beta,\gamma,\mu)=\Theta$. Finally, we demonstrate that $M(\lambda,\alpha,\beta,\gamma,\mu)$ and $M(\lambda,\alpha,\beta,\gamma,\mu)\otimes V(\epsilon,\xi,\eta)$ are new non-weight $\mathcal{SW}(0)$-modules.

Let $I(\epsilon,\xi,\eta)$ denote the left ideal of $U(\mathcal{SW}(0))$, which is generated by the element

$$\{f_i,q_i,h_i,z_i,p_i,e_i,d_i,p_0,e_0\mid i\in\mathbb{Z}_{\geqslant 1}\}\cup\{h_0-\epsilon,z_0-\xi,d_0-\eta\},$$

where $\epsilon,\xi,\eta\in\mathbb{C}$. Then, the Verma $\mathcal{SW}(0)$-module $\overline{V}(\epsilon,\xi,\eta)$ with the highest weight (ϵ,ξ,η) is the quotient module, i.e.,

$$\overline{V}(\epsilon,\xi,\eta)=U(\mathcal{SW}(0))/I(\epsilon,\xi,\eta).$$

According to the PBW Theorem, $\overline{V}(\epsilon,\xi,\eta)$ has a basis consisting of the vectors of the following form:

$$f_{-m_f}^{F_{-m_f}}\cdots f_0^{F_0}q_{-m_q}^{Q_{-m_q}}\cdots q_0^{Q_0}h_{-m_h}^{H_{-m_h}}\cdots h_{-1}^{H_{-1}}z_{-m_z}^{Z_{-m_z}}\cdots z_{-1}^{Z_{-1}}p_{-m_p}^{P_{-m_p}}\cdots p_{-1}^{P_{-1}}e_{-m_e}^{E_{-m_e}}\cdots e_{-1}^{E_{-1}}d_{-m_d}^{D_{-m_d}}\cdots d_{-1}^{D_{-1}}\cdot v_h,$$

where $v_h=1+I(\epsilon,\xi,\eta)$ and $D_{-1},\dots,D_{-m_d},E_{-1},\dots,E_{-m_e},P_{-1},\dots,P_{-m_p},Z_{-1},\dots,Z_{-m_z},H_{-1},\dots,H_{-m_h},Q_0,\dots,Q_{-m_q},F_0,\dots,F_{-m_f}\in\mathbb{Z}_{\geqslant 0}$. Thus, we obtain the irreducible highest weight module

$$V(\epsilon,\xi,\eta)=\overline{V}(\epsilon,\xi,\eta)/J,$$

where J is the unique maximal proper submodule of $\overline{V}(\epsilon,\xi,\eta)$.

Let $\lambda,\alpha,\mu\in\mathbb{C}^*$ and $\beta,\gamma,\epsilon,\xi,\eta\in\mathbb{C}$. In what follows, we assume $M(\lambda,\alpha,\beta,\gamma,\mu)=\Omega,\Delta,\Theta,\overline{\Omega}$ or $\overline{\Delta}$ constructed in Definition 2.3 and $V(\epsilon,\xi,\eta)$ is the irreducible highest weight module over $\mathcal{SW}(0)$.

Theorem 4.1. *The tensor product $\mathcal{SW}(0)$-module $M(\lambda,\alpha,\beta,\gamma,\mu)\otimes V(\epsilon,\xi,\eta)$ is irreducible, where $2\beta\notin\mathbb{Z}_{\geqslant 0}$ when $M(\lambda,\alpha,\beta,\gamma,\mu)=\Theta$.*

Proof. Assume that $W_{M(\lambda,\alpha,\beta,\gamma,\mu)}$ is a non-zero $\mathcal{SW}(0)$-submodule of $M(\lambda,\alpha,\beta,\gamma,\mu)\otimes V(\epsilon,\xi,\eta)$, where $M(\lambda,\alpha,\beta,\gamma,\mu)=\Omega,\Delta,\Theta(2\beta\notin\mathbb{Z}_{\geqslant 0}),\overline{\Omega}$ or $\overline{\Delta}$. It suffices to show that

$$W_{M(\lambda,\alpha,\beta,\gamma,\mu)}=M(\lambda,\alpha,\beta,\gamma,\mu)\otimes V(\epsilon,\xi,\eta).$$

From the structure of $V(\epsilon,\xi,\eta)$, we know that for any $v\in V(\epsilon,\xi,\eta)$, there exists a $K(v)\in\mathbb{Z}_{\geqslant 1}$ such that

$$x_m\cdot v=0,\quad \forall\, x_m\in\{f_m,q_m,h_m,z_m,p_m,e_m,d_m\mid m\geqslant K(v)\}.$$

Take a non-zero element

$$w=\sum_{i=0}^{r}a_i(t)s^i\otimes v_i\in W_{M(\lambda,\alpha,\beta,\gamma,\mu)},$$

where $a_i(t)\in\mathbb{C}[t]$, $v_i\in V(\epsilon,\xi,\eta)$, $a_r(t)\neq 0$, $v_r\neq 0$, and $r\in\mathbb{Z}_{\geqslant 0}$ is minimal. Based on the similar discussions to the proof of [11, Theorem 3.1] and the minimality of r, we have $r=0$, i.e.,

$$w=a_0(t)\otimes v_0\in W_{M(\lambda,\alpha,\beta,\gamma,\mu)}.$$

Fix this v_0 and let

$$P=\{a(s,t)\in\mathbb{C}[s,t]\mid a(s,t)\otimes v_0\in W_{M(\lambda,\alpha,\beta,\gamma,\mu)}\}.$$

Clearly, $a_0(t)\in P$. For any $k\in\mathbb{Z}_{\geqslant 1}$ and $m\geqslant K(v_0)$, one can inductively show that

$$\begin{aligned}
&\lambda^{-mk}\alpha^{-k}e_m^k\cdot(a_0(t)\otimes v_0)=a_0(t-2k)\otimes v_0\in W_\Omega, W_{\overline{\Omega}},\\
&\lambda^{-mk}\alpha^{-k}f_m^k\cdot(a_0(t)\otimes v_0)=a_0(t+2k)\otimes v_0\in W_\Delta, W_{\overline{\Delta}},\\
&\lambda^{-mk}\alpha^{-k}e_m^k\cdot(a_0(t)\otimes v_0)=\prod_{n=0}^{k-1}\left(\frac{t}{2}+\beta-n\right)a_0(t-2k)\otimes v_0\in W_\Theta,\\
&\lambda^{-mk}(-\alpha)^{k}f_m^k\cdot(a_0(t)\otimes v_0)=\prod_{n=0}^{k-1}\left(\frac{t}{2}-\beta+n\right)a_0(t+2k)\otimes v_0\in W_\Theta.
\end{aligned}$$

Now, we choose k sufficiently large such that

$$\begin{aligned}
&(a_0(t),a_0(t-2k))=1,\\
&(a_0(t),a_0(t+2k))=1,\\
&\left(\prod_{n=0}^{k-1}\left(\frac{t}{2}+\beta-n\right)a_0(t-2k),\prod_{n=0}^{k-1}\left(\frac{t}{2}-\beta+n\right)a_0(t+2k)\right)=1,
\end{aligned}$$

where $2\beta\notin\mathbb{Z}_{\geqslant 0}$. Then, it is inferred that $1\otimes v_0\in W_{M(\lambda,\alpha,\beta,\gamma,\mu)}$. Furthermore, for any $a(t)\in\mathbb{C}[t]$ and $m\geqslant K(v_0)$, the following formulae are valid by applying induction on $k\in\mathbb{Z}_{\geqslant 1}$:

$$\lambda^{-mk}h_m^k\cdot(a(t)\otimes v_0)=t^ka(t)\otimes v_0, \tag{4.1}$$

$$\lambda^{-mk}d_m^k\cdot(a(t)\otimes v_0)=\prod_{i=0}^{k-1}(s+m\gamma-mi)a(t)\otimes v_0. \tag{4.2}$$

Then, (4.1) gives us that $\mathbb{C}[t]\subseteq P$. Meanwhile, (4.2) indicates that P is stable under the multiplication by s. Then, we get

$$P=\mathbb{C}[s,t]=M(\lambda,\alpha,\beta,\gamma,a).$$

Next, let

$$Q=\{v\in V(\epsilon,\xi,\eta)\mid M(\lambda,\alpha,\beta,\gamma,\mu)\otimes v\in W_{M(\lambda,\alpha,\beta,\gamma,\mu)}\}.$$

Obviously, $v_0\in Q$. For any $v\in Q$ and $a(s,t)\in M(\lambda,\alpha,\beta,\gamma,\mu)$,

$$\mathcal{SW}(0)\cdot(a(s,t)\otimes v)=\mathcal{SW}(0)\cdot a(s,t)\otimes v+a(s,t)\otimes\mathcal{SW}(0)\cdot v\in W_{M(\lambda,\alpha,\beta,\gamma,\mu)}.$$

Then, $a(s,t)\otimes\mathcal{SW}(0)\cdot v\in W_{M(\lambda,\alpha,\beta,\gamma,\mu)}$, which yields that Q is a submodule of $V(\epsilon,\xi,\eta)$. Hence, $Q=V(\epsilon,\xi,\eta)$ due to the irreducibility of $V(\epsilon,\xi,\eta)$. Therefore, we have

$$W_{M(\lambda,\alpha,\beta,\gamma,\mu)}=M(\lambda,\alpha,\beta,\gamma,\mu)\otimes V(\epsilon,\xi,\eta).$$

We now complete the proof. □

Theorem 4.2. *Assume* $\lambda,\lambda_1,\alpha,\alpha_1,\mu,\mu_1\in\mathbb{C}^*,\beta,\beta_1,\gamma,\gamma_1,\epsilon,\epsilon_1,\xi,\xi_1,\eta,\eta_1\in\mathbb{C}$, *and* $M(\lambda,\alpha,\beta,\gamma,\mu)=\Omega,\Delta,\Theta(2\beta\notin\mathbb{Z}_{\geqslant 0}),\overline{\Omega}$ *or* $\overline{\Delta}$. *Let* $V(\epsilon,\xi,\eta)$ *and* $V(\epsilon_1,\xi_1,\eta_1)$ *be irreducible highest weight modules. Then, as* $\mathcal{SW}(b)$*-modules,* $M(\lambda,\alpha,\beta,\gamma,\mu)\otimes V(\epsilon,\xi,\eta)$ *and* $M(\lambda_1,\alpha_1,\beta_1,\gamma_1,\mu_1)\otimes V(\epsilon_1,\xi_1,\eta_1)$ *are isomorphic if and only if*

$$M(\lambda,\alpha,\beta,\gamma,\mu)\cong M(\lambda_1,\alpha_1,\beta_1,\gamma_1,\mu_1)\quad\textit{and}\quad V(\epsilon,\xi,\eta)\cong V(\epsilon_1,\xi_1,\eta_1).$$

Proof. The sufficiency is obvious. As for the necessity, if $M(\lambda,\alpha,\beta,\gamma,\mu)=\Omega,\Delta$ or $\Theta(2\beta\notin\mathbb{Z}_{\geqslant 0})$, then the conclusions are evident from Proposition 3.8 akin to those in [11, Theorem 4.1]. Subsequently, we focus on the case $M(\lambda,\alpha,\beta,\gamma,\mu)=\overline{\Omega}$. The other case can be treated in a similar way.

Let

$$\Phi:\overline{\Omega}\left(\lambda,\alpha,-\frac{3}{4},\gamma,\mu\right)\otimes V(\epsilon,\xi,\eta)\to\overline{\Omega}\left(\lambda_1,\alpha_1,-\frac{3}{4},\gamma_1,\mu_1\right)\otimes V(\epsilon_1,\xi_1,\eta_1)$$

be an isomorphism of $\mathcal{SW}(0)$-modules. Taking a non-zero element $v\in V(\epsilon,\xi,\eta)$, we may assume that

$$\Phi(1\otimes v)=\sum_{i=0}^{n}a_i(t)s^i\otimes w_i,$$

where $a_i(t)\in\mathbb{C}[t]$, $w_i\in V(\epsilon_1,\xi_1,\eta_1)$, $a_n(t)\neq 0$, and $w_n\neq 0$. Then, there exists a positive integer $K=\max\{K(v),K(w_i)\mid 1\leqslant i\leqslant n\}$ such that

$$x_m\cdot v=x_m\cdot w_i=0,\quad\forall\, x_m\in\{f_m,q_m,h_m,z_m,p_m,e_m,d_m\mid m\in\mathbb{Z}_{\geqslant K}\}.$$

Utilizing a similar proof to that of [11, Theorem 4.1], we obtain $\lambda=\lambda_1$, $\alpha=\alpha_1$, $\gamma=\gamma_1$, and $n=0$, which subsequently leads to $\Phi(1\otimes v)=1\otimes w_0$. Moreover, for any $m\in\mathbb{Z}_{\geqslant K}$, we have

$$\lambda^{-m}\mu^{-1}p_m\cdot(1\otimes v)=1\otimes v.\tag{4.3}$$

Upon applying Φ to both sides of (4.3), we obtain

$$\lambda^{-m}\lambda_1^m\mu^{-1}\mu_1(1\otimes w_0)=1\otimes w_0.$$

Along with $\lambda=\lambda_1$, we can get $\mu=\mu_1$. By Proposition 3.10, we immediately have

$$\overline{\Omega}\left(\lambda,\alpha,-\frac{3}{4},\gamma,\mu\right)\cong\overline{\Omega}\left(\lambda_1,\alpha_1,-\frac{3}{4},\gamma_1,\mu_1\right).$$

Furthermore, it is evident that there exists a linear injection

$$\tau:V(\epsilon,\xi,\eta)\to V(\epsilon_1,\xi_1,\eta_1)$$

such that

$$\Phi(1\otimes v)=1\otimes\tau(v),\quad\forall\, v\in V(\epsilon,\xi,\eta).\tag{4.4}$$

For $m\geqslant K$, the following two equations

$$\Phi(\lambda^{-m}d_m\cdot(1\otimes v))=\lambda^{-m}d_m\cdot\Phi(1\otimes v),$$
$$\Phi(\lambda^{-m}h_m\cdot(1\otimes v))=\lambda^{-m}h_m\cdot\Phi(1\otimes v)$$

respectively imply that

$$\Phi(s\otimes v)=s\otimes\tau(v),\tag{4.5}$$
$$\Phi(t\otimes v)=t\otimes\tau(v).\tag{4.6}$$

Moreover, for any $m\geqslant K$, according to (4.6) and $\Phi(\lambda^{-m}h_m\cdot(t\otimes v))=\lambda^{-m}h_m\cdot\Phi(t\otimes v)$, we obtain

$$\Phi(t^2\otimes v)=t^2\otimes\tau(v).\tag{4.7}$$

Equations (4.4)–(4.7) show that

$$\Phi((\mathcal{SW}(0)\cdot 1)\otimes v)=(\mathcal{SW}(0)\cdot 1)\otimes\tau(v),\quad \forall\, v\in V(\epsilon,\xi,\eta).$$

This, in conjunction with

$$\Phi(\mathcal{SW}(0)\cdot(1\otimes v))=\mathcal{SW}(0)\cdot\Phi(1\otimes v),\quad \forall\, v\in V(\epsilon,\xi,\eta),$$

leads to

$$\Phi(1\otimes\mathcal{SW}(0)\cdot v)=1\otimes(\mathcal{SW}(0)\cdot\tau(v)),\quad \forall\, v\in V(\epsilon,\xi,\eta).$$

Therefore,

$$\tau(\mathcal{SW}(0)\cdot v)=\mathcal{SW}(0)\cdot\tau(v),\quad \forall\, v\in V(\epsilon,\xi,\eta).$$

Consequently, τ is a non-zero $\mathcal{SW}(0)$-module homomorphism. Given that $V(\epsilon,\xi,\eta)$ and $V(\epsilon_1,\xi_1,\eta_1)$ are irreducible $\mathcal{SW}(0)$-modules, it follows that τ is an $\mathcal{SW}(0)$-module isomorphism. This completes the proof. □

Note that $\mathcal{SW}(0)$-modules $M(\lambda,\alpha,\beta,\gamma,\mu)$ and $M(\lambda,\alpha,\beta,\gamma,\mu)\otimes V(\epsilon,\xi,\eta)$ are non-weight modules. We now recall another well-known class of non-weight modules, namely, Whittaker modules. A Whittaker module over a Lie algebra $\mathfrak{g}$ with a triangular decomposition

$$\mathfrak{g}=\mathfrak{g}_-\oplus\mathfrak{g}_0\oplus\mathfrak{g}_+$$

is generated by a non-zero vector v such that $xv=\pi(x)v$ for all $x\in\mathfrak{g}_+$, where $\pi:\mathfrak{g}_+\to\mathbb{C}$ is a Lie algebra homomorphism. Whittaker modules were initially presented in [1,18].

Let $\underline{\theta}=(\theta_1,\theta_2,\ldots,\theta_6)\in\mathbb{C}^6$ and $J_{\underline{\theta}}$ be the left ideal of $U(\mathcal{SW}(0)_+)$ generated by the element

$$\{f_1-\theta_1,z_0-\theta_2,p_0-\theta_3,e_0-\theta_4,d_1-\theta_5,d_2-\theta_6,q_m,h_m,z_m,p_m,e_m,f_n,d_l\mid m\in\mathbb{Z}_{\geqslant 1},n\in\mathbb{Z}_{\geqslant 2},l\in\mathbb{Z}_{\geqslant 3}\}.$$

Denote $N_{\underline{\theta}}=U(\mathcal{SW}(0)_+)/J_{\underline{\theta}}$. Then the induced module

$$\mathrm{Ind}(N_{\underline{\theta}})=U(\mathcal{SW}(0))\otimes_{U(\mathcal{SW}(0)_+)}N_{\underline{\theta}}$$

is the universal Whittaker module over $\mathcal{SW}(0)$. Furthermore, every Whittaker module is a quotient of $\mathrm{Ind}(N_{\underline{\theta}})$.

Lemma 4.3. *Suppose that ϵ, ξ, and η are not all zero. Assume that $M(\lambda,\alpha,\beta,\gamma,\mu)=\Omega,\Delta,\Theta,\overline{\Omega}$ or $\overline{\Delta}$ constructed in Definition* 2.3 *and $V(\epsilon,\xi,\eta)$ is the irreducible highest weight module over $\mathcal{SW}(0)$. Let*

$$\omega_{l,m}^{(r)}=\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}d_{l-m-i}d_{m+i}\in U(\mathcal{SW}(0)),$$

where $r\in\mathbb{Z}_{\geqslant 0}$ and $l,m\in\mathbb{Z}$. Then, the following statements hold:

(1) *For any $i\in\mathbb{Z}$, d_i acts injectively on $M(\lambda,\alpha,\beta,\gamma,\mu)$ and $M(\lambda,\alpha,\beta,\gamma,\mu)\otimes V(\epsilon,\xi,\eta)$.*

(2) *For any $l,m\in\mathbb{Z}$, $r\in\mathbb{Z}_{\geqslant 3}$, and $g(s,t)\in M(\lambda,\alpha,\beta,\gamma,\mu)$, we have*

$$\omega_{l,m}^{(r)}(g(s,t))=0.$$

(3) *For any $r\in\mathbb{Z}_{\geqslant 3}$ and $0\neq g(s,t)\in M(\lambda,\alpha,\beta,\gamma,\mu)$, there exist $v\in V(\epsilon,\xi,\eta)$ and $l,m\in\mathbb{Z}$ such that*

$$\omega_{l,m}^{(r)}(g(s,t)\otimes v)\neq 0.$$

Proof. This follows from the similar proof of [11, Lemma 5.1]. We omit the details. □

Proposition 4.4. *Suppose that ϵ, ξ, and η are not all zero. Then, $M(\lambda,\alpha,\beta,\gamma,\mu)$ and $M(\lambda,\alpha,\beta,\gamma,\mu)$ $\otimes\, V(\epsilon,\xi,\eta)$ are new non-weight $\mathcal{SW}(0)$-modules.*

Proof. Suppose that W is a Whittaker module over $\mathcal{SW}(0)$ which is isomorphic to a quotient of $\mathrm{Ind}(N_{\underline{\theta}})$ for some $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_6) \in \mathbb{C}^6$. For any non-zero element $w \in W$, there exists an $i \in \mathbb{Z}_{\geqslant 1}$ such that d_i acts on w trivially. However, it is known by Lemma 4.3(1) that d_i acts injectively on $M(\lambda,\alpha,\beta,\gamma,\mu)$ and $M(\lambda,\alpha,\beta,\gamma,\mu)\otimes V(\epsilon,\xi,\eta)$. Hence, $W \not\cong M(\lambda,\alpha,\beta,\gamma,\mu)$ and $W \not\cong M(\lambda,\alpha,\beta,\gamma,\mu)\otimes V(\epsilon,\xi,\eta)$. Moreover, $M(\lambda,\alpha,\beta,\gamma,\mu) \not\cong M(\lambda,\alpha,\beta,\gamma,\mu) \otimes V(\epsilon,\xi,\eta)$ can be directly obtained from (2) and (3) in Lemma 4.3. Therefore, W, $M(\lambda,\alpha,\beta,\gamma,\mu)$, and $M(\lambda,\alpha,\beta,\gamma,\mu)\otimes V(\epsilon,\xi,\eta)$ are pairwise non-isomorphic. We complete the proof. □

5 Realization of tensor product modules as induced modules

In this section, we consider the tensor product modules $M(\lambda,\alpha,\beta,\gamma,\mu) \otimes \overline{V}(\epsilon,\xi,\eta)$ as induced modules from modules over certain subalgebras of $\mathcal{SW}(0)$, where $M(\lambda,\alpha,\beta,\gamma,\mu) = \Omega, \Delta, \Theta, \overline{\Omega}$ or $\overline{\Delta}$ constructed in Definition 2.3 and $\overline{V}(\epsilon,\xi,\eta)$ is the Verma module introduced in Section 4.

Fix $\lambda \in \mathbb{C}^*$, let

$$b_\lambda = \mathrm{span}_{\mathbb{C}}\{d_m - \lambda^m d_0, f_m, q_m, h_n, z_n, p_n, e_n \mid m \in \mathbb{Z}_{\geqslant 1}, n \in \mathbb{Z}_{\geqslant 0}\}.$$

Clearly, b_λ is a subalgebra of $\mathcal{SW}(0)$.

Definition 5.1. Let $\mathbb{C}[t]$ denote the polynomial algebra with respect to the variable t. For any $\lambda, \alpha, \mu \in \mathbb{C}^*$, $\beta, \gamma, \epsilon, \xi, \eta \in \mathbb{C}$, $m \in \mathbb{Z}_{\geqslant 1}, n \in \mathbb{Z}_{\geqslant 0}$, and $g(t) \in \mathbb{C}[t]$, we define the action of b_λ on $\mathbb{C}[t]$ as follows:

$$\begin{aligned}
\mathbb{C}[t]^{\Omega}: \ & (d_m - \lambda^m d_0) \circ g(t) = \lambda^m (m\gamma - \eta) g(t),\\
& f_m \circ g(t) = -\frac{\lambda^m}{\alpha}\left(\frac{t}{2} - \beta\right)\left(\frac{t}{2} + \beta + 1\right) g(t+2),\\
& q_m \circ g(t) = p_n \circ g(t) = 0, \quad h_n \circ g(t) = \lambda^n (t + \delta_{n,0}\epsilon) g(t),\\
& z_n \circ g(t) = \lambda^n \delta_{n,0} \xi g(t), \quad e_n \circ g(t) = \lambda^n \alpha g(t-2),\\
\mathbb{C}[t]^{\Delta}: \ & (d_m - \lambda^m d_0) \circ g(t) = \lambda^m (m\gamma - \eta) g(t),\\
& f_m \circ g(t) = \lambda^m \alpha g(t+2), \quad q_m \circ g(t) = p_n \circ g(t) = 0,\\
& h_n \circ g(t) = \lambda^n (t + \delta_{n,0}\epsilon) g(t), \quad z_n \circ g(t) = \lambda^n \delta_{n,0} \xi g(t),\\
& e_n \circ g(t) = -\frac{\lambda^n}{\alpha}\left(\frac{t}{2} + \beta\right) g\left(\frac{t}{2} - \beta - 1\right) g(t-2),\\
\mathbb{C}[t]^{\Theta}: \ & (d_m - \lambda^m d_0) \circ g(t) = \lambda^m (m\gamma - \eta) g(t),\\
& f_m \circ g(t) = -\frac{\lambda^m}{\alpha}\left(\frac{t}{2} - \beta\right) g(t+2),\\
& q_m \circ g(t) = p_n \circ g(t) = 0, \quad h_n \circ g(t) = \lambda^n (t + \delta_{n,0}\epsilon) g(t),\\
& z_n \circ g(t) = \lambda^n \delta_{n,0} \xi g(t), \quad e_n \circ g(t) = \lambda^n \alpha \left(\frac{t}{2} + \beta\right) g(t-2),\\
\mathbb{C}[t]^{\overline{\Omega}}: \ & (d_m - \lambda^m d_0) \circ g(t) = \lambda^m (m\gamma - \eta) g(t),\\
& f_m \circ g(t) = -\frac{\lambda^m}{\alpha}\left(\frac{t}{2} + \frac{3}{4}\right)\left(\frac{t}{2} + \frac{1}{4}\right) g(t+2),\\
& q_m \circ g(t) = -\frac{\lambda^m \mu}{2\alpha}\left(t + \frac{1}{2}\right) g(t+1),\\
& h_n \circ g(t) = \lambda^n (t + \delta_{n,0}\epsilon) g(t), \quad z_n \circ g(t) = \lambda^n \left(\frac{\mu^2}{2\alpha} + \delta_{n,0}\xi\right) g(t),\\
& e_n \circ g(t) = \lambda^n \alpha g(t-2), \quad p_n \circ g(t) = \lambda^n \mu g(t-1),\\
\mathbb{C}[t]^{\overline{\Delta}}: \ & (d_m - \lambda^m d_0) \circ g(t) = \lambda^m (m\gamma - \eta) g(t),
\end{aligned}$$

$$f_m \circ g(t) = \lambda^m \alpha g(t+2), \quad q_m \circ g(t) = \lambda^m \mu g(t+1),$$
$$h_n \circ g(t) = \lambda^n (t + \delta_{n,0}\epsilon) g(t), \quad z_n \circ g(t) = \lambda^n \Big(-\frac{\mu^2}{2\alpha} + \delta_{n,0}\xi \Big) g(t),$$
$$e_n \circ g(t) = -\frac{\lambda^n}{\alpha} \Big(\frac{t}{2} - \frac{3}{4}\Big)\Big(\frac{t}{2} - \frac{1}{4}\Big) g(t-2),$$
$$p_n \circ g(t) = \frac{\lambda^n \mu}{2\alpha}\Big(t - \frac{1}{2}\Big) g(t-1).$$

Proposition 5.2. *With the notations as in Definition* 5.1*, we have*

(1) $\mathbb{C}[t]^{\Omega}$, $\mathbb{C}[t]^{\Delta}$, $\mathbb{C}[t]^{\Theta}$, $\mathbb{C}[t]^{\overline{\Omega}}$, *and* $\mathbb{C}[t]^{\overline{\Delta}}$ *are* b_λ*-modules*;

(2) $\mathbb{C}[t]^{\Omega}$, $\mathbb{C}[t]^{\Delta}$, $\mathbb{C}[t]^{\overline{\Omega}}$ *and* $\mathbb{C}[t]^{\overline{\Delta}}$ *are irreducible. Moreover,* $\mathbb{C}[t]^{\Theta}$ *is irreducible if and only if* $2\beta \notin \mathbb{Z}_{\geqslant 0}$.

Proof. (1) This conclusion can be obtained through a direct calculation.

(2) Assume that $W^{M(\lambda,\alpha,\beta,\gamma,\mu)}$ is a non-zero b_λ-submodule of $\mathbb{C}[t]^{M(\lambda,\alpha,\beta,\gamma,\mu)}$, where $M(\lambda,\alpha,\beta,\gamma,\mu) = \Omega, \Delta, \Theta, \overline{\Omega}$ or $\overline{\Delta}$. It is sufficient to demonstrate that $1 \in W^{M(\lambda,\alpha,\beta,\gamma,\mu)}$. Take any non-zero element $g(t) \in W^{M(\lambda,\alpha,\beta,\gamma,\mu)}$. Based on Definition 5.1, the following equations hold by induction on $k \in \mathbb{Z}_{\geqslant 1}$:

$$f_1^k \circ g(t) = \lambda^k \alpha^k g(t+2k) \in W^{\Delta}, W^{\overline{\Delta}},$$
$$e_0^k \circ g(t) = \alpha^k g(t-2k) \in W^{\Omega}, W^{\overline{\Omega}},$$
$$f_1^k \circ g(t) = \lambda^k (-\alpha)^{-k} \prod_{i=0}^{k-1} \Big(\frac{t}{2} - \beta + i\Big) g(t+2k) \in W^{\Theta},$$
$$e_0^k \circ g(t) = \alpha^k \prod_{i=0}^{k-1} \Big(\frac{t}{2} + \beta - i\Big) g(t-2k) \in W^{\Theta}.$$

Choose k sufficiently large such that

$$(g(t), g(t-2k)) = 1, \quad (g(t), g(t+2k)) = 1,$$
$$\Big(\prod_{i=0}^{k-1}\Big(\frac{t}{2} + \beta - i\Big) g(t-2k), \prod_{i=0}^{k-1}\Big(\frac{t}{2} - \beta + i\Big) g(t+2k)\Big) = 1,$$

where $2\beta \notin \mathbb{Z}_{\geqslant 0}$. Then, $1 \in W^{M(\lambda,\alpha,\beta,\gamma,\mu)}$.

Next, assume $2\beta \in \mathbb{Z}_{\geqslant 0}$. It is straightforward to verify that the vector space

$$V = \mathbb{C}[t] \prod_{i=0}^{2\beta} \Big(\frac{t}{2} + \beta - i\Big)$$

is a proper submodule of W^{Θ}. The proof is now conclusive. □

From Proposition 5.2, we can obtain the induced $\mathcal{SW}(0)$-modules

$$\mathrm{Ind}(\mathbb{C}[t]^{M(\lambda,\alpha,\beta,\gamma,\mu)}) = U(\mathcal{SW}(0)) \otimes_{U(b_\lambda)} \mathbb{C}[t]^{M(\lambda,\alpha,\beta,\gamma,\mu)},$$

where $M(\lambda,\alpha,\beta,\gamma,\mu) = \Omega, \Delta, \Theta, \overline{\Omega}$ or $\overline{\Delta}$.

Theorem 5.3. *Keep the notations as above. Then, as* $\mathcal{SW}(0)$*-modules,*

$$M(\lambda,\alpha,\beta,\gamma,\mu) \otimes \overline{V}(\epsilon,\xi,\eta) \cong \mathrm{Ind}(\mathbb{C}[t]^{M(\lambda,\alpha,\beta,\gamma,\mu)}).$$

Proof. We only prove the case $M(\lambda,\alpha,\beta,\gamma,\mu) = \overline{\Omega}$, i.e.,

$$\overline{\Omega}\Big(\lambda, \alpha, -\frac{3}{4}, \gamma, \mu\Big) \otimes \overline{V}(\epsilon,\xi,\eta) \cong \mathrm{Ind}(\mathbb{C}[t]^{\overline{\Omega}}).$$

The remaining four cases are treated similarly. Define

$$\mathcal{U} = f_{-m_f}^{F_{-m_f}} \cdots f_0^{F_0} q_{-m_q}^{Q_{-m_q}} \cdots q_0^{Q_0} h_{-m_h}^{H_{-m_h}} \cdots h_{-1}^{H_{-1}} z_{-m_z}^{Z_{-m_z}} \cdots z_{-1}^{Z_{-1}} p_{-m_p}^{P_{-m_p}}$$

$$\begin{aligned}&\cdots p_{-1}^{P_{-1}}e_{-m_e}^{E_{-m_e}}\cdots e_{-1}^{E_{-1}}d_{-m_d}^{D_{-m_d}}\cdots d_0^{D_0},\\ \mathcal{V}=&f_{-m_f}^{F_{-m_f}}\cdots f_0^{F_0}q_{-m_q}^{Q_{-m_q}}\cdots q_0^{Q_0}h_{-m_h}^{H_{-m_h}}\cdots h_{-1}^{H_{-1}}z_{-m_z}^{Z_{-m_z}}\cdots z_{-1}^{Z_{-1}}p_{-m_p}^{P_{-m_p}}\\ &\cdots p_{-1}^{P_{-1}}e_{-m_e}^{E_{-m_e}}\cdots e_{-1}^{E_{-1}}d_{-m_d}^{D_{-m_d}}\cdots d_{-1}^{D_{-1}},\end{aligned}$$

where $D_0,\dots,D_{-m_d},E_{-1},\dots,E_{-m_e},P_{-1},\dots,P_{-m_p},Z_{-1},\cdots Z_{-m_z},H_{-1},\dots,H_{-m_h},Q_0,\dots,Q_{-m_q},F_0,$ $\dots,F_{-m_f}\in\mathbb{Z}_{\geqslant 0}$. According to the PBW Theorem, a basis of $\mathrm{Ind}(\mathbb{C}[t]^{\overline{\Omega}})$ is

$$\mathfrak{B}_1=\{\mathcal{U}\otimes t^i\mid i\in\mathbb{Z}_{\geqslant 0}\}$$

and a basis of $\overline{\Omega}(\lambda,\alpha,-\frac{3}{4},\gamma,\mu)\otimes\overline{V}(\epsilon,\xi,\eta)$ is

$$\mathfrak{B}_2=\{t^is^{D_0}\otimes\mathcal{V}\cdot v_h\mid i\in\mathbb{Z}_{\geqslant 0}\}.$$

We define the linear map

$$\psi:\mathrm{Ind}(\mathbb{C}[t]^{\overline{\Omega}})\rightarrow\overline{\Omega}\left(\lambda,\alpha,-\frac{3}{4},\gamma,a\right)\otimes\overline{V}(\eta,\epsilon,\xi,\theta)$$

given by $\psi(\mathcal{U}\otimes t^i)=\mathcal{U}(t^i\otimes v_h)$.

Claim 1. ψ is an $\mathcal{SW}(0)$-module homomorphism.

From Definition 5.1, we obtain that for any $m\in\mathbb{Z}_{\geqslant 1}$ and $i\in\mathbb{Z}_{\geqslant 0}$,

$$\begin{aligned}(d_m-\lambda^md_0)(t^i\otimes v_h)&=(d_m-\lambda^md_0)\cdot t^i\otimes v_h+t^i\otimes(d_m-\lambda^md_0)\cdot v_h\\&=\lambda^mm\gamma t^i\otimes v_h-\lambda^m\eta t^i\otimes v_h\\&=\lambda^m(m\gamma-\eta)t^i\otimes v_h\\&=((d_m-\lambda^md_0)\circ t^i)\otimes v_h.\end{aligned}$$

This implies that

$$\begin{aligned}\psi(\mathcal{U}d_m\otimes t^i)&=\psi(\lambda^m\mathcal{U}d_0\otimes t^i+\mathcal{U}(d_m-\lambda^md_0)\otimes t^i)\\&=\psi(\lambda^m\mathcal{U}d_0\otimes t^i+\mathcal{U}\otimes(d_m-\lambda^md_0)\circ t^i)\\&=\lambda^m\mathcal{U}d_0(t^i\otimes v_h)+\mathcal{U}((d_m-\lambda^md_0)\circ t^i\otimes v_h)\\&=\lambda^m\mathcal{U}d_0(t^i\otimes v_h)+\mathcal{U}(d_m-\lambda^md_0)(t^i\otimes v_h)\\&=\mathcal{U}d_m(t^i\otimes v_h).\end{aligned}$$

For any $x_m\in\{f_m,q_m\mid m\in\mathbb{Z}_{\geqslant 1}\}$ and $y_n\in\{h_n,z_n,p_n,e_n\mid n\in\mathbb{Z}_{\geqslant 0}\}$, we know that

$$\begin{aligned}\psi(\mathcal{U}x_m\otimes t^i)&=\psi(\mathcal{U}\otimes x_m\circ t^i)=\mathcal{U}(x_m\circ t^i\otimes v_h)=\mathcal{U}x_m(t^i\otimes v_h),\\ \psi(\mathcal{U}y_n\otimes t^i)&=\psi(\mathcal{U}\otimes y_n\circ t^i)=\mathcal{U}(y_n\circ t^i\otimes v_h)=\mathcal{U}y_n(t^i\otimes v_h).\end{aligned}$$

Using the PBW Theorem again, we obtain

$$\begin{aligned}x_j\mathcal{U}=&\sum_{j_a}X_{j_a}X'_{j_a}+\sum_{j_f}F_{j_f}F'_{j_f}f_{j_f}+\sum_{j_q}Q_{j_q}Q'_{j_q}q_{j_q}+\sum_{j_h}H_{j_h}H'_{j_h}h_{j_h}+\sum_{j_z}Z_{j_z}Z'_{j_z}z_{j_z}\\&+\sum_{j_p}P_{j_p}P'_{j_p}p_{j_p}+\sum_{j_e}E_{j_e}E'_{j_e}e_{j_e}+\sum_{j_d}D_{j_d}D'_{j_d}d_{j_d},\end{aligned}$$

where $x_j\in\{f_j,p_j,h_j,z_j,p_j,e_j,d_j\mid j\in\mathbb{Z}\}$, $X_{j_a},F_{j_f},Q_{j_q},H_{j_h},Z_{j_z},P_{j_p},E_{j_e},D_{j_d}\in\mathbb{C}$, $X'_{j_a}\otimes t^i$, $F'_{j_f}\otimes t^i$, $Q'_{j_q}\otimes t^i$, $H'_{j_h}\otimes t^i$, $Z'_{j_z}\otimes t^i$, $P'_{j_p}\otimes t^i$, $E'_{j_e}\otimes t^i$, $D'_{j_d}\otimes t^i\in\mathfrak{B}_1$, and $j_f,j_q,j_d\in\mathbb{Z}_{\geqslant 1}$, $j_h,j_z,j_p,j_e\in\mathbb{Z}_{\geqslant 0}$. Subsequently, it is evident that

$$\psi(x_j\mathcal{U}\otimes t^i)=\psi\Bigg(\sum_{j_a}X_{j_a}X'_{j_a}\otimes t^i+\sum_{j_f}F_{j_f}F'_{j_f}f_{j_f}\otimes t^i+\sum_{j_q}Q_{j_q}Q'_{j_q}q_{j_q}\otimes t^i+\sum_{j_h}H_{j_h}H'_{j_h}h_{j_h}\otimes t^i$$

$$
\begin{aligned}
&+\sum_{j_z} Z_{j_z}Z'_{j_z}z_{j_z}\otimes t^i+\sum_{j_p} P_{j_p}P'_{j_p}p_{j_p}\otimes t^i+\sum_{j_e} E_{j_e}E'_{j_e}e_{j_e}\otimes t^i+\sum_{j_d} D_{j_d}D'_{j_d}d_{j_d}\otimes t^i\Big)\\
&=\sum_{j_a} X_{j_a}X'_{j_a}(t^i\otimes v_h)+\sum_{j_f} F_{j_f}F'_{j_f}f_{j_f}(t^i\otimes v_h)+\sum_{j_q} Q_{j_q}Q'_{j_q}q_{j_q}(t^i\otimes v_h)\\
&+\sum_{j_h} H_{j_h}H'_{j_h}h_{j_h}(t^i\otimes v_h)+\sum_{j_z} Z_{j_z}Z'_{j_z}z_{j_z}(t^i\otimes v_h)+\sum_{j_p} P_{j_p}P'_{j_p}p_{j_p}(t^i\otimes v_h)\\
&+\sum_{j_e} E_{j_e}E'_{j_e}e_{j_e}(t^i\otimes v_h)+\sum_{j_d} D_{j_d}D'_{j_d}d_{j_d}(t^i\otimes v_h)\\
&=x_j\mathcal{U}(t^i\otimes v_h)=x_j\psi(\mathcal{U}\otimes t^i).
\end{aligned}
$$

Therefore, ψ is an $\mathcal{SW}(0)$-module homomorphism.

Claim 2. ψ is a surjection.

We need to show $\overline{\Omega}(\lambda,\alpha,-\frac{3}{4},\gamma,\mu)\otimes\overline{V}(\epsilon,\xi,\eta)\subseteq \mathrm{Im}(\psi)$. It is easy to see that

$$\psi(1\otimes t^i)=t^i\otimes v_h\in \mathrm{Im}(\psi)$$

for all $i\in\mathbb{Z}_{\geqslant 0}$. Meanwhile, for any $j\in\mathbb{Z}_{\geqslant 0}$, $\psi(d_0^j\otimes t^i)=d_0^j(t^i\otimes v_h)\in\mathrm{Im}(\psi)$, this results in $t^is^j\otimes v_h\in\mathrm{Im}(\psi)$, i.e.,

$$\overline{\Omega}\bigg(\lambda,\alpha,-\frac{3}{4},\gamma,\mu\bigg)\otimes v_h\subseteq\mathrm{Im}(\psi).$$

Letting $\mathcal{U}$ act on $\overline{\Omega}(\lambda,\alpha,-\frac{3}{4},\gamma,\mu)\otimes v_h$, we have $\overline{\Omega}(\lambda,\alpha,-\frac{3}{4},\gamma,\mu)\otimes\overline{V}(\epsilon,\xi,\eta)\subseteq\mathrm{Im}(\psi)$. Consequently, ψ is surjective.

Claim 3. ψ is a injection.

Let

$$
\begin{aligned}
\mathcal{V}'=\,&f_{-m'_f}^{F'_{-m'_f}}\cdots f_0^{F'_0}q_{-m'_q}^{Q'_{-m'_q}}\cdots q_0^{Q'_0}h_{-m'_h}^{H'_{-m'_h}}\cdots h_{-1}^{H'_{-1}}z_{-m'_z}^{Z'_{-m'_z}}\cdots z_{-1}^{Z'_{-1}}p_{-m'_p}^{P'_{-m'_p}}\cdots p_{-1}^{P'_{-1}}e_{-m'_e}^{E'_{-m'_e}}\\
&\cdots e_{-1}^{E'_{-1}}d_{-m'_d}^{D'_{-m'_d}}\cdots d_{-1}^{D'_{-1}},
\end{aligned}
$$

where $D'_{-1},\dots,D'_{-m'_d},E'_{-1},\dots,E'_{-m'_e},P'_{-1},\dots,P'_{-m'_p},Z'_{-1},\cdots Z'_{-m'_z},H'_{-1},\dots,H'_{-m'_h},Q'_0,\dots,Q'_{-m'_q},F'_0,\dots,F'_{-m'_f}\in\mathbb{Z}_{\geqslant 0}$. We now define a total order $\prec$ on $\mathfrak{B}_2$, i.e., $t^is^{D_0}\otimes(\mathcal{V}\cdot v_h)\prec t^{i'}s^{D'_0}\otimes(\mathcal{V}'\cdot v_h)$ if and only if

$$
\begin{aligned}
&(D_{-1},\dots,D_{-m_d},\overbrace{0,\dots,0}^{m'_d},E_{-1},\dots,E_{-m_e},\overbrace{0,\dots,0}^{m'_e},P_{-1},\dots,P_{-m_p},\overbrace{0,\dots,0}^{m'_p},Z_{-1},\dots,Z_{-m_z},\\
&\overbrace{0,\dots,0}^{m'_z},H_{-1},\dots,H_{-m_h},\overbrace{0,\dots,0}^{m'_h},Q_0,\dots,Q_{-m_q},\overbrace{0,\dots,0}^{m'_q},F_0,\dots,F_{-m_f},\overbrace{0,\dots,0}^{m'_f},D_0,i)\\
&<(D'_{-1},\dots,D'_{-m'_d},\overbrace{0,\dots,0}^{m_d},E'_{-1},\dots,E'_{-m'_e},\overbrace{0,\dots,0}^{m_e},P'_{-1},\dots,P'_{-m'_p},\overbrace{0,\dots,0}^{m_p},Z'_{-1},\dots,Z'_{-m'_z},\\
&\overbrace{0,\dots,0}^{m_z},H'_{-1},\dots,H'_{-m'_h},\overbrace{0,\dots,0}^{m_h},Q'_0,\dots,Q'_{-m'_q},\overbrace{0,\dots,0}^{m'_q},F'_0,\dots,F'_{-m'_f},\overbrace{0,\dots,0}^{m_f},D'_0,i'),
\end{aligned}
$$

where $(a_1,\dots,a_l)<(b_1,\dots,b_l)\Leftrightarrow\exists\,k>0$ such that $a_i=b_i$ for all $i<k$ and $a_k<b_k$.

Through calculation, we obtain

$$\mathcal{U}(t^i\otimes v_h)=t^is^{D_0}\otimes(\mathcal{V}\cdot v_h)+\text{lower terms},$$

which means that the set $\{\mathcal{U}(t^i\otimes v_h)\mid i\in\mathbb{Z}_{\geqslant 0}\}$ is also a basis of $\mathfrak{B}_2$. Thus, ψ is injective. We thus complete the proof. $\square$

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