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Multiplicative forms on Poisson groupoids

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Abstract First, we prove a decomposition formula for any multiplicative differential form on a Lie groupoid $\mathcal{G}$. Next, we prove that if $\mathcal{G}$ is a Poisson Lie groupoid, then the space $\Omega^\bullet_{\mathrm{mult}}(\mathcal{G})$ of multiplicative forms on $\mathcal{G}$ has a differential graded Lie algebra (DGLA) structure. Furthermore, when combined with $\Omega^\bullet(M)$, which is the space of forms on the base manifold M of $\mathcal{G}$, $\Omega^\bullet_{\mathrm{mult}}(\mathcal{G})$ forms a canonical DGLA crossed module. This supplements a previously known fact that multiplicative multi-vector fields on $\mathcal{G}$ form a DGLA crossed module with the Schouten algebra $\Gamma(\wedge^\bullet A)$ stemming from the Lie algebroid A of $\mathcal{G}$.

Keywords multiplicative form, Poisson groupoid, Lie algebra crossed module, characteristic pair

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1 Introduction

The motivation for this study derives from two primary sources. Firstly, we seek to build upon our prior research on multiplicative multi-vector fields on Lie groupoids [9]; our current focus is directed towards multiplicative forms. Secondly, we aim to investigate Poisson Lie groupoids, and in particular, the constituting multiplicative forms of their induced graded Lie algebras.

The concept of Lie groupoids was introduced by Ehresmann [15,16] in the late 1950s to describe smooth symmetries of a smooth family of objects, i.e., the collection of arrows is a manifold $\mathcal{G}$, the set of objects is a manifold M called the base, and all the structure maps of the groupoid are smooth. Taking sources of arrows defines the source map $s : \mathcal{G} \to M$, and similarly, one has the target map $t : \mathcal{G} \to M$, both being considered as part of the groupoid structures. Let us denote by $\mathcal{G} \rightrightarrows M$ for such a Lie groupoid. Its infinitesimal counterpart, i.e., the Lie algebroid of $\mathcal{G}$, is defined and denoted by $A := \ker(s_*)|_M$, i.e., vectors tangent to the s-fibers of $\mathcal{G}$ along M. The theory of Lie groupoids and Lie algebroids has become a far-reaching extension of the usual Lie theory, and it finds application in many areas of mathematics. The reader is referred to the texts [13, 28] for more useful information on this subject.

Geometric structures compatible with the groupoid structure are often called multiplicative. Multiplicative objects have attracted widespread attention because they can be regarded as geometric structures

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on differentiable stacks [33]. We refer to [21] for a thorough survey of different types of multiplicative structures on Lie groupoids defined and studied in the past decades. For this article, we hope that readers have some familiarities of works on multiplicative Poisson structures [25, 35, 36].

A multiplicative vector field on a Lie groupoid is a vector field generating a flow of local groupoid automorphisms [30]. In a recent work [3], Bonechi et al. have shown a canonical graded Lie algebra (GLA for short) crossed module structure on the space of multiplicative multi-vector fields of a groupoid—let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid over M and A its Lie algebroid. The aforementioned GLA crossed module is actually composed of a triple:

$$\Gamma(\wedge^\bullet A) \xrightarrow{T} \mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G}), \quad \text{where } T(u) := \overleftarrow{u} - \overrightarrow{u}, \quad \forall\, u \in \Gamma(\wedge^\bullet A). \tag{1.1}$$

Here, $\Gamma(\wedge^\bullet A)$ is the Schouten algebra of A, $\mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G})$ stands for the space of multiplicative multi-vector fields of $\mathcal{G}$, and the action of $\mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G})$ on $\Gamma(\wedge^\bullet A)$ is intrinsic (see Example 5.12). It is also proved in [3] that the homotopy equivalence class of the GLA crossed module $\Gamma(\wedge^\bullet A) \xrightarrow{T} \mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G})$ is invariant under Morita equivalence of Lie groupoids, and thus is considered as the multi-vector fields on the corresponding differentiable stack.

In our previous work [9], we have established a formula for multiplicative multi-vector fields (see Theorem 2.6)—any multiplicative k-vector field Π on a Lie groupoid $\mathcal{G} \rightrightarrows M$ can be decomposed into

$$\Pi_g = R_{g*}c([b_g]) + L_{[b_g]}\left(\frac{1-\mathrm{e}^{-D_\rho}}{D_\rho}(\pi)\right)_{s(g)}, \tag{1.2}$$

where $g \in \mathcal{G}$, b_g is a bisection through g, $c : \mathfrak{J}\mathcal{G} \to \wedge^k A$ is a 1-cocycle, $\pi \in \Gamma(TM \otimes (\wedge^{k-1}A))$ is a ρ-compatible $(k,0)$-tensor (see Definition 2.3), and D_ρ is a degree 0 derivation on $\wedge^\bullet(TM \oplus A)$. We call (c,π) a $(k,0)$-characteristic pair on $\mathcal{G}$. More facts about multiplicative multi-vector fields are recalled in Subsection 2.3.

In duality to multiplicative multi-vector fields, differential forms on Lie groupoids suitably compatible with the groupoid structure are referred to as multiplicative forms and are the main objects of interest in this paper. After their first appearance with the advent of symplectic groupoids [20, 35], a lot of interesting works on multiplicative forms of Lie groupoids have emerged. For example, a one-to-one correspondence between multiplicative forms (with certain coefficients) and Spencer operators on Lie algebroids is established in [14], from which we find a lot of inspiration.

Our first objective is to decompose multiplicative forms by drawing an analogy with (1.2) of multiplicative multi-vector fields. Crainic et al. [14] have discovered an important result in which a multiplicative k-form Θ on $\mathcal{G}$ can be characterized by a $(0,k)$-characteristic pair (e,θ). Here, e is a 1-cocycle of the jet groupoid $\mathfrak{J}\mathcal{G}$ valued in $\wedge^k T^*M$, i.e. $e \in Z^1(\mathfrak{J}\mathcal{G}, \wedge^k T^*M)$, and $\theta \in \Gamma(A^* \otimes (\wedge^{k-1}T^*M))$ is a ρ-compatible $(0,k)$-tensor (see Definition 3.1). Using this tool, we can analyze the constituent elements of multiplicative forms and determine the relationship between multiplicative forms Θ and (e,θ). Our first main Theorem 3.13 states that for any bisection b_g passing through $g \in \mathcal{G}$, a nice decomposition can be obtained, i.e.,

$$\Theta_g = R^*_{[b_g^{-1}]}\left(e[b_g] + \frac{\mathrm{e}^{D_{\rho^*}}-1}{D_{\rho^*}}(\theta)_{t(g)}\right). \tag{1.3}$$

Here, D_{ρ^*} is a degree 0 derivation $\wedge^\bullet(T^*M \oplus A^*) \to \wedge^\bullet(T^*M \oplus A^*)$ (see (3.3)).

The infinitesimal counterparts of multiplicative forms on Lie groupoids are certain structures on Lie algebroids, and they are called infinitesimally multiplicative (IM for short) forms [4]. Likewise, the infinitesimals of $(0,k)$-characteristic pairs on Lie groupoids are $(0,k)$-characteristic pairs on Lie algebroids (see Definition 3.20 and Proposition 3.21), and we show that they are equivalent to IM-forms (see Proposition 3.25).

Our second goal is to create a new object (in parallel to the aforementioned GLA crossed module)—a triple involving

$$\Omega^\bullet(M) \xrightarrow{J} \Omega^\bullet_{\mathrm{mult}}(\mathcal{G}), \quad \text{where } J(\gamma) := s^*\gamma - t^*\gamma, \quad \forall\, \gamma \in \Omega^\bullet(M). \tag{1.4}$$

Here, $\Omega^\bullet_{\rm mult}(\mathcal{G})$ refers to the space of multiplicative forms on $\mathcal{G}$, and s and t are the source and target maps of the $\mathcal{G}$ groupoid, respectively. However, as of now, the triple in (1.4) is only a morphism of cochain complexes, where the spaces $\Omega^\bullet(M)$ and $\Omega^\bullet_{\rm mult}(\mathcal{G})$ have the standard de Rham differentials. Our main result regarding this triple states that if $\mathcal{G}$ is a Poisson Lie groupoid, i.e., equipped with a multiplicative Poisson structure P, then the triple $\Omega^\bullet(M) \xrightarrow{J} \Omega^\bullet_{\rm mult}(\mathcal{G})$ becomes a differential graded Lie algebra (DGLA) crossed module.

Poisson Lie groupoids, which unify Poisson Lie groups [27] and symplectic groupoids, were introduced by Weinstein [35, 36]. The Poisson structure P on $\mathcal{G}$ upgrades the triple in (1.1) to a DGLA crossed module, and the two DGLA crossed modules (1.1) and (1.4) are related by a natural morphism. This is our second main Theorem 5.14.

Note that Ortiz and Waldron [32] had already discovered part of Theorem 5.14, which indicates that the data $\Omega^1(M) \xrightarrow{J} \Omega^1_{\rm mult}(\mathcal{G})$ form a Lie algebra crossed module. This elegant fact is reasserted in Theorem 5.5, and to ensure comprehensiveness, we present a proof employing our theory of characteristic pairs.

The rest of this paper is organized as follows. In Section 2, we provide an overview of the fundamental concepts of Lie groupoids, Lie algebroids, their corresponding jets, and multiplicative multi-vector fields. We also establish the relation between $(k, 0)$-characteristic pairs and multiplicative multi-vector fields. Next, in Section 3, we examine multiplicative forms, $(0, k)$-characteristic pairs on Lie groupoids and their interconnections, culminating in our main result, Theorem 3.13. We also explore the infinitesimal theories of $(0, k)$-characteristic pairs and IM forms on Lie algebroids. We then focus on transitive Lie groupoids and Lie algebroids. In Section 4, we derive a variety of essential formulas related to multiplicative multi-vector fields and forms. Finally, in Section 5, we investigate multiplicative forms on Poisson groupoids, and present our second main outcome, Theorem 5.14, along with its proof, which depends on propositions and lemmas developed earlier in the paper.

To make it more concise, we recommend consulting the following works related to Lie algebroid IM forms integrated to Lie groupoids: Bursztyn and Cabrera [4], Bursztyn et al. [5], and Cabrera et al. [7,8]. For further information on multiplicative tensors and their infinitesimals, please refer to Bursztyn and Drummond's work [6]. Additionally, we suggest exploring a relevant piece of work by Lean et al. [26] on multiplicative generalized complex structures that could be a valuable resource for further research.

List of conventions and notations. Throughout the paper, M stands for a smooth manifold and k denotes a *positive integer* (usually within the range $1 \leqslant k \leqslant \dim M + 1$). Furthermore, 'GLA' stands for 'graded Lie algebra', and 'DGLA' stands for 'differential graded Lie algebra'. Some commonly used symbols are listed below:

(1) $\mathrm{Sh}(p, q)$: the set of (p, q)-shuffles; a (p, q)-shuffle is a permutation σ of the set $\{1, 2, \ldots, p+q\}$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$;

(2) $T^\sharp$: the contraction map $U^* \to V$, $u^* \mapsto \iota_{u^*}T$ for a given tensor $T \in U \otimes V$;

(3) $\mathcal{G} \rightrightarrows M$: a Lie groupoid over M;

(4) $\mathfrak{X}^\bullet_{\rm mult}(\mathcal{G})$ $(\Omega^\bullet_{\rm mult}(\mathcal{G}))$: the space of multiplicative multi-vector fields (multiplicative forms) of $\mathcal{G}$;

(5) $(A, [-,-], \rho)$: a Lie algebroid with its bracket and anchor map; usually, A is the Lie algebroid of $\mathcal{G}$;

(6) b_g: a (local) bisection on $\mathcal{G}$ which passes through $g \in \mathcal{G}$;

(7) $[b_g]$: the first jet of b_g at $g \in \mathcal{G}$;

(8) $\mathfrak{J}\mathcal{G}$ $(\mathfrak{J}A)$: the jet groupoid of $\mathcal{G}$ (the jet Lie algebroid of A);

(9) $\jmath^1 u$: the section of jets in $\Gamma(\mathfrak{J}A)$ arising from $u \in \Gamma(A)$ (see (2.4));

(10) $\mathfrak{H}$ $(\mathfrak{h})$: the bundle of isotropy jet groups (the bundle of isotropy Lie algebras);

(11) Ad: the adjoint action of the jet groupoid $\mathfrak{J}\mathcal{G}$ on A and TM and also on $\wedge^k A$ and $\wedge^k TM$;

(12) $\mathrm{Ad}^\vee$: the coadjoint action of $\mathfrak{J}\mathcal{G}$ on A^*, T^*M and also on $\wedge^k A^*, \wedge^k T^*M$;

(13) D_{ρ^*}: a degree 0 derivation $\wedge^\bullet(T^*M \oplus A^*) \to \wedge^\bullet(T^*M \oplus A^*)$ (see (3.3));

(14) $\mathrm{CP}^\bullet(A)$: the space of characteristic pairs on A;

(15) $\mathrm{IM}^\bullet(A)$: the space of IM-forms on A.

2 Preliminaries

This section provides an overview of preliminary concepts such as Lie groupoids, Lie algebroids, their jets, and our notations and conventions regarding Lie groupoid and Lie algebroid modules. Additionally, we briefly recall multiplicative multi-vector fields and forms (see [13, 28] and our previous work [9]).

2.1 Lie algebroid and groupoid modules

A Lie algebroid is an $\mathbb{R}$-vector bundle $A \to M$ endowed with a Lie bracket $[\cdot, \cdot]$ in its space of sections $\Gamma(A)$, together with a bundle map $\rho: A \to TM$ called the **anchor**, such that $\rho: \Gamma(A) \to \mathfrak{X}^1(M)$ is a morphism of Lie algebras and

$$[u, fv] = f[u, v] + (\rho(u)f)v$$

holds for all $u, v \in \Gamma(A)$ and $f \in C^\infty(M)$.

By an ***A*-module**, we mean a vector bundle $E \to M$, which is endowed with an A-connection:

$$\nabla: \Gamma(A) \times \Gamma(E) \to \Gamma(E)$$

which is flat, i.e.,

$$\nabla_{[u,v]}e = \nabla_u \nabla_v e - \nabla_v \nabla_u e, \quad \forall\, u, v \in \Gamma(A), \quad e \in \Gamma(E).$$

Given an A-module E, we have the standard Chevalley-Eilenberg complex $(C^\bullet(A, E), d_A)$, where

$$C^\bullet(A, E) := \Gamma(\mathrm{Hom}(\wedge^\bullet A, E))$$

and the coboundary operator $d_A : C^n(A, E) \to C^{n+1}(A, E)$ is given by

$$\begin{aligned}&(d_A\lambda)(u_0, u_1, \ldots, u_n)\\&\quad = \sum_i (-1)^i \nabla_{u_i} \lambda(\ldots, \widehat{u}_i, \ldots) + \sum_{i<j} (-1)^{i+j} \lambda([u_i, u_j], \ldots, \widehat{u}_i, \ldots, \widehat{u}_j, \ldots)\end{aligned}$$

for all $u_0, \ldots, u_n \in \Gamma(A)$.

Let $\mathcal{G}$ be a Lie groupoid over a smooth manifold M with its source and target maps being denoted by $s : \mathcal{G} \to M$ and $t : \mathcal{G} \to M$, respectively. We use the short notation $\mathcal{G} \rightrightarrows M$ in which s and t are omitted to denote such a Lie groupoid. We treat the set of identities $M \hookrightarrow \mathcal{G}$ as a submanifold of $\mathcal{G}$. The groupoid multiplication of two elements g and r is denoted by gr, provided that $s(g) = t(r)$. The collection of such pairs (g, r), called composable pairs, is denoted by $\mathcal{G}^{(2)}$. The groupoid inverse map of $\mathcal{G}$ is denoted by $\mathrm{inv}: \mathcal{G} \to \mathcal{G}$. For $g \in \mathcal{G}$, its inverse $\mathrm{inv}(g)$ is also denoted by g^{-1}.

A **bisection** of $\mathcal{G}$ is a smooth splitting $b : M \to \mathcal{G}$ of the source map s (i.e., $sb = \mathrm{id}_M$) such that

$$\phi_b := tb : M \to M$$

is a diffeomorphism. The set of bisections of $\mathcal{G}$ forms a group which we denote by $\mathrm{Bis}(\mathcal{G})$ and its identity element is id_M. The multiplication bb' of b and $b' \in \mathrm{Bis}(\mathcal{G})$ is given by

$$(bb')(x) := b(\phi_{b'}(x))b'(x), \quad \forall\, x \in M.$$

A bisection b defines a diffeomorphism of $\mathcal{G}$ by left multiplication:

$$L_b : \mathcal{G} \to \mathcal{G}, \quad L_b(g) := b(t(g))g, \quad \forall\, g \in \mathcal{G}.$$

We call L_b the left translation by b. Similarly, b defines the right translation:

$$R_b : \mathcal{G} \to \mathcal{G}, \quad R_b(g) := gb(\phi_b^{-1}s(g)), \quad \forall\, g \in \mathcal{G}.$$

In particular, there is an induced map $R_b^! : M \to \mathcal{G}$ which is the restriction of R_b on M:

$$R_b^!(x) := R_b(x) = b(\phi_b^{-1}(x)), \quad \forall\, x \in M. \tag{2.1}$$

Clearly, $R_b^!$ is a section of the fibre bundle $\mathcal{G} \xrightarrow{t} M$.

A local bisection b passing through the point $b(x) = g$ on $\mathcal{G}$, where $x = s(g) \in U$, is also denoted by b_g, to emphasize the particular point g. *In the sequel, by saying a bisection through g, we mean a local bisection that passes through g.*

A (left) $\mathcal{G}$**-module** is a vector bundle $E \to M$ together with a smooth assignment: $g \mapsto \Phi_g$, where $g \in \mathcal{G}$ and $\Phi_g \in \mathrm{GL}(E_{s(g)}, E_{t(g)})$ (the set of isomorphisms from $E_{s(g)}$ to $E_{t(g)}$), satisfying

(1) $\Phi_x = \mathrm{id}_{E_x}$ for all $x \in M$;

(2) $\Phi_{gr} = \Phi_g \Phi_r$ for all composable pairs (g, r).

We recall the Lie algebroid $(A, \rho, [\ ,\ ]_A)$ of the Lie groupoid $\mathcal{G}$. In fact, $A = \ker s_*|_M$ and the anchor map $\rho:\ A \to TM$ is simply t_*. For $u, v \in \Gamma(A)$, the Lie bracket $[u, v]$ is determined by

$$\overrightarrow{[u, v]} = [\overrightarrow{u}, \overrightarrow{v}].$$

Here, $\overrightarrow{u}$ denotes the right-invariant vector field on $\mathcal{G}$ corresponding to u. In the meantime, the left-invariant vector field corresponding to u, denoted by $\overleftarrow{u}$, is related to $\overrightarrow{u}$ via

$$\overleftarrow{u} = -\mathrm{inv}_*(\overrightarrow{u}) = -\overleftarrow{\mathrm{inv}_* u}.$$

A $\mathcal{G}$-module E is also an A-module, i.e., there is an A-action on E:

$$\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$$

defined by

$$\nabla_u e = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \Phi_{\exp(-\epsilon u)} e, \quad \forall\, u \in \Gamma(A), \quad e \in \Gamma(E).$$

An n**-cochain** on $\mathcal{G}$ valued in the $\mathcal{G}$-module E is a smooth map $c : \mathcal{G}^{(n)} \to E$ such that

$$c(g_1, \dots, g_n) \in E_{t(g_1)}.$$

Denote by $C^n(\mathcal{G}, E)$ the space of n-cochains. The coboundary operator

$$d_{\mathcal{G}} : C^n(\mathcal{G}, E) \to C^{n+1}(\mathcal{G}, E)$$

is standard.

(1) For $n = 0$ and $\nu \in C^0(\mathcal{G}, E) = \Gamma(E)$, define

$$(d_{\mathcal{G}}\nu)(g) = \Phi_g \nu_{s(g)} - \nu_{t(g)}, \quad \forall\, g \in \mathcal{G}.$$

(2) For $c \in C^n(\mathcal{G}, E)$ $(n \geqslant 1)$ on $\mathcal{G}$, define

$$\begin{aligned}&(d_{\mathcal{G}}c)(g_0, g_1, \dots, g_n)\\ &\quad = \Phi_{g_0} c(g_1, \dots, g_n) + \sum_{i=0}^{n-1} (-1)^{i-1} c(g_0, \dots, g_i g_{i+1}, \dots, g_n) + (-1)^{n+1} c(g_0, \dots, g_{n-1}).\end{aligned}$$

A groupoid 1-cocycle c induces a Lie algebroid 1-cocycle $\widehat{c} : A \to E$ by the following formula. For each $u \in A_x$, choose a smooth curve $\gamma(\epsilon)$ in the s-fibre $s^{-1}(x)$ such that $\gamma'(0) = u$. Then, $\widehat{c}(u) \in E_x$ is defined by

$$\widehat{c}(u) := -\frac{d}{d\epsilon}\Big|_{\epsilon=0} \Phi^{-1}_{\gamma(\epsilon)} c(\gamma(\epsilon)) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} c(\gamma(\epsilon)^{-1}). \tag{2.2}$$

We call $\widehat{c}$ the infinitesimal of c. Note that our convention is slightly different from that in [1] (up to a minus sign). Also, it is easily verified that, for $\nu \in \Gamma(E)$, we have

$$\widehat{d_{\mathcal{G}}\nu} = d_A \nu. \tag{2.3}$$

2.2 The jet Lie algebroid and the jet Lie groupoid

The first jet space of a Lie algebroid A, denoted by $\mathfrak{J}A$, is also a Lie algebroid (see [10, 14]), which fits into an exact sequence of Lie algebroids:

$$0 \to \mathfrak{h} \xrightarrow{i} \mathfrak{J}A \xrightarrow{p} A \to 0.$$

Here,

$$\mathfrak{h} = T^*M \otimes A = \mathrm{Hom}(TM, A)$$

is a bundle of Lie algebras. We call $\mathfrak{h}$ the bundle of **isotropy jet Lie algebras**.

We have a natural lifting map $\jmath^1 : \Gamma(A) \to \Gamma(\mathfrak{J}A)$ that sends $u \in \Gamma(A)$ to its jet $\jmath^1 u \in \Gamma(\mathfrak{J}A)$:

$$(\jmath^1 u)_x = [u]_x, \quad \forall\, x \in M. \tag{2.4}$$

Moreover, the Lie bracket of $\Gamma(\mathfrak{J}A)$ is determined by the relation:

$$[\jmath^1 u_1, \jmath^1 u_2] = \jmath^1 [u_1, u_2].$$

The anchor of $\mathfrak{J}A$ is simply given by

$$\rho_{\mathfrak{J}A}(\jmath^1 u_1 + df \otimes u_2) = \rho(u_1).$$

The vector bundles A and TM are modules of the jet algebroid $\mathfrak{J}A$ via adjoint actions:

$$\mathrm{ad}_{\jmath^1 u} v = [u, v], \quad \mathrm{ad}_{df\otimes u} v = -\rho(v)(f)u,$$

and

$$\mathrm{ad}_{\jmath^1 u} X = [\rho(u), X], \quad \mathrm{ad}_{df\otimes u} X = -X(f)u,$$

where $u, v \in \Gamma(A)$, $f \in C^\infty(M)$, and $X \in \mathfrak{X}^1(M)$. Then, $\mathfrak{J}A$ also naturally acts on $\wedge^k A$, $\wedge^k TM$, $\wedge^k A^*$, and $\wedge^k T^*M$.

Let b_g and b'_g be two bisections through $g \in \mathcal{G}$. They are said to be equivalent at g if $b_{*x} = b'_{*x} : T_xM \to T_g\mathcal{G}$, where $x = s(g)$. The equivalence class, denoted by $[b]_x$, or $[b_g]$, is called the first jet of b_g.

The **(first) jet groupoid** $\mathfrak{J}\mathcal{G}$ of a Lie groupoid $\mathcal{G}$, consisting of such jets $[b_g]$, is a Lie groupoid over M. The source and target maps of $\mathfrak{J}\mathcal{G}$ are given by

$$s[b_g] = s(g), \quad t[b_g] = t(g).$$

The multiplication is given by

$$[b_g][b'_r] = [(bb')_{gr}]$$

for two bisections b_g and b'_r through g and r, respectively.

Given a bisection b_g through g, its first jet induces a left translation

$$L_{[b_g]} : T\mathcal{G}|_{t^{-1}(x)} \to T\mathcal{G}|_{t^{-1}(y)},$$

where $x = s(g)$ and $y = \phi_b(x) = t(g)$. Similarly, we have a right translation

$$R_{[b_g]} : T\mathcal{G}|_{s^{-1}(y)} \to T\mathcal{G}|_{s^{-1}(x)}.$$

We also note that when $L_{[b_g]}$ is restricted on $(\ker t_*)|_{t^{-1}(x)}$, it is exactly the left translation L_{g*}. Similarly, when restricted on $(\ker s_*)|_{s^{-1}(y)}$, $R_{[b_g]}$ is exactly the right translation R_{g*}.

Moreover, the restriction of $L_{[b_g]}$ to T_xM is exactly the tangent map $b_{*x} : T_xM \to T_g\mathcal{G}$, i.e.,

$$L_{[b_g]}(X) = b_{g*}(X) \quad \text{for } X \in T_xM.$$

A bisection b of $\mathcal{G}$ acts on $\mathcal{G}$ by conjugation

$$\mathrm{AD}_b(g) = R_{b^{-1}} L_b(g) = b(t(g))g(b(s(g)))^{-1},$$

which maps units to units and s-fibres to s-fibres (t-fibres to t-fibres as well). There is an induced action of the jet groupoid $\mathfrak{J}\mathcal{G}$ on A and TM.

Definition 2.1. There is an action of $\mathfrak{J}\mathcal{G}$, called the **adjoint action**, on A and TM:

$$\mathrm{Ad}_{[b_g]} : A_x \to A_{\phi_b(x)}, \quad T_xM \to T_{\phi_b(x)}M, \quad \text{where } x = s(g)$$

defined by

$$\mathrm{Ad}_{[b_g]}u = R_{g^{-1}*}L_{[b_g]}(u) = R_{g^{-1}*}(L_{g*}(u - \rho(u)) + b_*(\rho(u))), \quad u \in A_x$$

and

$$\mathrm{Ad}_{[b_g]}X = R_{[b_g]^{-1}}L_{[b_g]}(X) = \phi_{b*}(X), \quad X \in T_xM.$$

Then, we also have the adjoint actions of $\mathfrak{J}\mathcal{G}$ on $\wedge^k A$ and $\wedge^k TM$, all denoted by Ad. In the meantime, we have the **coadjoint actions** of $\mathfrak{J}\mathcal{G}$ on $\wedge^k A^*$ and $\wedge^k T^*M$ which we denote by $\mathrm{Ad}^\vee$, i.e.,

$$\mathrm{Ad}^\vee_{[b_g]} := (\mathrm{Ad}_{[b_g]^{-1}})^*.$$

Note that the Lie algebroid of $\mathfrak{J}\mathcal{G}$ is the jet Lie algebroid $\mathfrak{J}A$. By taking derivations, we get the adjoint and coadjoint actions of $\mathfrak{J}A$ introduced preciously.

The following exact sequence of groupoids can be easily established:

$$1 \to \mathfrak{H} \xrightarrow{I} \mathfrak{J}\mathcal{G} \xrightarrow{P} \mathcal{G} \to 1.$$

The space $\mathfrak{H}$ consists of jets $[h]$, where h is a bisection through $x \in M$. Let us call $\mathfrak{H}$ the bundle of **isotropy jet groups**.

For $[h] \in \mathfrak{H}_x$, there exists an $H : T_xM \to A_x$ such that

$$h_*(X) = H(X) + X, \quad X \in T_xM,$$

and

$$\phi_{h*} = t_*h_* = \mathrm{id} + \rho H \in \mathrm{GL}(T_xM).$$

Let us introduce

$$\underline{\mathrm{Hom}}(TM, A) := \{H \in \mathrm{Hom}(TM, A); \mathrm{id} + \rho H \in \mathrm{GL}(TM)\}.$$

Then, we have $\mathfrak{H} \cong \underline{\mathrm{Hom}}(TM, A)$. In the sequel, for $H \in \underline{\mathrm{Hom}}(T_xM, A_x)$, we write $[h] = \mathrm{id} + H \in \mathfrak{H}_x$ (see [9, 14, 24] for more details).

Now let us write down the explicit formulas for the translation of $\mathfrak{H}$ on $T\mathcal{G}|_M = A \oplus TM$ (see [9] for a proof of the following lemma).

Lemma 2.2. *If $[h] = \mathrm{id} + H \in \mathfrak{H}_x$, where $H \in \underline{\mathrm{Hom}}(T_xM, A_x)$, then the left and right translation maps*

$$L_{[h]}, R_{[h]} : A_x \oplus T_xM \to A_x \oplus T_xM$$

are given by

$$\begin{aligned} L_{[h]}(u + X) &= H(X) + u + H\rho(u) + X, \\ R_{[h]}(u + X) &= H(\mathrm{id} + \rho H)^{-1}(X) + u + (\mathrm{id} + \rho H)^{-1}X, \end{aligned} \tag{2.5}$$

where $u \in A_x$ and $X \in T_xM$.

Consequently, the adjoint action $\mathrm{Ad}_{[h]} : A_x \oplus T_xM \to A_x \oplus T_xM$ is given by

$$\mathrm{Ad}_{[h]}(u + X) = (\mathrm{id} + H\rho)(u) + (\mathrm{id} + \rho H)(X),$$

and the dual maps

$$R^*_{[h]}, L^*_{[h]} : A^*_x \oplus T^*_xM \to A^*_x \oplus T^*_xM$$

are respectively

$$\begin{aligned} L^*_{[h]}(\chi + \xi) &= \chi + \rho^* H^* \chi + \xi + H^*\chi, \\ R^*_{[h]}(\chi + \xi) &= \chi + (\mathrm{id} + \rho H)^{-1*}\xi + (\mathrm{id} + \rho H)^{-1*}H^*(\chi), \end{aligned} \tag{2.6}$$

where $\xi \in T^*_xM, \chi \in A^*_x$. In particular, the induced coadjoint action of $\mathfrak{H}$ on $\wedge^k T^*M$ is given by

$$\mathrm{Ad}^\vee_{[h]^{-1}} w = (\mathrm{Ad}_{[h]})^* w = (\mathrm{id} + \rho H)^{*\otimes k}(w), \quad \forall\, w \in \wedge^k T^*_xM. \tag{2.7}$$

2.3 Multiplicative k-vector fields and characteristic pairs of $(k,0)$-type

We first recall ρ-compatible $(k,0)$-tensors introduced in [9].

Definition 2.3. Let $k \geqslant 1$ be an integer. A ***ρ*-compatible $(k,0)$-tensor** is a section

$$\pi \in \Gamma(TM \otimes (\wedge^{k-1}A)),$$

which satisfies

$$\iota_{\rho^*\xi}\iota_\eta \pi = -\iota_{\rho^*\eta}\iota_\xi \pi, \quad \forall\, \xi, \eta \in \Omega^1(M).$$

We need a particular operator D_ρ which was first studied in [11]—a degree 0 derivation on $\wedge^\bullet(TM \oplus A)$ determined by its $(TM \oplus A)$-to-$(TM \oplus A)$ part:

$$D_\rho(X+u) = \rho(u), \quad \forall\, X \in TM, \quad u \in A.$$

Hence D_ρ maps $(\wedge^p TM) \otimes (\wedge^q A)$ to $(\wedge^{p+1}TM) \otimes (\wedge^{q-1}A)$. For $\pi \in TM \otimes (\wedge^{k-1}A)$, we introduce

$$B\pi = \frac{1-\mathrm{e}^{-D_\rho}}{D_\rho}(\pi) = \pi - \frac{1}{2!}D_\rho \pi + \frac{1}{3!}D_\rho^2 \pi + \cdots + \frac{(-1)^{k-1}}{k!}D_\rho^{k-1}\pi \in \wedge^k(TM \oplus A). \tag{2.8}$$

Note that the term is $D_\rho^j \pi \in (\wedge^{j+1}TM) \otimes (\wedge^{k-1-j}A)$.

Recall that A, the Lie algebroid of $\mathcal{G}$, is a module of the jet groupoid $\mathfrak{J}\mathcal{G}$ via the adjoint action (see Definition 2.1). Therefore, $\mathfrak{J}\mathcal{G}$ also acts on $\wedge^k A$. Hence we have a coboundary operator

$$d_{\mathfrak{J}\mathcal{G}} : C^n(\mathfrak{J}\mathcal{G}, \wedge^k A) \to C^{n+1}(\mathfrak{J}\mathcal{G}, \wedge^k A).$$

We denote by $Z^1(\mathfrak{J}\mathcal{G}, \wedge^k A)$ the set of 1-cocycles $c : \mathfrak{J}\mathcal{G} \to \wedge^k A$.

Definition 2.4. Let $k \geqslant 1$ be an integer. A **$(k,0)$-characteristic pair** on $\mathcal{G}$ is a pair

$$(c, \pi) \in Z^1(\mathfrak{J}\mathcal{G}, \wedge^k A) \times \Gamma(TM \otimes (\wedge^{k-1}A)),$$

where π is ρ-compatible and when c is restricted to $\mathfrak{H}$, it satisfies

$$c([h_x]) = (B\pi)_x - L_{[h_x]}(B\pi)_x, \quad \forall\, [h_x] \in \mathfrak{H}_x, \quad x \in M. \tag{2.9}$$

This paper relies heavily on our previous work [9] on multiplicative k-vector fields of Lie groupoids. Before diving into the specific details of these fields, however, we see that it is important to review some introductory information about the tangent and cotangent Lie groupoids. For a more comprehensive understanding, please refer to [12,21]. The Lie groupoid $\mathcal{G} \rightrightarrows M$ has a corresponding tangent bundle $T\mathcal{G}$ that is a Lie groupoid over TM with its structure maps determined by the tangent maps of $\mathcal{G}$'s structure maps. Similarly, the cotangent bundle $T^*\mathcal{G}$ is a Lie groupoid over A^*, with source and target maps given by s and $t : T^*\mathcal{G} \to A^*$, respectively,

$$\langle s(\alpha_g), u\rangle = \langle \alpha_g, \overleftarrow{u}_g\rangle = \langle \alpha_g, L_{g*}(u - \rho(u))\rangle \quad \text{and} \quad \langle t(\alpha_g), u\rangle = \langle \alpha_g, \overrightarrow{u}_g\rangle = \langle \alpha_g, R_{g*}u\rangle \tag{2.10}$$

for all $\alpha_g \in T_g^*\mathcal{G}$ and $u \in A_{s(g)}$ (or $A_{t(g)}$). The multiplication of composable elements $\alpha_g \in T_g^*\mathcal{G}$ and $\beta_r \in T_r^*\mathcal{G}$ is defined by

$$(\alpha_g \cdot \beta_r)(X_g \cdot Y_r) = \alpha_g(X_g) + \beta_r(Y_r), \quad \forall\, (X_g, Y_r) \in (T\mathcal{G})^{(2)}.$$

We remark that both $T\mathcal{G} \rightrightarrows TM$ and $T^*\mathcal{G} \rightrightarrows A^*$ are special cases of vector bundle (VB)-groupoids [18]. Taking $T^*\mathcal{G}$, we have the law of interchange:

$$(\gamma_1 + \gamma_3) \cdot (\gamma_2 + \gamma_4) = \gamma_1 \cdot \gamma_2 + \gamma_3 \cdot \gamma_4 \tag{2.11}$$

for all $\gamma_1, \gamma_3 \in T_g^*\mathcal{G}$ and $\gamma_2, \gamma_4 \in T_r^*\mathcal{G}$ on the premise that $(g, r) \in \mathcal{G}^{(2)}$ and $(\gamma_1, \gamma_2), (\gamma_3, \gamma_4) \in (T^*\mathcal{G})^{(2)}$.

In the literature, various definitions or characterizations of multiplicative multi-vector fields can be found (see [19,21]). In this article, we use the following one.

Definition 2.5 (See [19, 21]). Let $k \geqslant 1$ be an integer. A k-vector field $\Pi \in \mathfrak{X}^k(\mathcal{G})$ on $\mathcal{G}$ is called **multiplicative** if

$$\begin{array}{ccc} \oplus^{k-1}T^*\mathcal{G} & \xrightarrow{\Pi^\sharp} & T\mathcal{G} \\ \downdownarrows & & \downdownarrows \\ \oplus^{k-1}A^* & \xrightarrow{\pi^\sharp} & TM \end{array}$$

is a Lie groupoid morphism. Here, the groupoid $\oplus^{k-1}T^*\mathcal{G} \rightrightarrows \oplus^{k-1}A^*$ is the direct sum of $T^*\mathcal{G} \rightrightarrows A^*$, $\Pi^\sharp$ is defined by contraction:

$$(\alpha^1, \dots, \alpha^{k-1}) \mapsto \Pi(\alpha^1, \dots, \alpha^{k-1}, \cdot),$$

and $\pi^\sharp$ is similar:

$$(\eta^1, \dots, \eta^{k-1}) \mapsto (-1)^{k-1}\pi(\cdot, \eta^1, \dots, \eta^{k-1}),$$

where $\pi = \mathrm{pr}_{\Gamma(TM \otimes \wedge^{k-1}A)}(\Pi|_M)$, a part of $\Pi|_M$ by taking the projection

One of the main results in [9] is the following theorem.

Theorem 2.6. *There is a one-to-one correspondence between multiplicative k-vector fields Π on a Lie groupoid $\mathcal{G} \rightrightarrows M$ and $(k, 0)$-characteristic pairs (c, π) on $\mathcal{G}$ such that*

$$\Pi_g = R_{g*}c([b_g]) + L_{[b_g]}(B\pi)_{s(g)} \tag{2.12}$$

holds for all $g \in \mathcal{G}$ and bisections b_g through g, where $B\pi$ is given by (2.8).

An equivalent form of this formula was found by Iglesias-Ponte et al. as early as ten years ago in their draft of the work [19]. However, in their final published version [19], they did not write it.

The second element $\pi \in \Gamma(TM \otimes (\wedge^{k-1}A))$ of the characteristic pair of Π is indeed the $\Gamma(TM \otimes (\wedge^{k-1}A))$-component of Π's restriction on M, i.e.,

$$\Pi|_M \in \sum_{i+j=k} \Gamma((\wedge^i TM) \otimes (\wedge^j A)).$$

We call π the **leading term** of $\Pi \in \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G})$. The first element of the pair, i.e., the 1-cocycle $c : \mathfrak{J}\mathcal{G} \to \wedge^k A$, can be determined by Π:

$$c([b_g]) = R_{g*}^{-1}(\Pi_g - L_{[b_g]}(B\pi)_{s(g)}). \tag{2.13}$$

Example 2.7. The characteristic pair (c, π) of an exact multiplicative k-vector field $\Pi = \overleftarrow{u} - \overrightarrow{u}$ for $u \in \Gamma(\wedge^k A)$ is given by

$$c = d_{\mathfrak{J}\mathcal{G}}u, \quad \pi = -D_\rho(u).$$

3 Multiplicative k-forms and characteristic pairs of $(0, k)$-type

3.1 Multiplicative k-forms

As usual, we denote by $\mathcal{G} \rightrightarrows M$ a Lie groupoid over M and $\Omega^k(\mathcal{G}) = \Gamma(\wedge^k T^*\mathcal{G})$ the space of differential k-forms (k-forms for short, $k \geqslant 1$). A **multiplicative** k-form on $\mathcal{G}$ is an element $\Theta \in \Omega^k(\mathcal{G})$ satisfying

$$m^*\Theta = \mathrm{pr}_1^*\Theta + \mathrm{pr}_2^*\Theta, \tag{3.1}$$

where the maps m and $\mathrm{pr}_i : \mathcal{G}^{(2)} \to \mathcal{G}$ are, respectively, the groupoid multiplication and the projection from $\mathcal{G}^{(2)}$ to its i-th summand (see [4, 5, 14]). We denote by $\Omega^k_{\mathrm{mult}}(\mathcal{G})$ the space of multiplicative k-forms on $\mathcal{G}$.

By default, a multiplicative 0-form is one and the same as a smooth multiplicative function on $\mathcal{G}$, i.e., $f : \mathcal{G} \to \mathbb{R}$ which is a morphism of Lie groupoids:

$$f(gr) = f(g) + f(r), \quad \forall\, (g, r) \in \mathcal{G}^{(2)}.$$

For $k \geqslant 1$, we have an equivalent characterization: $\Theta \in \Omega^k(\mathcal{G})$ is multiplicative if and only if

$$\Theta^\sharp : \oplus^{k-1} T\mathcal{G} \to T^*\mathcal{G}$$

is a Lie groupoid morphism. For more knowledge on this topic, please refer to [21].

3.2 ρ-compatible $(0,k)$-tensors

Let $A \to M$ be a vector bundle which is equipped with a bundle map $\rho : A \to TM$.

Definition 3.1. Let $k \geqslant 1$ be an integer. A **ρ-compatible $(0,k)$-tensor** is vector bundle map $\theta : A \to \wedge^{k-1} T^*M$ such that

$$\iota_{\rho(u)}\theta(u) = 0, \quad \forall\, u \in A.$$

Equivalently, it is an element $\theta \in \Gamma(A^* \otimes (\wedge^{k-1} T^*M))$ subject to the following property:

$$\iota_{\rho(v)}\iota_u\theta = -\iota_{\rho(u)}\iota_v\theta, \quad \forall\, u, v \in A. \tag{3.2}$$

We also define

$$D_{\rho^*} : \wedge^\bullet(A^* \oplus T^*M) \to \wedge^\bullet(A^* \oplus T^*M) \tag{3.3}$$

as a degree 0 derivation such that

$$D_{\rho^*}(\xi + \alpha) = \rho^*(\xi), \quad \forall\, \alpha \in A^*, \quad \xi \in T^*M.$$

In particular, D_{ρ^*} maps $\wedge^i A^* \otimes (\wedge^j T^*M)$ to $\wedge^{i+1} A^* \otimes (\wedge^{j-1} T^*M)$.

Example 3.2. In the particular case where $k = 1$, a ρ-compatible $(0,1)$-tensor is simply a section $\theta \in \Gamma(A^*)$. No other conditions are needed. If $k = 2$, then a ρ-compatible $(0,2)$-tensor is an element $\theta \in \Gamma(A^* \otimes T^*M)$ subject to the condition $\theta(u, \rho(u)) = 0$ for all $u \in \Gamma(A)$. In other words,

$$(\mathrm{id} \otimes \rho^*)\theta \in \Gamma(A^* \otimes A^*)$$

is skew-symmetric.

Example 3.3. Given any $\gamma \in \Omega^k(M)$, the tensor $D_{\rho^*}\gamma \in \Gamma(A^* \otimes (\wedge^{k-1} T^*M))$ is ρ-compatible. In fact, we have

$$\iota_{\rho(v)}\iota_u D_{\rho^*}\gamma = \iota_{\rho(v)}\iota_{\rho(u)}\gamma = -\iota_{\rho(u)}\iota_{\rho(v)}\gamma = -\iota_{\rho(u)}\iota_v D_{\rho^*}\gamma.$$

Lemma 3.4. *Given a ρ-compatible $(0,k)$-tensor $\theta \in \Gamma(A^* \otimes (\wedge^{k-1} T^*M))$ and an integer $j \geqslant 1$, we have*

$$\iota_{\rho(u)} D_{\rho^*}^{j-1}\theta = D_{\rho^*}^j(\iota_u\theta) = \frac{1}{j+1}\iota_u D_{\rho^*}^j\theta, \quad \forall\, u \in A. \tag{3.4}$$

Proof. It is direct to check the following equations:

$$\iota_u D_{\rho^*} - D_{\rho^*}\iota_u = \iota_{\rho(u)}, \tag{3.5}$$

$$\iota_{\rho(u)} D_{\rho^*} = D_{\rho^*}\iota_{\rho(u)}. \tag{3.6}$$

Now we prove (3.4) by induction. When $j = 1$, it follows from the definition of ρ-compatibility, i.e., (3.2), that $\iota_{\rho(u)}\theta = D_{\rho^*}(\iota_u\theta)$. By (3.5), we get

$$(\iota_u D_{\rho^*} - D_{\rho^*}\iota_u)\theta = \iota_{\rho(u)}\theta = D_{\rho^*}(\iota_u\theta)$$

and thus

$$D_{\rho^*}(\iota_u\theta) = \frac{1}{2}\iota_u D_{\rho^*}\theta.$$

Assume that (3.4) holds for $j \geqslant 1$. Then, using (3.6), we have

$$\iota_{\rho(u)}(D_{\rho^*}^j\theta) = (D_{\rho^*}\iota_{\rho(u)})(D_{\rho^*}^{j-1}\theta) = D_{\rho^*}D_{\rho^*}^j(\iota_u\theta) = D_{\rho^*}^{j+1}(\iota_u\theta).$$

Besides, applying (3.5) and (3.6), we have

$$\begin{aligned} D^{j+1}_{\rho^*}(\iota_u\theta) &= \frac{1}{j+1}(D_{\rho^*}\iota_u D^j_{\rho^*})\theta \\ &= \frac{1}{j+1}(\iota_u D_{\rho^*} - \iota_{\rho(u)})(D^j_{\rho^*}\theta) \\ &= \frac{1}{j+1}(\iota_u(D^{j+1}_{\rho^*}\theta) - D^{j+1}_{\rho^*}(\iota_u\theta)), \end{aligned}$$

which implies that

$$D^{j+1}_{\rho^*}(\iota_u\theta) = \frac{1}{j+2}\iota_u D^{j+1}_{\rho^*}\theta.$$

Thus (3.4) also holds for $j+1$, and the assertion is proved by induction. □

For the convenience of future use, we define a bundle map

$$B : A^* \otimes (\wedge^{k-1}T^*M) \to \bigoplus_{i=1}^{k} \wedge^i A^* \otimes (\wedge^{k-i}T^*M)$$

by

$$B\theta := \frac{\mathrm{e}^{D_{\rho^*}} - 1}{D_{\rho^*}}(\theta) = \theta + \frac{1}{2}D_{\rho^*}\theta + \frac{1}{3!}D^2_{\rho^*}\theta + \cdots + \frac{1}{k!}D^{k-1}_{\rho^*}\theta. \tag{3.7}$$

Note that the term $D^{j-1}_{\rho^*}\theta$ sits in $\Gamma(\wedge^j A^* \otimes \wedge^{k-j}T^*M)$ (for $1 \leqslant j \leqslant k$).

Proposition 3.5. *If $\theta \in \Gamma(A^* \otimes (\wedge^{k-1}T^*M))$ is ρ-compatible, then the following statements are true*:

(1) *For all $u \in A$, we have*

$$\iota_u(B\theta) = \iota_{\rho(u)}(B\theta) + \iota_u\theta. \tag{3.8}$$

(2) *For all $u_1, \dots, u_j \in A_x$ $(1 \leqslant j \leqslant k)$ and $X_{j+1}, \dots, X_k \in T_xM$, we have*

$$\begin{aligned} &(D^{j-1}_{\rho^*}\theta)(u_1, \dots, u_j, X_{j+1}, \dots, X_k) \\ &\quad = j!\theta(u_1, \rho(u_2), \rho(u_3), \dots, \rho(u_j), X_{j+1}, \dots, X_k). \end{aligned} \tag{3.9}$$

(3) *For all $u_1 + X_1, \dots, u_k + X_k \in A \oplus TM$, we have*

$$\begin{aligned} &(B\theta)(u_1 + X_1, \dots, u_k + X_k) \\ &= \sum_{j=1}^{k} \sum_{\sigma \in \mathrm{Sh}(j,k-j)} \mathrm{sgn}(\sigma)\theta(u_{\sigma_1}, \rho(u_{\sigma_2}), \dots, \rho(u_{\sigma_j}), X_{\sigma_{j+1}}, \dots, X_{\sigma_k}) \qquad (3.10\text{a}) \\ &= \theta(u_1, \rho(u_2) + X_2, \rho(u_3) + X_3, \dots, \rho(u_k) + X_k) \\ &\quad + \theta(X_1, u_2, \rho(u_3) + X_3, \rho(u_4) + X_4, \dots, \rho(u_k) + X_k) \\ &\quad + \theta(X_1, X_2, u_3, \rho(u_4) + X_4, \rho(u_5) + X_5, \dots, \rho(u_k) + X_k) \\ &\quad + \cdots + \theta(X_1, X_2, \dots, X_{k-1}, u_k). \qquad (3.10\text{b}) \end{aligned}$$

Proof. By the definition of $B\theta$ in (3.7), we do direct computations:

$$\begin{aligned} \iota_u(B\theta) &= \iota_u\theta + \iota_u\left(\frac{1}{2}D_{\rho^*}\theta + \frac{1}{3!}D^2_{\rho^*}\theta + \cdots + \frac{1}{k!}D^{k-1}_{\rho^*}\theta\right) \\ &= \iota_u\theta + \iota_{\rho(u)}\left(\theta + \frac{1}{2}D_{\rho^*}\theta + \frac{1}{3!}D^2_{\rho^*}\theta + \cdots + \frac{1}{(k-1)!}D^{k-2}_{\rho^*}\theta\right). \end{aligned}$$

In the last step, we repetitively used (3.4). The above terms turn to the desired right-hand side of (3.8).

By the definition of D_{ρ^*}, we have

$$(D^{j-1}_{\rho^*}\theta)(u_1, \dots, u_j, X_{j+1}, \dots, X_k)$$

$$= (j-1)! \sum_{i=1}^{j} (-1)^{i+1} \theta(u_i, \rho(u_1), \dots, \widehat{\rho(u_i)}, \dots, \rho(u_j), X_{j+1}, \dots, X_k),$$

where $u_i \in A_x$ and $X_i \in T_x M$. Then, by the ρ-compatibility condition of θ, the term

$$\begin{aligned}
&(-1)^{i+1} \theta(u_i, \rho(u_1), \dots, \widehat{\rho(u_i)}, \dots, \rho(u_j), X_{j+1}, \dots, X_k) \\
&\quad = (-1)^{i} \theta(u_1, \rho(u_i), \rho(u_2), \dots, \widehat{\rho(u_i)}, \dots, \rho(u_j), X_{j+1}, \dots, X_k) \\
&\quad = \theta(u_1, \rho(u_2), \rho(u_3), \dots, \rho(u_j), X_{j+1}, \dots, X_k).
\end{aligned}$$

This proves (3.9) and we then get

$$\begin{aligned}
(B\theta)(u_1, \dots, u_j, X_{j+1}, \dots, X_k) &= \frac{1}{j!} (D_{\rho^*}^{j-1} \theta)(u_1, \dots, u_j, X_{j+1}, \dots, X_k) \\
&= \theta(u_1, \rho(u_2), \rho(u_3), \dots, \rho(u_j), X_{j+1}, \dots, X_k).
\end{aligned}$$

From these relations, it is immediate to derive (3.10a) and (3.10b). □

3.3 From multiplicative forms to ρ-compatible tensors

Let $k \geqslant 1$ be an integer and $\Theta \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$ be multiplicative. We first notice that M is isotropic with respect to Θ, i.e., $\Theta_x(X_1, \dots, X_k) = 0$ for $X_i \in T_x M$. In fact, this can be easily seen by applying (3.1) to the tangent vectors $(X_1, X_1), \dots, (X_k, X_k)$ (at $(x, x) \in \mathcal{G}^{(2)}$) and noticing that $X_i = m_*(X_i, X_i)$. We then know that $\Theta|_M$ has no summand in the space $\Gamma(\wedge^k T^* M)$, or

$$\Theta|_M \in \bigoplus_{i=1}^{k} \Gamma(\wedge^i A^* \otimes (\wedge^{k-i} T^* M)).$$

We now claim that all the $\Gamma(\wedge^i A^* \otimes (\wedge^{k-i} T^* M))$-summands of $\Theta|_M$ $(i = 2, 3, \dots, k)$ are decided by $\theta \in \Gamma(A^* \otimes (\wedge^{k-1} T^* M))$ which is the $\Gamma(A^* \otimes (\wedge^{k-1} T^* M))$-component of $\Theta|_M$. The element θ will be called the **leading term** of $\Theta \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$.

Proposition 3.6. *Let $\Theta \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$ be a multiplicative k-form on $\mathcal{G}$ $(k \geqslant 1)$. Then, we have the following facts*:

(1) *Its leading term $\theta := \mathrm{pr}_{\Gamma(A^* \otimes (\wedge^{k-1} T^* M))} \Theta|_M$ is a ρ-compatible $(0, k)$-tensor.*

(2) *The restriction of Θ on M is given by*

$$\Theta|_M = B\theta \tag{3.11}$$

(*see* (3.7) *for the definition of $B\theta$*).

In fact, this proposition is covered by Crainic-Salazar-Struchiner's result [14, Proposition 4.1] (see also Proposition 3.11, Theorem 3.13, and Corollary 3.14 in the sequel), which is highly nontrivial. However, we give a direct proof using some basic techniques. Before that, let us state the following fact.

Lemma 3.7. *Given $k \geqslant 1$ and $\Theta \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$, for any $u \in \Gamma(A)$, the term $\iota_{u-\rho(u)}(\Theta|_M) \in \Omega^{k-1}(\mathcal{G})|_M$ has no summand in the space $\Gamma(\wedge^i A^* \otimes (\wedge^{k-1-i} T^* M))$ $(i = 1, 2, \dots, k-1)$.*

Proof. Consider the particular point $(g, s(g)) \in \mathcal{G}^{(2)}$ and the following tangent vectors at $(g, s(g))$:

$$(0_g, (u - \rho(u))_{s(g)}), \ (X_1, s_* X_1), \dots, (X_{k-1}, s_* X_{k-1}).$$

Here, $X_i \in T_g \mathcal{G}$. Notice the following facts:

$$m_*(0_g, (u - \rho(u))_{s(g)}) = L_{g*}(u - \rho(u))_{s(g)} = \overleftarrow{u}_g, \quad m_*(X_i, s_* X_i) = X_i.$$

Substituting these vectors to (3.1), we obtain

$$\Theta_g(\overleftarrow{u}_g, X_1, \dots, X_{k-1}) = \Theta_{s(g)}((u - \rho(u))_{s(g)}, s_* X_1, \dots, s_* X_{k-1}).$$

Especially, if $g = x = s(g) \in M$, the above equation implies the desired assertion. □

This lemma is indeed saying that $\iota_{u-\rho(u)}(\Theta|_M)$ is a $(k-1)$-form on M. Let us denote it by

$$\omega = \iota_{u-\rho(u)}(\Theta|_M) \in \Omega^{k-1}(M),$$

and then the term $\iota_{\overleftarrow{u}}\Theta \in \Omega^{k-1}(\mathcal{G})$ is a left-invariant $(k-1)$-form, and the following identity holds:

$$\iota_{\overleftarrow{u}}\Theta = s^*\omega. \tag{3.12}$$

Similar to the proof above, by considering the tangent vectors $(u_{t(g)}, 0_g), (t_*X_1, X_1), \dots, (t_*X_{k-1}, X_{k-1})$ at the point $(t(g), g) \in \mathcal{G}^{(2)}$, we get

$$\Theta_g(\overrightarrow{u}_g, X_1, \dots, X_{k-1}) = \Theta_{t(g)}(u_{t(g)}, t_*X_1, \dots, t_*X_{k-1}).$$

Then, by the fact that M is isotropic with respect to Θ, the right-hand side of the above equation becomes

$$\Theta_{t(g)}((u-\rho(u))_{t(g)}, t_*X_1, \dots, t_*X_{k-1}) = \omega_{t(g)}(t_*X_1, \dots, t_*X_{k-1}).$$

This shows that the $(k-1)$-form $\iota_{\overrightarrow{u}}\Theta$ is right-invariant, and the following identity holds:

$$\iota_{\overrightarrow{u}}\Theta = t^*\omega.$$

We now turn to the proof of Proposition 3.6.

Proof of Proposition 3.6. First, since M is isotropic with respect to Θ, it has no summand in $\wedge^k T^*M$. Thus $\Theta|_M$ is expressed as

$$\Theta|_M = \theta^{1,k-1} + \theta^{2,k-2} + \cdots + \theta^{k,0},$$

where $\theta^{i,k-i} \in \Gamma(\wedge^i A^* \otimes (\wedge^{k-i}T^*M))$. Next, Lemma 3.7 tells us that $\iota_{u-\rho(u)}\Theta|_M \in \Gamma(\wedge^{k-1}T^*M)$, which signifies that

$$\iota_{\rho(u)}\theta^{i,k-i} = \iota_u\theta^{i+1,k-i-1}, \quad i = 1, 2, \dots, k-1. \tag{3.13}$$

By Lemma 3.7, we also have $\iota_{u-\rho(u)}\iota_v\Theta|_M = 0$ for $u, v \in \Gamma(A)$. This implies that $\iota_u\iota_v\Theta|_M = \iota_{\rho(u)}\iota_v\Theta|_M$. Thus we obtain

$$\iota_u\iota_{\rho(v)}\theta^{1,k-1} = -\mathrm{pr}_{\Gamma(\wedge^{k-2}T^*M)}(\iota_v\iota_u\Theta|_M) = \mathrm{pr}_{\Gamma(\wedge^{k-2}T^*M)}(\iota_u\iota_v\Theta|_M) = -\iota_v\iota_{\rho(u)}\theta^{1,k-1},$$

i.e., $\theta^{1,k-1}$ is ρ-compatible. Then, we claim that

$$\theta^{i,k-i} = \frac{1}{i!}D_{\rho^*}^{i-1}\theta, \quad \text{where } \theta := \theta^{1,k-1},$$

which yields the desired equation (3.11). The proof is by induction for i—it is true for $i = 1$. Assume that the assertion holds for i. By (3.13) and Lemma 3.4, we find

$$\iota_u\theta^{i+1,k-i-1} = \iota_{\rho(u)}\theta^{i,k-i} = \frac{1}{i!}\iota_{\rho(u)}D_{\rho^*}^{i-1}\theta = \frac{1}{i!}\frac{1}{i+1}\iota_u D_{\rho^*}^{i}\theta = \frac{1}{(i+1)!}\iota_u D_{\rho^*}^{i}\theta$$

for all $u \in \Gamma(A)$. Then, we get

$$\theta^{i+1,k-i-1} = \frac{1}{(i+1)!}D_{\rho^*}^{i}\theta.$$

This completes the proof. □

Remark 3.8. By (3.9) and (3.11), we have

$$\begin{aligned}\Theta_x(u_1, \dots, u_j, X_{j+1}, \dots, X_k) &= \frac{1}{j!}(D_{\rho^*}^{j-1}\theta)(u_1, \dots, u_j, X_{j+1}, \dots, X_k)\\ &= \theta(u_1, \rho(u_2), \rho(u_3), \dots, \rho(u_j), X_{j+1}, \dots, X_k).\end{aligned}$$

In fact, this relation has already appeared in [14, Lemma 4.2] (or [8, Remark 2]). So one can treat (3.11) as a compact form of their result.

Remark 3.9. For $\Theta \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$, by the definition of the source map s of $T^*\mathcal{G} \rightrightarrows A^*$ as in (2.10) and (3.12), we see that $s(\Theta_g)$ only depends on $\Theta_{s(g)}$. So by Proposition 3.6(2) and (3.7), we have

$$s(\Theta) = s(\Theta|_M) = s(B\theta) = \frac{1}{k!}D_{\rho^*}^{k-1}\theta \in \Gamma(\wedge^k A^*).$$

Similarly, we have $t(\Theta) = \frac{1}{k!}D_{\rho^*}^{k-1}\theta$. In particular, we have $s(\Theta) = t(\Theta) = \theta \in \Gamma(A^*)$ for $\Theta \in \Omega^1_{\mathrm{mult}}(\mathcal{G})$.

3.4 Groupoid $(0,k)$-characteristic pairs

3.4.1 *Characterizations of multiplicative forms*

We need a result due to Crainic et al. [14] which we now recall. They used it to prove the correspondence of multiplicative forms and Spencer operators.

In [14], a special type of pairs (e,θ) was introduced, where

(1) e is a 1-cocycle of the jet groupoid $\mathfrak{J}\mathcal{G}$ valued in $\wedge^k T^*M$, i.e., $e \in Z^1(\mathfrak{J}\mathcal{G}, \wedge^k T^*M)$, and

(2) $\theta \in \Gamma(A^* \otimes (\wedge^{k-1}T^*M))$ is a ρ-compatible $(0,k)$-tensor.

Here, $k \geqslant 1$ is an integer, and moreover, e and θ are subject to the following two conditions:

$$\begin{aligned} &e[b'_g](\mathrm{Ad}_{[b'_g]}X_1, \dots, \mathrm{Ad}_{[b'_g]}X_k) - e[b_g](\mathrm{Ad}_{[b_g]}X_1, \dots, \mathrm{Ad}_{[b_g]}X_k) \\ &\quad = \sum_{i=1}^{k} (-1)^{i+1} \theta((b'_g \ominus b_g)X_i, \mathrm{Ad}_{[b_g]}X_1, \dots, \mathrm{Ad}_{[b_g]}X_{i-1}, \mathrm{Ad}_{[b'_g]}X_{i+1}, \dots, \mathrm{Ad}_{[b'_g]}X_k), \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\theta(\mathrm{Ad}_{[b_g]}u, \mathrm{Ad}_{[b_g]}X_1, \dots, \mathrm{Ad}_{[b_g]}X_{k-1}) - \theta(u, X_1, \dots, X_{k-1}) \\ &\quad = e[b_g](\mathrm{Ad}_{[b_g]}\rho(u), \mathrm{Ad}_{[b_g]}X_1, \dots, \mathrm{Ad}_{[b_g]}X_{k-1}) \end{aligned} \tag{3.15}$$

for all bisections b_g and b'_g passing through g, $u \in A_{s(g)}$ and $X_1, \dots, X_k \in T_{s(g)}M$. Here, Ad is the natural adjoint action of $\mathfrak{J}\mathcal{G}$ on TM and

$$b'_g \ominus b_g := R_{g^{-1}*}(b'_{g*} - b_{g*}) : T_{s(g)}M \to A_{t(g)}.$$

For convenience, we give such pairs a notion.

Definition 3.10. A pair (e,θ) as described above is called a **$(0,k)$-characteristic pair** on the Lie groupoid $\mathcal{G}$.

We reinterpret the compatibility conditions of e and θ, i.e., (3.14) and (3.15), from a different aspect (see Proposition 3.19).

A multiplicative function (0-form) on $\mathcal{G}$ is exactly a Lie groupoid 1-cocycle valued in the trivial $\mathcal{G}$-module $M \times \mathbb{R}$. In other words, we have

$$\Omega^0_{\mathrm{mult}}(\mathcal{G}) = Z^1(\mathcal{G}, M \times \mathbb{R}).$$

We wish to see what data characterize the space $\Omega^k_{\mathrm{mult}}(\mathcal{G})$ for $k \geqslant 1$. The result we state below gives an answer. Indeed, it is a particular instance of [14, Proposition 4.1].

Proposition 3.11. *There is a one-to-one correspondence between multiplicative k-forms $\Theta \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$ and $(0,k)$-characteristic pairs (e,θ) on $\mathcal{G}$ such that for all the bisections b_g passing through a point $g \in \mathcal{G}$, one has*

$$\begin{aligned} &\Theta_g(R_{[b_g]}(u_1 + X_1), \dots, R_{[b_g]}(u_k + X_k)) \\ &\quad = e[b_g](X_1, \dots, X_k) + \sum_{j=1}^{k} \sum_{\sigma \in \mathrm{Sh}(j,k-j)} \mathrm{sgn}(\sigma)\ \theta(u_{\sigma_1}, \rho(u_{\sigma_2}), \dots, \rho(u_{\sigma_j}), X_{\sigma_{j+1}}, \dots, X_{\sigma_k}), \end{aligned} \tag{3.16}$$

where $u_i + X_i \in A_{t(g)} \oplus T_{t(g)}M$ $(i = 1, \dots, k)$. Here, we use the identification

$$T_g\mathcal{G} = R_{[b_g]}(A_{t(g)} \oplus T_{t(g)}M),$$

and the notation $\mathrm{Sh}(p,q)$ stands for the set of (p,q)-shuffles.

Remark 3.12. From this one-to-one correspondence, we see that all the multiplicative k-forms with $k \geqslant \dim M + 2$ on $\mathcal{G}$ are trivial. In particular, if the Lie groupoid happens to be a Lie group G (i.e., M is a single point), then all the multiplicative k-forms on G with $k \geqslant 2$ are trivial.

For this reason, we always assume that $1 \leqslant k \leqslant \dim M + 1$ in the sequel.

Upon the correspondence of this proposition, we refer to (e,θ) as the $(0,k)$-characteristic pair *of* the multiplicative k-form $\Theta\in\Omega^k_{\mathrm{mult}}(\mathcal{G})$. In addition, we can reformulate the relation between (e,θ) and Θ in a compact manner.

Theorem 3.13. *Let $\Theta\in\Omega^k(\mathcal{G})$ be multiplicative and (e,θ) be its corresponding $(0,k)$-characteristic pair. Then, for any bisection b_g passing through $g\in\mathcal{G}$, we have*

$$\Theta_g = R^*_{[b_g^{-1}]}(e[b_g]+B\theta_{t(g)}). \tag{3.17}$$

In this formula, $B\theta$ is defined by (3.7) and $R^*_{[b_g^{-1}]}: T^*_{t(g)}\mathcal{G}\to T^*_g\mathcal{G}$ is the dual map of the right translation $R_{[b_g^{-1}]}: T_g\mathcal{G}\to T_{t(g)}\mathcal{G}$.

Proof. (3.17) is just a variation of (3.16), a result by Crainic et al. [14]. In fact, by (3.10a), we can reformulate (3.16):

$$\begin{aligned}&\Theta_g(R_{[b_g]}(u_1+X_1),\dots,R_{[b_g]}(u_k+X_k))\\ &\quad = e[b_g](X_1,\dots,X_k)+B\theta(u_1+X_1,\dots,u_k+X_k),\end{aligned}$$

which proves the desired equation (3.17). □

Corollary 3.14. *Given a multiplicative k-form $\Theta\in\Omega^k_{\mathrm{mult}}(\mathcal{G})$, its corresponding $(0,k)$-characteristic pair (e,θ) is determined by the following methods*:

(1) *As a map $e:\mathfrak{J}\mathcal{G}\to\wedge^k T^*M$, one has*

$$e[b] = R_b^{!*}\Theta \tag{3.18}$$

for all bisections $b: M\to\mathcal{G}$ of $\mathcal{G}$. Here, we treat $[b]: M\to\mathfrak{J}\mathcal{G}$ as a section of the fibre bundle $\mathfrak{J}\mathcal{G}\xrightarrow{s} M$, and $R_b^!: M\to\mathcal{G}$ is defined by (2.1).

(2) *As a section of the vector bundle $A^*\otimes(\wedge^{k-1}T^*M)$, we have*

$$\theta = \mathrm{pr}_{\Gamma(A^*\otimes(\wedge^{k-1}T^*M))}\Theta|_M. \tag{3.19}$$

Moreover, $\Theta|_M$ is identically $B\theta$.

Proof. In (3.17), if we take the trivial bisection $b=i_M: M\hookrightarrow\mathcal{G}$, then $e[b]=0$ and we get

$$\Theta|_M = B\theta$$

and hence (3.19). Also, from (3.17), we have

$$e[b_g] = R^*_{[b_g]}\Theta_g - B\theta_{t(g)}.$$

Taking projections of both sides to $\wedge^k T^*_{t(g)}M$, we obtain (3.18). □

Example 3.15. For a multiplicative 1-form $\Theta\in\Omega^1_{\mathrm{mult}}(\mathcal{G})$, the corresponding characteristic pair (e,θ) with $e\in Z^1(\mathfrak{J}\mathcal{G},T^*M)$ and $\theta\in\Gamma(A^*)$ is determined as follows:

$$\langle e[b_g], X_{t(g)}\rangle = \langle\Theta_g, b_{g*}\phi^{-1}_{b_g *}X_{t(g)}\rangle,\quad \theta=\Theta|_M,\quad \forall\, X_{t(g)}\in T_{t(g)}M.$$

We then have the relation $\Theta_g = R^*_{[b_g^{-1}]}(e[b_g]+\theta_{t(g)})$ for all the bisections b_g passing through $g\in\mathcal{G}$.

Example 3.16. Given a multiplicative function $f\in\Omega^0_{\mathrm{mult}}(\mathcal{G})$, we have $df\in\Omega^1_{\mathrm{mult}}(\mathcal{G})$. The corresponding $(0,1)$-characteristic pair (e,θ) of df is explained below. First, as f can be regarded as a Lie groupoid 1-cocycle $\mathcal{G}\to M\times\mathbb{R}$, its infinitesimal $\hat{f}: A\to M\times\mathbb{R}$ is a Lie algebroid 1-cocycle. Indeed, we can treat $\hat{f}$ as in $\Gamma(A^*)$, which reads

$$\hat{f}(u) = -\overrightarrow{u}(f)|_M,\quad \forall\, u\in\Gamma(A).$$

(1) The element $e\in Z^1(\mathfrak{J}\mathcal{G},T^*M)$ is determined by

$$e[b] = R_b^{!*}(df) = d(f\circ R_b^!)\in\Omega^1(M) \tag{3.20}$$

for all the bisections $b : M \to \mathcal{G}$ of $\mathcal{G}$. Here, we treat $[b] : M \to \mathfrak{J}\mathcal{G}$ as a section of the fibre bundle $\mathfrak{J}\mathcal{G} \xrightarrow{s} M$.

(2) The element $\theta \in \Gamma(A^*)$ coincides with $(-\hat{f})$.

Example 3.17. Following Remark 3.12, on a Lie group G with its Lie algebra $\mathfrak{g} := T_eG$, only multiplicative 1-forms could be nontrivial, and $(0,1)$-characteristic pairs on G are of the form $(0,\theta)$, where $\theta \in \mathfrak{g}^*$ is G-invariant, i.e., $\mathrm{Ad}^{\vee}_g\theta = \theta$ for all $g \in G$. Any $\Theta \in \Omega^1_{\mathrm{mult}}(\mathcal{G})$ stems from such a θ. In fact, from Θ, one may take $\theta := \Theta|_e \in \mathfrak{g}^*$ which is necessarily G-invariant, and from θ one can recover Θ as $\Theta(g) = R^*_{g^{-1}}\theta$ for all $g \in G$.

Lemma 3.18. *Let $\gamma \in \Omega^k(M)$ be a k-form on the base manifold M. It corresponds to an exact k-form on the Lie groupoid $\mathcal{G}$:*

$$J(\gamma) = s^*\gamma - t^*\gamma \in \Omega^k(\mathcal{G}),$$

which is multiplicative and the corresponding $(0,k)$-characteristic pair is given by

$$(e = d_{\mathfrak{J}\mathcal{G}}(\gamma), \theta = -D_{\rho^*}\gamma),$$

where $d_{\mathfrak{J}\mathcal{G}} : \Omega^k(M) \to Z^1(\mathfrak{J}\mathcal{G}, \wedge^kT^*M)$ is the differential of $\mathfrak{J}\mathcal{G}$ with respect to its coadjoint action on $\wedge^kT^*M$.

Proof. In fact, by Corollary 3.14 we have

$$e[b] = R^{!*}_b(s^*\gamma - t^*\gamma) = \mathrm{Ad}^{\vee}_{[b]}\gamma - \gamma = (d_{\mathfrak{J}\mathcal{G}}\gamma)[b]$$

for all bisections $b \in \mathrm{Bis}(\mathcal{G})$, and

$$\begin{aligned}\theta(u, X_1, \dots, X_{k-1}) &= (s^*\gamma - t^*\gamma)|_M(u, X_1, \dots, X_{k-1})\\ &= -\gamma(\rho(u), X_1, \dots, X_{k-1})\\ &= -(D_{\rho^*}\gamma)(u, X_1, \dots, X_{k-1})\end{aligned}$$

for all $u \in \Gamma(A)$ and $X_i \in \mathfrak{X}^1(M)$. □

3.4.2 *Another description of groupoid $(0,k)$-characteristic pairs*

Proposition 3.19. *Let $e \in Z^1(\mathfrak{J}\mathcal{G}, \wedge^kT^*M)$ be a 1-cocycle and $\theta \in \Gamma(A^* \otimes (\wedge^{k-1}T^*M))$ be ρ-compatible. They form a $(0,k)$-characteristic pair (e,θ) on $\mathcal{G}$ if and only if*

$$e[h] = R^*_{[h]}(B\theta) - B\theta, \quad \forall\, [h] \in \mathfrak{H}, \tag{3.21}$$

$$D_{\rho^*} \circ e = -d_{\mathfrak{J}\mathcal{G}}\theta. \tag{3.22}$$

Here,

$$d_{\mathfrak{J}\mathcal{G}} : \Gamma(A^* \otimes (\wedge^{k-1}T^*M)) \to C^1(\mathfrak{J}\mathcal{G}, A^* \otimes (\wedge^{k-1}T^*M))$$

is the differential of $\mathfrak{J}\mathcal{G}$ with coefficients in $A^* \otimes (\wedge^{k-1}T^*M)$.

One notes that the right-hand side of (3.21) lands in $\Gamma(\wedge^k(T^*M \oplus A^*))$. So, before we give the proof, we need to explain why it only has the $\Gamma(\wedge^kT^*M)$-component, or

$$\iota_v(R^*_{[h]}(B\theta) - B\theta) = 0, \quad \forall\, v \in A_x. \tag{3.23}$$

In fact, one can examine that for all $u_i + X_i \in A_x \oplus T_xM$,

$$\begin{aligned}&(\iota_v(R^*_{[h]}(B\theta)))(u_1 + X_1, \dots, u_{k-1} + X_{k-1})\\ &\quad= (B\theta)(R_{[h]}v, R_{[h]}(u_1 + X_1), \dots, R_{[h]}(u_{k-1} + X_{k-1}))\\ &\quad= (B\theta)(v, u_1 + H(\mathrm{id} + \rho H)^{-1}X_1 + (\mathrm{id} + \rho H)^{-1}X_1, \dots, u_{k-1}\\ &\qquad + H(\mathrm{id} + \rho H)^{-1}X_{k-1} + (\mathrm{id} + \rho H)^{-1}X_{k-1}) \text{ (by (2.5))}\\ &\quad= \theta(v, \rho(u_1) + X_1, \dots, \rho(u_{k-1}) + X_{k-1}) \text{ (by (3.10b))}.\end{aligned}$$

The last line is independent of $[h]$ and hence (3.23) is valid.

Proof of Proposition 3.19. We first show that (3.22) is equivalent to (3.15). In fact, (3.22) reads

$$D_{\rho^*}(e[b_g]) = \theta - \mathrm{Ad}^{\vee}_{[b_g]}\theta, \quad \forall\, [b_g] \in \mathfrak{J}_g\mathcal{G}.$$

When applied to arguments $u \in A_{t(g)}$ and $X_i \in T_{t(g)}M$, the above equality becomes

$$e[b_g](\rho(u), X_1, \dots, X_{k-1}) = \theta(u, X_1, \dots, X_{k-1}) - \theta(\mathrm{Ad}_{[b_g^{-1}]}\rho(u), \mathrm{Ad}_{[b_g^{-1}]}X_1, \dots, \mathrm{Ad}_{[b_g^{-1}]}X_{k-1}),$$

which is clearly the same relation as (3.15).

We then unravel (3.21) and it suffices to consider an arbitrary $[h] = \mathrm{id} + H \in \mathfrak{H}_x$ (where $H \in \underline{\mathrm{Hom}}(T_xM, A_x)$) and all $X_1, \dots, X_k \in T_xM$. Substituting them into (3.21), we get

$$\begin{aligned} e[h]&(X_1, X_2, \dots, X_k) \\ &= (R^*_{[h]}(B\theta) - B\theta)(X_1, \dots, X_k) \\ &= (B\theta)(R_{[h]}X_1, \dots, R_{[h]}X_k) \\ &= (B\theta)(H(\mathrm{id}+\rho H)^{-1}X_1 + (\mathrm{id}+\rho H)^{-1}X_1, \dots, H(\mathrm{id}+\rho H)^{-1}X_k + (\mathrm{id}+\rho H)^{-1}X_k) \\ &= \theta(H(\mathrm{id}+\rho H)^{-1}X_1, X_2, \dots, X_k) \\ &\quad + \theta((\mathrm{id}+\rho H)^{-1}X_1, H(\mathrm{id}+\rho H)^{-1}X_2, X_3, \dots, X_k) \\ &\quad + \cdots + \theta((\mathrm{id}+\rho H)^{-1}X_1, \dots, (\mathrm{id}+\rho H)^{-1}X_{k-1}, H(\mathrm{id}+\rho H)^{-1}X_k). \end{aligned} \tag{3.24}$$

Here, in the last step, we use (3.10b). So to finish the proof, it suffices to show the equivalence of (3.14) $\Leftrightarrow$ (3.24) (which $\Leftrightarrow$ (3.21)).

(1) (3.14) $\Rightarrow$ (3.24): Consider the particular point $g = x \in M$. Take two bisections b_x and b'_x passing through x, where b_x is the trivial identity section. So we know that $[b_x] = \mathrm{id} + 0_x$ is the unit of the group $\mathfrak{H}_x$, and we suppose that $[b'_x] = [h] = \mathrm{id} + H$ for some $H \in \underline{\mathrm{Hom}}(T_xM, A_x)$. Then, we have $\mathrm{Ad}_{[h]}X_i = (\mathrm{id} + \rho H)X_i$ (for $X_i \in T_xM$), $e[b_x] = 0$ (since e is a 1-cocycle), and

$$b'_x \ominus b_x = R_{x^{-1}*}(b'_{x*} - b_{x*}) = H \quad \text{as a map } T_xM \to A_x.$$

Therefore, in this particular case, from (3.14), we have

$$\begin{aligned} e[h]&((\mathrm{id}+\rho H)X_1, (\mathrm{id}+\rho H)X_2, \dots, (\mathrm{id}+\rho H)X_k) \\ &= \theta(H(X_1), (\mathrm{id}+\rho H)X_2, \dots, (\mathrm{id}+\rho H)X_k) \\ &\quad + \theta(X_1, H(X_2), (\mathrm{id}+\rho H)X_3, \dots, (\mathrm{id}+\rho H)X_k) \\ &\quad + \cdots + \theta(X_1, X_2, \dots, X_{k-1}, H(X_k)), \end{aligned}$$

which is just a variation of (3.24).

(2) (3.24) $\Rightarrow$ (3.14): For two bisections b_g and b'_g passing through g, there exists some

$$[h] = \mathrm{id} + H \in \mathfrak{H}_{t(g)}$$

such that $[b'_g] = [h] \cdot [b_g]$. Therefore, the left-hand side of (3.14) becomes

$$\begin{aligned} &e[b'_g](\mathrm{Ad}_{[b'_g]}X_1, \dots, \mathrm{Ad}_{[b'_g]}X_k) - e[b_g](\mathrm{Ad}_{[b_g]}X_1, \dots, \mathrm{Ad}_{[b_g]}X_k) \\ &\quad = e([h][b_g])(\mathrm{Ad}_{[h]}\mathrm{Ad}_{[b_g]}X_1, \dots, \mathrm{Ad}_{[h]}\mathrm{Ad}_{[b_g]}X_k) - e[b_g](\mathrm{Ad}_{[b_g]}X_1, \dots, \mathrm{Ad}_{[b_g]}X_k). \end{aligned}$$

Using the cocycle condition $e([h][b_g]) = e[h] + \mathrm{Ad}^{\vee}_{[h]}e[b_g]$ and $\mathrm{Ad}_{[h]}X = (\mathrm{id} + \rho H)X$ (for $X \in T_{t(g)}M$), we see that

$$\text{the left-hand side of (3.14)} = e[h]((\mathrm{id}+\rho H)\mathrm{Ad}_{[b_g]}X_1, \dots, (\mathrm{id}+\rho H)\mathrm{Ad}_{[b_g]}X_k).$$

On the other hand, we have

$$b'_g \ominus b_g = R_{g^{-1}*}(b'_{g*} - b_{g*}) = H \circ \mathrm{Ad}_{[b_g]} \quad \text{as a map } T_{s(g)}M \to A_{t(g)}.$$

So we get

$$
\begin{aligned}
&\text{the right-hand side of (3.14)}\\
&\quad = \theta(H\mathrm{Ad}_{[b_g]}X_1, (\mathrm{id}+\rho H)\mathrm{Ad}_{[b_g]}X_2, \ldots, (\mathrm{id}+\rho H)\mathrm{Ad}_{[b_g]}X_k)\\
&\qquad + \theta(\mathrm{Ad}_{[b_g]}X_1, H\mathrm{Ad}_{[b_g]}X_2, (\mathrm{id}+\rho H)\mathrm{Ad}_{[b_g]}X_3, \ldots, (\mathrm{id}+\rho H)\mathrm{Ad}_{[b_g]}X_k)\\
&\qquad + \cdots + \theta(\mathrm{Ad}_{[b_g]}X_1, \mathrm{Ad}_{[b_g]}X_2, \ldots, \mathrm{Ad}_{[b_g]}X_{k-1}, H\mathrm{Ad}_{[b_g]}X_k).
\end{aligned}
$$

From these, we see that if (3.24) holds, then the two sides of (3.14) match. □

3.5 Lie algebroid $(0,k)$-characteristic pairs and IM-forms

Definition 3.20. Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid over M and $k \geqslant 1$ be an integer. A **$(0,k)$-characteristic pair** on A is a pair (μ,θ), where $\mu \in Z^1(\mathfrak{J}A, \wedge^k T^*M)$ is a Lie algebroid 1-cocycle, and $\theta \in \Gamma(A^* \otimes (\wedge^{k-1}T^*M))$ is a ρ-compatible $(0,k)$-tensor, and they are subject to the following two conditions:

(i) For all $H \in \mathfrak{h} = \mathrm{Hom}(TM, A)$, we have

$$\mu(H) = -(H^* \otimes \mathrm{id}_{T^*M}^{\otimes(k-1)})\theta. \tag{3.25}$$

(ii) The identity

$$D_{\rho^*} \circ \mu = -d_{\mathfrak{J}A}\theta, \tag{3.26}$$

i.e.,

$$\iota_{\rho(v)}\mu(\jmath^1 u) = \iota_{[u,v]}\theta - \mathcal{L}_{\rho(u)}\iota_v\theta \tag{3.27}$$

holds for all $u, v \in \Gamma(A)$.

Recall here

$$d_{\mathfrak{J}A} : \Gamma(A^* \otimes (\wedge^{k-1}T^*M)) \to C^1(\mathfrak{J}A, A^* \otimes (\wedge^{k-1}T^*M))$$

is the Lie algebroid differential with coefficients in the $\mathfrak{J}A$-module $A^* \otimes (\wedge^{k-1}T^*M)$.

Another way expressing the condition (3.25) is that the relation

$$\mu(df \otimes u) = -df \wedge \iota_u\theta \tag{3.28}$$

holds for all $H = df \otimes u \in T^*M \otimes A$.

Indeed, such pairs (μ,θ) are also introduced in [14]. We call them Lie algebroid characteristic pairs because they can be viewed as the infinitesimal counterpart of groupoid $(0,k)$-characteristic pairs as in Definition 3.10.

Proposition 3.21. *If (e,θ) is a $(0,k)$-characteristic pair on a Lie groupoid $\mathcal{G}$, then the pair $(\widehat{e},\theta)$, where $\widehat{e} \in Z^1(\mathfrak{J}A, \wedge^k T^*M)$ is the infinitesimal 1-cocycle of e, is a $(0,k)$-characteristic pair on the Lie algebroid A of $\mathcal{G}$.*

Proof. We use Proposition 3.19 which describes the conditions of e and θ. Clearly, (3.22) together with the definition of the infinitesimal (2.2) and the relation (2.3) implies the second condition in Definition 3.20.

We now derive (3.25) from (3.21). Consider $H \in \mathrm{Hom}(TM, A)$ and a curve of isotropy jets $[h(\epsilon)] = \mathrm{id} + \epsilon H \in \mathfrak{H}$. Then, we can compute $\widehat{e}(H)$ according to (2.2):

$$
\begin{aligned}
\widehat{e}(H) &= -\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathrm{Ad}^\vee_{[h(\epsilon)]^{-1}} e[h(\epsilon)]\\
&= -\frac{d}{d\epsilon}\bigg|_{\epsilon=0} (\mathrm{id} + \epsilon\rho H)^{*\otimes k}(R^*_{(\mathrm{id}+\epsilon H)}B\theta - B\theta).
\end{aligned} \tag{3.29}
$$

Here, we have used (2.7). Observing the fact that $(R^*_{(\mathrm{id}+\epsilon H)}B\theta - B\theta) \in \wedge^k T^*M$ (see explanation after Proposition 3.19 or (3.23)), we have

$$\begin{aligned}(\mathrm{id}+\epsilon\rho H)^{*\otimes k}(R^*_{(\mathrm{id}+\epsilon H)}B\theta - B\theta) &= (\mathrm{id}+\epsilon\rho H)^{*\otimes k}\circ \mathrm{pr}_{\Gamma(\wedge^k T^*M)}(R^*_{(\mathrm{id}+\epsilon H)}B\theta - B\theta)\\ &= (\mathrm{id}+\epsilon\rho H)^{*\otimes k}\circ \mathrm{pr}_{\Gamma(\wedge^k T^*M)}\circ R^*_{(\mathrm{id}+\epsilon H)}(B\theta)\\ &= ((\mathrm{id}+\epsilon\rho H)^{*}\circ \mathrm{pr}_{\Gamma(T^*M)}\circ R^*_{(\mathrm{id}+\epsilon H)})^{\otimes k}B\theta. \end{aligned} \tag{3.30}$$

We then recall from (2.6) the formula of $R^*_{(\mathrm{id}+\epsilon H)}$ and derive the following expression of the composition of three maps:

$$(\mathrm{id}+\epsilon\rho H)^{*}\circ \mathrm{pr}_{\Gamma(T^*M)}\circ R^*_{(\mathrm{id}+\epsilon H)} : \begin{cases} A^*\to T^*M, & \chi\mapsto \epsilon(\mathrm{id}+\epsilon\rho H)^{-1*}\circ H^*\chi \mapsto \epsilon H^*\chi,\\ T^*M\to T^*M, & \xi\mapsto (\mathrm{id}+\epsilon\rho H)^{-1*}\xi\mapsto \xi. \end{cases}$$

So we are able to continue from (3.29) and (3.30), getting

$$\widehat{e}(H) = -\frac{d}{d\epsilon}\Big|_{\epsilon=0}(\mathrm{id}_{T^*M}\oplus \epsilon H^*)^{\otimes k}B\theta = -(H^*\otimes \mathrm{id}^{\otimes(k-1)}_{T^*M})\theta.$$

In the last step, we have used the following relation:

$$(\mathrm{id}_{T^*M}\oplus \epsilon H^*)^{\otimes k}B\theta \equiv \epsilon(H^*\otimes \mathrm{id}^{\otimes(k-1)}_{T^*M})\theta \mod \epsilon^2,$$

which is easily seen. □

The following lemma is the Lie algebroid version of Lemma 3.18.

Lemma 3.22. *Let A be a Lie algebroid over M with the anchor map $\rho : A\to TM$. To every $\gamma\in\Omega^k(M)$, there is an associated $(0,k)$-characteristic pair $(\mu_\gamma,\theta_\gamma)$ on A, where $\mu_\gamma : \mathfrak{J}A\to\wedge^k T^*M$ and $\theta_\gamma\in\Gamma(A^*\otimes(\wedge^{k-1}T^*M))$ are defined respectively by*

$$\mu_\gamma = d_{\mathfrak{J}A}(\gamma) \quad \textit{and} \quad \theta_\gamma = -D_{\rho^*}\gamma.$$

Proof. By Example 3.3, we see that θ_γ is ρ-compatible. Following the definition, we have

$$\mu_\gamma(\jmath^1 u) = \mathcal{L}_{\rho(u)}\gamma, \quad \forall\, u\in\Gamma(A).$$

So we can examine

$$\mu_\gamma(df\otimes u) = \mu_\gamma(\jmath^1(fu) - f\jmath^1 u) = \mathcal{L}_{f\rho(u)}\gamma - f\mathcal{L}_{\rho(u)}\gamma = df\wedge\iota_{\rho(u)}\gamma = -df\wedge\iota_u\theta_\gamma,$$

which verifies (3.28). Also, we check that

$$\iota_{\rho(v)}\mu_\gamma(\jmath^1 u) = \iota_{\rho(v)}\mathcal{L}_{\rho(u)}\gamma = -\iota_{[\rho(u),\rho(v)]}\gamma + \mathcal{L}_{\rho(u)}\iota_{\rho(v)}\gamma = \iota_{[u,v]}\theta_\gamma - \mathcal{L}_{\rho(u)}\iota_v\theta_\gamma,$$

which fulfills Definition 3.20(ii). This proves that $(\mu_\gamma,\theta_\gamma)$ is a $(0,k)$-characteristic pair on A. □

It turns out that Lie algebroid $(0,k)$-characteristic pairs are variations of the well-known notion of IM-forms (abbreviated from *infinitesimally multiplicative*).

Definition 3.23 (See [4])**.** Let $k\geqslant 1$ be an integer. An **IM *k*-form** on a Lie algebroid $A\to M$ is a pair (ν,θ) of vector bundle maps $\nu : A\to\wedge^k T^*M$ and $\theta : A\to\wedge^{k-1}T^*M$ satisfying the following conditions:

(1) $\iota_{\rho(u)}\theta(v) = -\iota_{\rho(v)}\theta(u)$,
(2) $\theta[u,v] = \mathcal{L}_{\rho(u)}\theta(v) - \iota_{\rho(v)}d\theta(u) - \iota_{\rho(v)}\nu(u)$, and
(3) $\nu[u,v] = \mathcal{L}_{\rho(u)}\nu(v) - \iota_{\rho(v)}d\nu(u)$

for all $u,v\in\Gamma(A)$.

Example 3.24. Consider the $k=1$ case. An IM 1-form is a pair (ν,θ) formed by $\nu: A\to T^*M$ (seen as in $\Gamma(A^*\otimes T^*M)$) and $\theta\in\Gamma(A^*)$ such that

$$(\mathrm{id}\otimes\rho^*)\nu = d_A\theta \quad \text{and} \quad \mathcal{L}_{\rho(u)}(\nu) = D_{\rho^*}d\nu(u), \quad u\in\Gamma(A),$$

where $d_A:\Gamma(A^*)\to\Gamma(\wedge^2 A^*)$ is the differential of the Lie algebroid A.

Proposition 3.25. *There is a one-to-one correspondence between the set of* $(0,k)$*-characteristic pairs* (μ,θ) *on a Lie algebroid* A *and the set of IM* k*-forms* (ν,θ) *such that*

$$\nu(u) = -\mu(\jmath^1 u) - d\iota_u\theta, \quad \forall\, u\in\Gamma(A).$$

The proof of this proposition is a direct verification, which we omit.

Note that we have assumed that $k\geqslant 1$ in the above discussions. For the extreme case of a multiplicative 0-form, i.e., a multiplicative function $f\in\Omega^0_{\mathrm{mult}}(\mathcal{G}) = Z^1(\mathcal{G}, M\times\mathbb{R})$, its infinitesimal is the Lie algebroid 1-cocycle $\hat{f}$ (see Example 3.16). So we can simply define **IM 0-forms** on a Lie algebroid A to be Lie algebroid 1-cocycles of A.

3.6 The transitive case

A Lie groupoid $\mathcal{G}$ is called transitive if given any two points in the base manifold, there is at least one element in $\mathcal{G}$ connecting them. A Lie algebroid A over M is called transitive if its anchor map $\rho: A\to TM$ is surjective. It is standard that the Lie algebroid of a transitive Lie groupoid is transitive.

Lemma 3.26. *If* A *is transitive and* $k\geqslant 2$*, then any* ρ*-compatible* $(0,k)$*-tensor* $\theta\in\Gamma(A^*\otimes(\wedge^{k-1}T^*M))$ *is determined uniquely by some* $\gamma\in\Omega^k(M)$ *such that* $\theta = D_{\rho^*}\gamma$ *(see Example* 3.3*).*

Proof. As ρ is surjective, given θ, we can define γ by the relation

$$\iota_X\gamma = \iota_u\theta, \quad \forall\, X\in TM \text{ and } u\in\Gamma(A) \text{ such that } \rho(u)=X.$$

The ρ-compatibility property of θ guarantees that γ is well-defined when $k\geqslant 2$. It is clear that the above relation is equivalent to $\theta = D_{\rho^*}\gamma$. Uniqueness of γ is thus apparent. □

Proposition 3.27. *Let* $\mathcal{G}$ *be a transitive Lie groupoid over* M*.*

(1) *If* $k\geqslant 2$*, then all the* $(0,k)$*-characteristic pairs on* $\mathcal{G}$ *are of the form* $(d_{\mathfrak{J}\mathcal{G}}(\gamma), -D_{\rho^*}\gamma)$ *as described by Lemma* 3.18*.*

(2) *If* $k\geqslant 2$*, then all the multiplicative* k*-forms on* $\mathcal{G}$ *are exact, i.e., they are of the form* $s^*\gamma - t^*\gamma$ *for* $\gamma\in\Omega^k(M)$*.*

(3) *Given any* $\theta\in\Gamma(A^*)$ *satisfying the condition*

$$\iota_v(d_{\mathfrak{J}\mathcal{G}}\theta) = 0, \quad \forall\, v\in\ker\rho, \tag{3.31}$$

there exists a unique 1*-cocycle* $e_\theta: \mathfrak{J}\mathcal{G}\to T^*M$ *such that the pair* (e_θ,θ) *is a* $(0,1)$*-characteristic pair on* $\mathcal{G}$*. Moreover, all the* $(0,1)$*-characteristic pairs on* $\mathcal{G}$ *arise from this construction.*

(4) *Every* θ *satisfying* (3.31) *gives rise to a multiplicative* 1*-form* Θ *on* $\mathcal{G}$ *such that*

$$\Theta_g(R_{[b_g]}(u+X)) = \theta_{t(g)}(u+v) - \theta_{s(g)}(\mathrm{Ad}_{[b_g]^{-1}}v)$$

for all $u\in A_{t(g)}$*,* $X\in T_{t(g)}M$*, the bisection* b_g *passing through* $g\in\mathcal{G}$*, and* $v\in A_{t(g)}$ *satisfying* $\rho(v)=X$*. All the multiplicative* 1*-forms* Θ *on* $\mathcal{G}$ *are of this form.*

Proof. (1) Let (e,θ) be a $(0,k)$-characteristic pair on $\mathcal{G}$. Following Lemma 3.26, since θ is ρ-compatible and $\mathcal{G}$ is transitive, we have $\theta = D_{\rho^*}\gamma$, where $\gamma\in\Omega^k(M)$. Then, the compatibility condition (3.22) becomes $D_{\rho^*}\circ e = -D_{\rho^*}\circ d_{\mathfrak{J}\mathcal{G}}\gamma$. So e is indeed the negative of $d_{\mathfrak{J}\mathcal{G}}\gamma$ for ρ being surjective.

Statement (2) is implied by (1) due to the one-to-one correspondence established by Proposition 3.11.

(3) First, as we have (3.31), $d_{\mathfrak{J}\mathcal{G}}\theta$ is indeed a map $\mathfrak{J}\mathcal{G}\to \mathrm{Im}\rho^*$. Second, since ρ^* is injective, we can define e_θ via the relation

$$\rho^*\circ e_\theta = -d_{\mathfrak{J}\mathcal{G}}\theta. \tag{3.32}$$

To see that e_θ is a 1-cocycle, one notices that the right-hand side of (3.32) is a 1-cocycle valued in A^*, and $T^*M \xrightarrow{\rho^*} A^*$ is compatible with the actions of $\mathfrak{J}\mathcal{G}$ on T^*M and A^*.

Now we examine that (e_θ, θ) is a $(0,1)$-characteristic pair on $\mathcal{G}$. The given $\theta \in \Gamma(A^*)$ is certainly ρ-compatible. One of the conditions we need is (3.22) which now becomes (3.32).

The other condition we need, i.e., (3.21), now reads

$$e_\theta[h] = R^*_{[h]}(\theta) - \theta, \quad \forall\, [h] \in \mathfrak{H}.$$

Indeed, it is implied by (3.32) as ρ^* is injective. Finally, given any $(0,1)$-characteristic pair (e,θ) on $\mathcal{G}$, one easily finds that e must be of the form e_θ.

Following the result of the statement (3), one can use Proposition 3.11 to show the statement (4) directly. □

Remark 3.28. Our earlier work [9] presents some results on the structure of multiplicative multi-vector fields on transitive Lie groupoids.

We now turn to Lie algebroid characteristic pairs and IM forms. The statements (2) and (4) in the following proposition have already appeared in [4, Remark 3.5].

Proposition 3.29. *Let A be a transitive Lie algebroid over M.*

(1) *If $k \geqslant 2$, then all the $(0,k)$-characteristic pairs on A are of the form $(\mu_\gamma = d_{\mathfrak{J}A}(\gamma), \theta_\gamma = -D_{\rho^*}\gamma)$ as described by Lemma 3.22, where $\gamma \in \Omega^k(M)$.*

(2) *If $k \geqslant 2$, then all the IM k-forms are of the form $(\nu_\gamma, \theta_\gamma = -D_{\rho^*}\gamma)$, where $\gamma \in \Omega^k(M)$ and $\nu_\gamma : A \to \wedge^k T^*M$ is determined by the formula*

$$\nu_\gamma(u) = \iota_{\rho(u)} d\gamma, \quad \forall\, u \in \Gamma(A).$$

(3) *Given any $\theta \in \Gamma(A^*)$ satisfying the condition*

$$\iota_v(d_A\theta) = 0, \quad \forall\, v \in \ker\rho, \tag{3.33}$$

there exists a unique 1-cocycle $\mu_\theta \colon \mathfrak{J}A \to T^*M$ such that the pair (μ_θ, θ) is a $(0,1)$-characteristic pair on A. The element μ_θ is defined by the relation

$$\rho^* \circ \mu_\theta(\jmath^1 u) = -\mathcal{L}_u\theta, \quad \forall\, u \in \Gamma(A).$$

Moreover, all the $(0,1)$-characteristic pairs on A arise from this construction.

(4) *Given any θ satisfying (3.33), there exists a unique $\nu_\theta : A \to \wedge^k T^*M$ such that (ν_θ, θ) is an IM 1-form. The element ν_θ is defined by the relation*

$$\rho^* \circ \nu_\theta(u) = \iota_u(d_A\theta), \quad \forall\, u \in \Gamma(A).$$

Moreover, all the IM 1-forms of A are of this form.

The proof of this proposition is completely similar to the previous one, so we omit it.

4 The complex of multiplicative forms

4.1 The de Rham differential

It is easily verified that the standard de Rham differential $d : \Omega^\bullet(\mathcal{G}) \to \Omega^{\bullet+1}(\mathcal{G})$ maps multiplicative k-forms to multiplicative $(k+1)$-forms. In plain terms, $(\Omega^\bullet_{\mathrm{mult}}(\mathcal{G}), d)$ is a subcomplex of $(\Omega^\bullet(\mathcal{G}), d)$. As $\Omega^\bullet_{\mathrm{mult}}(\mathcal{G})$ corresponds to groupoid $(0,\bullet)$-characteristic pairs, it is tempting to describe d in terms of characteristic pairs as well.

First, given $f \in \Omega^0_{\mathrm{mult}}(\mathcal{G})$ (a multiplicative function on $\mathcal{G}$), we have $df \in \Omega^1_{\mathrm{mult}}(\mathcal{G})$ which corresponds to the $(0,1)$-characteristic pair (e,θ) as described in Example 3.16.

Second, for all $k \geqslant 1$, we characterise $d : \Omega^k_{\mathrm{mult}}(\mathcal{G}) \to \Omega^{k+1}_{\mathrm{mult}}(\mathcal{G})$ as follows.

Proposition 4.1. *Given a multiplicative k-form* $\Theta \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$ *which corresponds to the* $(0,k)$*-characteristic pair* (e,θ)*, the* $(0,k+1)$*-characteristic pair of* $d\Theta \in \Omega^{k+1}_{\mathrm{mult}}(\mathcal{G})$*, denoted by* $(\tilde{e},\tilde{\theta})$*, is given as follows*:

(1) *The map* $\tilde{e} : \mathfrak{J}\mathcal{G} \to \wedge^{k+1}T^*M$ *is determined by*

$$\tilde{e}[b] = d(e[b]) \tag{4.1}$$

for all the bisections $b : M \to \mathcal{G}$ *of* $\mathcal{G}$*. Here, we treat* $[b] : M \to \mathfrak{J}\mathcal{G}$ *as a section of the fibre bundle* $\mathfrak{J}\mathcal{G} \xrightarrow{s} M$*,* $e[b]$ *as in* $\Omega^k(M)$*, and* $d : \Omega^k(M) \to \Omega^{k+1}(M)$ *is the standard de Rham differential.*

(2) *The section* $\tilde{\theta} \in \Gamma(A^* \otimes (\wedge^k T^*M))$ *is determined by*

$$\iota_u\tilde{\theta} = -d(\iota_u\theta) - \widehat{e}(\jmath^1 u) \tag{4.2}$$

for all $u \in \Gamma(A)$*. Here,* $\widehat{e} \in Z^1(\mathfrak{J}A, \wedge^k T^*M)$ *is the infinitesimal of the* 1*-cocycle* $e \in Z^1(\mathfrak{J}\mathcal{G}, \wedge^k T^*M)$*, and* $\jmath^1 : \Gamma(A) \to \Gamma(\mathfrak{J}A)$ *is the lifting map defined by* (2.4).

Proof. By Corollary 3.14, we have

$$\tilde{e}[b] = R_b^{!*}(d\Theta) = d(R_b^{!*}(\Theta)) = d(e[b]),$$

which proves (4.1).

Also by Corollary 3.14, we have

$$\tilde{\theta} = \mathrm{pr}_{\Gamma(A^*\otimes(\wedge^k T^*M))}(d\Theta)|_M.$$

Therefore, to find $\tilde{\theta}$ at $x \in M$, we need to consider arbitrary $u_x \in A_x$ and $X_{1x}, \ldots, X_{kx} \in T_xM$, and to evaluate

$$\tilde{\theta}_x(u_x, X_{1x}, \ldots, X_{kx}) = (d\Theta)(u_x, X_{1x}, \ldots, X_{kx}). \tag{4.3}$$

We extend u_x to a smooth section $u \in \Gamma(A)$ which has its exponential $\exp \epsilon u \in \mathrm{Bis}(\mathcal{G})$ (for $|\epsilon|$ sufficiently small). The flow of the right-invariant vector field $\overrightarrow{u} \in \mathfrak{X}^1(\mathcal{G})$ is given by $L_{\exp \epsilon u}$. The map

$$\phi(\epsilon) = t \circ \exp \epsilon u : M \to M$$

is indeed the flow of $\rho(u) = t_*\overrightarrow{u} \in \mathfrak{X}^1(M)$.

Using the flow $\phi(\epsilon)$, one is able to find extensions $X_i \in \mathfrak{X}^1(M)$ of X_{ix} such that $X_i|_{\phi(\epsilon)x} = \phi(\epsilon)_*X_{ix}$.

We then use the flow $L_{\exp \epsilon u}$ of $\overrightarrow{u}$ to extend $X_i \in \mathfrak{X}^1(M)$ to $\tilde{X}_i \in \mathfrak{X}^1(\mathcal{G}^o)$, $\mathcal{G}^o$ being a small neighbourhood of the identity section $M \subset \mathcal{G}$, such that

$$\tilde{X}_i|_{\exp \epsilon u(y)} = L_{\exp \epsilon u *}X_i|_y, \quad \forall\, y \in M.$$

It follows that $[\overrightarrow{u}, \tilde{X}_i] = 0$ and

$$\tilde{X}_i|_{\exp \epsilon u(x)} = L_{\exp \epsilon u *}X_{ix} = R^!_{\exp \epsilon u *}X_i|_{\phi(\epsilon)x}. \tag{4.4}$$

We are now ready to compute

$$\begin{aligned}
(4.3) &= (d\Theta)|_x(\overrightarrow{u}, \tilde{X}_1, \ldots, \tilde{X}_k) \\
&= \sum_{i=1}^k (-1)^i X_{ix}(\Theta(\overrightarrow{u}, \ldots, \widehat{\tilde{X}_i}, \ldots)) - \sum_{i<j} (-1)^{i+j}\Theta_x(\overrightarrow{u}, [\tilde{X}_i, \tilde{X}_j], \ldots) \\
&\quad + \overrightarrow{u}_x(\Theta(\tilde{X}_1, \ldots, \tilde{X}_k)) \\
&= \sum_{i=1}^k (-1)^i X_{ix}((\iota_u\theta)(\ldots, \widehat{X_i}, \ldots)) - \sum_{i<j} (-1)^{i+j}(\iota_u\theta)_x([X_i, X_j], \ldots) \\
&\quad + \overrightarrow{u}_x(\Theta(\tilde{X}_1, \ldots, \tilde{X}_k)).
\end{aligned}$$

The first two terms add to $(-d(\iota_u\theta))_x(X_1,\dots,X_k)$, while the third one is

$$\begin{aligned}
&\frac{d}{d\epsilon}\Big|_{\epsilon=0}\Theta_{\exp\epsilon u(x)}(\tilde{X}_1,\dots,\tilde{X}_k)\\
&=\frac{d}{d\epsilon}\Big|_{\epsilon=0}R^{!*}_{\exp\epsilon u}\Theta_{\exp\epsilon u(x)}(X_1|_{\phi(\epsilon)x},\dots,X_k|_{\phi(\epsilon)x})\ \text{(by (4.4))}\\
&=\frac{d}{d\epsilon}\Big|_{\epsilon=0}e[\exp\epsilon u](X_1|_{\phi(\epsilon)x},\dots,X_k|_{\phi(\epsilon)x})\ \text{(by Corollary 3.14)}\\
&=\frac{d}{d\epsilon}\Big|_{\epsilon=0}((\mathrm{Ad}^{\vee}_{\exp\epsilon u})^{-1}e[\exp\epsilon u])(X_{1x},\dots,X_{kx})\\
&=\frac{d}{d\epsilon}\Big|_{\epsilon=0}(\mathrm{Ad}^{\vee}_{\exp(\epsilon\jmath^1u)})^{-1}(e\circ\exp(\epsilon\jmath^1u))(X_{1x},\dots,X_{kx})\\
&=-\widehat{e}(\jmath^1u)(X_{1x},\dots,X_{kx}),
\end{aligned}$$

where $\widehat{e}\in Z^1(\mathfrak{J}A,\wedge^kT^*M)$, the infinitesimal of $e\in Z^1(\mathfrak{J}\mathcal{G},\wedge^kT^*M)$, is computed according to its definition formula (2.2). This proves (4.2). □

Example 4.2. Let $\Theta\in\Omega^2(\mathcal{G})$ be a presymplectic structure, i.e., Θ is a closed and multiplicative (but not necessarily nondegenerate) 2-form. Suppose that Θ corresponds to the $(0,2)$-characteristic pair (e,θ), where $e\in Z^2(\mathfrak{J}\mathcal{G},\wedge^2T^*M)$ and $\theta\in\Gamma(A^*\otimes T^*M)$. Then, by Proposition 4.1, we have

$$d(e[b])=0,\quad \widehat{e}(\jmath^1u)=-d(\iota_u\theta),\quad \forall\, u\in\Gamma(A),$$

where $\widehat{e}\in Z^1(\mathfrak{J}A,\wedge^2T^*M)$ is the infinitesimal of e. Thus, a $(0,2)$-characteristic pair (e,θ) of a presymplectic structure Θ satisfies

$$\theta(u,\rho(v))=-\theta(v,\rho(u)),\quad u,v\in\Gamma(A),$$

and the infinitesimal $\widehat{e}$ of e is determined by θ via $\widehat{e}(\jmath^1u)=-d(\iota_u\theta)$. Moreover, Θ and (e,θ) are related by the formula

$$\Theta_g(R_{[b_g]}(u_1+X_1),R_{[b_g]}(u_2+X_2))=e[b_g](X_1,X_2)+\theta(u_1,X_2)-\theta(u_2,X_1)+\theta(u_1,\rho(u_2)).$$

By taking derivations, we see that multiplicative functions on $\mathcal{G}$ correspond to Lie algebroid 1-cocycles of the Lie algebroid A, and $(0,k)$-characteristic pairs on $\mathcal{G}$ correspond to $(0,k)$-characteristic pairs on A (see Proposition 3.21), and also to IM k-forms of A (see Proposition 3.25). Then, we are naturally led to considering the infinitesimal version of Proposition 4.1.

Proposition 4.3. *Let A be a Lie algebroid. Denote by $\mathrm{CP}^0(A)=Z^1(A,M\times\mathbb{R})$ the set of Lie algebroid 1-cocycles and by $\mathrm{CP}^k(A)$ the set of $(0,k)$-characteristic pairs on A for $k\geqslant 1$. Then,*

$$\mathrm{CP}^\bullet(A)=\bigoplus_{j=0}^{\dim M+1}\mathrm{CP}^j(A)$$

admits a canonical cochain complex structure with the differential expressed as follows:

(1) *The differential $d:\mathrm{CP}^0(A)\to\mathrm{CP}^1(A)$ is simply $d(c)=(\mu_c,-c)$ for all 1-cocycles $c\in Z^1(A,M\times\mathbb{R})$, where $\mu_c:\mathfrak{J}A\to T^*M$ is given by $\mu_c(\jmath^1u)=d(c(u))$ (see Example* 3.16).

(2) *For $1\leqslant k\leqslant\dim M$, the differential $d:\mathrm{CP}^k(A)\to\mathrm{CP}^{k+1}(A)$ is given by $d(\mu,\theta)=(\tilde{\mu},\tilde{\theta})$, where*

$$\tilde{\mu}(\jmath^1u):=d\mu(\jmath^1u),\quad \iota_u\tilde{\theta}:=-d(\iota_u\theta)-\mu(\jmath^1u),\quad\forall\, u\in\Gamma(A).$$

We have a direct corollary following the one-to-one correspondence established by Proposition 3.25.

Corollary 4.4. *Denote by $\mathrm{IM}^0(A)=Z^1(A,M\times\mathbb{R})$ the set of Lie algebroid 1-cocycles and by $\mathrm{IM}^k(A)$ the set of IM k-forms of A for $k\geqslant 1$. Then,*

$$\mathrm{IM}^\bullet(A)=\bigoplus_{j=0}^{\dim M+1}\mathrm{IM}^j(A)$$

admits a canonical cochain complex structure with the differential as described below:

(1) *The differential* $d: \mathrm{CP}^0(A) \to \mathrm{CP}^1(A)$ *is simply* $d(c) = (0, -c)$ *for all* 1*-cocycles* $c \in Z^1(A, M \times \mathbb{R})$.

(2) *For* $1 \leqslant k \leqslant \dim M$, *the differential* $d: \mathrm{CP}^k(A) \to \mathrm{CP}^{k+1}(A)$ *is given by* $d(\nu, \theta) = (0, \nu)$.

We note that the above fact has already appeared in [4].

4.2 The Cartan calculus

In this subsection, useful Cartan formulas are introduced to describe the interaction between multiplicative multi-vector fields and forms on a Lie groupoid $G \rightrightarrows M$. The notation used is consistent with the earlier sections.

Lemma 4.5. *Let* $\Pi \in \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G})$ $(k \geqslant 1)$ *and* $\alpha \in \Omega^1_{\mathrm{mult}}(\mathcal{G})$ *be given.*

(1) *Their contraction is also multiplicative, i.e.,* $\iota_\alpha \Pi \in \mathfrak{X}^{k-1}_{\mathrm{mult}}(\mathcal{G})$.

(2) *For* $\gamma \in \Omega^1(M)$, *we have* $\iota_{s^*\gamma}\Pi = \overleftarrow{\iota_\gamma \pi}$ *and* $\iota_{t^*\gamma}\Pi = \overrightarrow{\iota_\gamma \pi}$, *where* $\pi \in \Gamma(TM \otimes (\wedge^{k-1}A))$ *is the leading term of* Π.

(3) *For* $u \in \Gamma(\wedge^k A)$, *we have* $\iota_\alpha \overleftarrow{u} = \overleftarrow{\iota_a u}$ *and* $\iota_\alpha \overrightarrow{u} = \overrightarrow{\iota_a u}$, *where* $a \in \Gamma(A^*)$ *is the leading term of* α.

Proof. To show that $\iota_\alpha \Pi \in \mathfrak{X}^{k-1}_{\mathrm{mult}}(\mathcal{G})$, we need two facts.

(i) An n-vector field $\Gamma \in \mathfrak{X}^n(\mathcal{G})$ is multiplicative if and only if

$$\Gamma_{gr}(\alpha^1_g \cdot \beta^1_r, \ldots, \alpha^n_g \cdot \beta^n_r) = \Gamma_g(\alpha^1_g, \ldots, \alpha^n_g) + \Gamma_r(\beta^1_r, \ldots, \beta^n_r), \quad \forall\, (g, r) \in \mathcal{G}^{(2)}$$

for all the composable pairs $(\alpha^i_g \in T^*_g\mathcal{G}, \beta^i_r \in T^*_r\mathcal{G})$ $(i = 1, \ldots, n)$ (see [19, Proposition 2.7]).

(ii) A 1-form $\alpha \in \Omega^1(\mathcal{G})$ is multiplicative if and only if $\alpha: \mathcal{G} \to T^*\mathcal{G}$ is a groupoid morphism. This is explained in Subsection 3.1 (see also [21]).

Since our Π and α are both multiplicative, we have $\alpha_{gr} = \alpha_g \cdot \alpha_r$ for $(g, r) \in \mathcal{G}^{(2)}$ by (ii) and

$$\begin{aligned}
(\iota_\alpha \Pi)_{gr}(\alpha^1_g \cdot \beta^1_r, \ldots, \alpha^{k-1}_g \cdot \beta^{k-1}_r) &= \Pi_{gr}(\alpha_{gr}, \alpha^1_g \cdot \beta^1_r, \ldots, \alpha^{k-1}_g \cdot \beta^{k-1}_r) \\
&\overset{\text{(ii)}}{=} \Pi_{gr}(\alpha_g \cdot \alpha_r, \alpha^1_g \cdot \beta^1_r, \ldots, \alpha^{k-1}_g \cdot \beta^{k-1}_r) \\
&\overset{\text{(i)}}{=} \Pi_g(\alpha_g, \alpha^1_g, \ldots, \alpha^{k-1}_g) + \Pi_r(\alpha_r, \beta^1_r, \ldots, \beta^{k-1}_r) \\
&= (\iota_\alpha \Pi)_g(\alpha^1_g, \ldots, \alpha^{k-1}_g) + (\iota_\alpha \Pi)_r(\beta^1_r, \ldots, \beta^{k-1}_r).
\end{aligned}$$

By (i) again, the above relation proves the assertion $\iota_\alpha \Pi \in \mathfrak{X}^{k-1}_{\mathrm{mult}}(\mathcal{G})$.

Next, we show the equality $\iota_{s^*\gamma}\Pi = \overleftarrow{\iota_\gamma \pi}$. In fact, by Π being multiplicative, we have a groupoid morphism

$$\begin{array}{ccc}
\oplus^{k-1} T^*\mathcal{G} & \xrightarrow{\Pi^\sharp} & T\mathcal{G} \\
\downdownarrows & & \downdownarrows \\
\oplus^{k-1} A^* & \xrightarrow{\pi^\sharp} & TM.
\end{array}$$

Here, $\oplus^{k-1}T^*\mathcal{G}$ denotes the Whitney sum of $(k-1)$ copies of the vector bundle $T^*\mathcal{G}$, and it is treated as a Lie groupoid over $\oplus^{k-1}A^*$. For more details, see [6].

Following the above diagram, we have the relation $s_* \circ \Pi^\sharp = \pi^\sharp \circ (\oplus^{k-1}s)$. Using this, we can examine the relation

$$\begin{aligned}
(\iota_{s^*\gamma}\Pi)(\alpha^1, \ldots, \alpha^{k-1}) &= (-1)^{k-1}\langle \Pi^\sharp(\alpha^1, \ldots, \alpha^{k-1}), s^*\gamma \rangle \\
&= (-1)^{k-1}\langle s_*\Pi^\sharp(\alpha^1, \ldots, \alpha^{k-1}), \gamma \rangle \\
&= (-1)^{k-1}\langle \pi^\sharp(s(\alpha^1), \ldots, s(\alpha^{k-1})), \gamma \rangle = (\iota_\gamma \pi)(s(\alpha^1), \ldots, s(\alpha^{k-1})) \\
&= \overleftarrow{\iota_\gamma \pi}(\alpha^1, \ldots, \alpha^{k-1}).
\end{aligned}$$

The last step is due to the definition of $s: T^*\mathcal{G} \to A^*$ as in (2.10). The other equality $\iota_{t^*\gamma}\Pi = \overrightarrow{\iota_\gamma \pi}$ is approached similarly.

We finally show $\iota_\alpha \overleftarrow{u} = \overleftarrow{\iota_a u}$ (the other one is similar). For this, we need the fact that $s(\alpha) = a$ (see Remark 3.9). Then, we have

$$\begin{aligned}(\iota_\alpha \overleftarrow{u})(\alpha^1, \ldots, \alpha^{k-1}) &= u(s(\alpha), s(\alpha^1), \ldots, s(\alpha^{k-1}))\\ &= u(a, s(\alpha^1), \ldots, s(\alpha^{k-1}))\\ &= (\iota_a u)(s(\alpha^1), \ldots, s(\alpha^{k-1}))\\ &= \overleftarrow{\iota_a u}(\alpha^1, \ldots, \alpha^{k-1}),\end{aligned}$$

as desired. □

Example 4.6. For a Poisson Lie groupoid $(\mathcal{G}, P)$ (see the next section for more details), the Hamiltonian vector field $X_f := P^\sharp(df)$ of a multiplicative function $f \in C^\infty(\mathcal{G})$ is a multiplicative vector field, by Lemma 4.5(1) and the fact that $df \in \Omega^1_{\mathrm{mult}}(\mathcal{G})$.

Example 4.7. Consider an exact 2-vector field $P \in \mathfrak{X}^2_{\mathrm{mult}}(\mathcal{G})$ which is of the form $P = \overleftarrow{u} - \overrightarrow{u}$, where $u \in \Gamma(\wedge^2 A)$. Then, for any $\alpha \in \Omega^1_{\mathrm{mult}}(\mathcal{G})$, by Lemma 4.5(3), we have

$$P^\sharp(\alpha) = \overleftarrow{\iota_a u} - \overrightarrow{\iota_a u} \in \mathfrak{X}^1_{\mathrm{mult}}(\mathcal{G}),$$

where $a \in \Gamma(A^*)$ is the leading term of α. In general, the contraction of α to every exact k-vector field yields an exact $(k-1)$-vector field.

We have a lemma parallel to the previous one.

Lemma 4.8. *Suppose that $\Theta \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$ and $X \in \mathfrak{X}^1_{\mathrm{mult}}(\mathcal{G})$ are given.*

(1) *Their contraction is also multiplicative, i.e., $\iota_X \Theta \in \Omega^{k-1}_{\mathrm{mult}}(\mathcal{G})$.*

(2) *For $u \in \Gamma(A)$, we have $\iota_{\overleftarrow{u}}\Theta = s^*(\iota_u \theta)$ and $\iota_{\overrightarrow{u}}\Theta = t^*(\iota_u\theta)$, where $\theta \in \Gamma(A^* \otimes (\wedge^{k-1} T^*M))$ is the leading term of Θ.*

(3) *For $\gamma \in \Omega^k(M)$, we have $\iota_X s^*\gamma = s^*(\iota_x \gamma)$ and $\iota_X t^*\gamma = t^*(\iota_x\gamma)$, where $x \in \mathfrak{X}^1(M)$ is the leading term of X.*

The proof is omitted. Note that the part of the statement (2) is reminiscent of Lemma 3.7.

At this point, we see that the de Rham differential d of forms, the contraction ι_X by a multiplicative vector field $X \in \mathfrak{X}^1_{\mathrm{mult}}(\mathcal{G})$, and the Lie derivative via Cartan's formula $\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X$ all preserve $\Omega^\bullet_{\mathrm{mult}}(\mathcal{G})$.

For a multiplicative 2-vector field $P \in \mathfrak{X}^2_{\mathrm{mult}}(\mathcal{G})$, we see that the map $P^\sharp$ maps a multiplicative 1-form Θ to a multiplicative vector field $P^\sharp(\Theta)$. Thereby, if Θ corresponds to the $(0,1)$-characteristic pair (e, θ), $P^\sharp(\Theta)$ should correspond to a $(1,0)$-characteristic pair $(c_{P^\sharp(\Theta)}, \pi_{P^\sharp(\Theta)})$, where

$$e \in Z^1(\mathfrak{J}\mathcal{G}, T^*M), \quad \theta \in \Gamma(A^*), \quad c_{P^\sharp(\Theta)} \in Z^1(\mathfrak{J}\mathcal{G}, A), \quad \text{and} \quad \pi_{P^\sharp(\Theta)} \in \mathfrak{X}^1(M).$$

To find the explicit relations between these data, we assume that $P \in \mathfrak{X}^2_{\mathrm{mult}}(\mathcal{G})$ corresponds to the $(2,0)$-characteristic pair (c_P, p), where $c_P \in Z^1(\mathfrak{J}\mathcal{G}, \wedge^2 A)$ and $p \in \Gamma(TM \otimes A)$.

Proposition 4.9. *With assumptions as above, 1-cocycle $c_{P^\sharp(\Theta)} : \mathfrak{J}\mathcal{G} \to A$ is given by*

$$c_{P^\sharp(\Theta)}([b_g]) = (\mathrm{Ad}_{[b_g]} p)^\sharp \circ e([b_g]) + \iota_\theta \circ c_P([b_g])$$

and $\pi_{P^\sharp(\Theta)} \in \mathfrak{X}^1(M)$ by $p^\sharp(\theta)$, i.e.,

$$\pi_{P^\sharp(\Theta)}(f) = -p(df, \theta), \quad \forall\, f \in C^\infty(M).$$

Proof. The formula for $\pi_{P^\sharp(\Theta)}$ can be found by its definition:

$$\pi_{P^\sharp(\Theta)} = \mathrm{pr}_{\Gamma(TM)} P^\sharp(\Theta)|_M = \mathrm{pr}_{\Gamma(TM)} P^\sharp|_M(\Theta|_M) = p^\sharp(\theta).$$

Then, according to (2.12) and (3.17), we have

$$P_g = R_{g*} c_P([b_g]) + L_{[b_g]}\left(p - \frac{1}{2} D_\rho p\right), \quad P^\sharp(\Theta)_g = R_{g*} c_{P^\sharp\Theta}([b_g]) + L_{[b_g]}(p^\sharp \theta),$$

and

$$\Theta_g = R^*_{[b_g^{-1}]}(e[b_g] + \theta).$$

Then, we have

$$\begin{aligned} c_{P^\sharp(\Theta)}([b_g]) &= R_{g^{-1}*}(P^\sharp(\Theta)_g - L_{[b_g]}(p^\sharp\theta)) \\ &= R_{[b_g^{-1}]*}P^\sharp(R^*_{[b_g^{-1}]}(e[b_g]+\theta)) - \mathrm{Ad}_{[b_g]}p^\sharp\theta \\ &= (R_{[b_g^{-1}]*}P_g)^\sharp(e[b_g]+\theta) - \mathrm{Ad}_{[b_g]}p^\sharp\theta \\ &= \left(c_P([b_g]) + \mathrm{Ad}_{[b_g]}\left(p - \frac{1}{2}D_\rho p\right)\right)^\sharp (e[b_g]+\theta) - \mathrm{Ad}_{[b_g]}p^\sharp\theta \\ &= \iota_\theta c_P([b_g]) + (\mathrm{Ad}_{[b_g]}p)^\sharp e([b_g]) + (\mathrm{Ad}_{[b_g]}p)^\sharp(\theta) - \frac{1}{2}(\mathrm{Ad}_{[b_g]}D_\rho p)^\sharp(e[b_g]) - \mathrm{Ad}_{[b_g]}(p^\sharp\theta). \end{aligned}$$

To prove the desired formula for $c_{P^\sharp(\Theta)}([b_g])$, it remains to show that the last three terms in the last line above cancel out. In fact, by applying (3.15), we have

$$e[b_g](\mathrm{Ad}_{[b_g]}\rho(u)) = \theta(\mathrm{Ad}_{[b_g]}u) - \theta(u), \quad u \in \Gamma(A).$$

We may write $p = X \otimes u$ for $X \in \mathfrak{X}^1(M)$ and $u \in \Gamma(A)$, and then we have

$$\begin{aligned} &(\mathrm{Ad}_{[b_g]}p)^\sharp(\theta) - \frac{1}{2}(\mathrm{Ad}_{[b_g]}D_\rho p)^\sharp(e[b_g]) - \mathrm{Ad}_{[b_g]}(p^\sharp\theta) \\ &\quad = \theta(\mathrm{Ad}_{[b_g]}u)\mathrm{Ad}_{[b_g]}X - e[b_g](\mathrm{Ad}_{[b_g]}\rho(u))\mathrm{Ad}_{[b_g]}X - \theta(u)\mathrm{Ad}_{[b_g]}X \\ &\quad = 0, \end{aligned}$$

where we have used the ρ-compatibility of p. □

Example 4.10. For a Poisson Lie group (G, P) (see [27]), by Example 3.17, the characteristic pair of a multiplicative 1-form Θ is $(0, \theta)$ such that $\Theta(g) = R^*_{g^{-1}}\theta$, where $\theta \in \mathfrak{g}^*$ is G-invariant. Also, the characteristic pair of P is $(c_P, 0)$, where $c_P \in Z^1(G, \wedge^2\mathfrak{g})$ is subject to $c_P(g) = R_{g^{-1}*}P_g$. The characteristic pair of $P^\sharp(\Theta)$ is $(c, 0)$, where $c \in Z^1(G, \mathfrak{g})$ is determined by

$$c(g) = R_{g^{-1}*}(P^\sharp\Theta) = R_{g^{-1}*}P^\sharp(R^*_{g^{-1}}\theta) = (R_{g^{-1}*}P_g)(\theta) = \iota_\theta c_P(g).$$

Example 4.11. By Lemma 3.18, the characteristic pair of an exact multiplicative 1-form $\Theta = s^*\gamma - t^*\gamma$ for $\gamma \in \Omega^1(M)$ is $(e = d_{\mathfrak{J}\mathcal{G}}\gamma, \theta = -\rho^*\gamma)$, where ρ^* is the dual map of the anchor ρ of the Lie algebroid A. Based on Lemma 4.5, we know that

$$P^\sharp(\Theta) = \overleftarrow{\iota_\gamma p} - \overrightarrow{\iota_\gamma p},$$

and its characteristic pair (c, π) is given by

$$c = d_{\mathfrak{J}\mathcal{G}}(\iota_\gamma p) \in Z^1(\mathfrak{J}\mathcal{G}, A), \quad \pi = -\rho(\iota_\gamma p) \in \mathfrak{X}^1(M)$$

by Example 2.7. One can also show these identities by utilization of Proposition 4.9.

5 Multiplicative forms on Poisson groupoids

5.1 Multiplicative 1-forms on Poisson groupoids

Consider a smooth manifold N and a bivector field $P \in \mathfrak{X}^2(N)$. One can define a skew-symmetric bracket $[\cdot, \cdot]_P$ on $\Omega^1(N)$ given by

$$[\alpha, \beta]_P = \mathcal{L}_{P^\sharp\alpha}\beta - \mathcal{L}_{P^\sharp\beta}\alpha - dP(\alpha, \beta) = d(\iota_{P^\sharp\alpha}\beta) + \iota_{P^\sharp\alpha}d\beta - \iota_{P^\sharp\beta}d\alpha, \quad \forall\, \alpha, \beta \in \Omega^1(N), \tag{5.1}$$

and an anchor map $P^\sharp : T^*N \to TN$, $\alpha \mapsto \iota_\alpha P$. We have two formulas (see [22])

$$[\alpha_1, [\alpha_2, \alpha_3]_P]_P + \text{c.p.} = -\frac{1}{2}L_{[P,P](\alpha_1,\alpha_2,\cdot)}\alpha_3 + \text{c.p.} + d([P, P](\alpha_1, \alpha_2, \alpha_3)), \quad \forall\, \alpha_i \in \Omega^1(N) \tag{5.2}$$

and

$$P^\sharp[\alpha_1,\alpha_2]_P - [P^\sharp\alpha_1, P^\sharp\alpha_2] = \frac{1}{2}[P,P](\alpha_1,\alpha_2), \quad \forall\, \alpha_i \in \Omega^1(N), \tag{5.3}$$

where c.p. is the cyclic permutation.

A Poisson manifold is a pair (N,P), where N is a smooth manifold and $P \in \mathfrak{X}^2(N)$ is a bivector field subject to $[P,P]=0$. Due to (5.2) and (5.3), we see that T^*N is a Lie algebroid when equipped with the bracket $[\,\boldsymbol{\cdot}\,,\boldsymbol{\cdot}\,]_P$ and the anchor $P^\sharp$.

Recall that a **Poisson groupoid** is a Lie groupoid $\mathcal{G}$ with a multiplicative bivector field $P \in \mathfrak{X}^2_{\mathrm{mult}}(\mathcal{G})$ such that $[P,P]=0$ (see [29,36]). In this section, we study the space $\Omega^1_{\mathrm{mult}}(\mathcal{G})$ of multiplicative 1-forms on a Poisson groupoid. We first show that $\Omega^1_{\mathrm{mult}}(\mathcal{G})$ carries a natural Lie algebra structure.

Theorem 5.1. *For a Poisson Lie groupoid $(\mathcal{G},P)$, the space of multiplicative* 1*-forms $\Omega^1_{\mathrm{mult}}(\mathcal{G})$ is a Lie subalgebra of the Lie algebra $(\Omega^1(\mathcal{G}),[\,\boldsymbol{\cdot}\,,\boldsymbol{\cdot}\,]_P)$.*

Proof. For $\Theta_1,\Theta_2 \in \Omega^1_{\mathrm{mult}}(\mathcal{G})$, we wish to show that

$$[\Theta_1,\Theta_2]_P = d(\iota_{P^\sharp\Theta_1}\Theta_2) + \iota_{P^\sharp\Theta_1}d\Theta_2 - \iota_{P^\sharp\Theta_2}d\Theta_1 \tag{5.4}$$

is also multiplicative. In fact, this follows from Lemmas 4.5(1) and 4.8(1), and the fact that the de Rham differentials of multiplicative forms are still multiplicative. □

In general, it is hard to explicitly calculate the Lie bracket on $\Omega^1_{\mathrm{mult}}(\mathcal{G})$ in terms of characteristic pairs. However, when M is a single point, we have the following fact.

Example 5.2. Suppose that we are working with a Poisson Lie group (G,P). According to Example 3.17, $\Omega^1_{\mathrm{mult}}(G)$ is in one-to-one correspondence with the set of G-invariant elements $\theta \in \mathfrak{g}^*$. In specific, $\Theta \in \Omega^1_{\mathrm{mult}}(G)$ corresponds to $\theta := \Theta|_e$ and conversely $\Theta(g) = R^*_{g^{-1}}\theta (= L^*_{g^{-1}}\theta)$ for all $g \in G$. Let Θ_1 and $\Theta_2 \in \Omega^1_{\mathrm{mult}}(G)$ be arising from G-invariants θ_1 and $\theta_2 \in \mathfrak{g}^*$, respectively. To get $[\Theta_1,\Theta_2]_P$, it suffices to compute $[\Theta_1,\Theta_2]_P|_e$. According to the defining equation (5.4), for all $u \in \mathfrak{g} = T_eG$, we have

$$\langle [\Theta_1,\Theta_2]_P|_e, u\rangle = \overrightarrow{u}|_e(P(\Theta_1,\Theta_2)) = \langle (\mathcal{L}_{\overrightarrow{u}}P)|_e, \theta_1\wedge\theta_2\rangle.$$

It is a standard fact that $\mathcal{L}_{\overrightarrow{u}}P$ coincides with $(-\overrightarrow{d_*u})$, where $d_* : \wedge^\bullet\mathfrak{g} \to \wedge^{\bullet+1}\mathfrak{g}$ stems from the Lie bialgebra $(\mathfrak{g},\mathfrak{g}^*)$ induced by the Poisson Lie group (G,P). Therefore, we get

$$\langle [\Theta_1,\Theta_2]_P|_e, u\rangle = -\langle d_*u, \theta_1\wedge\theta_2\rangle = \langle u, [\theta_1,\theta_2]_*\rangle.$$

Here, $[\cdot,\cdot]_*$ denotes the Lie bracket on $\mathfrak{g}^*$. So we conclude that as Lie algebras, $\Omega^1_{\mathrm{mult}}(G)$ is isomorphic to the Lie subalgebra of $\mathfrak{g}^*$ consisting of G-invariant elements.

Let us recall the notion of Lie algebra crossed modules.

Definition 5.3 (See [17])**.** A Lie algebra crossed module consists of a pair of Lie algebras ϑ and $\mathfrak{g}$, and a morphism of Lie algebras $\phi : \vartheta \to \mathfrak{g}$ such that $\mathfrak{g}$ acts on ϑ by derivations and satisfies for all $x \in \mathfrak{g}$ and $u,v \in \vartheta$,

(1) $\phi(u) \triangleright v = [u,v]$;

(2) $\phi(x \triangleright u) = [x,\phi(u)]$,

where $\triangleright$ denotes the $\mathfrak{g}$-action on ϑ.

We write $(\vartheta \xrightarrow{\phi} \mathfrak{g})$ to denote a Lie algebra crossed module.

Definition 5.4. A morphism (f,F): $(\vartheta \xrightarrow{\phi} \mathfrak{g}) \to (\vartheta' \xrightarrow{\phi'} \mathfrak{g}')$ of Lie algebra crossed modules consists of two Lie algebra morphisms $f : \vartheta \to \vartheta'$ and $F : \mathfrak{g} \to \mathfrak{g}'$, which fit into the following commutative diagram:

$$\begin{array}{ccc} \vartheta & \xrightarrow{f} & \vartheta' \\ \phi\downarrow & & \downarrow\phi' \\ \mathfrak{g} & \xrightarrow{F} & \mathfrak{g}', \end{array}$$

and satisfy $f(x \triangleright u) = F(x) \triangleright' f(u)$ for $x \in \mathfrak{g}$ and $u \in \vartheta$.

A typical instance of Lie algebra crossed modules arising from Lie groupoids is illustrated in [3] (see also [2, 32]). Given a Lie groupoid $\mathcal{G}$ with its Lie algebroid A, the triple $(\Gamma(A) \xrightarrow{T} \mathfrak{X}^1_{\mathrm{mult}}(\mathcal{G}))$, where $T: u \mapsto \overleftarrow{u} - \overrightarrow{u}$, consists of a Lie algebra crossed module. Here, the action of $\mathfrak{X}^1_{\mathrm{mult}}(\mathcal{G})$ on $\Gamma(A)$ is determined by $\overleftarrow{X \triangleright u} = [X, \overleftarrow{u}]$ (or $\overrightarrow{X \triangleright u} = [X, \overrightarrow{u}]$). In analogy to T, we have a map of vector spaces (following Lemma 3.18)

$$\Omega^1(M) \xrightarrow{J} \Omega^1_{\mathrm{mult}}(\mathcal{G}), \quad \gamma \mapsto s^*\gamma - t^*\gamma.$$

In the case of a Poisson Lie groupoid $(\mathcal{G}, P)$, a canonical Lie algebra crossed module structure can be found that underlies both the aforementioned vector spaces and the linear map J. This key finding is attributed to Ortiz and Waldron [32]. Their investigation demonstrated that the set of multiplicative sections of any $\mathcal{LA}$-groupoid possesses the structure of a strict Lie 2-algebra, which is presented by using a crossed module. By applying this general outcome to the specific situation of the cotangent bundle of a Poisson groupoid (which is highlighted in [32, Example 7.3]), we arrive at the desired structure. To provide further background, we rephrase this fact and present an alternative approach to it.

Theorem 5.5 (See [32])**.** *Continue to use notations as above. Endow $\Omega^1(M)$ with the Lie bracket $[\cdot,\cdot]_{\underline{P}}$ and $\Omega^1_{\mathrm{mult}}(\mathcal{G})$ with $[\cdot,\cdot]_P$. Then,*

(1) *there exists a Lie algebra action $\cdot \triangleright \cdot : \Omega^1_{\mathrm{mult}}(\mathcal{G}) \otimes \Omega^1(M) \to \Omega^1(M)$ such that*

$$[\Theta, s^*\gamma]_P = s^*(\Theta \triangleright \gamma), \quad \forall\, \Theta \in \Omega^1_{\mathrm{mult}}(\mathcal{G}), \quad \gamma \in \Omega^1(M);$$

(2) *the triple*

$$(\Omega^1(M) \xrightarrow{J} \Omega^1_{\mathrm{mult}}(\mathcal{G}))$$

forms a Lie algebra crossed module, where J is defined by $J(\gamma) = s^*\gamma - t^*\gamma$.

We prove this theorem more directly by utilizing the theory of characteristic pairs explained in Section 3. In what follows, for the Poisson structure $P \in \mathfrak{X}^2_{\mathrm{mult}}(\mathcal{G})$ on $\mathcal{G}$, we denote by $p \in \Gamma(TM \otimes A)$ the leading term of P.

We need the following technical lemma.

Lemma 5.6. *Let P be in $\mathfrak{X}^2_{\mathrm{mult}}(\mathcal{G})$ and $p \in \Gamma(TM \otimes A)$ be the leading term of P. For all $\Theta \in \Omega^1_{\mathrm{mult}}(\mathcal{G})$ and $\gamma \in \Omega^1(M)$, one has*

$$[\Theta, s^*\gamma]_P = s^*(\widehat{e}(\jmath^1\iota_\gamma p) + \iota_{p^\sharp(\theta)}(d\gamma)) \quad \text{and} \quad [\Theta, t^*\gamma]_P = t^*(\widehat{e}(\jmath^1\iota_\gamma p) + \iota_{p^\sharp(\theta)}(d\gamma)), \tag{5.5}$$

where (e,θ) with $e \in Z^1(\mathfrak{J}\mathcal{G}, T^*M)$ and $\theta \in \Gamma(A^*)$ is the $(0,1)$-characteristic pair of Θ and $\widehat{e} \in Z^1(\mathfrak{J}A, T^*M)$ is the infinitesimal of e. Moreover, if $(\mathcal{G}, P)$ is a Poisson groupoid, then the map

$$\cdot \triangleright \cdot : \Omega^1_{\mathrm{mult}}(\mathcal{G}) \otimes \Omega^1(M) \to \Omega^1(M)$$

defined by

$$\Theta \triangleright \gamma = \widehat{e}(\jmath^1\iota_\gamma p) + \iota_{p^\sharp(\theta)}(d\gamma), \tag{5.6}$$

is a Lie algebra action.

Proof. The leading term of $P^\sharp(\Theta)$ is $p^\sharp(\theta)$ (by Proposition 4.9). By (1) and (2) of Lemma 4.5, we have $P^\sharp\Theta \in \mathfrak{X}^1_{\mathrm{mult}}(\mathcal{G})$ and

$$\iota_{P^\sharp\Theta}(s^*\gamma) = \iota_{s^*\gamma}P^\sharp\Theta = s^*(\iota_\gamma p^\sharp(\theta)) = -s^*(p(\gamma,\theta)).$$

Similarly, from $s_*P^\sharp\Theta = p^\sharp(\theta)$, we obtain

$$\iota_{P^\sharp\Theta}(s^*d\gamma) = s^*(\iota_{s_*P^\sharp\Theta}(d\gamma)) = s^*(\iota_{p^\sharp(\theta)}(d\gamma)).$$

Using these two identities, we can compute

$$[\Theta, s^*\gamma]_P = \mathcal{L}_{P^\sharp\Theta}(s^*\gamma) - \iota_{P^\sharp(s^*\gamma)}d\Theta$$

$$
\begin{aligned}
&= d\iota_{P^\sharp\Theta}(s^*\gamma) + \iota_{P^\sharp\Theta}(s^* d\gamma) - \iota_{\overleftarrow{\iota_\gamma p}} d\Theta \text{ (by Lemma 4.5(2))} \\
&= -s^*(d(p(\gamma,\theta))) + s^*(\iota_{p^\sharp(\theta)}(d\gamma)) - s^*(\widehat{d\Theta}(\iota_\gamma p)) \text{ (by Lemma 4.8(2))}.
\end{aligned}
$$

Here, $\widehat{d\Theta} = \mathrm{pr}_{\Gamma(A^*\otimes T^*M)}(d\Theta)|_M$ is the leading term of $d\Theta$. By Proposition 4.1, we have

$$
\widehat{d\Theta}(\iota_\gamma p) = -d(\theta(\iota_\gamma p)) - \widehat{e}(\jmath^1 \iota_\gamma p),
$$

where $\widehat{e} \in Z^1(\mathfrak{J}A, T^*M)$ is the infinitesimal of $e \in Z^1(\mathfrak{J}\mathcal{G}, T^*M)$. So we get the first equality of (5.5):

$$
[\Theta, s^*\gamma]_P = s^*(\widehat{e}(\jmath^1\iota_\gamma p) + \iota_{p^\sharp(\theta)}(d\gamma)) = s^*(\Theta \triangleright \gamma).
$$

The other one is proved in a similar manner.

Furthermore, if P is Poisson, then by the Jacobi identity of $[\,\cdot\,,\,\cdot\,]_P$, we have

$$
[\Theta', [\Theta, s^*\gamma]_P]_P + [\Theta, [s^*\gamma, \Theta']_P]_P + [s^*\gamma, [\Theta', \Theta]_P]_P = 0, \quad \forall\, \Theta, \Theta' \in \Omega^1_{\mathrm{mult}}(\mathcal{G}), \quad \gamma \in \Omega^1(M).
$$

It follows immediately that

$$
s^*(\Theta' \triangleright (\Theta \triangleright \gamma) - \Theta \triangleright (\Theta' \triangleright \gamma) - [\Theta', \Theta]_P \triangleright \gamma) = 0.
$$

Since s^* is injective, we prove that the map $\triangleright$ defines an action of $\Omega^1_{\mathrm{mult}}(\mathcal{G})$ on $\Omega^1(M)$. □

We also need a standard fact.

Lemma 5.7 (See [36]). *The source map $s : \mathcal{G} \to M$ is a Poisson map and the target map $t : \mathcal{G} \to M$ is an anti-Poisson map. Moreover, for any $\gamma, \eta \in \Omega^1(M)$, we have $[s^*\gamma, t^*\eta]_P = 0$. As a direct consequence, we have*

$$
[s^*\gamma - t^*\gamma, s^*\eta - t^*\eta]_P = s^*[\gamma, \eta]_{\underline{P}} - t^*[\gamma, \eta]_{\underline{P}}, \quad \forall\, \gamma, \eta \in \Omega^1(M). \tag{5.7}
$$

Recall that the base manifold M is equipped with an induced Poisson structure $\underline{P} = s_*P \in \mathfrak{X}^2(M)$ (see [36]). By our formula (2.8), we have $\underline{P} = -\frac{1}{2}(1 \otimes \rho)p$.

We now finish the proof of Theorem 5.5.

Proof of Theorem 5.5. Statement (1) is proved by Lemma 5.6. For (2), we note that $J : \Omega^1(M) \to \Omega^1_{\mathrm{mult}}(\mathcal{G})$ is a morphism of Lie algebras (by (5.7)). To prove that $\Omega^1(M) \xrightarrow{J} \Omega^1_{\mathrm{mult}}(\mathcal{G})$ is a Lie algebra crossed module, it suffices to show

$$
[\gamma, \gamma']_{\underline{P}} = (J\gamma) \triangleright \gamma' \quad \text{and} \quad J(\Theta \triangleright \gamma) = [\Theta, J(\gamma)]_P.
$$

In fact, the first one follows from

$$
s^*[\gamma, \gamma']_{\underline{P}} = [s^*\gamma, s^*\gamma']_P = [s^*\gamma - t^*\gamma, s^*\gamma']_P = s^*((J\gamma) \triangleright \gamma')
$$

(by Lemma 5.7 and the definition of $\triangleright$). The second is a direct consequence of Lemma 5.6. □

Proposition 5.8. *Let $(\mathcal{G}, P)$ be a Poisson groupoid and $p \in \Gamma(TM \otimes A)$ be the leading term of P. The map*

$$
P^\sharp : (\Omega^1_{\mathrm{mult}}(\mathcal{G}), [\,\cdot\,,\,\cdot\,]_P) \to (\mathfrak{X}^1_{\mathrm{mult}}(\mathcal{G}), [\,\cdot\,,\,\cdot\,]), \quad \alpha \mapsto \iota_\alpha P
$$

is a Lie algebra morphism. Moreover, the pair $(P^\sharp, p^\sharp)$ constitutes a morphism of Lie algebra crossed modules:

$$
\begin{array}{ccc}
\Omega^1(M) & \xrightarrow{\;p^\sharp\;} & \Gamma(A) \\
{\scriptstyle J}\downarrow & & \downarrow{\scriptstyle T} \\
\Omega^1_{\mathrm{mult}}(\mathcal{G}) & \xrightarrow{\;P^\sharp\;} & \mathfrak{X}^1_{\mathrm{mult}}(\mathcal{G}),
\end{array}
$$

where T is defined by $u \mapsto \overleftarrow{u} - \overrightarrow{u}$ for $u \in \Gamma(A)$.

Proof. We verify that $(P^\sharp, p^\sharp)$ is a morphism of Lie algebra crossed modules. First, $P^\sharp$ is a Lie algebra morphism. Second, by Lemma 4.5(2), we have

$$P^\sharp \circ J(\gamma) = P^\sharp(s^*\gamma - t^*\gamma) = \overleftarrow{p^\sharp\gamma} - \overrightarrow{p^\sharp\gamma} = T \circ p^\sharp(\gamma), \quad \gamma \in \Omega^1(M).$$

So $P^\sharp \circ J = T \circ p^\sharp$ holds true. Next, we show the relation

$$p^\sharp(\Theta \triangleright \gamma) = (P^\sharp\Theta) \triangleright (p^\sharp\gamma).$$

In fact, using Lemma 4.5(2) again, we have

$$\overleftarrow{p^\sharp(\Theta \triangleright \gamma)} = P^\sharp s^*(\Theta \triangleright \gamma) = P^\sharp[\Theta, s^*\gamma]_P = [P^\sharp\Theta, \overleftarrow{p^\sharp\gamma}] = \overleftarrow{(P^\sharp\Theta) \triangleright (p^\sharp\gamma)}.$$

Then, we obtain the desired relation because the left translation is injective. The fact that $p^\sharp$ is a Lie algebra morphism follows from direct verification:

$$p^\sharp[\gamma, \gamma']_{\underline{P}} = p^\sharp(J\gamma \triangleright \gamma') = (P^\sharp J\gamma) \triangleright (p^\sharp\gamma') = (Tp^\sharp\gamma) \triangleright (p^\sharp\gamma') = [p^\sharp\gamma, p^\sharp\gamma']_A.$$

This completes the proof. □

Example 5.9. Suppose that a Poisson manifold $(M, \underline{P})$ admits a symplectic groupoid $(\mathcal{G}, \omega)$ which integrates the Lie algebroid T^*M arising from the Poisson structure $\underline{P}$. In this case, the pair $((\omega^\sharp)^{-1}, \mathrm{id})$ forms an isomorphism of Lie algebra crossed modules:

$$\begin{array}{ccc} \Omega^1(M) & \xrightarrow{\mathrm{id}} & \Omega^1(M) \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle T} \\ \Omega^1_{\mathrm{mult}}(\mathcal{G}) & \xrightarrow{(\omega^\sharp)^{-1}} & \mathfrak{X}^1_{\mathrm{mult}}(\mathcal{G}). \end{array}$$

5.2 The DGLA of multiplicative forms on a Poisson groupoid

On a general Poisson manifold (N, P), the space of all the degree forms $\Omega^\bullet(N) = \bigoplus_{k=0}^n \Omega^k(N)$, where $n = \dim(N)$, admits a graded Lie bracket known as the Schouten-Nijenhuis bracket which is extended by the Leibniz rule from the Lie bracket (5.1) of 1-forms $\Omega^1(N)$, and also denoted by $[\,\cdot\,,\,\cdot\,]_P$. So we have a GLA $(\Omega^\bullet(N), [\,\cdot\,,\,\cdot\,]_P)$. Equipped with the de Rham differential d, the triple $(\Omega^\bullet(N), [\,\cdot\,,\,\cdot\,]_P, d)$ is a DGLA. In fact, we have

$$d[\alpha, \beta]_P = [d\alpha, \beta]_P + (-1)^{k-1}[\alpha, d\beta]_P, \quad \forall\, \alpha \in \Omega^k(N), \quad \beta \in \Omega^l(N) \tag{5.8}$$

(see [34]). Also, the induced map

$$\wedge^\bullet P^\sharp : (\Omega^\bullet(N), [\,\cdot\,,\,\cdot\,]_P, d) \to (\mathfrak{X}^\bullet(N), [\,\cdot\,,\,\cdot\,], [P, \,\cdot\,])$$

defined by

$$(\wedge^k P^\sharp)(\alpha_1 \wedge \cdots \wedge \alpha_k) = P^\sharp(\alpha_1) \wedge \cdots \wedge P^\sharp(\alpha_k)$$

is a morphism of DGLAs. In other words, $\wedge^\bullet P^\sharp$ is a morphism of GLAs and a cochain map:

$$(\wedge^{\bullet+1} P^\sharp)(d\alpha) = [P, (\wedge^\bullet P^\sharp)\alpha], \quad \forall\, \alpha \in \Omega^\bullet(N).$$

Here, we take the convention that when $\bullet = 0$, $\wedge^0 P^\sharp$ reduces to the identity map $C^\infty(N) \to C^\infty(N)$.

The notation $\wedge^\bullet P^\sharp$ should not be confused with $P^\sharp$, which represents the standard contraction (see (5.10)).

On a Poisson Lie groupoid $(\mathcal{G}, P)$, it is natural to expect that $\Omega^\bullet_{\mathrm{mult}}(\mathcal{G})$ also admits a DGLA structure $([\,\cdot\,,\,\cdot\,]_P, d)$. We prove this fact and find some more interesting conclusions. Let us first recall the notion of GLA crossed modules (also known as $\mathbb{Z}$-graded Lie 2-algebras).

Definition 5.10 (See [3]). A GLA crossed module $(\vartheta \xrightarrow{\phi} \mathfrak{g})$ consists of a pair of GLAs ϑ and $\mathfrak{g}$, and a morphism of GLAs $\phi : \vartheta \to \mathfrak{g}$ such that $\mathfrak{g}$ acts on ϑ and satisfies for all $x, y \in \mathfrak{g}$ and $u, v \in \vartheta$,

(1) $\phi(u) \triangleright v = [u, v]$;

(2) $\phi(x \triangleright u) = [x, \phi(u)]$,

where $\triangleright$ denotes the $\mathfrak{g}$-action on ϑ.

What we need is an enhanced version of this definition.

Definition 5.11. A DGLA crossed module is a GLA crossed module $(\vartheta \xrightarrow{\phi} \mathfrak{g})$ as defined above, where ϑ and $\mathfrak{g}$ are both DGLAs, $\phi : \vartheta \to \mathfrak{g}$ is a morphism of DGLAs, and the action $\triangleright$ of $\mathfrak{g}$ on ϑ is compatible with the relevant differentials:

$$d_\vartheta(x \triangleright u) = (d_{\mathfrak{g}} x) \triangleright u + (-1)^{|x|} x \triangleright d_\vartheta u, \quad \forall\, x \in \mathfrak{g}, \quad u \in \vartheta.$$

Morphisms of GLA and DGLA crossed modules are defined in the same fashions as those of Definition 5.4.

Example 5.12 (See [3]). Let $\mathcal{G}$ be a Lie groupoid. The space $\mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G})$ of multiplicative multi-vector fields on $\mathcal{G}$ is a graded vector space (not an algebra). It constitutes a GLA (after degree shifts), the Schouten bracket being its structure map. Indeed, we have a GLA crossed module

$$\Gamma(\wedge^\bullet A) \xrightarrow{T} \mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G}), \quad u \mapsto \overleftarrow{u} - \overrightarrow{u},$$

where $X \triangleright u \in \Gamma(\wedge^{k+l-1} A)$ is determined by the relation

$$\overleftarrow{X \triangleright u} = [X, \overleftarrow{u}] \ (\text{or } \overrightarrow{X \triangleright u} = [X, \overrightarrow{u}]), \quad X \in \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G}), \quad u \in \Gamma(\wedge^l A).$$

Note that we regard $\Gamma(\wedge^0 A)$ as $C^\infty(M)$ and $\mathfrak{X}^0_{\mathrm{mult}}(\mathcal{G})$ as multiplicative functions on $\mathcal{G}$. The action of $\mathfrak{X}^0_{\mathrm{mult}}(\mathcal{G})$ on $\Gamma(\wedge^0 A)$ is simply trivial.

Example 5.13. Continuing the above example, if we are given a multiplicative Poisson bivector field P on the Lie groupoid $\mathcal{G}$, then $(\mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G}), [\,\cdot\,,\,\cdot\,], [P,\,\cdot\,])$ becomes a DGLA equipped with the differential $[P,\,\cdot\,] : \mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G}) \to \mathfrak{X}^{\bullet+1}_{\mathrm{mult}}(\mathcal{G})$. It also induces a differential $\delta_P : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+1} A)$ defined by

$$\overleftarrow{\delta_P u} = [P, \overleftarrow{u}], \quad \forall\, u \in \Gamma(\wedge^\bullet A)$$

so that $(\Gamma(\wedge^\bullet A), [\,\cdot\,,\,\cdot\,]_A, \delta_P)$ is a DGLA. Now, the GLA crossed module

$$\Gamma(\wedge^\bullet A) \xrightarrow{T} \mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G}), \quad u \mapsto \overleftarrow{u} - \overrightarrow{u}$$

in Example 5.12 is indeed a DGLA crossed module. To see it, we need to show

$$T(\delta_P u) = [P, Tu] \quad \text{and} \quad \delta_P(X \triangleright u) = [P, X] \triangleright u + (-1)^{k-1} X \triangleright \delta_P u, \quad \forall\, X \in \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G}).$$

Let us examine these two equations. We have

$$T(\delta_P u) = \overleftarrow{\delta_P u} - \overrightarrow{\delta_P u} = [P, \overleftarrow{u}] - [P, \overrightarrow{u}] = [P, Tu]$$

and

$$\begin{aligned} \overleftarrow{\delta_P(X \triangleright u)} &= \overleftarrow{[P, X \triangleright u]} = [P, \overleftarrow{X \triangleright u}] = [P, [X, \overleftarrow{u}]] \\ &= [[P, X], \overleftarrow{u}] + (-1)^{k-1}[X, [P, \overleftarrow{u}]] \\ &= \overleftarrow{[P, X] \triangleright u} + (-1)^{k-1} \overleftarrow{X \triangleright (\delta_P u)}, \end{aligned}$$

where the graded Jacobi identity of the Schouten bracket is applied.

We present our main result, which notably improves upon the Ortiz-Waldron Theorem 5.5.

Theorem 5.14. *Let $(\mathcal{G}, P)$ be a Poisson Lie groupoid.*

(1) *With respect to the graded Lie bracket $[\cdot,\cdot]_P$ and the de Rham differential d, the space $\Omega^\bullet_{\mathrm{mult}}(\mathcal{G})$ is a sub DGLA of $\Omega^\bullet(\mathcal{G})$.*

(2) *Endow $\Omega^\bullet(M)$ with the graded Lie bracket $[\cdot,\cdot]_{\underline{P}}$ and the de Rham differential d, where $\underline{P}$ is the Poisson structure on M induced from $(\mathcal{G}, P)$. The triple*

$$(\Omega^\bullet(M) \xrightarrow{J} \Omega^\bullet_{\mathrm{mult}}(\mathcal{G}))$$

consists of a DGLA crossed module, where J is defined by

$$J(\gamma) := s^*\gamma - t^*\gamma, \quad \forall\, \gamma \in \Omega^\bullet(M) \tag{5.9}$$

and the action map $\triangleright$ of $\Omega^\bullet_{\mathrm{mult}}(\mathcal{G})$ on $\Omega^\bullet(M)$ is uniquely determined by the relation

$$s^*(\Theta \triangleright \gamma) = [\Theta, s^*\gamma]_P, \quad \forall\, \Theta \in \Omega^k_{\mathrm{mult}}(\mathcal{G}), \quad \gamma \in \Omega^l(M).$$

(3) *The map $\wedge^\bullet P^\sharp$ sends multiplicative k-forms on $\mathcal{G}$ to multiplicative k-vector fields, and thereby,*

$$\wedge^\bullet P^\sharp : (\Omega^\bullet_{\mathrm{mult}}(\mathcal{G}), [\cdot,\cdot]_P, d) \to (\mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G}), [\cdot,\cdot], [P,\cdot])$$

is a morphism of DGLAs. (When $\bullet = 0$, we treat $\wedge^0 P^\sharp$ as the identity map on the space of multiplicative functions on $\mathcal{G}$.)

(4) *The map $\wedge^\bullet P^\sharp$ together with $\wedge^\bullet p^\sharp$ is a morphism of DGLA crossed modules:*

$$\begin{array}{ccc} \Omega^\bullet(M) & \xrightarrow{\wedge^\bullet p^\sharp} & \Gamma(\wedge^\bullet A) \\ \downarrow J & & \downarrow T \\ \Omega^\bullet_{\mathrm{mult}}(\mathcal{G}) & \xrightarrow{\wedge^\bullet P^\sharp} & \mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G}), \end{array}$$

where $p = \mathrm{pr}_{\Gamma(TM\otimes A)} P|_M \in \Gamma(TM \otimes A)$ is the leading term of P. (When $\bullet = 0$, we treat $\wedge^0 p^\sharp$ as the identity map on $C^\infty(M)$.)

We should note that the wedge product of multiplicative forms is not multiplicative in general. So one can *not* deduce that the graded Lie bracket $[\cdot,\cdot]_P$ on $\Omega^\bullet_{\mathrm{mult}}(\mathcal{G})$ is extended from the one on $\Omega^1_{\mathrm{mult}}(\mathcal{G})$.

To prove Theorem 5.14, we need to set up some basic formulas and facts. For a bivector field $P \in \mathfrak{X}^2(N)$, we define

$$P^\sharp : \Omega^k(N) \to \Omega^{k-1}(N) \otimes \mathfrak{X}^1(N)$$

by

$$P^\sharp(\alpha_1 \wedge \cdots \wedge \alpha_k) := \sum_{i=1}^k (-1)^{i+k} \alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_k \otimes P^\sharp(\alpha_i). \tag{5.10}$$

Then, for $\alpha \in \Omega^k(N)$ and $\beta \in \Omega^l(N)$, define $\iota_{P^\sharp\alpha}\beta \in \Omega^{k+l-2}(N)$ by

$$\begin{aligned} &\iota_{P^\sharp(\alpha_1\wedge\cdots\wedge\alpha_k)}(\beta_1 \wedge \cdots \wedge \beta_l) \\ &\quad = \sum_{i=1}^k (-1)^{i+k} \alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_k \wedge \iota_{P^\sharp(\alpha_i)}(\beta_1 \wedge \cdots \wedge \beta_l) \\ &\quad = \sum_{i,j} (-1)^{i+k+j-1} (\iota_{P^\sharp\alpha_i}\beta_j) \alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_k \wedge \beta_1 \wedge \cdots \wedge \widehat{\beta_j} \wedge \cdots \wedge \beta_l. \end{aligned} \tag{5.11}$$

For every k-form $\alpha \in \Omega^k(N)$, we denote by $\alpha^\sharp : \wedge^{k-1} TN \to T^*N$ the map

$$\alpha^\sharp(X_1, \ldots, X_{k-1}) = \alpha(X_1, \ldots, X_{k-1}, \cdot), \quad X_i \in \mathfrak{X}^1(N).$$

With notations as above, we can verify the following identity. For all $X_1,\dots,X_{k+l-3}\in\mathfrak{X}^1(N)$, one has

$$\begin{aligned}&(\iota_{P^\sharp\alpha}\beta)^\sharp(X_1,\dots,X_{k+l-3})\\&\quad=\sum_{\sigma\in\mathrm{Sh}(k-1,l-2)}(-1)^\sigma\beta^\sharp(P^\sharp\alpha^\sharp(X_{\sigma_1},\dots,X_{\sigma_{k-1}}),X_{\sigma_k},\dots,X_{\sigma_{k+l-3}})\\&\qquad-(-1)^{kl}\sum_{\tau\in\mathrm{Sh}(l-1,k-2)}(-1)^\tau\alpha^\sharp(P^\sharp\beta^\sharp(X_{\tau_1},\dots,X_{\tau_{l-1}}),X_{\tau_l},\dots,X_{\tau_{k+l-3}}).\end{aligned}\tag{5.12}$$

Lemma 5.15. *On a Poisson manifold (N,P), for $\alpha\in\Omega^k(N)$ and $\beta\in\Omega^l(N)$, we have*

$$[\alpha,\beta]_P=\iota_{P^\sharp\alpha}d\beta+(-1)^{k-1}d\iota_{P^\sharp\alpha}\beta-(-1)^{(k-1)(l-1)}\iota_{P^\sharp\beta}d\alpha,\tag{5.13}$$

where $\iota_{P^\sharp\alpha}\beta$ is defined by (5.11).

In the existing literature, a more common formula of $[\cdot,\cdot]_P$ is of the form

$$[\alpha,\beta]_P=(-1)^{k-1}(\mathcal{L}_P(\alpha\wedge\beta)-\mathcal{L}_P(\alpha)\wedge\beta)-\alpha\wedge\mathcal{L}_P\beta,\quad\alpha\in\Omega^k(N),\quad\beta\in\Omega^l(N)$$

(see [23]). Here, $\mathcal{L}_P:\Omega^n(N)\to\Omega^{n-1}(N)$ is defined by $\mathcal{L}_P=\iota_P\circ d-d\circ\iota_P$, and $\iota_P:\Omega^n(N)\to\Omega^{n-2}(N)$ is the contraction. The bracket $[\cdot,\cdot]_P$ is also known as the *Koszul bracket*. From the formula as described above, one can prove (5.13). For completeness, we sketch a direct proof of (5.13).

Proof of Lemma 5.15. If $\alpha=\alpha_1\wedge\cdots\wedge\alpha_k$ and $\beta=\beta_1\wedge\cdots\wedge\beta_l$, then by the Leibniz rule, we have

$$\begin{aligned}[\alpha,\beta]_P&=[\alpha_1\wedge\cdots\wedge\alpha_k,\beta_1\wedge\cdots\wedge\beta_l]_P\\&=\sum_{i,j}(-1)^{i+j}[\alpha_i,\beta_j]_P\wedge\alpha_1\wedge\cdots\widehat{\alpha_i}\wedge\cdots\wedge\widehat{\beta_j}\wedge\cdots\wedge\beta_l\\&=\sum_{i,j}(-1)^{i+j}(\iota_{P^\sharp\alpha_i}d\beta_j+d\iota_{P^\sharp\alpha_i}\beta_j-\iota_{P^\sharp\beta_j}d\alpha_i)\wedge\alpha_1\wedge\cdots\widehat{\alpha_i}\wedge\cdots\wedge\widehat{\beta_j}\wedge\cdots\wedge\beta_l.\end{aligned}$$

By the definition of $P^\sharp$, we have

$$\begin{aligned}\iota_{P^\sharp\alpha}d\beta&=\sum_{i,j}(-1)^{i+k+j-1}\iota_{\alpha_1\wedge\cdots\wedge\widehat{\alpha_i}\wedge\cdots\wedge\alpha_k\otimes P^\sharp\alpha_i}(\beta_1\wedge\cdots\wedge d\beta_j\wedge\cdots\wedge\beta_l)\\&=\sum_{i,j}(-1)^{i+k+j-1}\Big((-1)^{k-1}(\iota_{P^\sharp\alpha_i}d\beta_j)\wedge\alpha_1\wedge\cdots\wedge\widehat{\alpha_i}\wedge\cdots\wedge\widehat{\beta_j}\wedge\cdots\wedge\beta_l\\&\quad+\sum_{p>j}(-1)^p(\iota_{P^\sharp\alpha_i}\beta_p)\alpha_1\wedge\cdots\wedge\widehat{\alpha_i}\wedge\cdots\wedge d\beta_j\wedge\cdots\wedge\widehat{\beta_p}\wedge\cdots\wedge\beta_l\\&\quad+\sum_{p<j}(-1)^{p-1}(\iota_{P^\sharp\alpha_i}\beta_p)\alpha_1\wedge\cdots\wedge\widehat{\alpha_i}\wedge\cdots\wedge\widehat{\beta_p}\wedge\cdots\wedge d\beta_j\wedge\cdots\wedge\beta_l\Big),\\d\iota_{P^\sharp\alpha}\beta&=\sum_{i,j}(-1)^{i+j+k-1}d((\iota_{P^\sharp\alpha_i}\beta_j)\wedge\alpha_1\wedge\cdots\wedge\widehat{\alpha_i}\wedge\cdots\wedge\widehat{\beta_j}\wedge\cdots\wedge\beta_l)\\&=\sum_{i,j}(-1)^{i+j+k-1}\Big(d(\iota_{P^\sharp\alpha_i}\beta_j)\wedge\alpha_1\wedge\cdots\wedge\widehat{\alpha_i}\wedge\cdots\wedge\widehat{\beta_j}\wedge\cdots\wedge\beta_l\\&\quad+\sum_{p<i}(-1)^{p-1}(\iota_{P^\sharp\alpha_i}\beta_j)\alpha_1\wedge\cdots\wedge d\alpha_p\wedge\cdots\wedge\widehat{\alpha_i}\wedge\cdots\widehat{\beta_j}\wedge\cdots\wedge\beta_l\\&\quad+\sum_{p>i}(-1)^p(\iota_{P^\sharp\alpha_i}\beta_j)\alpha_1\wedge\cdots\wedge\widehat{\alpha_i}\wedge\cdots\wedge d\alpha_p\cdots\wedge\widehat{\beta_j}\wedge\cdots\wedge\beta_l\\&\quad+\sum_{p<j}(-1)^{p+k}(\iota_{P^\sharp\alpha_i}\beta_j)\alpha_1\wedge\cdots\wedge\widehat{\alpha_i}\wedge\cdots\wedge d\beta_p\wedge\cdots\wedge\widehat{\beta_j}\wedge\cdots\wedge\beta_l\\&\quad+\sum_{p>j}(-1)^{p+k-1}(\iota_{P^\sharp\alpha_i}\beta_j)\alpha_1\wedge\cdots\wedge\widehat{\alpha_i}\wedge\cdots\wedge\widehat{\beta_j}\wedge\cdots\wedge d\beta_p\wedge\cdots\wedge\beta_l\Big),\end{aligned}$$

and

$$
\begin{aligned}
\iota_{P^\sharp\beta}d\alpha &= \sum_{i,j}(-1)^{j+l+i-1}\iota_{\beta_1\wedge\cdots\wedge\widehat{\beta_j}\wedge\cdots\wedge\beta_l\otimes P^\sharp\beta_j}(\alpha_1\wedge\cdots\wedge d\alpha_i\wedge\cdots\wedge\alpha_k)\\
&= \sum_{i,j}(-1)^{j+l+i-1}\Big((-1)^{l-1}(\iota_{P^\sharp\beta_j}d\alpha_i)\wedge\beta_1\wedge\cdots\wedge\widehat{\beta_j}\wedge\cdots\wedge\widehat{\alpha_i}\wedge\cdots\wedge\alpha_k\\
&\quad+\sum_{p<i}(-1)^{p-1}(\iota_{P^\sharp\beta_j}\alpha_p)\beta_1\wedge\cdots\wedge\widehat{\beta_j}\wedge\cdots\wedge\widehat{\alpha_p}\wedge\cdots\wedge d\alpha_i\wedge\cdots\wedge\alpha_k\\
&\quad+\sum_{p>i}(-1)^{p}(\iota_{P^\sharp\beta_j}\alpha_p)\beta_1\wedge\cdots\wedge\widehat{\beta_j}\wedge\cdots\wedge d\alpha_i\wedge\cdots\wedge\widehat{\alpha_p}\wedge\cdots\wedge\alpha_k\Big)\\
&= \sum_{i,j}(-1)^{j+l+i-1}\Big((-1)^{k(l-1)}(\iota_{P^\sharp\beta_j}d\alpha_i)\wedge\alpha_1\wedge\cdots\wedge\widehat{\alpha_i}\wedge\cdots\wedge\beta_1\wedge\cdots\wedge\widehat{\beta_j}\wedge\cdots\wedge\beta_l\\
&\quad+\sum_{p<i}(-1)^{p-1+(l-1)k}(\iota_{P^\sharp\beta_j}\alpha_p)\alpha_1\wedge\cdots\wedge\widehat{\alpha_p}\wedge\cdots\wedge d\alpha_i\wedge\cdots\wedge\widehat{\beta_j}\wedge\cdots\wedge\beta_l\\
&\quad+\sum_{p>i}(-1)^{p+(l-1)k}(\iota_{P^\sharp\beta_j}\alpha_p)\alpha_1\wedge\cdots\wedge d\alpha_i\wedge\cdots\widehat{\alpha_p}\wedge\cdots\wedge\widehat{\beta_j}\wedge\cdots\wedge\beta_l\Big).
\end{aligned}
$$

Taking the summation of these formulas, we see that the second and third terms of $\iota_{P^\sharp\alpha}d\beta$ cancel out with the fourth and fifth terms of $(-1)^{k-1}d\iota_{P^\sharp\alpha}\beta$, and the second and third terms of $-(-1)^{(k-1)(l-1)}\iota_{P^\sharp\beta}d\alpha$ cancel out with the third and second terms of $(-1)^{k-1}d\iota_{P^\sharp\alpha}\beta$. Combining these calculations is the formula that we expect. □

Proposition 5.16. *Let $\mathcal{G}$ be a Lie groupoid. For all the multiplicative forms $\alpha\in\Omega^k_{\mathrm{mult}}(\mathcal{G})$ and $\beta\in\Omega^l_{\mathrm{mult}}(\mathcal{G})$, if $P\in\mathfrak{X}^2_{\mathrm{mult}}(\mathcal{G})$, then the contraction $\iota_{P^\sharp\alpha}\beta\in\Omega^{k+l-2}(\mathcal{G})$ defined by* (5.11) *is also multiplicative.*

Proof. To prove that the $(k+l-2)$-form $\iota_{P^\sharp\alpha}\beta$ is multiplicative, it suffices to show that

$$(\iota_{P^\sharp\alpha}\beta)^\sharp:\oplus^{k+l-3}T\mathcal{G}\to T^*\mathcal{G}$$

is a Lie groupoid morphism. Here, $\oplus^{k+l-3}T\mathcal{G}$ is the Whitney sum of $T\mathcal{G}$, and a Lie groupoid over $\oplus^{k+l-3}TM$. As α, β, and P are all multiplicative, the three maps

$$\alpha^\sharp:\oplus^{k-1}T\mathcal{G}\to T^*\mathcal{G},\quad \beta^\sharp:\oplus^{l-1}T\mathcal{G}\to T^*\mathcal{G},\quad\text{and}\quad P^\sharp:T^*\mathcal{G}\to T\mathcal{G}$$

are all groupoid morphisms. Thus the compositions

$$\oplus^{k+l-3}T\mathcal{G}=\oplus^{k-1}T\mathcal{G}\ \oplus\ (\oplus^{l-2}T\mathcal{G})\xrightarrow{\alpha^\sharp\oplus\mathrm{Id}}T^*\mathcal{G}\ \oplus\ (\oplus^{l-2}T\mathcal{G})\xrightarrow{P^\sharp\oplus\mathrm{Id}}T\mathcal{G}\ \oplus\ (\oplus^{l-2}T\mathcal{G})\xrightarrow{\beta^\sharp}T^*\mathcal{G}$$

and

$$\oplus^{k+l-3}T\mathcal{G}=\oplus^{l-1}T\mathcal{G}\ \oplus\ (\oplus^{k-2}T\mathcal{G})\xrightarrow{\beta^\sharp\oplus\mathrm{Id}}T^*\mathcal{G}\ \oplus\ (\oplus^{k-2}T\mathcal{G})\xrightarrow{P^\sharp\oplus\mathrm{Id}}T\mathcal{G}\ \oplus\ (\oplus^{k-2}T\mathcal{G})\xrightarrow{\alpha^\sharp}T^*\mathcal{G}$$

are both Lie groupoid morphisms as well. By (5.12), $(\iota_{P^\sharp\alpha}\beta)^\sharp$ is the summation of a series of the above two compositions. Based on the interchange law (2.11) of $T^*\mathcal{G}$, it is also a Lie groupoid morphism. □

Lemma 5.17. *For all the integers k and $\Theta\in\Omega^k_{\mathrm{mult}}(\mathcal{G})$, we have $(\wedge^kP^\sharp)\Theta\in\mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G})$.*

Proof. For any $\alpha^i_g\in T^*_g\mathcal{G}$ and $\beta^i_r\in T^*_r\mathcal{G}$ such that $s(\alpha^i_g)=t(\beta^i_r)$, since $P^\sharp:T^*\mathcal{G}\to T\mathcal{G}$ is a Lie groupoid morphism and Θ is multiplicative, we have

$$
\begin{aligned}
((\wedge^kP^\sharp)\Theta)(\alpha^1_g\cdot\beta^1_r,\dots,\alpha^k_g\cdot\beta^k_r) &= (-1)^k\Theta(P^\sharp(\alpha^1_g\cdot\beta^1_r),\dots,P^\sharp(\alpha^k_g\cdot\beta^k_r))\\
&= (-1)^k\Theta(P^\sharp(\alpha^1_g)\cdot P^\sharp(\beta^1_r),\dots,P^\sharp(\alpha^k_r)\cdot P^\sharp(\beta^k_r))\\
&= (-1)^k\Theta(P^\sharp(\alpha^1_g),\dots,P^\sharp(\alpha^k_g))+(-1)^k\Theta(P^\sharp(\beta^1_r),\dots,P^\sharp(\beta^k_r))\\
&= (\wedge^kP^\sharp)(\Theta)(\alpha^1_g,\dots,\alpha^k_g)+(\wedge^kP^\sharp)(\Theta)(\beta^1_r,\dots,\beta^k_r).
\end{aligned}
$$

This property implies that $(\wedge^kP^\sharp)\Theta\in\mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G})$. □

We are ready to finish the proof of Theorem 5.14.

Proof of Theorem 5.14. We first prove the statement (1). By Lemma 5.15, Proposition 5.16, and the fact that the de Rham differential preserves multiplicativity, the graded Lie bracket $[\cdot,\cdot]_P$ is closed on multiplicative forms. Thus $\Omega^\bullet_{\mathrm{mult}}(\mathcal{G}) \subset \Omega^\bullet(\mathcal{G})$ is a graded Lie subalgebra and a sub DGLA as well.

For (2), we need a fact that for $\alpha \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$ and $\gamma \in \Omega^l(M)$, there exists a unique $(k+l-1)$-form $\omega \in \Omega^{k+l-1}(M)$ such that

$$[\alpha, s^*\gamma]_P (= \iota_{P^\sharp\alpha} ds^*\gamma + d\iota_{P^\sharp\alpha} s^*\gamma - \iota_{P^\sharp s^*\gamma} d\alpha) = s^*\omega. \tag{5.14}$$

To see it, we need to show that for all $X \in \ker s_*$ and $Y_i \in \mathfrak{X}^1(\mathcal{G})$ $(i = 1, \dots, k+l-2)$,

$$[\alpha, s^*\gamma]_P (X, Y_1, \dots, Y_{k+l-2}) = 0. \tag{5.15}$$

Indeed, by (5.12), we have

$$\begin{aligned}
&(\Omega^1(\mathcal{G}) \ni)\ (\iota_{P^\sharp\alpha} s^*\gamma)^\sharp (X, Y_1, \dots, Y_{k+l-4}) \\
&= \sum_{\sigma \in \mathrm{Sh}(k-2,l-2)} (-1)^\sigma (s^*\gamma)^\sharp (P^\sharp \alpha^\sharp (X, Y_{\sigma_1}, \dots, Y_{\sigma_{k-2}}), Y_{\sigma_{k-1}}, \dots, Y_{\sigma_{k+l-4}}) \\
&\quad - (-1)^{kl} \sum_{\tau \in \mathrm{Sh}(l-1,k-3)} (-1)^\tau \alpha^\sharp (P^\sharp s^*\gamma (Y_{\tau_1}, \dots, Y_{\tau_{l-1}}), X, Y_{\tau_l}, \dots, Y_{\tau_{k+l-4}}).
\end{aligned}$$

We claim that all the terms above vanish. For this, we examine that

$$\begin{aligned}
&(s^*\gamma)^\sharp (P^\sharp \alpha^\sharp (X, Y_{\sigma_1}, \dots, Y_{\sigma_{k-2}}), Y_{\sigma_{k-1}}, \dots, Y_{\sigma_{k+l-4}}) \\
&\quad = s^*(\gamma^\sharp (s_* P^\sharp \alpha^\sharp (X, Y_{\sigma_1}, \dots, Y_{\sigma_{k-2}}), Y_{\sigma_{k-1}}, \dots, Y_{\sigma_{k+l-4}})) \\
&\quad = s^*(\gamma^\sharp (P^\sharp \alpha^\sharp (s_*X, s_*Y_{\sigma_1}, \dots, s_*Y_{\sigma_{k-2}}), s_*Y_{\sigma_{k-1}}, \dots, s_*Y_{\sigma_{k+l-4}})) = 0,
\end{aligned}$$

where we have used the facts that $P^\sharp$ and $\alpha^\sharp$ are Lie groupoid morphisms, which commute with the source maps, and $s_*X = 0$. Similarly, we can verify that

$$\begin{aligned}
&\alpha^\sharp (P^\sharp s^*\gamma (Y_{\tau_1}, \dots, Y_{\tau_{l-1}}), X, Y_{\tau_l}, \dots, Y_{\tau_{k+l-4}}) \\
&\quad = (-1)^{k+l} \langle P^\sharp s^*\gamma (Y_{\tau_1}, \dots, Y_{\tau_{l-1}}), \alpha^\sharp (X, Y_{\tau_l}, \dots, Y_{\tau_{k+l-4}}) \rangle \\
&\quad = -(-1)^{k+l} \langle \gamma (Y_{\tau_1}, \dots, Y_{\tau_{l-1}}), s_* P^\sharp \alpha^\sharp (X, Y_{\tau_l}, \dots, Y_{\tau_{k+l-4}}) \rangle \\
&\quad = -(-1)^{k+l} \langle \gamma (Y_{\tau_1}, \dots, Y_{\tau_{l-1}}), P^\sharp \alpha^\sharp (s_*X, s_*Y_{\tau_l}, \dots, s_*Y_{\tau_{k+l-4}}) \rangle = 0.
\end{aligned}$$

So we have

$$(\iota_{P^\sharp\alpha} s^*\gamma)^\sharp (X, Y_1, \dots, Y_{k+l-4}) = 0.$$

Now, due to the expression of $[\alpha, s^*\gamma]_P$, we obtain the desired (5.15).

Once we obtain ω which is subject to (5.14), we can define the action of α on γ by setting $\alpha \triangleright \gamma := \omega$. Thanks to the graded Jacobi identity of $[\cdot,\cdot]_P$ and the injectivity of s^*, we see that $\triangleright$ defines an action of the GLA $\Omega^\bullet_{\mathrm{mult}}(\mathcal{G})$ on $\Omega^\bullet(M)$. Moreover, by (5.8), we have

$$d[\alpha, s^*\gamma]_P = [d\alpha, s^*\gamma]_P + (-1)^{k-1} [\alpha, s^* d\gamma]_P,$$

which implies

$$d(\alpha \triangleright \gamma) = (d\alpha) \triangleright \gamma + (-1)^{k-1} \alpha \triangleright (d\gamma)$$

as s^* is injective. So $\triangleright$ is compatible with the differentials. One further checks that $(\Omega^\bullet(M) \xrightarrow{J} \Omega^\bullet_{\mathrm{mult}}(\mathcal{G}))$ defines a DGLA crossed module.

The space of multiplicative forms $\Omega^\bullet_{\mathrm{mult}}(\mathcal{G}) \subset \Omega^\bullet(\mathcal{G})$ is preserved by the de Rham differential and the space of multiplicative multi-vector fields $\mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G}) \subset \mathfrak{X}^\bullet(\mathcal{G})$ is closed under the Schouten bracket. So the statement (3) follows from Lemma 5.17 and the fact that

$$\wedge^\bullet P^\sharp : (\Omega^\bullet(\mathcal{G}), [\cdot,\cdot]_P, d) \to (\mathfrak{X}^\bullet(\mathcal{G}), [\cdot,\cdot], [P,\cdot])$$

is a morphism of DGLAs. Finally, the statements (1)–(3) together with Lemma 4.5 imply (4). □

Example 5.18. Let M be a smooth manifold. The cotangent bundle $T^*M \to M$ is an abelian Lie group bundle, and can be regarded as a special Lie groupoid; its source and target maps are both the bundle projection $T^*M \to M$ and the multiplication is the fiberwise addition. With the canonical symplectic form $\omega = d\alpha$, α being the canonical Liouville-Poincaré 1-form, $T^*M \to M$ is a symplectic Lie groupoid. Let us take the standard local coordinates (q^i, p^j) of T^*M, where q^i is the coordinate on M and p^j that of the fibre. Then, one can write $\omega = dq^i \wedge dp_i$.

As the groupoid multiplication of T^*M is given by the fiberwise addition, multiplicative multi-vector fields and multiplicative forms are indeed linear multi-vector fields [19] and linear forms [4], respectively. So according to [4], a multi-vector field $\Pi \in \mathfrak{X}^k(T^*M)$ is multiplicative if it is locally of the form

$$\Pi = \frac{1}{k!}\Pi^{i_1\cdots i_k}_j(q)p^j\frac{\partial}{\partial p^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial p^{i_k}} + \frac{1}{(k-1)!}\Pi^{i_1\cdots i_{k-1},j}(q)\frac{\partial}{\partial p^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial p^{i_{k-1}}}\wedge\frac{\partial}{\partial q^j}.$$

Similarly, a k-form $\Theta \in \Omega^k(T^*M)$ is multiplicative if it is of the form

$$\Theta = \frac{1}{k!}\Theta_{i_1\cdots i_k,j}(q)p^j dq^{i_1}\wedge\cdots\wedge dq^{i_k} + \frac{1}{(k-1)!}\Theta_{i_1\cdots i_{k-1},j}(q)dq^{i_1}\wedge\cdots\wedge dq^{i_{k-1}}\wedge dp^j.$$

As $\omega^\sharp : T(T^*M) \to T^*(T^*M)$ maps $\frac{\partial}{\partial p^i}$ to $-dq^i$ and $\frac{\partial}{\partial q^i}$ to dp^i, we see that $\omega^\sharp$ establishes an isomorphism between $\mathfrak{X}^k_{\mathrm{mult}}(T^*M)$ and $\Omega^k_{\mathrm{mult}}(T^*M)$.

Next, we find the Lie algebra crossed module and the GLA crossed module structures stemming from the Poisson Lie groupoid $\mathcal{G} = T^*M \to M$. The Poisson structure is

$$P = \frac{\partial}{\partial q^i}\wedge\frac{\partial}{\partial p^i}$$

corresponding to the earlier symplectic structure ω.

(1) Since source and target maps s and t are one and the same, the map J is just trivial:

$$\Omega^1(M) \xrightarrow{J=0} \Omega^1_{\mathrm{mult}}(\mathcal{G}).$$

The Lie bracket on $\Omega^1(M)$ is also trivial, whereas the Lie bracket on $\Omega^1_{\mathrm{mult}}(\mathcal{G})$ is listed below:

$$\begin{aligned}
&[\Theta_{i,j}(q)p^j dq^i, \Theta'_{a,b}(q)p^b dq^a]_P = \Theta_{i,j}(q)\Theta'_{a,i}(q)p^j dq^a - \Theta'_{a,b}(q)\Theta_{i,a}(q)p^b dq^i,\\
&[\Theta_{i,j}(q)p^j dq^i, \Theta'_l(q)dp^l]_P = \Theta'_l(q)\frac{\partial\Theta_{i,j}(q)}{\partial q^l}p^j dq^i,\\
&[\Theta_k(q)dp^k, \Theta'_l(q)dp^l]_P = -\Theta_k(q)\frac{\partial\Theta'_l(q)}{\partial q^k}dp^l + \Theta'_l(q)\frac{\partial\Theta_k(q)}{\partial q^l}dp^k.
\end{aligned}$$

The action of $\Omega^1_{\mathrm{mult}}(\mathcal{G})$ on $\Omega^1(M)$ is given by

$$(\Theta_{i,j}(q)p^j dq^i + \Theta_l(q)dp^l) \triangleright (\gamma_k(q)dq^k) = -\gamma_k(q)\Theta_{i,k}(q)dq^i - \Theta_l(q)\frac{\partial\gamma_k(q)}{\partial q^l}dq^k.$$

(2) For the same reasons, we have the trivial map $\Omega^\bullet(M) \xrightarrow{J=0} \Omega^\bullet_{\mathrm{mult}}(\mathcal{G})$ and trivial graded Lie bracket on $\Omega^\bullet(M)$. The graded Lie bracket on $\Omega^\bullet_{\mathrm{mult}}(\mathcal{G})$ is as described below.

Let $I = \{i_1, i_2, \ldots, i_k\}$ be a multi-index and $dq^I = dq^{i_1}\wedge\cdots\wedge dq^{i_k}$ be a k-form on M. Similarly, let $A = \{a_1, a_2, \ldots, a_l\}$ and $dq^A = dq^{a_1}\wedge\cdots\wedge dq^{a_l}$ be an l-form. Denote by I_s the multi-index by removing i_s from I. The notation A_t is similar. We have computed the following:

$$\begin{aligned}
&[\Theta_{I,j}(q)p^j dq^I, \Theta'_{A,b}(q)p^b dq^A]_P\\
&\quad = (-1)^{k-s}\Theta_{I,j}(q)\Theta'_{A,i_s}(q)p^j dq^{I_s}\wedge dq^A - (-1)^{l-t}\Theta'_{A,b}(q)\Theta_{I,a_t}(q)p^b dq^{A_t}\wedge dq^I,\\
&[\Theta_{I,j}(q)p^j dq^I, \Theta'_{L,b}(q)dq^L\wedge dp^b]_P\\
&\quad = (-1)^{l-s}\Theta'_{L,b}(q)\Theta_{I,l_s}(q)dq^{L_s}\wedge dp^b\wedge dq^I - \Theta'_{L,b}(q)\frac{\partial\Theta_{I,j}(q)}{\partial q^b}p^j dq^L\wedge dq^I,
\end{aligned}$$

$$
\begin{aligned}
&[\Theta_{K,a}(q)dq^K \wedge dp^a, \Theta'_{L,b}(q)dq^L \wedge dp^b]_P \\
&\quad = -\Theta_{K,a}(q)\frac{\partial \Theta'_{L,b}(q)}{\partial q^a}dq^K \wedge dq^L \wedge dp^b + \Theta'_{L,b}(q)\frac{\partial \Theta_{K,a}(q)}{\partial p^b}dq^L \wedge dq^K \wedge dp^a.
\end{aligned}
$$

The action of $\Omega^k_{\mathrm{mult}}(\mathcal{G})$ on $\Omega^l(M)$ is given by

$$
\begin{aligned}
&(\Theta_{I,j}(q)p^j dq^I + \Theta_{K,a}(q)dq^K \wedge dp^a) \triangleright (\gamma_L(q)dq^L) \\
&\quad = -(-1)^{l-s}\gamma_L(q)\Theta_{I,l_s}(q)dq^{L_s} \wedge dq^I - \Theta_{K,a}(q)\frac{\partial \gamma_L(q)}{\partial q^a}dq^K \wedge dq^L.
\end{aligned}
$$

We can also explicitly write the Schouten bracket on multiplicative multi-vector fields, which are omitted.

Finally, as the infinitesimal counterpart of Theorem 5.14(1), we know that $\mathrm{IM}^\bullet(A)$, the space of IM-forms of the Lie algebroid of $\mathcal{G}$, carries a graded Lie bracket structure. Hence $\mathrm{IM}^\bullet(A)$ is a DGLA provided that the groupoid $\mathcal{G}$ is Poisson (see Corollary 4.4). The correspondence between Poisson Lie groupoids and Lie bialgebroids [29, 31] suggests that a canonical DGLA structure on $\mathrm{IM}^\bullet(A)$ can be derived from any Lie bialgebroid (A, A^*). This will be explored in future research.

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