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The character of Thurston's circle packings

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Abstract We introduce the character of Thurston's circle packings in the hyperbolic background geometry. Consequently, some quite simple criteria are obtained for the existence of hyperbolic circle packings. For example, if a closed surface X admits a circle packing with all the vertex degrees $d_i \geqslant 7$, then it admits a unique complete hyperbolic metric so that the triangulation graph of the circle packing is isotopic to a geometric decomposition of X. This criterion is sharp due to the fact that any closed hyperbolic surface admits no triangulations with all $d_i \leqslant 6$. As a corollary, we obtain a new proof of the uniformization theorem for closed surfaces with genus $g \geqslant 2$; moreover, any hyperbolic closed surface has a geometric decomposition. To obtain our results, we use Chow-Luo's combinatorial Ricci flow as a fundamental tool.

Keywords character, circle packings, combinatorial Ricci flow

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1 Background

Thurston's hyperbolization theorem for 3-manifolds is one of the truly great mathematical discoveries of the twentieth century. It establishes a deep and strong link between the geometry and topology of 3-manifolds and the algebra of discrete groups of $\mathrm{Isom}(\mathbb{H}^3)$. Proving Thurston's hyperbolization for Haken 3-manifolds requires three major tools: the existence of hierarchies in Haken manifolds which allows us to cut a Haken manifold into polyhedra, Andreev's theorem which allows us to give these polyhedra hyperbolic structures, and the skinning lemma which allows us to glue the pieces together again.

The beautiful theorem of Andreev [1, 2] states that if we make a topological model for a polyhedron and choose candidate dihedral angles (angles between the faces) that are at most $\pi/2$, then there are simply verifiable conditions that tell us whether there exists a hyperbolic polyhedron with the assigned angles. Furthermore, if such a polyhedron exists, it is unique. Andreev's theorem provides a complete characterization of compact hyperbolic polyhedra with non-obtuse dihedral angles and is essential for proving Thurston's hyperbolization theorem for Haken 3-manifolds.

Thurston [38] observed a very deep connection between circle packings and hyperbolic polyhedra. Given a convex hyperbolic polyhedron in the hyperbolic 3-space $\mathbb{B}^3$, the boundaries of the oriented hyperbolic

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planes containing its faces form a circle packing on the sphere $\partial\mathbb{B}^3$. This circle packing records all the information of the original polyhedron: its combinatorial type is exactly dual to the 1-skeleton of the polyhedron, and the exterior intersection angles between the circles are equal to the dihedral angles between the faces of the hyperbolic polyhedron. Thurston's circle packing theorem regards the existence and uniqueness of circle packings on higher-genus surfaces with a prescribed combinatorial type and non-obtuse exterior intersection angles. Thurston pointed out that the sphere version of his theorem followed from Andreev's theorem. For related results on circle packings and hyperbolic polyhedra, we refer to the works of Marden and Rodin [29], de Verdière [11], Bowditch [6], Hodgson and Rivin [23], Rivin [31, 32], Bao and Bonahon [3], Bobenko and Springborn [5], Leibon [26], Rousset [34], Schlenker [35], and others. Thurston also posed a conjecture regarding the convergence of infinitesimal hexagonal tangent circle packings to conformal mappings, which was proved by Rodin and Sullivan [33]. From then on circle packings have played significant roles in the study of low-dimensional geometry and topology, complex analysis (such as the famous Koebe uniformization conjecture for circle domains; see, e.g., [20–22]), and various problems in combinatorics [27, 36], discrete and computational geometry [8, 10, 37], minimal surfaces [4], and many others.

A circle packing $\mathcal{P} = \{C_v : v \in V\}$ on a surface is a collection of circles with a particular combinatorial structure. Let X be a closed surface with a triangulation $\mathcal{T} = (V, E, F)$, where V, E, and F are the sets of vertices, edges, and triangles, respectively. Assume that μ is a Riemannian metric on X with a constant curvature. Given a weight $\Phi : E \to [0, \pi/2]$, a circle packing $\mathcal{P}$ on (X, μ) is called $(\mathcal{T}, \Phi)$-*type* if there exists a geodesic triangulation of $\mathcal{T}_\mu$ on (X, μ) isotopic to $\mathcal{T}$ such that the circle C_v is centered at $\mathcal{T}_\mu(v)$ and for any edge $e \in E$, the two circles C_v and C_u, which correspond to the vertices v and u of e, intersect at an angle $\Phi(e)$. A fundamental question arises naturally: does there exist a $(\mathcal{T}, \Phi)$-type circle packing $\mathcal{P}$ on (X, μ), and is it unique when it exists? A celebrated answer to this question is the following circle packing theorem due to Thurston [38, Chapter 13, Theorem 13.7.1].

Theorem 1.1 (See [38, Chapter 13, Theorem 13.7.1])**.** *Let $\mathcal{T}$ be a triangulation of a closed surface X of genus $g > 0$ and $\Phi : E \to [0, \pi/2]$ be a function satisfying the following conditions*:

(T_1) *If e_1, e_2, and e_3 form a null-homotopic closed path in $\mathcal{T}$, and $\sum_{l=1}^{3} \Phi(e_l) \geqslant \pi$, then these edges form the boundary of a triangle of $\mathcal{T}$.*

(T_2) *If e_1, e_2, e_3, and e_4 form a null-homotopic closed path, and $\sum_{l=1}^{4} \Phi(e_l) = 2\pi$, then e_1, e_2, e_3, and e_4 form the boundary of the union of two adjacent triangles.*

Then, there is a constant curvature metric μ on X such that (X, μ) supports a $(\mathcal{T}, \Phi)$-type circle packing $\mathcal{P}$. Moreover, the pair $(\mu, \mathcal{P})$ is unique up to isometries if $g > 1$ and up to similarities if $g = 1$.

The sphere version of the above theorem follows from Koebe [25] and Andreev [1, 2]. Together they are often named Koebe-Andreev-Thurston's circle packing theorem. In this paper, we call the above theorem *Thurston's circle packing theorem* for simplicity. Of course, this will not weaken Koebe and Andreev's contributions. Thurston's theorem gives a complete criterion for the existence of circle packings. Note that it does not assume the existence of the metric μ *a priori*. Consequently, there is a uniquely determined geometric decomposition of (X, μ) isotopic to $\mathcal{T}$ so that the edges are geodesics.

2 Main results

Theorem 1.1 gives a wonderful criterion for the existence of circle packings with constant curvatures. However, Thurston's criteria (T_1) and (T_2) are extremely difficult to verify for a general weight $\Phi \in [0, \pi/2]$ since the angle structure Φ and the combinatorial structure of $\mathcal{T}$ are globally intertwined together on X. Inspired by the works by Ge et al. [18], Ge and Hua [14], and Feng et al. [13] on 3-dimensional geometric triangulations, we introduce the characters of circle packings to overcome this difficulty. Consequently, we found some quite simple criteria for the existence of Thurston's circle packings. First, we have the following theorem.

Theorem 2.1. *Let X be a closed surface. If X admits a triangulation $\mathcal{T}$ with degree $d \geqslant 7$ at each vertex, then for any given constant weight $\Phi : E \to [0, \pi/2]$, there exists a unique complete hyperbolic* *metric μ on X so that (X,μ) supports a $(\mathcal{T},\Phi)$-type circle packing $\mathcal{P}$.*

The requirement that $d \geqslant 7$ is sharp in the sense that every circle packing $\mathcal{P}$ with all the vertex degrees $d \leqslant 6$ cannot be supported on any closed surface X of genus $g \geqslant 2$, and every circle packing $\mathcal{P}$ with all the vertex degrees $d \geqslant 7$ cannot be supported on the sphere or torus (see Section 3 for details).

The requirement that Φ is a constant in Theorem 2.1 is too restrictive, and it can be released to some extent. Theorem 2.1 can be generalized to the following.

Theorem 2.2. *Let X be a closed surface. If X admits a triangulation $\mathcal{T}$ with degree $d \geqslant 7$ at each vertex, then for any weight $\Phi \in [\arccos\eta, \arccos\xi] \subset [0,\pi/2]$, where $0 \leqslant \xi \leqslant \eta \leqslant 1$ are arbitrarily chosen so that $\eta < (2\cos\frac{2\pi}{7} - 1 + \xi)/(2 - 2\cos\frac{2\pi}{7})$, there exists a unique complete hyperbolic metric μ on X so that (X,μ) supports a $(\mathcal{T},\Phi)$-type circle packing $\mathcal{P}$.*

Assume that Φ is a constant, or Φ takes values in $[0, 0.33\pi]$ or in $[0.4\pi, \pi/2]$, respectively. They all satisfy the assumption in Theorem 2.2 by Corollary 3.8. However, under a slightly stronger condition than $d \geqslant 7$, we do not need any restrictions on Φ.

Theorem 2.3. *Let X be a closed surface. If X admits a triangulation $\mathcal{T}$ with degree $d \geqslant 9$ at each vertex, then for any given weight $\Phi : E \to [0,\pi/2]$, there exists a unique complete hyperbolic metric μ on X such that (X,μ) supports a $(\mathcal{T},\Phi)$-type circle packing $\mathcal{P}$.*

Compared with Thurston's conditions (T_1) and (T_2), our degree conditions do not respect the angle structure Φ. This seems quite amazing. The two degree criteria (i.e., $d \geqslant 7$ for constant weights or $d \geqslant 9$ for arbitrary weights) are both the special cases of the "*character criteria*" $\mathcal{L}(\mathcal{T},\Phi)_i > 2\pi$ for each vertex $i \in V$, where

$$\mathcal{L}(\mathcal{T},\Phi)_i = \sum_{\triangle ijk \in F} \arccos\left(\frac{1+\cos\Phi_{ij}+\cos\Phi_{ik}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ik}}}\right), \tag{2.1}$$

and Φ_{mn} denotes the value of Φ on the edge $e_{mn} \in E$. Let $N = |V|$ be the number of vertices in the triangulation $\mathcal{T}$. We call

$$\mathcal{L}(\mathcal{T},\Phi) = (\mathcal{L}(\mathcal{T},\Phi)_1, \ldots, \mathcal{L}(\mathcal{T},\Phi)_N)$$

the *character* of the weighted triangulation $(\mathcal{T},\Phi)$ on X. We show in Section 3 that Theorems 2.1–2.3 are all the special cases of the following theorem.

Theorem 2.4. *Let X be a closed surface with a triangulation $\mathcal{T}$ and a weight $\Phi : E \to [0,\pi/2]$. If the character $\mathcal{L}(\mathcal{T},\Phi)_i > 2\pi$ at all the vertices, then $\chi(X) < 0$ and there exists a unique complete hyperbolic metric μ on X such that (X,μ) supports a $(\mathcal{T},\Phi)$-type circle packing $\mathcal{P}$. On the other hand, if the character $\mathcal{L}(\mathcal{T},\Phi)_i \leqslant 2\pi$ at all the vertices, then there exists no such $(\mathcal{T},\Phi)$-type circle packings.*

The distinct feature of the characters $\mathcal{L}(\mathcal{T},\Phi)_i$ is their *localities*. They can be verified locally and separately at each vertex. However, Thurston's conditions (T_1) and (T_2) are intertwined together, which are global and cannot be verified separately.

As an application of Theorem 2.1, we reprove the following well-known uniformization theorem. Combining Theorem 2.1 in the case of weight $\Phi \equiv 0$ with the fact that every orientable closed surface of genus $g \geqslant 2$ admits a triangulation with degree $d = 7$ at each vertex, we obtain the uniformization theorem for surfaces with higher genus.

Corollary 2.5. *Every closed surface of genus $g \geqslant 2$ admits a complete hyperbolic metric.*

Moreover, we reprove the following geometric decomposition theorem for hyperbolic surfaces. We refer to Martelli's book [30] for more details on geometric decompositions.

Corollary 2.6. *A closed surface X of genus $g \geqslant 2$ admits infinitely many geometric decompositions. For any triangulation $\mathcal{T}$ with degree $d > 6$ at each vertex, there exists a geometric decomposition on X isotopic to $\mathcal{T}$ so that the edges are geodesics.*

Theorems 2.1–2.4 all deal with circle packings on closed hyperbolic surfaces. Generally, one may consider circle packings on a topological surface: let X be a closed surface with a triangulation $\mathcal{T}$ and a weight $\Phi : E \to [0,\pi/2]$. Consider the *circle packing metric* defined as follows: to each vertex $v_i \in V$

assign a number $r_i > 0$. Realize each edge e_{ij} joining v_i to v_j by a hyperbolic segment of length $l_{ij} = \cosh^{-1}(\cosh r_i \cosh r_j + \sinh r_i \sinh r_j \cos \Phi(e_{ij}))$. Thus, one can realize each triangle $\triangle v_i v_j v_k$ by a hyperbolic triangle of edge lengths l_{ij}, l_{jk}, and l_{ki}. The triangle is formed by the centers of three circles of radii r_i, r_j, and r_k intersecting at angles $\Phi(e_{ij})$, $\Phi(e_{jk})$, and $\Phi(e_{ki})$. This produces a *hyperbolic cone metric* on the surface X with singularities at the vertices. Let A_i be the cone angle at the vertex v_i, which is the sum of all the inner angles having the vertex v_i. The combinatorial Gaussian curvature K_i at v_i is defined to be $K_i = 2\pi - A_i$. Thus, every circle packing metric $r = (r_1, \dots, r_N)$ gives on X a hyperbolic cone metric with singularities described by $K = (K_1, \dots, K_N)$. Consider the curvature map

$$K = K(r) : r \mapsto K, \quad r \in \mathbb{R}^N_{>0}.$$

The famous Koebe-Andreev-Thurston theorem (see, e.g., [29] for a proof) says that the curvature map $r \mapsto K$ is injective and its image set $K(\mathbb{R}^N_{>0})$ is a bounded convex polytope

$$K(\mathbb{R}^N_{>0}) = \bigcap_{I \subset V} \left\{ x = (x_1, \dots, x_N) : \sum_{i \in I} x_i > -\sum_{(e,v) \in Lk(I)} (\pi - \Phi(e)) + 2\pi\chi(F_I) \right\},$$

where I is taken over all the proper subsets of V, F_I is the subcomplex whose vertices are in I, and $Lk(I)$ is the set of pairs (e, v) of an edge e and a vertex v satisfying the following: (i) the end points of e are not in I; (ii) v is in I; (iii) e and v form a triangle.

Using the character (2.1) defined above, we give a new observation of the image set.

Theorem 2.7. *Considering the weighted triangulation $(\mathcal{T}, \Phi)$ on a closed surface X, we have*

$$\prod_{i \in V} (2\pi - \mathcal{L}(\mathcal{T}, \Phi)_i, 2\pi) \subset K(\mathbb{R}^N_{>0}) \tag{2.2}$$

and

$$\prod_{i \in V} (-\infty, 2\pi - \mathcal{L}(\mathcal{T}, \Phi)_i] \cap K(\mathbb{R}^N_{>0}) = \emptyset. \tag{2.3}$$

Remark 2.8. Using Corollaries 3.5, 3.7, and 3.8, Proposition 3.6 in Section 3, and the conclusions in Theorem 2.7, we obtain some quite simple estimates for the image set $K(\mathbb{R}^N_{>0})$ under certain "degree-type" criteria. For example, if all $d_i \geqslant 9$, then $[-0.07\pi, 2\pi)^N \subset K(\mathbb{R}^N_{>0})$. For more examples and details, see Corollaries 8.3–8.5.

Thurston's conditions (T_1) and (T_2) have been generalized to some other settings (see, e.g., [5, 15–17, 19, 24, 29, 31, 32, 35, 39, 40]). However, all the existence criteria are essentially the same as (T_1) and (T_2). It seems that our "character-type" or "degree-type" criteria are the first conditions totally different from Thurston's conditions (T_1) and (T_2) in the literature. To obtain our results, we use Chow-Luo's [9] combinatorial Ricci flow as a fundamental tool. We also borrow the techniques developed in [13, 14, 18] to control the flow. The main difficulty in the proofs of our results is to establish the compactness of the solution to the flow. To circumvent the difficulty, we thoroughly study the geometry of the basic building blocks, i.e., hyperbolic triangles. Particularly, we establish some comparison principles for inner angles at both the longest and the shortest circle packing components, which is the key to establishing the compactness of the solution.

The rest of this paper is organized as follows. In Section 3, we recall Chow-Luo's combinatorial Ricci flow and state our main theorem, i.e., Theorem 4.1, which covers Theorem 2.4. In Section 4, we establish some comparison principles for the interior angles of hyperbolic triangles. In Sections 5 and 6, we study the combinatorial Ricci flow by characters and further prove Theorem 4.1. In the final Section 7, we prove Theorem 2.7 by further investigations into the prescribed combinatorial Ricci flow.

3 Circle packings and their characters

Circle packings provide a bridge between the combinatorics on the one hand and the geometry on the other hand. Let X be a closed surface with a triangulation $\mathcal{T} = (V, E, F)$, where V, E, and F denote the sets of vertices, edges, and faces, respectively. A *weight* on the triangulation is defined to be a function $\Phi : E \to [0, \pi/2]$. Throughout this paper, a function defined on vertices is an N-dimensional column vector, where $N = |V|$ is the number of vertices. Moreover, all the vertices $v_1, \ldots, v_N$ are abbreviated as $1, \ldots, N$ for simplicity. Next, we study the circle packings on a topological surface from the perspective of metrics.

Endow X with hyperbolic (resp. flat) cone metrics as follows. A *circle packing metric* based on $(X, \mathcal{T}, \Phi)$ is a function $r : V \to (0, +\infty)$. We consider $r = (r_1, \ldots, r_N) \in \mathbb{R}^N_{>0}$ as a point in $\mathbb{R}^N_{>0} = (0, +\infty)^N$ in the whole paper. Geometrically, a circle packing metric r assigns a circle with radius r_i at every vertex i, and realizes each edge $e_{ij} \in E$ joining i to j by a hyperbolic (resp. Euclidean) segment of length

$$l_{ij} = \cosh^{-1}(\cosh r_i \cosh r_j + \sinh r_i \sinh r_j \cos \Phi(e_{ij})) \tag{3.1}$$

$$(\text{resp. } l_{ij} = \sqrt{r_i^2 + r_j^2 + 2r_i r_j \cos \Phi(e_{ij})}). \tag{3.2}$$

Thurston [38, Lemma 13.7.2] once noted that for each triangle $\triangle ijk \in F$, the three edge lengths l_{ij}, l_{jk}, and l_{ik} satisfy triangle inequalities. Thus, one can realize each triangle $\triangle ijk \in F$ by a hyperbolic (resp. Euclidean) triangle of edge lengths l_{ij}, l_{jk}, and l_{ik}. The triangle is formed by the centers of three circles of radii r_i, r_j, and r_k intersecting at angles Φ_{ij}, Φ_{jk}, and Φ_{ik}, where Φ_{mn} denotes the value of Φ on the edge $e_{mn} \in E$. This produces a *hyperbolic cone metric* (resp. *flat cone metric*) on the surface X with singularities at the vertices. Denote by θ_i^{jk} (or by θ_i when there is no confusion) the inner angle at a vertex i in the triangle $\triangle ijk \in F$. Let $A_i = \sum_{\triangle ijk \in F} \theta_i^{jk}$ be the cone angle at the vertex i, which is equal to the sum of inner angles at i for all triangles incident to i. The *combinatorial Gaussian curvature* K_i at i is defined as

$$K_i = 2\pi - A_i = 2\pi - \sum_{\triangle ijk \in F} \theta_i^{jk}. \tag{3.3}$$

Then, for a given $(\mathcal{T}, \Phi)$-type circle packing $\mathcal{P}$, every circle packing metric $r = (r_1, \ldots, r_N)$ produces a hyperbolic (resp. flat) cone metric on X with cone singularities A_i centered on each $i \in V$. There are no singularities on $X \setminus V$ by the constructions. At a particular vertex i, $A_i = 2\pi$ means that there is no singularity at i. Hence, a circle packing metric r_{ze} with $K(r_{ze}) = 0$ is particularly meaningful in the sense that there are no singularities on X, even on V: it produces a complete hyperbolic (resp. flat) metric μ on X. It is $\mathcal{T}$-type and has exterior intersection angles given by Φ. Moreover, those hyperbolic (resp. Euclidean) triangles form a geodesic triangulation of (X, μ). Specifically, when it produces a hyperbolic metric, we name the unique circle packing r_{ze} a *hyperbolic circle packing*.

Remark 3.1. By abuse of language, a circle packing metric r is also called a circle packing for brevity. The original $(\mathcal{T}, \Phi)$-type circle packing $\mathcal{P}$ may be considered as the equivalent class of all the circle packing metrics $r \in \mathbb{R}^N_{>0}$. We say our setting is in the hyperbolic background if we consider hyperbolic cone metrics constructed by gluing hyperbolic triangles with hyperbolic edge lengths (3.1). The Euclidean background is defined similarly.

Definition 3.2. Let X be a closed surface with a triangulation $\mathcal{T}$ and a weight $\Phi \in [0, \pi/2]$. For any circle packing based on $(X, \mathcal{T}, \Phi)$, define the character $\mathcal{L}(\mathcal{T}, \Phi)_i$ at each $i \in V$ by

$$\mathcal{L}(\mathcal{T}, \Phi)_i = \sum_{\triangle ijk \in F} \arccos\left(\frac{1 + \cos \Phi_{ij} + \cos \Phi_{ik} - \cos \Phi_{jk}}{2\sqrt{1 + \cos \Phi_{ij}}\sqrt{1 + \cos \Phi_{ik}}}\right), \tag{3.4}$$

where the sum runs over all the triangles having a vertex i. The character of $(X, \mathcal{T}, \Phi)$ is

$$\mathcal{L}(\mathcal{T}, \Phi) = (\mathcal{L}(\mathcal{T}, \Phi)_1, \ldots, \mathcal{L}(\mathcal{T}, \Phi)_N).$$

It seems that the character is just defined for a weighted triangulation $(\mathcal{T}, \Phi)$ on X and has no relation to any circle packings. However, both the combinatorial structure $\mathcal{T}$ and the angle structure Φ come from a circle packing $\mathcal{P}$ on X. Recall that a circle packing $\mathcal{P} = \{C_v : v \in V\}$ on (S, μ) is of $(\mathcal{T}, \Phi)$-type if there exists a geodesic triangulation of $\mathcal{T}_\mu(S,\mu)$ isotopic to $\mathcal{T}$ such that the circle C_v is centered at $\mathcal{T}_\mu(v)$ and for any edge $e \in E$, the two circles C_v and C_u, which correspond to the vertices v and u of e, intersect at an angle $\Phi(e)$. Hence, the weighted triangulation $(\mathcal{T},\Phi)$ records all the information of a $(\mathcal{T},\Phi)$-type circle packing $\mathcal{P}$, i.e., the combinatorial structure given by $\mathcal{T}$ and the angle structure given by Φ. Hence, the character $(\mathcal{L}(\mathcal{T},\Phi)_i)_{i\in V}$ is indeed an invariant of all the $(\mathcal{T},\Phi)$-type circle packings $\mathcal{P}$ on a closed surface X.

Proposition 3.3. *Given a weighted triangulated surface $(X,\mathcal{T},\Phi)$, for each vertex $i \in V$, the character $\mathcal{L}(\mathcal{T},\Phi)_i$ is exactly the cone angle A_i of a circle packing metric $r = (1,\dots,1)$ in the Euclidean background.*

Proof. Assume $r_i = 1$ for each vertex $i \in V$. Then, by (3.2), $l_{ij} = \sqrt{2+2\cos\Phi_{ij}}$ for each edge $e_{ij} \in E$. Considering a triangle $\triangle ijk \in F$, we see that

$$\cos\theta_i^{jk} = \frac{1+\cos\Phi_{ij}+\cos\Phi_{ik}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ik}}}.$$

Then, by the definition of the character (3.4), we obtain $\mathcal{L}(\mathcal{T},\Phi)_i = \sum_{\triangle ijk\in F}\theta_i^{jk} = A_i$. □

Proposition 3.4. *Denote the average character by $\mathcal{L}_{av} = \sum_{i\in V}\mathcal{L}(\mathcal{T},\Phi)_i/N$. Then,*

$$\mathcal{L}_{av} = 2\pi\left(1-\frac{\chi(X)}{N}\right).$$

Consequently, $\mathcal{L}_{av} > 2\pi$ if $\chi(X) < 0$, $\mathcal{L}_{av} = 2\pi$ if $\chi(X) = 0$, and $\mathcal{L}_{av} < 2\pi$ if $\chi(X) > 0$.

Proof. Note that $2\pi - \mathcal{L}(\mathcal{T},\Phi)_i$ is the combinatorial curvature K_i of a particular packing $r \equiv 1$ in the Euclidean background by Proposition 3.3. The conclusion follows from the following combinatorial Gauss-Bonnet formula [9]:

$$\sum_{i\in V}K_i = 2\pi\chi(X)$$

in the Euclidean background. We can also prove this proposition directly. For simplicity, we write

$$\gamma_i^{jk} = \arccos\left(\frac{1+\cos\Phi_{ij}+\cos\Phi_{ik}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ik}}}\right).$$

Consider a particular triangle $\triangle ijk \in F$ with edge lengths $l_{mn} = \sqrt{2+2\cos\Phi_{mn}}$, where an edge e_{mn} is chosen from $\{e_{ij}, e_{jk}, e_{ki}\}$. Note that γ_i^{jk} is just the inner angle at the vertex i. Hence, we have $\gamma_i^{jk} + \gamma_j^{ik} + \gamma_k^{ij} = \pi$. Since X is closed and $\mathcal{T}$ is a triangulation, we have $2|E| = 3|F|$, and it follows that

$$\begin{aligned}\sum_{i\in V}\mathcal{L}(\mathcal{T},\Phi)_i &= \sum_{i\in V}\sum_{\triangle ijk\in F}\gamma_i^{jk}\\ &= \sum_{\triangle ijk\in F}(\gamma_i^{jk}+\gamma_j^{ik}+\gamma_k^{ij})\\ &= \pi|F| = 2\pi(|E|-|F|) = 2\pi N - 2\pi\chi(X).\end{aligned}$$

This completes the proof. □

Obviously, Theorem 2.1 follows from Theorem 2.4 and the following corollary.

Corollary 3.5. *Given a weighted triangulated surface $(X,\mathcal{T},\Phi)$ such that $\Phi : E \to [0,\pi/2]$ is a constant, if $d_i \geqslant 7$ at a vertex $i \in V$, then $\mathcal{L}(\mathcal{T},\Phi)_i > 2\pi$.*

Proof. If Φ takes a constant, then $\mathcal{L}(\mathcal{T},\Phi)_i = \sum_{\triangle ijk\in F}\arccos\frac{1}{2} = \frac{\pi}{3}d_i \geqslant \frac{7\pi}{3}$ by (3.4). □

Proposition 3.6. *Given a weighted triangulated surface $(X,\mathcal{T},\Phi)$ with $\Phi : E \to [0,\pi/2]$, for each vertex $i \in v$, the character has a lower bound*

$$\mathcal{L}(\mathcal{T},\Phi)_i \geqslant d_i\arccos\frac{3}{4} \approx \frac{2.07\pi}{9}d_i.$$

Proof. For arbitrary $a, b \in [0,1]$, it is easy to show that $4a^2 + 4b^2 \leqslant 5 + a + b + ab$. It follows that

$$\frac{(1+a+b)^2}{(1+a)(1+b)} \leqslant \frac{9}{4}.$$

Now considering a triangle $\triangle ijk \in F$ and substituting $a = \cos\Phi_{ij}$ and $b = \cos\Phi_{ik}$ into the above inequality, we get

$$\left(\frac{1+\cos\Phi_{ij}+\cos\Phi_{ik}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ik}}}\right)^2 \leqslant \frac{1}{4}\frac{(1+\cos\Phi_{ij}+\cos\Phi_{ik})^2}{(1+\cos\Phi_{ij})(1+\cos\Phi_{ik})} \leqslant \frac{9}{16},$$

which implies

$$0 \leqslant \frac{1+\cos\Phi_{ij}+\cos\Phi_{ik}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ik}}} \leqslant \frac{3}{4}. \tag{3.5}$$

This completes the proof. □

Theorem 2.3 follows from Theorem 2.4 and the following corollary.

Corollary 3.7. *Given a weighted triangulated surface $(X, \mathcal{T}, \Phi)$ with $\Phi : E \to [0, \pi/2]$, if $d_i \geqslant 9$ at a vertex $i \in V$, then $\mathcal{L}(\mathcal{T}, \Phi)_i > 2\pi$.*

By thorough analysis of the proof of Proposition 3.6, Corollary 3.5 can be improved. Moreover, Theorem 2.2 follows from Theorem 2.4 and the following corollary.

Corollary 3.8. *Given a weighted triangulated surface $(X, \mathcal{T}, \Phi)$, assume that the weight satisfies $\Phi \in [0, 0.33\pi]$, or $\Phi \in [0.4\pi, \pi/2]$, or more generally, $\Phi \in [\arccos\eta, \arccos\xi] \subset [0, \pi/2]$, where $0 \leqslant \xi \leqslant \eta \leqslant 1$ are arbitrarily chosen so that $\eta < (2\cos\frac{2\pi}{7} - 1 + \xi)/(2 - 2\cos\frac{2\pi}{7})$. If $d_i \geqslant 7$ at a vertex $i \in V$, then $\mathcal{L}(\mathcal{T}, \Phi)_i > 2\pi$.*

Proof. Define $\lambda = 2\cos\frac{2\pi}{7}$. For arbitrary $a, b \in [\xi, \eta] \subset [0,1]$, it is easy to prove

$$1 + a + b - \lambda\sqrt{1+a}\sqrt{1+b} \leqslant 1 + 2\eta - \lambda(1+\eta).$$

Furthermore, assume $\eta < \frac{\lambda-1+\xi}{2-\lambda}$, which is equivalent to $1 + 2\eta - \lambda(1+\eta) < \xi$. Then,

$$1 + a + b - \lambda\sqrt{1+a}\sqrt{1+b} < \xi.$$

Assume $\Phi \in [\arccos\eta, \arccos\xi] \subset [0, \pi/2]$. Then, $\xi \leqslant \cos\Phi_{ij}, \cos\Phi_{ik}, \cos\Phi_{jk} \leqslant \eta$. Substituting $a = \cos\Phi_{ij}$ and $b = \cos\Phi_{ik}$ into the above inequality, we get

$$1 + \cos\Phi_{ij} + \cos\Phi_{ik} - \lambda\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ik}} < \xi \leqslant \cos\Phi_{jk}.$$

Then, it follows that

$$0 \leqslant \frac{1+\cos\Phi_{ij}+\cos\Phi_{ik}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ik}}} < \frac{\lambda}{2} = \cos\frac{2\pi}{7},$$

and furthermore,

$$\arccos\left(\frac{1+\cos\Phi_{ij}+\cos\Phi_{ik}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ik}}}\right) > \frac{2\pi}{7}.$$

If $d_i \geqslant 7$, then we get the conclusion

$$\mathcal{L}(\mathcal{T}, \Phi)_i > \sum_{\triangle ijk \in F} \frac{2\pi}{7} \geqslant 7 \cdot \frac{2\pi}{7} = 2\pi.$$

Specially, choose $[\xi, \eta] = [0, 0.309]$ or $[\xi, \eta] = [0.509, 1]$. Direct calculations show that they all satisfy the assumption $\eta < \frac{\lambda-1+\xi}{2-\lambda}$, $[\arccos 1, \arccos 0.509] = [0, 0.33\pi]$, and $[\arccos 0.309, \arccos 0] = [0.4\pi, \pi/2]$. Hence, we finish the proof. □

4 Combinatorial Ricci flows

The combinatorial Ricci flows were introduced by Chow and Luo [9], which are the analog of Hamilton's Ricci flow in the combinatorial setting. They obtained a new proof of the existence part of Thurston's circle packing theorem, showing that the combinatorial Ricci flows produce solutions which converge exponentially fast to Thurston's circle packings on surfaces. Since then, the combinatorial Ricci flows have provided an effective algorithm for finding complete hyperbolic metrics and are one of the main tools for finding geometric structures on surfaces and 3-manifolds. Luo [28] also initiated a program to hyperbolize 3-manifolds with boundaries by the combinatorial Ricci flows. The program was formulated more clearly and carried forward extensively by the series of works of Ge and his collaborators. Particularly for a 3-manifold with a boundary of higher genus, we refer to [13], where a hyperbolic metric and an ideal geometric decomposition were obtained under suitable combinatorial conditions.

We recall Chow-Luo's theory briefly. Let X be a closed surface with a triangulation $\mathcal{T}$ and a weight $\Phi \in [0, \pi/2]$. Consider a smooth family of circle packings $r(t) \subset \mathbb{R}^N_{>0}$ based on $(X, \mathcal{T}, \Phi)$, which evolves according to the combinatorial Ricci flow

$$\frac{dr_i(t)}{dt} = -K_i r_i, \quad i \in V \tag{4.1}$$

in the Euclidean background. For any initial circle packing $r(0) \in \mathbb{R}^N_{>0}$, a solution to the combinatorial Ricci flow (4.1) is called *convergent* if $\lim_{t\to\infty} r_i(t) = r_i(\infty) \in \mathbb{R}_{>0}$ exists for all $i \in V$. As a consequence, $\lim_{t\to\infty} K_i(t) = K_i(\infty)$ exists for all i since the curvature map $K(r) = (K_1(r), \dots, K_N(r))$ is a smooth map of r. A convergent solution $r(t)$ is called *convergent exponentially fast* if there are positive constants c_1 and c_2 so that for all time $t \geqslant 0$ and all the vertices $i \in V$, $\|r_i(t) - r_i(\infty)\| \leqslant c_1 \mathrm{e}^{-c_2 t}$ and $\|K_i(t) - K_i(\infty)\| \leqslant c_1 \mathrm{e}^{-c_2 t}$. Chow and Luo [9] proved that in the Euclidean background, the solution to the normalized combinatorial Ricci flow

$$\frac{dr_i}{dt} = \left(\frac{2\pi\chi(X)}{N} - K_i\right) r_i, \quad i \in V, \tag{4.2}$$

which is related to (4.1) by a change of scales, exists for all time. Moreover, the solution converges if and only if there exists a circle packing with the constant curvature $2\pi\chi(X)/N$, and in this case, the solution converges exponentially fast to a constant curvature circle packing (which is unique up to scaling).

We use the following combinatorial Ricci flow in the hyperbolic background

$$\frac{dr_i(t)}{dt} = -K_i \sinh r_i, \quad i \in V \tag{4.3}$$

as the main tool to obtain our results in this paper. It provides a useful tool to deform any initial circle packing $r(0)$ to a hyperbolic one. Indeed, in the case $\chi(X) < 0$, Chow and Luo proved that the solution to (4.3) exists for all time $t \geqslant 0$ and converges if and only if Thurston's conditions (T_1) and (T_2) are satisfied. Furthermore, if it converges, then it converges exponentially fast to a circle packing r_{ze} with $K_i = 0$ at all the vertices $i \in V$, i.e., a hyperbolic circle packing. In this case, the hyperbolic circle packing r_{ze} is unique by the rigidity part of Thurston's circle packing theorem.

Thurston's criteria (T_1) and (T_2) are globally intertwined together. It is generally not easy to verify directly. Using the character $\mathcal{L}(\mathcal{T}, \Phi)$ introduced in Definition 3.2, we give some simple criteria for the existence of Thurston's hyperbolic circle packings next.

Theorem 4.1. *Given a weighted triangulated closed surface $(X, \mathcal{T}, \Phi)$ with the weight $\Phi : E \to [0, \pi/2]$, consider the hyperbolic background setting.*

(a) *If the character $\mathcal{L}(\mathcal{T}, \Phi)_i > 2\pi$ at all the vertices, then $\chi(X) < 0$ and there exists a unique hyperbolic circle packing r_{ze} based on $(X, \mathcal{T}, \Phi)$. Moreover, the solution $r(t)$ to the combinatorial Ricci flow* (4.3) *converges exponentially fast to r_{ze} for any initial circle packing $r(0) \in \mathbb{R}^N_{>0}$. Consequently, r_{ze} determines a unique complete hyperbolic metric μ on X such that (X, μ) supports a geometric decomposition isotopic to $\mathcal{T}$, and each edge connecting two adjacent vertices is a hyperbolic geodesic.*

(b) *If the character* $\mathcal{L}(\mathcal{T}, \Phi)_i \leqslant 2\pi$ *at all the vertices, then* $\chi(X) \geqslant 0$*. Consequently, there exist no hyperbolic circle packings based on* $(X, \mathcal{T}, \Phi)$*. In this case, any solution* $r(t)$ *to* (4.3) *satisfies* $r(t) \to 0$ *when* t *tends to* $+\infty$.

Sketch of Proof of Theorem 4.1. We prove Theorem 4.1 in the following Sections 4–6. First, we establish some comparison principles in Section 4. In Section 5, we prove the long-time existence of the solutions to the combinatorial Ricci flow (4.3). In Section 6, we prove the lower bound estimate for the solution to (4.3), and then by Theorem 7.1 and Proposition 7.2, we can prove Theorem 4.1(a). At last, we prove Theorem 4.1(b) by Theorem 7.3. □

Proof of Theorem 2.4 *via Theorem* 4.1. It is obvious that Theorem 4.1 covers Theorem 2.4. If the character $\mathcal{L}(\mathcal{T}, \Phi)_i > 2\pi$ at all the vertices, then by Theorem 4.1(a), the solution $r(t)$ to the combinatorial Ricci flow (4.3) converges to a unique hyperbolic circle packing r_{ze}, which determines a unique complete hyperbolic metric μ on X. If the character $\mathcal{L}(\mathcal{T}, \Phi)_i \leqslant 2\pi$ at all the vertices, by Theorem 4.1(b), any solution $r(t)$ to (4.3) satisfies $r(t) \to 0$, which means that $r(t)$ cannot converge when t tends to $+\infty$. Finally, by Chow-Luo's result [9] on the combinatorial Ricci flow (4.3), the solution $r(t)$ to the combinatorial Ricci flow (4.3) converges if and only if there exists a hyperbolic circle packing. □

Remark 4.2. The condition $\Phi : E \to [0, \pi/2]$ can be released to a weight $\Phi : E \to [0, \pi)$ with $\cos\Phi_{lm} + \cos\Phi_{mn}\cos\Phi_{ln} \geqslant 0$ for each triangle $\triangle ijk \in F$, where $\{l, m, n\}$ is any rearrangement of $\{i, j, k\}$.

Assume that $\Phi \in [0, \pi/2]$ is a constant, and Theorem 4.1 becomes very brief, interesting, and enlightening. In this case, $\mathcal{L}(\mathcal{T}, \Phi)_i = d_i\pi/3$ by (3.4), and hence if each vertex degree $d_i > 6$ (or equivalently, $d_i \geqslant 7$), then $\mathcal{L}(\mathcal{T}, \Phi)_i > 2\pi$ for all the vertices. Thus, we obtain the following pure geometrical-topological result.

Corollary 4.3. *For any triangulation* $\mathcal{T}$ *on a closed surface* X *with degree* $d \geqslant 7$ *at each vertex, there exists a unique complete hyperbolic metric* μ *on* X *so that* (X, μ) *supports a geometric decomposition isotopic to* $\mathcal{T}$.

Note that any orientable closed surface of genus $g \geqslant 2$ admits a triangulation with $d = 7$ at each vertex by [12]. The lower bound 7 is sharp in two aspects. On the one side, any orientable closed surface X of genus $g \geqslant 2$ admits no triangulations with $d_i \leqslant 6$ at each vertex. This is a well-known result and can be shown by purely combinatorial methods. For example, if $d_i \leqslant 6$ for all $i \in V$, then by $2|E| = 3|F| = \sum_{i\in V} d_i \leqslant 6N$, one gets $N \geqslant |F|/2$. Hence, the Euler characteristic $\chi(X) = N - |E| + |F| \geqslant |F|/2 - 3|F|/2 + |F| = 0$ contradicts the fact that the Euler characteristic $\chi(X) = 2(1-g) < 0$. On the other side, there is another well-known fact that any orientable closed surface X of genus $g \leqslant 1$ admits no triangulations with $d_i \geqslant 7$ at each vertex, which can be shown similarly as follows. If $d_i \geqslant 7$ for each vertex $i \in V$, then by $2|E| = 3|F| = \sum_{i\in V} d_i \geqslant 7N$, one gets $N \leqslant 3|F|/7$. In this case, the Euler characteristic $\chi(X) = N - |E| + |F| \leqslant 3|F|/7 - 3|F|/2 + |F| = -|F|/14 < 0$ contradicts the fact that the Euler characteristic $\chi(X) = 2(1-g) \geqslant 0$.

For the mixing-degree case, i.e., some degrees are larger than 6, while some degrees are no more than 6, we expect that the expectation (or the arithmetical mean) condition $\mathbb{E}(d) = \sum_{i=1}^N d_i/N > 6$ with respect to degrees of vertices will play an essential role in the existence of Thurston's hyperbolic circle packings.

A triangulation on X is called *equivelar* if every vertex has the same degree. It is well known that every closed orientable surface has an equivelar triangulation. For example, the torus admits a 6-equivelar triangulation, and the sphere can admit three equivelar triangulations, i.e., the 3-equivelar (tetrahedron), 4-equivelar (octahedron), and 5-equivelar (icosahedron) triangulations. Any orientable closed surface of genus $g \geqslant 2$ admits a d-equivelar triangulation with $d \geqslant 7$. Indeed, all such surfaces have a universal covering space $\mathbb{H}^2$ and a d-equivelar geometric triangulation of the hyperbolic plane $\mathbb{H}^2$ is generated by the Schwarz triangle group $\Delta(3, d)$ of reflections across the three sides of a hyperbolic triangle with interior angles $\pi/2$, $\pi/3$, and π/d. However, non-orientable surfaces may have no equivelar triangulation such as the non-orientable surface with Euler characteristic -1. As a consequence of Theorem 4.1, we have the following results for equivelar triangulations.

Corollary 4.4. *Assume that a closed surface X admits a d-equivelar weighted triangulation $(\mathcal{T}, \Phi)$, where $\Phi : E \to [0, \pi/2]$ is a constant. Then, for $d \geqslant 7$, there exists a unique hyperbolic circle packing r_{ze}, while for $d \leqslant 6$, there exists no hyperbolic circle packing r_{ze}.*

5 Geometry of hyperbolic triangles

In this section, we study the geometry of hyperbolic triangles, which are the basic building blocks of the weighted triangulated surface $(X, \mathcal{T}, \Phi)$. For any $\{v_i, v_j, v_k\} \subset \mathbb{H}^2$, we denote by $\triangle ijk$ the corresponding hyperbolic triangle in $\mathbb{H}^2$. Obviously, the interior angle $\theta_i = \theta_i(\vec{r})$ at each vertex i is a smooth function of $\vec{r} = (r_i, r_j, r_k)$. By the hyperbolic cosine law, we have

$$\cos\theta_i(\vec{r}) = \frac{\cosh l_{ij} \cosh l_{ik} - \cosh l_{jk}}{\sinh l_{ij} \sinh l_{ik}}.$$

Thurston once obtained the following lemma.

Lemma 5.1 (See [38, Lemma 13.7.3]). *For a weighted triangulated closed surface $(X, \mathcal{T}, \Phi)$ whose weight satisfies $\Phi : E \to [0, \pi/2]$, in the hyperbolic background geometry $\mathbb{H}^2$, one has $\partial\theta_i/\partial r_i < 0$, $\partial\theta_i/\partial r_j > 0$ for $i \neq j$, and $\partial(\theta_i + \theta_j + \theta_k)/\partial r_i < 0$.*

Choosing a special radius vector $\vec{r} = t\vec{1} = (t, t, t)$, we get the following monotonicity proposition for the angle function $\theta_i(t\vec{1})$.

Proposition 5.2. *The angle function $t \mapsto \theta_i(t\vec{1})$ is continuously differentiable and strictly decreasing in $(0, +\infty)$.*

Proof. Set $f(t) = \cos\theta_i(t\vec{1})$. Then, we have

$$\begin{aligned}
\cosh l_{ij} &= \cosh^2 t + \sinh^2 t \cos\Phi_{ij} = \sinh^2 t(1 + \cos\Phi_{ij}) + 1,\\
\cosh l_{ik} &= \cosh^2 t + \sinh^2 t \cos\Phi_{ik} = \sinh^2 t(1 + \cos\Phi_{ik}) + 1,\\
\cosh l_{jk} &= \cosh^2 t + \sinh^2 t \cos\Phi_{jk} = \sinh^2 t(1 + \cos\Phi_{jk}) + 1.
\end{aligned}$$

By a tedious but direct computation, we have

$$f(t) = \frac{(1 + \cos\Phi_{ij})(1 + \cos\Phi_{ik})\sinh^2 t + (1 + \cos\Phi_{ij} + \cos\Phi_{ik} - \cos\Phi_{jk})}{f_1(t) f_2(t)}, \tag{5.1}$$

where

$$\begin{aligned}
f_1(t) &= \sqrt{(1 + \cos\Phi_{ij})^2 \sinh^2 t + 2(1 + \cos\Phi_{ij})},\\
f_2(t) &= \sqrt{(1 + \cos\Phi_{ik})^2 \sinh^2 t + 2(1 + \cos\Phi_{ik})}.
\end{aligned}$$

By taking the derivative, we have

$$f'(t) = \frac{a(\Phi)\sinh(2t)[b(\Phi)\sinh^2 t + c(\Phi)]}{f_1^3(t) f_2^3(t)}, \tag{5.2}$$

where

$$\begin{aligned}
a(\Phi) &= (1 + \cos\Phi_{ij})(1 + \cos\Phi_{ik}),\\
b(\Phi) &= (1 + \cos\Phi_{ij})(1 + \cos\Phi_{ik})(1 + \cos\Phi_{jk}),\\
c(\Phi) &= (1 + \cos\Phi_{jk})(2 + \cos\Phi_{ij} + \cos\Phi_{ik}) - (\cos\Phi_{ij} - \cos\Phi_{ik})^2.
\end{aligned}$$

Note the weight function $\Phi : E \to [0, \pi/2]$, and hence $\cos\Phi_{ij}, \cos\Phi_{ik}, \cos\Phi_{jk} \in [0, 1]$. This implies that $f_1(t) > 0$, $f_2(t) > 0$, $a(\Phi) > 0$, $b(\Phi) > 0$, and

$$c(\Phi) = (1 + \cos\Phi_{jk})(2 + \cos\Phi_{ij} + \cos\Phi_{ik}) - (\cos\Phi_{ij} - \cos\Phi_{ik})^2$$

$$= 2 + \cos\Phi_{jk}(2 + \cos\Phi_{ij} + \cos\Phi_{ik}) + 2\cos\Phi_{ij}\cos\Phi_{ik}$$
$$+ \cos\Phi_{ij}(1 - \cos\Phi_{ij}) + \cos\Phi_{ik}(1 - \cos\Phi_{ik}) \geqslant 2.$$

Therefore, $f'(t) > 0$ and $f(t)$ is strictly increasing, which is equivalent to that $\theta_i(t\vec{1})$ is strictly decreasing. □

Remark 5.3. We thank the referees for pointing out that a stronger version of Proposition 5.2 has already appeared in the work of Bowers and Stephenson [7, Lemma 2.3].

Proposition 5.4. *For any weight function $\Phi : E \to [0, \pi/2]$, the following limits exist:*

$$\lim_{t\to 0}\theta_i(t\vec{1}) = \arccos\left(\frac{1 + \cos\Phi_{ij} + \cos\Phi_{ik} - \cos\Phi_{jk}}{2\sqrt{1 + \cos\Phi_{ij}}\sqrt{1 + \cos\Phi_{ik}}}\right),$$

where $\vec{1} = (1, 1, 1)$, and $\lim_{t\to+\infty}\theta_i(t\vec{1}) = 0$. As a special case, if the weight $\Phi : E \to [0, \pi/2]$ is a constant, we obtain $\lim_{t\to 0}\theta_i(t\vec{1}) = \frac{\pi}{3}$.

Proof. Taking limits directly in the formula (5.1), we get the first two limits. In the case where $\Phi : E \to [0, \pi/2]$ is a constant, we set $\Phi_{ij} \equiv \varphi$, $\forall\, e_{ij} \in E$. By (5.1), we have

$$f(t) = \frac{(1 + \cos\varphi)^2\sinh^2 t + (1 + \cos\varphi)}{(1 + \cos\varphi)^2\sinh^2 t + 2(1 + \cos\varphi)}.$$

Then, the third limit can follow from direct computations. □

Remark 5.5. For an arbitrary weight $\Phi : E \to [0, \pi/2]$, by the key estimate (3.5), we have

$$\lim_{t\to 0}\theta_i(t\vec{1}) \geqslant \arccos\frac{3}{4} > 0.23\pi.$$

Now we are ready to prove the following comparison principle for inner angles which is key to proofs of our main theorems.

Lemma 5.6. *For a fixed hyperbolic triangle $\triangle_{ijk}$, if $r_i = \min\{r_i, r_j, r_k\}$, then $\theta_i(\vec{r}) \geqslant \theta_i(r_i\vec{1})$, while if $r_i = \max\{r_i, r_j, r_k\}$, then $\theta_i(\vec{r}) \leqslant \theta_i(r_i\vec{1})$.*

Proof. Let $\sigma : [0, 1] \to \mathbb{R}^3_{>0}$ be a curve defined as

$$\sigma(s) = (r_i(s), r_j(s), r_k(s)) := (1 - s)\vec{r} + sr_i\vec{1}, \quad \forall\, s \in [0, 1],$$

which connects the two points $\vec{r}, r_i\vec{1} \in \mathbb{R}^3_{>0}$ and

$$r_i(s) = r_i, \quad r_j(s) = r_j + s(r_i - r_j), \quad r_k(s) = r_k + s(r_i - r_k), \quad \forall\, s \in [0, 1].$$

If $r_i = \min\{r_i, r_j, r_k\}$, then by Lemma 5.1,

$$\frac{d}{ds}(\theta_i(\sigma(s))) = (r_i - r_j)\frac{\partial\theta_i}{\partial r_j}(\sigma(s)) + (r_i - r_k)\frac{\partial\theta_i}{\partial r_k}(\sigma(s)) < 0. \tag{5.3}$$

Hence, $\theta_i(\sigma(s))$ is decreasing in $[0, 1]$. This yields that

$$\theta_i(\vec{r}) = \theta_i(\sigma(0)) \geqslant \theta_i(\sigma(1)) = \theta_i(r_i\vec{1}).$$

Similarly, if $r_i = \max\{r_i, r_j, r_k\}$, then by Lemma 5.1,

$$\frac{d}{ds}(\theta_i(\sigma(s))) = (r_i - r_j)\frac{\partial\theta_i}{\partial r_j}(\sigma(s)) + (r_i - r_k)\frac{\partial\theta_i}{\partial r_k}(\sigma(s)) > 0. \tag{5.4}$$

Hence, $\theta_i(\sigma(s))$ is increasing in $[0, 1]$. This yields that

$$\theta_i(\vec{r}) = \theta_i(\sigma(0)) \leqslant \theta_i(\sigma(1)) = \theta_i(r_i\vec{1}).$$

This completes the proof. □

6 Long-time existence

Chow and Luo [9] proved the long-time existence of the solutions to the combinatorial Ricci flow (4.3) by the maximum principle. In this section, we give an alternative proof based on our comparison principle. We need the following calculus lemma introduced by Ge and Hua [14]. For a continuous function $f : [0,+\infty) \to \mathbb{R}$ and any $C \in \mathbb{R}$, the upper level set of f at C is defined as $\{f > C\} := \{t \in [0,+\infty) : f(t) > C\}$. The lower level set $\{f < C\}$ is defined similarly.

Lemma 6.1 (See [14, Lemma 3.9]). *Let $f : [0,+\infty) \to \mathbb{R}$ be a locally Lipschitz function. Suppose that there is a constant C_1 such that $f'(t) \leqslant 0$ for $t \in \{f > C_1\}$ a.e. Then,*

$$f(t) \leqslant \max\{f(0), C_1\}, \quad \forall\, t \in [0,+\infty).$$

Similarly, if $f'(t) \geqslant 0$ for $t \in \{f < C_1\}$ a.e., then

$$f(t) \geqslant \min\{f(0), C_1\}, \quad \forall\, t \in [0,+\infty).$$

To estimate the inner angles in hyperbolic geometry, we need the following lemma in [9] on a hyperbolic triangle $\triangle ijk$ in $\mathbb{H}^2$.

Lemma 6.2 (See [9, Lemma 3.5]). *For any $\epsilon > 0$, there exists a constant $C_2 = C_2(\epsilon)$ such that when $r_i > C_2$, the inner angle θ_i in the hyperbolic triangle $\triangle ijk$ is smaller than ϵ.*

Chow and Luo [9] once obtained the following proposition. We give an alternative proof here.

Proposition 6.3 (See [9, Corollary 3.6]). *Let $r(t)$ be a solution to the combinatorial Ricci flow* (4.3)*. Then, there exists a positive constant $C_3 = C_3(\mathcal{T}, r(0)) > 0$ depending on the triangulation $\mathcal{T}$ and the initial data $r(0)$ such that $r_i(t) \leqslant C_3$ for all $i \in V$.*

Proof. Set $f(t) := \max_{m\in V} r_m(t)$. Then, $g(t)$ is a locally Lipschitz function and for $t \in [0,+\infty)$ a.e., there exists an $i \in V$ depending on t such that

$$f(t) = r_i(t), \quad f'(t) = r_i'(t). \tag{6.1}$$

Let C_2 be the constant determined in Lemma 6.2 such that for any hyperbolic triangle $\triangle ijk$, if $r_i \geqslant C_2$, then $\theta_i \leqslant \pi/\max_{m\in V} d_m$. We claim that $f'(t) \leqslant 0$ for $t \in \{f > C_2\}$ a.e.

Let $t \in [0,+\infty)$ and $i \in V$ satisfying (6.1) and $t \in \{f > C_2\}$. Then, for any hyperbolic triangle $\triangle ijk$ incident to i realized by the circle packing $r(t)$, $\theta_i(r(t)) \leqslant \pi/\max_{m\in V} d_m$. Hence, by the definition of the combinatorial Gaussian curvature,

$$K_i(r(t)) = 2\pi - \sum_{\triangle ijk\in F} \theta_i(r(t)) > \pi.$$

Then, by the combinatorial Ricci flow (4.3), $f'(t) = r_i'(t) = -K_i \sinh r_i < 0$. This proves the claim. Then, the proposition follows from Lemma 6.1. □

We provide an alternative proof for the long-time existence of the solution to (4.3).

Proposition 6.4 (See [9, Proposition 3.4]). *For any initial circle packing $r(0) \in \mathbb{R}^N_{>0}$, the solution $r(t)$ to the combinatorial Ricci flow* (4.3) *exists for all time $t \in [0,+\infty)$.*

Proof. Since all the functions in the equation (4.3) are smooth and locally Lipschitz continuous, there is a unique solution $r(t)$ with $t \in [0,T)$ and $0 < T \leqslant +\infty$ by the classical ordinary differential equation (ODE) theory. Rewrite (4.3) as $d\ln(\tanh(\frac{r_i}{2}))/dt = -K_i$. Note $|K_i| \leqslant 2\pi \max_{m\in V}(d_m + 1) =: C$. Thus,

$$\tanh\left(\frac{r_i(0)}{2}\right)\mathrm{e}^{-Ct} \leqslant \tanh\left(\frac{r_i(t)}{2}\right) \leqslant \tanh\left(\frac{r_i(0)}{2}\right)\mathrm{e}^{Ct},$$

which implies that $r_i(t)$ cannot go to 0 in finite time. On the other hand, by Proposition 6.3, there exists a constant C_3 depending on the initial data $r(0)$ and the triangulation $\mathcal{T}$ such that $r_i(t) \leqslant C_3$. Therefore, by the extension theorem of solutions in ODE theory, the solution to the combinatorial Ricci flow (4.3) exists for all $t \in [0,+\infty)$. □

7 The proof of Theorem 4.1

In this section, we give the proof of the main theorem, i.e., Theorem 4.1. First, we prove the lower bound estimate for the solution to the combinatorial Ricci flow (4.3).

Theorem 7.1. *Let X be a closed surface with a given weighted triangulation $(\mathcal{T},\Phi)$. Assume the character $\mathcal{L}(\mathcal{T},\Phi)_i > 2\pi$ for all $i \in V$. Let $r(t)$ be a solution to the combinatorial Ricci flow* (4.3)*. Then, there is a positive constant $C = C(\mathcal{T},\Phi,r(0)) > 0$ depending only on the weighted triangulation $(\mathcal{T},\Phi)$ and the initial data $r(0)$ such that $r_i(t) \geqslant C$ for all the vertices and all time.*

Proof. Set $g(t) := \min_{m\in V} r_m(t)$. Then, $g(t)$ is a locally Lipschitz function and for $t \in [0,+\infty)$ a.e., there exists a special vertex $i \in V$ depending on t such that

$$g(t) = r_i(t), \quad g'(t) = r_i'(t). \tag{7.1}$$

For the particular vertex i, by Propositions 5.2 and 5.4, $\theta_i(t\vec{1})$ is continuously differentiable and

$$\lim_{t\to 0}\theta_i(t\vec{1}) = \arccos\left(\frac{1+\cos\Phi_{ij}+\cos\Phi_{ik}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ik}}}\right).$$

Hence, for any positive constant ϵ_0, small enough and to be determined later, there exists a constant $C = C(\mathcal{T},\Phi,r(0)) > 0$ such that for any $t \leqslant C$,

$$\theta_i(t\vec{1}) \geqslant \arccos\left(\frac{1+\cos\Phi_{ij}+\cos\Phi_{ik}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ik}}}\right) - \epsilon_0.$$

We claim that $g'(t) \geqslant 0$ for $t \in \{g < C\}$ a.e.

Let $t \in [0,+\infty)$ and $i \in V$ satisfying (7.1) and $t \in \{g < C\}$. Then, for any hyperbolic triangle $\triangle ijk$ incident to i realized by the circle packing $r(t)$,

$$r_i(t) = \min\{r_i(t), r_j(t), r_k(t)\} < C.$$

Set $\vec{r}(t) = (r_i(t), r_j(t), r_k(t))$, and then by Lemma 5.6,

$$\theta_i(\vec{r}(t)) \geqslant \theta_i(r_i(t)\vec{1}) \geqslant \arccos\left(\frac{1+\cos\Phi_{ij}+\cos\Phi_{ik}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ik}}}\right) - \epsilon_0.$$

Under the assumption $\mathcal{L}(\mathcal{T},\Phi)_j > 2\pi$ for all $j \in V$, we choose the constant ϵ_0 so that

$$0 < \epsilon_0 < \frac{\min_{j\in V}(\mathcal{L}(\mathcal{T},\Phi)_j - 2\pi)}{\max_{j\in V} d_j}.$$

Note that ϵ_0 depends on the data of the weighted triangulation $(\mathcal{T},\Phi)$ on X. It follows that

$$\begin{aligned}
K_i(r(t)) &= 2\pi - \sum_{\triangle ijk\in F}\theta_i(\vec{r}(t))\\
&\leqslant 2\pi - \sum_{\triangle ijk\in F}\left(\arccos\left(\frac{1+\cos\Phi_{ij}+\cos\Phi_{ik}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ik}}}\right) - \epsilon_0\right)\\
&= \epsilon_0\cdot d_i - (\mathcal{L}(\mathcal{T},\Phi)_i - 2\pi)\\
&\leqslant \epsilon_0\cdot\max_{j\in V} d_j - \min_{j\in V}(\mathcal{L}(\mathcal{T},\Phi)_j - 2\pi)\\
&< 0.
\end{aligned}$$

By (7.1) and the combinatorial Ricci flow (4.3), $g'(t) = r_i'(t) = -K_i\sinh r_i \geqslant 0$. This proves the claim. Then, the theorem follows from the claim and Lemma 6.1. □

Let $r(t)$ be a solution to the combinatorial Ricci flow (4.3). Then, by Theorem 7.1 and Proposition 6.3, there exist positive constants C_1 and C_2 depending on the weighted triangulation $(\mathcal{T}, \Phi)$ and the initial data $r(0)$ such that

$$C_1 \leqslant r_i(t) \leqslant C_2, \quad \forall\, i \in V, \quad t \in [0, +\infty).$$

This is equivalent to saying that $r(t)$ lies in the compact region in $\mathbb{R}^N_{>0}$. Then, the existence part of Theorem 4.1 is obtained by the following proposition proved by Chow and Luo [9].

Proposition 7.2 (See [9, Proposition 3.7]). *Suppose that $r(t)$ for $t \in [0, +\infty)$ is a solution to the combinatorial Ricci flow* (4.3) *in the hyperbolic background geometry so that the set $\{r(t) \mid t \in [0, +\infty)\}$ lies in a compact region in $\mathbb{R}^N_{>0}$. Then, there exists a hyperbolic circle packing r_{ze} in $\mathbb{R}^N_{>0}$, and $r(t)$ converges exponentially fast to r_{ze}.*

Proof of Theorem 4.1. Obviously, the uniqueness part of Theorem 4.1 is a consequence of the rigidity part of the Koebe-Andreev-Thurston theorem, i.e., the curvature map $r \mapsto K$ is injective. Finally, we prove the nonexistence part of Theorem 4.1.

Assume $\mathcal{L}(\mathcal{T}, \Phi)_i \leqslant 2\pi$ at each vertex $i \in V$. Hence, the average character $\mathcal{L}_{av} \leqslant 2\pi$. By Proposition 3.4, we get $\chi(X) \geqslant 0$. For any hyperbolic triangle $\triangle ijk$, denote by $\mathrm{Area}(\triangle ijk)$ its hyperbolic area. Then, $\mathrm{Area}(X) = \sum_{\triangle ijk \in F} \mathrm{Area}(\triangle ijk)$. The combinatorial Gauss-Bonnet formula [9] in the hyperbolic background says

$$\sum_{i \in V} K_i = 2\pi\chi(X) + \mathrm{Area}(X).$$

Hence, there is no circle packing r with curvature zero. By Chow-Luo's result [9] that the combinatorial Ricci flow (4.3) converges if and only if there exists a circle packing r_{ze} with curvature zero, we get the nonexistence part of Theorem 4.1. □

If $\mathcal{L}(\mathcal{T}, \Phi)_i < 2\pi$ for all the vertices, we can even show more, i.e., any initial circle packing shrinks to a point along the combinatorial Ricci flow (4.3). We have the following theorem.

Theorem 7.3. *If the character $\mathcal{L}(\mathcal{T}, \Phi)_i < 2\pi$ or $\mathcal{L}(\mathcal{T}, \Phi)_i \leqslant 2\pi$ for a constant weight at each vertex, then the solution $r(t)$ to the combinatorial Ricci flow* (4.3) *satisfies $r(t) \to 0$ when t tends to $+\infty$.*

Proof. We claim that if the character $\mathcal{L}(\mathcal{T}, \Phi)_i < 2\pi$ or $\mathcal{L}(\mathcal{T}, \Phi)_i \leqslant 2\pi$ for a constant weight at each vertex, then for any circle packing r, there exists a vertex i such that $K_i \geqslant C_0$, where $C_0 = C_0(\mathcal{T}, \Phi) > 0$ is a sufficiently small positive constant depending on the weighted triangulation $(\mathcal{T}, \Phi)$.

Let i be the vertex such that $r_i = \max_{j \in V} r_j$. For any hyperbolic triangle $\triangle ijk$ incident to i, we have $r_i = \max\{r_i, r_j, r_k\}$. Setting $\vec{r} = (r_i, r_j, r_k)$, by Lemma 5.6, we have

$$\theta_i(\vec{r}) \leqslant \theta_i(r_i \vec{1}). \tag{7.2}$$

If the character $\mathcal{L}(\mathcal{T}, \Phi)_j < 2\pi$ for each vertex $j \in V$, we choose the constant ϵ_0 depending only on the data of the weighted triangulation $(\mathcal{T}, \Phi)$ on X so that

$$0 < \epsilon_0 < \frac{\min_{j \in V}(2\pi - \mathcal{L}(\mathcal{T}, \Phi)_j)}{\max_{j \in V} d_j}.$$

Set

$$C_0 = \min_{j \in V}(2\pi - \mathcal{L}(\mathcal{T}, \Phi)_j) - \epsilon_0 \cdot \max_{j \in V} d_j > 0.$$

By Propositions 5.2 and 5.4, $\theta_i(t\vec{1})$ is continuously differentiable and

$$\lim_{t \to 0} \theta_i(t\vec{1}) = \arccos\left(\frac{1 + \cos\Phi_{ij} + \cos\Phi_{ik} - \cos\Phi_{jk}}{2\sqrt{1 + \cos\Phi_{ij}}\sqrt{1 + \cos\Phi_{ik}}}\right).$$

Therefore, for the positive constant ϵ_0, there exists a constant $C > 0$ such that for any $t \leqslant C$,

$$\theta_i(t\vec{1}) \leqslant \arccos\left(\frac{1 + \cos\Phi_{ij} + \cos\Phi_{ik} - \cos\Phi_{jk}}{2\sqrt{1 + \cos\Phi_{ij}}\sqrt{1 + \cos\Phi_{ik}}}\right) + \epsilon_0.$$

Choose any t_0 satisfying that $0 < t_0 < \min\{r_i, C\}$. Combining (7.2) and Proposition 5.2 which states that $t \mapsto \theta_i(t\vec{1})$ is strictly decreasing, we have

$$\theta_i(\vec{r}) \leqslant \theta_i(r_i\vec{1}) < \theta_i(t_0\vec{1}) \leqslant \arccos\left(\frac{1+\cos\Phi_{ij}+\cos\Phi_{ik}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ik}}}\right)+\epsilon_0. \tag{7.3}$$

It follows that

$$\begin{aligned}
K_i(r) &= 2\pi - \sum_{\triangle ijk\in F}\theta_i(\vec{r})\\
&\geqslant 2\pi - \sum_{\triangle ijk\in F}\left(\arccos\left(\frac{1+\cos\Phi_{ij}+\cos\Phi_{ik}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ik}}}\right)+\epsilon_0\right)\\
&= (2\pi - \mathcal{L}(\mathcal{T},\Phi)_i) - \epsilon_0\cdot d_i\\
&\geqslant C_0.
\end{aligned}$$

On the other hand, if the weight function $\Phi : E \to [0, \frac{\pi}{2}]$ is a constant, then $\mathcal{L}(\mathcal{T},\Phi)_i = \frac{\pi}{3}d_i$ so that the assumption $\mathcal{L}(\mathcal{T},\Phi)_i \leqslant 2\pi$ is equivalent to $d_i \leqslant 6$ at each vertex. By Propositions 5.2 and 5.4, we know that the angle function $t \mapsto \theta_i(t\vec{1})$ is continuously differentiable and

$$\lim_{t\to 0}\theta_i(t\vec{1}) = \frac{\pi}{3}, \quad \lim_{t\to+\infty}\theta_i(t\vec{1}) = 0.$$

Suppose that $C_0 > 0$ is any fixed sufficiently small positive number and set $\delta_0 = \frac{C_0}{6}$. Then, by the intermediate value theorem and above limits, there exists a t_0 $(0 < t_0 < r_i)$ such that

$$\theta_i(t_0\vec{1}) = \frac{\pi}{3} - \delta_0 > 0.$$

Noting that $t \mapsto \theta_i(t\vec{1})$ is strictly decreasing, by (7.2), we have

$$\theta_i(\vec{r}) \leqslant \theta_i(r_i\vec{1}) < \theta_i(t_0\vec{1}) = \frac{\pi}{3} - \delta_0.$$

Since $d_i \leqslant 6$,

$$K_i = 2\pi - \sum_{\triangle ijk\in F}\theta_i(\vec{r}) \geqslant 2\pi - 6\left(\frac{\pi}{3}-\delta_0\right) = C_0.$$

This completes the proof of the claim. Let $r(t)$ be a solution to the combinatorial Ricci flow (4.3). Set $r_M(t) := \max_{i\in V} r_i(t)$. Then, for $t \in [0,+\infty)$ a.e., there exists an $i \in V$ depending on t such that $r_M(t) = r_i(t)$ and $r'_M(t) = r'_i(t)$. By the claim above, for the circle packing $r(t)$, we have $K_i(r(t)) \geqslant C_0$, where $C_0 > 0$ is a sufficiently small positive constant depending on $(\mathcal{T},\Phi)$. By the combinatorial Ricci flow (4.3), for $t \in [0,+\infty)$ a.e.,

$$r'_M(t) = r'_i(t) \leqslant -C_0\sinh(r_i(t)) = -C_0\sinh(r_M(t)),$$

which is equivalent to $(\ln\tanh\frac{r_M(t)}{2})' \leqslant -C_0$. Integrating both the sides from 0 to t, we have $\tanh\frac{r_M(t)}{2} \leqslant \tanh\frac{r_M(0)}{2}\mathrm{e}^{-C_0 t}$. Hence, $r(t) \to 0$ as $t \to \infty$. This completes the proof of Theorem 7.3. □

8 The prescribed flow

In this final section, we derive Theorem 2.7. We call a given $\bar{K} = (\bar{K_1},\ldots,\bar{K_N})$ a prescribed combinatorial Gaussian curvature. We want to know if there is a circle packing $\bar{r}$ with the curvature $K(\bar{r}) = \bar{K}$. This is called a prescribed circle packing problem. By Thurston's rigidity theorem, the prescribed circle packing $\bar{r}$ is unique if it exists. By this terminology, a hyperbolic circle packing is just a packing with prescribed curvature zero.

Obviously, if the prescribed circle packing problem has a solution, then the prescribed curvature $\bar{K}$ lies in $K(\mathbb{R}^N_{>0})$, which is the image set of the curvature map K and vice versa. It has been proved in [9] that the prescribed circle packing problem has a solution, i.e., there is a circle packing $\bar{r}$ with the curvature $K(\bar{r}) = \bar{K}$, if and only if the following prescribed combinatorial Ricci flow converges:

$$\frac{dr_i}{dt} = (\bar{K}_i - K_i)\sinh r_i. \tag{8.1}$$

From the famous Koebe-Andreev-Thurston theorem, the image set $K(\mathbb{R}^N_{>0})$ of the curvature map $r \mapsto K$ is a bounded convex polytope. Using the data of $(\mathcal{T}, \Phi)$ on X, we see that it is described by a set of linear inequalities:

$$K(\mathbb{R}^N_{>0}) = \bigcap_{I \subset V} \left\{ x \in \mathbb{R}^N : \sum_{i \in I} x_i > -\sum_{(e,v)\in Lk(I)} (\pi - \Phi(e)) + 2\pi\chi(F_I) \right\}, \tag{8.2}$$

where I is taken over all the proper subsets of V, F_I is the subcomplex whose vertices are in I, and $Lk(I)$ is the set of pairs (e, v) of an edge e and a vertex v satisfying the following: (i) the end points of e are not in I; (ii) v is in I; (iii) e and v form a triangle.

(8.2) describes the image set of K completely. Despite its accuracy, it is global and not easy to verify. In this section, we further investigate the prescribed flow (8.1) and prove the following theorem.

Theorem 8.1. *Consider a given weighted triangulation $(\mathcal{T}, \Phi)$ on a closed surface X. If*

$$2\pi - \mathcal{L}(\mathcal{T}, \Phi)_i < \bar{K}_i < 2\pi$$

at each vertex $i \in V$, then the solution to the prescribed flow (8.1) *converges exponentially fast to a packing $\bar{r}$ with the curvature $K(\bar{r}) = \bar{K}$. Thus, $\bar{r}$ is the solution of the prescribed circle packing problem. On the other hand, if $\bar{K}_i \leqslant 2\pi - \mathcal{L}(\mathcal{T}, \Phi)_i$ at each vertex $i \in V$, then the prescribed flow* (8.1) *cannot converge. Hence, the prescribed circle packing problem has no solutions, and equivalently, there is no circle packing $\bar{r}$ with the curvature $K(\bar{r}) = \bar{K}$.*

Proof. The proof of [9, Proposition 3.4] can be used here to show that $\coth \frac{r_i}{2} \leqslant C e^{2\pi t}$, where C is some constant depending on $(X, \mathcal{T}, \Phi)$, $r(0)$, and $\bar{K}$. Thus, $r_i(t)$ remains bounded away from 0 as long as time t is bounded. By the following Claim 1, for any initial $r(0)$, there exists a unique long-time solution $r(t), t \in [0, \infty)$ under the assumption $\bar{K}_i < 2\pi$. To show the convergence part, we just need to show $r(t) \subset\subset \mathbb{R}^N_{>0}$. We show this by proving the following Claims 1 and 2.

Claim 1. If $\bar{K}_i < 2\pi$, $\forall\, i \in V$, then all $r_i(t)$'s are bounded from above uniformly.

For any $0 < \epsilon_0 < \min_i(2\pi - \bar{K}_i)/\max_i d_i$, there is a constant $C = C(\mathcal{T}, \Phi, \epsilon_0) > 0$ by Lemma 6.2 so that whenever $r_i \geqslant C$, then $\theta_i \leqslant \epsilon_0$. It follows at each vertex $i \in V$ that

$$K_i = 2\pi - \sum_{\triangle ijk \in F} \theta_i > 2\pi - d_i\epsilon_0 > \bar{K}_i.$$

If at time t and a vertex i, $r_i(t) = C$, then we have $dr_i(t)/dt = (\bar{K}_i - K_i)\sinh r_i < 0$, which implies that $r_i(t)$ is strictly decreasing whenever it attains the constant C. Hence, C is a uniform upper bound for all $r_i(t)$, which proves Claim 1.

Claim 2. If $\bar{K}_i > 2\pi - \mathcal{L}(\mathcal{T}, \Phi)_i$, $\forall\, i \in V$, then all $r_i(t)$'s have a positive lower bound.

For any given constant $0 < \epsilon_0 < \min_i(\bar{K}_i - (2\pi - \mathcal{L}(\mathcal{T}, \Phi)_i))/\max_i d_i$, there exists a constant $C = C(\mathcal{T}, \Phi, \epsilon_0) > 0$ such that whenever $r_i \leqslant C$,

$$\theta_i(r_i, r_j, r_k) \geqslant \arccos\left(\frac{1 + \cos\Phi_{ij} + \cos\Phi_{ik} - \cos\Phi_{jk}}{2\sqrt{1 + \cos\Phi_{ij}}\sqrt{1 + \cos\Phi_{ik}}}\right) - \epsilon_0.$$

It follows that $K_i = 2\pi - \sum_{\triangle ijk \in F} \theta_i < 2\pi - \mathcal{L}(\mathcal{T}, \Phi)_i + d_i\epsilon_0$, and then

$$\bar{K}_i - K_i > \bar{K}_i - (2\pi - \mathcal{L}(\mathcal{T}, \Phi)_i) - d_i\epsilon_0 > 0.$$

If at time t and a vertex i, $r_i(t) = C$, then we have $dr_i(t)/dt = (\bar{K}_i - K_i)\sinh r_i > 0$, which implies that $r_i(t)$ is strictly increasing whenever it attains the constant C. Hence, C is a uniform lower bound for all $r_i(t)$, which proves Claim 2.

Finally, we show the nonconvergence part. Assume $\bar{K}_i \leqslant 2\pi - \mathcal{L}(\mathcal{T}, \Phi)_i$ at each vertex $i \in V$, which is equivalent to $\mathcal{L}(\mathcal{T}, \Phi)_i \leqslant 2\pi - \bar{K}_i$ at each vertex $i \in V$. Hence, the average character $\mathcal{L}_{av} \leqslant 2\pi - (\sum_{i\in V} \bar{K}_i)/N$. By Proposition 3.4, we get $\sum_{i\in V} \bar{K}_i \leqslant 2\pi\chi(X)$. The combinatorial Gauss-Bonnet formula [9] in the hyperbolic background says $\sum_{i\in V} K_i = 2\pi\chi(X) + \mathrm{Area}(X)$. Hence, there is no circle packing $\bar{r}$ with the curvature $K(\bar{r}) = \bar{K}$. We complete the proof. □

Consequently, we can derive Theorem 2.7, which is restated as the following corollary.

Corollary 8.2. *Considering the weighted triangulated surface* $(X, \mathcal{T}, \Phi)$, *we have*

$$\prod_{i\in V}(2\pi - \mathcal{L}(\mathcal{T}, \Phi)_i,\ 2\pi) \subset K(\mathbb{R}^N_{>0}) \tag{8.3}$$

and

$$\prod_{i\in V}(-\infty,\ 2\pi - \mathcal{L}(\mathcal{T}, \Phi)_i] \cap K(\mathbb{R}^N_{>0}) = \emptyset. \tag{8.4}$$

Combining Corollary 8.2 and our "degree-type" criteria, i.e., Corollaries 3.5, 3.7, and 3.8, we have some quite interesting results for the image set.

Corollary 8.3. *Assume that the weight function* $\Phi : E \to [0, \pi/2]$ *is a constant. Then,*

$$\prod_{i\in V}\left(2\pi - \frac{d_i}{3}\pi, 2\pi\right) \subset K(\mathbb{R}^N_{>0})$$

and

$$\prod_{i\in V}\left(-\infty, 2\pi - \frac{d_i}{3}\pi\right] \cap K(\mathbb{R}^N_{>0}) = \emptyset.$$

Corollary 8.4. *Assume the weight* $\Phi \in [0, \pi/2]$ *and* $d_i \geqslant 9$ *for each vertex* $i \in V$. *Then,*

$$[-0.07\pi, 2\pi)^N \subset \left(2\pi - 9\arccos\frac{3}{4}, 2\pi\right)^N \subset K(\mathbb{R}^N_{>0}).$$

Corollary 8.5. *Assume* $d_i \geqslant 7$ *for each vertex* $i \in V$. *If further assume the weight function* $\Phi : E \to [0, \pi/2]$ *satisfies that* Φ *is a constant, or* $\Phi \in [0, 0.33\pi]$, *or* $\Phi \in [0.4\pi, \pi/2]$, *or more generally,* $\Phi \in [\arccos\eta, \arccos\xi] \subset [0, \pi/2]$, *where* $0 \leqslant \xi \leqslant \eta \leqslant 1$ *are arbitrarily chosen so that*

$$\eta < \left(2\cos\frac{2\pi}{7} - 1 + \xi\right)\bigg/\left(2 - 2\cos\frac{2\pi}{7}\right),$$

then $[0,\ 2\pi)^N \subset K(\mathbb{R}^N_{>0})$.

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