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Quantitative Green's function estimates for lattice quasi-periodic Schrödinger operators

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Abstract In this paper, we establish quantitative Green's function estimates for some higher-dimensional lattice quasi-periodic (QP) Schrödinger operators. The resonances in the estimates can be described via a pair of symmetric zeros of certain functions, and the estimates apply to the sub-exponential-type non-resonance conditions. As the application of quantitative Green's function estimates, we prove both the arithmetic version of Anderson localization and the finite volume version of $(\frac{1}{2}-)$-Hölder continuity of the integrated density of states (IDS) for such QP Schrödinger operators. This gives an affirmative answer to Bourgain's problem in Bourgain (2000).

Keywords Hölder continuity of the IDS, quantitative Green's function estimates, quasi-periodic Schrödinger operators, arithmetic Anderson localization, multi-scale analysis

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1 Introduction

Consider the quasi-periodic (QP) Schrödinger operators

$$H = \Delta + \lambda V(\theta + n\omega)\delta_{n,n'} \quad \text{on } \mathbb{Z}^d, \tag{1.1}$$

where Δ is the discrete Laplacian, $V : \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \to \mathbb{R}$ is the potential, and $n\omega = (n_1\omega_1, \ldots, n_d\omega_d)$. Typically, we call $\theta \in \mathbb{T}^d$ the phase, $\omega \in [0,1]^d$ the frequency and $\lambda \in \mathbb{R}$ the coupling. Particularly, if $V = 2\cos 2\pi\theta$ and $d = 1$, then the operators (1.1) become the famous almost Mathieu operators (AMOs).

Over the past decades, the study of spectral and dynamical properties of lattice QP Schrödinger operators has been one of the central themes in mathematical physics. Of particular importance is the phenomenon of Anderson localization (i.e., the pure point spectrum with exponentially decaying eigenfunctions). Determining the nature of the spectrum and the eigenfunction properties of (1.1) can be viewed as a small divisor problem, which depends sensitively on features of λ, V, ω, θ and d. Then, substantial progress has been made following Green's function estimates based on a Kolmogorov-Arnold-Moser (KAM)-type multi-scale analysis (MSA) of Fröhlich and Spencer [17]. More precisely, Sinai [34]

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first proved the Anderson localization for a class of 1D QP Schrödinger operators with a C^2 cosine-like potential assuming the Diophantine frequency[1)]. The proof focuses on eigenfunction parametrization, and the resonances are overcome via a KAM iteration scheme. Independently, Fröhlich et al. [18] extended the celebrated method of Fröhlich and Spencer [17] originating from the random Schrödinger operator case to the QP one and obtained a similar Anderson localization result to [34]. The proof, however, uses estimates of finite volume Green's functions based on the MSA and the eigenvalue variations. Both [34] and [18] were inspired essentially by the arguments of [17]. Eliasson [16] applied a reducibility method based on KAM iterations to general Gevrey QP potentials and established the pure point spectrum for corresponding Schrödinger operators. All these 1D results are perturbative in the sense that the required perturbation strength depends heavily on the Diophantine frequency (i.e., localization holds for $|\lambda| \geqslant \lambda_0(V,\omega) > 0$). The great breakthrough was made by Jitomirskaya [24, 25], in which the nonperturbative methods for controlling Green's functions (see [26]) were developed first for AMOs. Nonperturbative methods can avoid the use of multi-scale schemes and eigenvalue variations. This will allow effective (even optimal in many cases) and independent-of-ω estimates on λ_0. In addition, such methods can provide an arithmetic version of Anderson localization, which means the removed sets on both ω and θ when obtaining localization have an explicit arithmetic description (see [25, 28] for details). In contrast, the current perturbation methods seem to only provide some certain measure or complexity bounds on these sets. Later, Bourgain and Jitomirskaya [11] extended some results of [25] to the exponential long-range hopping case (thus the absence of the Lyapunov exponent) and obtained both nonperturbative and arithmetic Anderson localization. Significantly, Bourgain and Goldstein [9] generalized nonperturbative Green's function estimates of Jitomirskaya [25] by introducing the new ingredients of semi-algebraic set theory and subharmonic function estimates, and established the nonperturbative Anderson localization[2)] for general analytic QP potentials. The localization results of [9] hold for arbitrary $\theta \in \mathbb{T}$ and a.e. Diophantine frequencies (the permitted set of frequencies depends on θ), and there seems to be no arithmetic version of Anderson localization results in this case. We mention that the Anderson localization can also be obtained via reducibility arguments based on the Aubry duality [3, 27].

If one increases the lattice dimensions of QP operators, the proof of Anderson localization becomes significantly difficult. In this setting, Chulaevsky and Dinaburg [12] and Dinaburg [14] first extended the results of Sinai [34] to the exponential long-range operator with a C^2 cosine-type potential on $\mathbb{Z}^d$ for arbitrary $d \geqslant 1$. However, in this case, the localization holds without an explicit arithmetic description on θ. Subsequently, the remarkable work of Bourgain et al. [10] established the Anderson localization for the general analytic QP Schrödinger operators with $(n,\theta,\omega) \in \mathbb{Z}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$ via Green's function estimates. In [10], they first proved the large deviation theorem (LDT) for finite volume Green's functions by combining MSA, matrix-valued Cartan's estimates, and semi-algebraic set theory. Then, by using further semi-algebraic arguments together with the LDT, they proved the Anderson localization for all $\theta \in \mathbb{T}^2$ and ω in a set of positive measures (depending on θ). While the restrictions of the frequencies in the LDT are purely arithmetic and do not depend on the choice of potentials, in order to obtain the Anderson localization, we need to remove an additional frequency set of positive measures. The proof of [10] is essentially two-dimensional, and its generation to higher dimensions is significantly difficult. In 2007, Bourgain [8] successfully extended the results of [10] to arbitrary dimensions, and one of his key ideas is allowing the restrictions of frequencies to depend on the potential by means of delicate semi-algebraic set analysis when proving the LDT for Green's functions. In other words, for the proof of the LDT in [8], there have already been additional restrictions on the frequencies, which depend on the potential V and are thus not arithmetic. The results of [8] have been largely generalized by Jitomirskaya et al. [29] to the case of both multi-frequencies in arbitrary dimensions and exponential long-range hopping. Very

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[1)] We say $\omega \in \mathbb{R}$ satisfies the Diophantine condition if there are $\tau > 1$ and $\gamma > 0$ so that

$$\|k\omega\| = \inf_{l\in\mathbb{Z}} |l - k\omega| \geqslant \frac{\gamma}{|k|^\tau}, \quad \forall\, k \in \mathbb{Z} \setminus \{0\}.$$

[2)] That is Anderson localization assuming the positivity of the Lyapunov exponent. In the present context, by nonperturbative Anderson localization, we mean localization if $|\lambda| \geqslant \lambda_0 = \lambda_0(V) > 0$ with λ_0 being independent of ω.

recently, Ge and You [19] applied a reducibility argument to higher-dimensional long-range QP operators with the cosine potential and proved the first arithmetic Anderson localization assuming the Diophantine frequency.

Definitely, LDT-type Green's function estimate methods are powerful to deal with higher-dimensional QP Schrödinger operators with general analytic potentials. However, such methods do not provide detailed information on Green's functions and eigenfunctions that may be extracted by purely perturbative methods based on the Weierstrass preparation theorem. As evidence, in the celebrated work [5], Bourgain first developed the method of [4] further to obtain the finite volume version of $(\frac{1}{2}-)$-Hölder continuity of the integrated density of states (IDS) for AMOs. The proof shows that Green's functions can be controlled via certain quadratic polynomials, and the resonances are completely determined by zeros of these polynomials. Using this method yields a surprising quantitative result on the Hölder exponent of the IDS, since the celebrated method of Goldstein and Schlag [21] which is nonperturbative and works for more general potentials does not seem to provide explicit information on the Hölder exponent. In 2009, by using the KAM reducibility method of Eliasson [15], Amor [1] obtained the first $\frac{1}{2}$-Hölder continuity result of the IDS for 1D and multi-frequency QP Schrödinger operators with small analytic potentials and Diophantine frequencies. Later, the one-frequency result of Amor was largely generalized by Avila and Jitomirskaya [2] to the nonperturbative case via the quantitative almost reducibility and localization method. In the regime of the positive Lyapunov exponent, Goldstein and Schlag [22] successfully proved the $(\frac{1}{2m}-)$-Hölder continuity of the IDS for 1D and one-frequency QP Schrödinger operators with potentials given by analytic perturbations of certain trigonometric polynomials of degree $m \geqslant 1$. This celebrated work provides the finite volume version of estimates on the IDS. We remark that the Hölder continuity of the IDS for 1D and multi-frequency QP Schrödinger operators with large potentials is hard to prove. In [21], by using the LDT for the transfer matrix and the avalanche principle, Goldstein and Schlag showed the weak Hölder continuity (see (1.2)) of the IDS for 1D and multi-frequency QP Schrödinger operators assuming the positivity of the Lyapunov exponent and strong Diophantine frequencies. The weak Hölder continuity of the IDS for higher-dimensional QP Schrödinger operators has been established in [8, 30, 33]. Very recently, Ge et al. [20] proved the $(\frac{1}{2m}-)$-Hölder continuity of the IDS for higher-dimensional QP Schrödinger operators with small exponential long-range hopping and trigonometric-polynomial (of degree m) potentials via the reducibility argument. By the Aubry duality, they can obtain the $(\frac{1}{2m}-)$-Hölder continuity of the IDS for 1D and multi-frequency QP operators with finite-range hopping.

Of course, the references mentioned above are far from being complete, and we refer the reader to [7, 13, 31] for more recent results on the study of both Anderson localization and the Hölder regularity of the IDS for lattice QP Schrödinger operators.

1.1 Bourgain's problems

Remarkable Green's function estimates of [5] should not be restricted to the proof of $(\frac{1}{2}-)$-Hölder regularity of the IDS for AMOs only. In fact, Bourgain [5, p. 89] made three comments on the possible extensions of his method:

(1) One may also recover the Anderson localization results from [18, 34] in the perturbative case.

(2) One may hope that it may be combined with nonperturbative arguments in the spirit of [9, 21] to establish $(\frac{1}{2}-)$-Hölder regularity assuming positivity of the Lyapunov exponent only.

(3) It may also allow progress in the multi-frequency case (perturbative or nonperturbative), where regularity estimates of the form (0.28)[3)] are the best obtained so far.

An extension of (2) has been accomplished by Goldstein and Schlag [22]. The answer to the extension of (1) is highly nontrivial due to the following reasons:

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3) That is a weak Hölder continuity estimate

$$|\mathcal{N}(E)-\mathcal{N}(E')| \leqslant \mathrm{e}^{-(\log\frac{1}{|E-E'|})^{\zeta}}, \quad \zeta \in (0,1), \tag{1.2}$$

where $\mathcal{N}(\cdot)$ denotes the IDS.

• Green's function on **good** sets (see Section 3 for details) only has a sub-exponential off-diagonal decay estimate rather than an exponential one required in the proof of Anderson localization.

• At the s-th iteration step ($s \geqslant 1$), the resonances of [5] are characterized as

$$\min\{\|\theta + k\omega - \theta_{s,1}\|, \|\theta + k\omega - \theta_{s,2}\|\} \leqslant \delta_s \sim \delta_0^{C^s}, \quad C > 1.$$

However, the symmetry information of $\theta_{s,1}$ and $\theta_{s,2}$ is missing. Actually, in [5], it might be $\theta_{s,1} + \theta_{s,2} \neq 0$ because of the construction of resonant blocks.

• If one tries to extend the method of Bourgain [5] to higher lattice dimensions, there comes a new difficulty: the resonant blocks at each iteration step could not be cubes similar to the intervals that appear in the 1D case.

To extend the method of Bourgain [5] to higher lattice dimensions and recover the Anderson localization, one has to address the above issues, which is our main motivation for this paper.

1.2 Main results

In this paper, we study the QP Schrödinger operators on $\mathbb{Z}^d$:

$$H(\theta) = \varepsilon\Delta + \cos 2\pi(\theta + n \cdot \omega)\delta_{n,n'}, \quad \varepsilon > 0, \tag{1.3}$$

where the discrete Laplacian Δ is defined as

$$\Delta(n, n') = \delta_{\|n-n'\|_1,1}, \quad \|n\|_1 := \sum_{i=1}^{d} |n_i|.$$

For the diagonal part of (1.3), we have $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\omega \in [0,1]^d$ and

$$n \cdot \omega = \sum_{i=1}^{d} n_i \omega_i.$$

Throughout the paper, we assume that $\omega \in \mathcal{R}_{\tau,\gamma}$ for some $0 < \tau < 1$ and $\gamma > 0$ with

$$\mathcal{R}_{\tau,\gamma} = \Big\{\omega \in [0,1]^d : \|n \cdot \omega\| = \inf_{l \in \mathbb{Z}} |l - n \cdot \omega| \geqslant \gamma e^{-\|n\|^\tau}, \, \forall\, n \in \mathbb{Z}^d \setminus \{0\}\Big\}, \tag{1.4}$$

where

$$\|n\| := \sup_{1 \leqslant i \leqslant d} |n_i|.$$

We aim to extend the method of Bourgain [5] to higher lattice dimensions and establish quantitative Green's function estimates assuming (1.4). As the application, we prove the arithmetic version of Anderson localization and the finite volume version of $(\frac{1}{2}-)$-Hölder continuity of the IDS for (1.3).

1.2.1 *Quantitative Green's function estimates*

The first main result of this paper is a quantitative version of Green's function estimates, which will imply both arithmetic Anderson localization and the finite volume version of $(\frac{1}{2}-)$-Hölder continuity of the IDS. The estimates on Green's function are based on multi-scale induction arguments.

Let $\Lambda \subset \mathbb{Z}^d$ and denote by R_Λ the restriction operator. Given $E \in \mathbb{R}$, Green's function (if existing) is defined by

$$T_\Lambda^{-1}(E;\theta) = (H_\Lambda(\theta) - E)^{-1}, \quad H_\Lambda(\theta) = R_\Lambda H(\theta) R_\Lambda.$$

Recall that $\omega \in \mathcal{R}_{\tau,\gamma}$ and $\tau \in (0,1)$. We fix a constant $c > 0$ so that

$$1 < c^{20} < \frac{1}{\tau}.$$

At the s-th iteration step, let δ_s^{-1} (resp. N_s) describe the resonance strength (resp. the size of resonant blocks) defined by

$$N_{s+1} = \left[\left|\log\frac{\gamma}{\delta_s}\right|^{\frac{1}{c^5\tau}}\right], \quad \left|\log\frac{\gamma}{\delta_{s+1}}\right| = \left|\log\frac{\gamma}{\delta_s}\right|^{c^5}, \quad \delta_0 = \varepsilon^{\frac{1}{10}},$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$.

If $a \in \mathbb{R}$, let

$$\|a\| = \mathrm{dist}(a, \mathbb{Z}) = \inf_{l\in\mathbb{Z}} |l - a|.$$

For $z = a + \sqrt{-1}b \in \mathbb{C}$ with $a, b \in \mathbb{R}$, define

$$\|z\| = \sqrt{|b|^2 + \|a\|^2}.$$

Denote by $\mathrm{dist}(\cdot,\cdot)$ the distance induced by the supremum norm on $\mathbb{R}^d$. Then, we have the following theorem.

Theorem 1.1. *Let $\omega \in \mathcal{R}_{\tau,\gamma}$. Then, there is some $\varepsilon_0 = \varepsilon_0(d,\tau,\gamma) > 0$ so that for $0 < \varepsilon \leqslant \varepsilon_0$ and $E \in [-2, 2]$, there exists a sequence $\{\theta_s = \theta_s(E)\}_{s=0}^{s'} \subset \mathbb{C}$ ($s' \in \mathbb{N} \cup \{+\infty\}$) with the following properties. Fix any $\theta \in \mathbb{T}$. If a finite set $\Lambda \subset \mathbb{Z}^d$ is s-**good** (see $(\mathbf{e})_s$ of Statement 3.1 for the definition of s-**good** sets, and Section 3 for the definitions of $\{\theta_s\}_{s=0}^{s'}$ and the sets P_s, Q_s and $\widetilde{\Omega}_k^s$), then*

$$\begin{aligned}
&\|T_\Lambda^{-1}(E;\theta)\| < \delta_{s-1}^{-3} \sup_{\{k\in P_s : \widetilde{\Omega}_k^s \subset \Lambda\}} \|\theta + k\cdot\omega - \theta_s\|^{-1} \cdot \|\theta + k\cdot\omega + \theta_s\|^{-1} < \delta_s^{-3},\\
&|T_\Lambda^{-1}(E;\theta)(x,y)| < \mathrm{e}^{-\frac{1}{4}|\log\varepsilon|\cdot\|x-y\|_1} \quad \text{for } \|x-y\| > N_s^{c^3}.
\end{aligned}$$

In particular, for any finite set $\Lambda \subset \mathbb{Z}^d$, there exists some $\widetilde{\Lambda}$ satisfying

$$\Lambda \subset \widetilde{\Lambda} \subset \{k \in \mathbb{Z}^d : \mathrm{dist}(k, \Lambda) \leqslant 50N_s^{c^2}\}$$

so that if

$$\min_{k\in\widetilde{\Lambda}^*} \min_{\sigma=\pm1} (\|\theta + k\cdot\omega + \sigma\theta_s\|) \geqslant \delta_s,$$

then

$$\begin{aligned}
&\|T_{\widetilde{\Lambda}}^{-1}(E;\theta)\| \leqslant \delta_{s-1}^{-3}\delta_s^{-2} \leqslant \delta_s^{-3},\\
&|T_{\widetilde{\Lambda}}^{-1}(E;\theta)(x,y)| \leqslant \mathrm{e}^{-\frac{1}{4}|\log\varepsilon|\cdot\|x-y\|} \quad \text{for } \|x-y\| > N_s^{c^3},
\end{aligned}$$

where

$$\widetilde{\Lambda}^* = \left\{k \in \frac{1}{2}\mathbb{Z}^d : \mathrm{dist}(k, \widetilde{\Lambda}) \leqslant \frac{1}{2}\right\}.$$

Let us refer to Section 3 for a complete description of Green's function estimates.

1.2.2 *Arithmetic Anderson localization and Hölder continuity of the IDS*

As the application of quantitative Green's function estimates, we first prove the following arithmetic version of Anderson localization for $H(\theta)$. Let $\tau_1 > 0$ and define

$$\Theta_{\tau_1} = \{(\theta,\omega) \in \mathbb{T}\times\mathcal{R}_{\tau,\gamma} : \|2\theta + n\cdot\omega\| \leqslant \mathrm{e}^{-\|n\|^{\tau_1}} \text{ holds for finitely many } n \in \mathbb{Z}^d\}.$$

We have the following theorem.

Theorem 1.2. *Let $H(\theta)$ be given by* (1.3) *and let $0 < \tau_1 < \tau$. Then, there exists some $\varepsilon_0 = \varepsilon_0(d,\tau,\gamma) > 0$ such that if $0 < \varepsilon \leqslant \varepsilon_0$, then for $(\theta,\omega) \in \Theta_{\tau_1}$, $H(\theta)$ satisfies the Anderson localization.*

Remark 1.3. It is easy to check both $\mathrm{mes}(\mathbb{T} \setminus \Theta_{\tau_1,\omega}) = 0$ and $\mathrm{mes}(\mathcal{R}_{\tau,\gamma} \setminus \Theta_{\tau_1,\theta}) = 0$, where

$$\Theta_{\tau_1,\omega} = \{\theta \in \mathbb{T} : (\theta,\omega) \in \Theta_{\tau_1}\}, \quad \Theta_{\tau_1,\theta} = \{\omega \in \mathcal{R}_{\tau,\gamma} : (\theta,\omega) \in \Theta_{\tau_1}\},$$

and $\mathrm{mes}(\cdot)$ denotes the Lebesgue measure. Thus, Anderson localization can be established either by fixing $\omega \in \mathcal{R}_{\tau,\gamma}$ and removing θ in the spirit of [25], or by fixing $\theta \in \mathbb{T}$ and removing ω in the spirit of [9, 10].

The second application is a proof of the finite volume version of $(\frac{1}{2}-)$-Hölder continuity of the IDS for $H(\theta)$. For a finite set Λ, denote by $\#\Lambda$ the cardinality of Λ. Let

$$\mathcal{N}_\Lambda(E;\theta) = \frac{1}{\#\Lambda}\#\{\lambda \in \sigma(H_\Lambda(\theta)) : \lambda \leqslant E\}$$

and denote by

$$\mathcal{N}(E) = \lim_{N\to\infty} \mathcal{N}_{\Lambda_N}(E;\theta) \tag{1.5}$$

the IDS, where $\Lambda_N = \{k \in \mathbb{Z}^d : \|k\| \leqslant N\}$ for $N > 0$. It is well known that the limit in (1.5) exists and is independent of θ for a.e. θ.

Theorem 1.4. *Let $H(\theta)$ be given by (1.3) and let $\omega \in \mathcal{R}_{\tau,\gamma}$. Then, there exists some $\varepsilon_0 = \varepsilon_0(d,\tau,\gamma) > 0$ such that if $0 < \varepsilon \leqslant \varepsilon_0$, then for any small $\mu > 0$ and $0 < \eta < \eta_0(d,\tau,\gamma,\mu)$, we have for sufficiently large N depending on η,*

$$\sup_{\theta\in\mathbb{T},E\in\mathbb{R}} (\mathcal{N}_{\Lambda_N}(E+\eta;\theta) - \mathcal{N}_{\Lambda_N}(E-\eta;\theta)) \leqslant \eta^{\frac{1}{2}-\mu}. \tag{1.6}$$

In particular, the IDS is Hölder continuous with exponent ι for any $\iota \in (0, \frac{1}{2})$.

Let us give some remarks on our results.

(1) Green's function estimates can be extended to the exponential long-range hopping case and may not be restricted to the cosine potential. Except for the proof of arithmetic Anderson localization and the finite volume version of $(\frac{1}{2}-)$-Hölder regularity of the IDS, the quantitative Green's function estimates should have potential applications in other problems, such as the estimates of Lebesgue measure of the spectrum, dynamical localization, the estimates of level spacings of eigenvalues and the finite volume version of localization. We can even expect fine results in dealing with Melnikov's persistency problem (see [4]) by employing Green's function estimates.

(2) As mentioned previously, Ge and You [19] proved the first arithmetic Anderson localization result for higher-dimensional QP operators with the exponential long-range hopping and the cosine potential via their reducibility method. Our result is valid for frequencies satisfying the sub-exponential non-resonance condition (see (1.4)) of Rüssmann type [32], which slightly generalizes the Diophantine-type localization result of [19]. While the Rüssmann-type condition is sufficient for the use of the classical KAM method, it is not clear whether such a condition still suffices for the validity of the MSA method. Definitely, the localization result of both [19] and the present work is perturbative[4)]. Finally, since our proof of arithmetic Anderson localization is based on Green's function estimates, we can improve it to obtain the finite volume version of Anderson localization as that obtained in [23].

(3) Apparently, using the Aubry duality together with Amor's result [1] has already led to the $\frac{1}{2}$-Hölder continuity of the IDS for higher-dimensional QP operators with small exponential long-range hopping and the cosine potential assuming Diophantine frequencies. So our result of $(\frac{1}{2}-)$-Hölder continuity is weaker than that of [1] in the Diophantine frequency case. However, we want to emphasize that the method of Amor seems only valid for estimating the limit $\mathcal{N}(E)$ and provides no precise information on the finite volume quantity $\mathcal{N}_\Lambda(E;\theta)$. In this context, our result (see (1.6)) is also new as it gives a uniform upper bound on the number of eigenvalues inside a small interval. In addition, our result also improves the upper bound on the number of eigenvalues of Schlag [33, Proposition 2.2] in the special case where the potential is given by the cosine function.

[4)] Bourgain [6] has proven that the nonperturbative localization cannot be expected in dimensions $d \geqslant 2$. More precisely, consider $H^{(2)} = \lambda\Delta + 2\cos 2\pi(\theta + n\cdot\omega)\delta_{n,n'}$ on $\mathbb{Z}^2$. Using Aubry duality together with the result of Bourgain [6] yields that for any $\lambda \neq 0$, there exists a set $\Omega \subset \mathbb{T}^2$ of positive measures with the following property, i.e., for $\omega \in \Omega$, there exists a set $\Theta \subset \mathbb{T}$ of positive measures, s.t. for $\theta \in \Theta$, $H^{(2)}$ does not satisfy Anderson localization.

1.3 Notations and the structure of the paper

• Given $A \in \mathbb{C}$ and $B \in \mathbb{C}$, we write $A \lesssim B$ (resp. $A \gtrsim$ B) if there is some $C = C(d, \tau, \gamma) > 0$ depending only on d, τ and γ so that $|A| \leqslant C|B|$ (resp. $|A| \geqslant C|B|$). We also define

$$A \sim B \Leftrightarrow \frac{1}{C} < \left|\frac{A}{B}\right| < C,$$

and for some $D > 0$,

$$A \overset{D}{\sim} B \Leftrightarrow \frac{1}{CD} < \left|\frac{A}{B}\right| < CD.$$

• The determinant of a matrix M is denoted by $\det M$.
• For $n \in \mathbb{R}^d$, let

$$\|n\|_1 := \sum_{i=1}^{d} |n_i| \quad \text{and} \quad \|n\| := \sup_{1 \leqslant i \leqslant d} |n_i|.$$

Denote by $\operatorname{dist}(\cdot, \cdot)$ the distance induced by $\|\cdot\|$ on $\mathbb{R}^d$, and define

$$\operatorname{diam} \Lambda = \sup_{k,k' \in \Lambda} \|k - k'\|.$$

Given $n \in \mathbb{Z}^d$, $\Lambda_1 \subset \frac{1}{2}\mathbb{Z}^d$ and $L > 0$, define

$$\Lambda_L(n) = \{k \in \mathbb{Z}^d : \|k - n\| \leqslant L\}$$

and

$$\Lambda_L(\Lambda_1) = \{k \in \mathbb{Z}^d : \operatorname{dist}(k, \Lambda_1) \leqslant L\}.$$

In particular, write $\Lambda_L = \Lambda_L(0)$.

• Assume $\Lambda' \subset \Lambda \subset \mathbb{Z}^d$. Define the relative boundaries as $\partial_\Lambda^+ \Lambda' = \{k \in \Lambda : \operatorname{dist}(k, \Lambda') = 1\}$, $\partial_\Lambda^- \Lambda' = \{k \in \Lambda : \operatorname{dist}(k, \Lambda \setminus \Lambda') = 1\}$ and $\partial_\Lambda \Lambda' = \{(k, k') : \|k - k'\| = 1, k \in \partial_\Lambda^- \Lambda', k' \in \partial_\Lambda^+ \Lambda'\}$.

• Let $\Lambda \subset \mathbb{Z}^d$ and let $T : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ be a linear operator. Define $T_\Lambda = R_\Lambda T R_\Lambda$, where R_Λ is the restriction operator. Denote by $\langle \cdot, \cdot \rangle$ the standard inner product on $\ell^2(\mathbb{Z}^d)$. Set $T_\Lambda(x, y) = \langle \delta_x, T_\Lambda \delta_y \rangle$ for $x, y \in \Lambda$. By $\|T_\Lambda\|$, we mean the standard operator norm of T_Λ. The spectrum of the operator T is denoted by $\sigma(T)$. Finally, I typically denotes the identity operator.

The rest of this paper is organized as follows. The key ideas of the proof are introduced in Section 2. The proofs of Theorems 1.1, 1.2 and 1.4 are presented in Sections 3–5, respectively. Some useful estimates can be found in the appendixes.

2 Key ideas of the proof

The main scheme of our proof is definitely adapted from Bourgain [5]. The key ingredient of the proof in [5] is that the resonances in dealing with Green's function estimates can be completely determined by the roots of some quadratic polynomials. The polynomials were produced in a Fröhlich-Spencer-type MSA induction procedure. However, in the estimates of Green's functions restricted to the resonant blocks, Bourgain directly applied Cramer's rule and provided estimates on certain determinants. It turns out these determinants can be well controlled via estimates of previous induction steps, the Schur complement argument, and the Weierstrass preparation theorem. It is the preparation-type technique that yields the desired quadratic polynomials. We emphasize that this new method of Bourgain is fully free from eigenvalue variations or eigenfunction parametrization.

However, in order to extend the method to achieve an arithmetic version of Anderson localization in higher dimensions, we need some new ideas as follows:

• The off-diagonal decay of Green's function obtained by Bourgain [5] is sub-exponential rather than exponential, which is not sufficient for a proof of Anderson localization. We resolve this issue by modifying the definitions of the resonant blocks $\Omega_k^s \subset \widetilde{\Omega}_k^s \subset \mathbb{Z}^d$ and allowing

$$\text{diam } \Omega_k^s \sim (\text{diam } \widetilde{\Omega}_k^s)^\rho, \quad 0 < \rho < 1.$$

This sublinear bound is crucial for the proof of exponential off-diagonal decay. In the argument of Bourgain, it actually requires that $\rho = 1$. Another issue we want to highlight is that Bourgain just provided outputs by iterating the resolvent identity in many places of the paper [5] but did not present the details. This motivates us to write down the whole iteration arguments that are also important to the exponential decay estimate.

• To prove Anderson localization, one has to eliminate the energy $E \in \mathbb{R}$ that appears in Green's function estimates by removing θ or ω further. Moreover, if one wants to prove an arithmetic version of Anderson localization, a geometric description of resonances (i.e., the symmetry of zeros of certain functions appearing as the perturbations of quadratic polynomials in the present context) is essential. Precisely, at the s-th iteration step, using the Weierstrass preparation theorem, Bourgain [5] has shown the existence of zeros $\theta_{s,1}(E)$ and $\theta_{s,2}(E)$, but provided no symmetry information. Indeed, the symmetry property of $\theta_{s,1}(E)$ and $\theta_{s,2}(E)$ relies highly on that of resonant blocks $\widetilde{\Omega}_k^s$. However, in the construction of $\widetilde{\Omega}_k^s$ in [5], the symmetry property is missing. In this paper, we prove

$$\theta_{s,1}(E) + \theta_{s,2}(E) = 0.$$

The main idea is that we reconstruct $\widetilde{\Omega}_k^s$ so that it is symmetrical about k and allow the center $k \in \frac{1}{2}\mathbb{Z}^d$.

• In the construction of resonant blocks [5], the property that

$$\widetilde{\Omega}_{k'}^{s'} \cap \widetilde{\Omega}_k^s \neq \emptyset \Rightarrow \widetilde{\Omega}_{k'}^{s'} \subset \widetilde{\Omega}_k^s \quad \text{for } s' < s \tag{2.1}$$

plays a central role. In the 1D case, $\widetilde{\Omega}_k^s$ can be defined as an interval so that (2.1) holds. This interval structure of $\widetilde{\Omega}_k^s$ plays an important role in the usage of the resolvent identity. However, to generalize this argument to higher dimensions, one needs to give up the "interval" structure of $\widetilde{\Omega}_k^s$ in order to fulfill the property (2.1). As a result, the geometric description of $\widetilde{\Omega}_k^s$ becomes significantly complicated, and the estimates relying on the resolvent identity remain unclear. We address this issue by proving that $\widetilde{\Omega}_k^s$ can be constructed to satisfy (2.1) and stay in some enlarged cubes such as

$$\Lambda_{N_s^{c^2}} \subset \widetilde{\Omega}_k^s - k \subset \Lambda_{N_s^{c^2} + 50N_{s-1}^{c^2}}.$$

• We want to mention that in the estimates of zeros for some perturbations of quadratic polynomials, we use the standard Rouché theorem rather than the Weierstrass preparation theorem as in [5]. This technical modification avoids controlling the first-order derivatives of determinants and significantly simplifies the proof.

The proofs of Theorems 1.2 and 1.4 follow from the estimates in Theorem 1.1.

3 Quantitative Green's function estimates

It holds that the spectrum $\sigma(H(\theta)) \subset [-2, 2]$ since $\|H(\theta)\| \leqslant 1 + 2d\varepsilon < 2$ if $0 < \varepsilon < \frac{1}{2d}$. In this section, we fix

$$\theta \in \mathbb{T}, \quad E \in [-2, 2].$$

Write

$$E = \cos 2\pi\theta_0$$

with $\theta_0 \in \mathbb{C}$. Consider

$$T(E;\theta) = H(\theta) - E = D_n\delta_{n,n'} + \varepsilon\Delta, \tag{3.1}$$

where

$$D_n = \cos 2\pi(\theta + n\cdot\omega) - E. \tag{3.2}$$

For simplicity, we may omit the dependence of $T(E;\theta)$ on E and θ below.

We use a multi-scale analysis induction to provide estimates of Green's functions. Of particular importance is the analysis of resonances, which will be described by zeros of certain functions appearing as perturbations of some quadratic polynomials. Roughly speaking, at the s-th iteration step, the set $Q_s \subset \frac{1}{2}\mathbb{Z}^d$ of singular sites will be completely described by a pair of symmetric zeros of certain functions, i.e.,

$$Q_s = \bigcup_{\sigma=\pm1}\{k\in P_s : \|\theta + k\cdot\omega + \sigma\theta_s\| < \delta_s\}.$$

While Green's functions restricted to Q_s cannot generally be well controlled, the algebraic structure of Q_s combined with the non-resonance condition of ω may lead to the fine separation property of singular sites. As a result, one can cover Q_s with a new generation of resonant blocks $\widetilde{\Omega}_k^{s+1} (k\in P_{s+1})$. It turns out that one can control $\|T^{-1}_{\widetilde{\Omega}_k^{s+1}}\|$ via zeros $\pm\theta_{s+1}$ of some new functions which are also perturbations of quadratic polynomials in the sense that

$$\det T_{\widetilde{\Omega}_k^{s+1}} \sim \delta_s^{-2}\|\theta + k\cdot\omega - \theta_{s+1}\|\cdot\|\theta + k\cdot\omega + \theta_{s+1}\|.$$

The key point is that while $\widetilde{\Omega}_k^{s+1}$ intersects Q_s some $T^{-1}_{\widetilde{\Omega}_k^{s+1}}$ becomes controllable[5)] at the $(s+1)$-th step. Moreover, the completely uncontrollable singular sites form the $(s+1)$-th singular sites, i.e.,

$$Q_{s+1} = \bigcup_{\sigma=\pm1}\{k\in P_{s+1} : \|\theta + k\cdot\omega + \sigma\theta_{s+1}\| < \delta_{s+1}\}.$$

Now, we turn to the statement of our main result on multi-scale-type Green's function estimates. Define the induction parameters as follows:

$$N_{s+1} = \left[\left|\log\frac{\gamma}{\delta_s}\right|^{\frac{1}{c^5\tau}}\right],\quad \left|\log\frac{\gamma}{\delta_{s+1}}\right| = \left|\log\frac{\gamma}{\delta_s}\right|^{c^5}.$$

Thus

$$N_s^{c^5} - 1 \leqslant N_{s+1} \leqslant (N_s+1)^{c^5}.$$

We first introduce the following statement.

Statement 3.1 ($\mathcal{P}_s$ $(s\geqslant 1)$)**.** Let

$$Q^{\pm}_{s-1} = \{k\in P_{s-1} : \|\theta + k\cdot\omega \pm \theta_{s-1}\| < \delta_{s-1}\},\quad Q_{s-1} = Q^+_{s-1}\cup Q^-_{s-1}, \tag{3.3}$$

$$\widetilde{Q}^{\pm}_{s-1} = \{k\in P_{s-1} : \|\theta + k\cdot\omega \pm \theta_{s-1}\| < \delta_{s-1}^{\frac{1}{100}}\},\quad \widetilde{Q}_{s-1} = \widetilde{Q}^+_{s-1}\cup \widetilde{Q}^-_{s-1}. \tag{3.4}$$

We distinguish the following two cases:

$$(\mathbf{C}1)_{s-1}\quad \mathrm{dist}(\widetilde{Q}^-_{s-1}, Q^+_{s-1}) > 100N_s^c \tag{3.5}$$

and

$$(\mathbf{C}2)_{s-1}\quad \mathrm{dist}(\widetilde{Q}^-_{s-1}, Q^+_{s-1}) \leqslant 100N_s^c. \tag{3.6}$$

Let

$$\mathbb{Z}^d \ni l_{s-1} = \begin{cases} 0 & \text{if (3.5) holds,}\\ i_{s-1} - j_{s-1} & \text{if (3.6) holds,}\end{cases}$$

5) Even more general sets, e.g., the $(s+1)$-**good** sets remain true.

where $i_{s-1}\in Q_{s-1}^{+}$ and $j_{s-1}\in \widetilde{Q}_{s-1}^{-}$ such that $\|i_{s-1}-j_{s-1}\|\leqslant 100N_s^c$ in $(\mathbf{C}2)_{s-1}$. Set $\Omega_k^0=\{k\}$ $(k\in\mathbb{Z}^d)$. Let $\Lambda\subset\mathbb{Z}^d$ be a finite set. We say Λ is $(s-1)$-**good** if and only if

$$\begin{cases} k'\in Q_{s'}, \quad \widetilde{\Omega}_{k'}^{s'}\subset\Lambda, \quad \widetilde{\Omega}_{k'}^{s'}\subset\Omega_k^{s'+1}\Rightarrow\widetilde{\Omega}_k^{s'+1}\subset\Lambda \quad \text{for } s'<s-1,\\ \{k\in P_{s-1}:\widetilde{\Omega}_k^{s-1}\subset\Lambda\}\cap Q_{s-1}=\emptyset. \end{cases} \tag{3.7}$$

$(\mathbf{a})_s$ There is $P_s\subset\frac{1}{2}\mathbb{Z}^d$ so that the following holds. In the case $(\mathbf{C}1)_{s-1}$, we have

$$P_s=Q_{s-1}\subset\left\{k\in\mathbb{Z}^d+\frac{1}{2}\sum_{i=0}^{s-1}l_i:\min_{\sigma=\pm1}\|\theta+k\cdot\omega+\sigma\theta_{s-1}\|<\delta_{s-1}\right\}. \tag{3.8}$$

For the case $(\mathbf{C}2)_{s-1}$, we have

$$\begin{aligned} &P_s\subset\left\{k\in\mathbb{Z}^d+\frac{1}{2}\sum_{i=0}^{s-1}l_i:\|\theta+k\cdot\omega\|<3\delta_{s-1}^{\frac{1}{100}}\right\},\\ &\text{or } P_s\subset\left\{k\in\mathbb{Z}^d+\frac{1}{2}\sum_{i=0}^{s-1}l_i:\|\theta+k\cdot\omega+\frac{1}{2}\|<3\delta_{s-1}^{\frac{1}{100}}\right\}. \end{aligned} \tag{3.9}$$

For every $k\in P_s$, we can find resonant blocks $\Omega_k^s,\widetilde{\Omega}_k^s\subset\mathbb{Z}^d$ with the following properties. If (3.5) holds, then

$$\Lambda_{N_s}(k)\subset\Omega_k^s\subset\Lambda_{N_s+50N_{s-1}^{c^2}}(k),$$
$$\Lambda_{N_s^c}(k)\subset\widetilde{\Omega}_k^s\subset\Lambda_{N_s^c+50N_{s-1}^{c^2}}(k),$$

and if (3.6) holds, then

$$\Lambda_{100N_s^c}(k)\subset\Omega_k^s\subset\Lambda_{100N_s^c+50N_{s-1}^{c^2}}(k),$$
$$\Lambda_{N_s^{c^2}}(k)\subset\widetilde{\Omega}_k^s\subset\Lambda_{N_s^{c^2}+50N_{s-1}^{c^2}}(k).$$

These resonant blocks are constructed to satisfy the following two properties:

$(\mathbf{a}1)_s$
$$\begin{cases} \Omega_k^s\cap\widetilde{\Omega}_{k'}^{s'}\neq\emptyset\ (s'<s)\Rightarrow\widetilde{\Omega}_{k'}^{s'}\subset\Omega_k^s,\\ \widetilde{\Omega}_k^s\cap\widetilde{\Omega}_{k'}^{s'}\neq\emptyset\ (s'<s)\Rightarrow\widetilde{\Omega}_{k'}^{s'}\subset\widetilde{\Omega}_k^s,\\ \operatorname{dist}(\widetilde{\Omega}_k^s,\widetilde{\Omega}_{k'}^s)>10\operatorname{diam}\widetilde{\Omega}_k^s \quad \text{for } k\neq k'\in P_s. \end{cases} \tag{3.10}$$

$(\mathbf{a}2)_s$ The translation of $\widetilde{\Omega}_k^s$,

$$\widetilde{\Omega}_k^s-k\subset\mathbb{Z}^d+\frac{1}{2}\sum_{i=0}^{s-1}l_i,$$

is independent of $k\in P_s$ and symmetrical about the origin.

$(\mathbf{b})_s$ Q_{s-1} is covered by Ω_k^s $(k\in P_s)$ in the sense that for every $k'\in Q_{s-1}$, there exists a $k\in P_s$ such that

$$\widetilde{\Omega}_{k'}^{s-1}\subset\Omega_k^s. \tag{3.11}$$

$(\mathbf{c})_s$ For each $k\in P_s$, $\widetilde{\Omega}_k^s$ contains a subset $A_k^s\subset\Omega_k^s$ with $\#A_k^s\leqslant 2^s$ such that $\widetilde{\Omega}_k^s\setminus A_k^s$ is $(s-1)$-**good**. Moreover, A_k^s-k is independent of k and is symmetrical about the origin.

$(\mathbf{d})_s$ There is a $\theta_s=\theta_s(E)\in\mathbb{C}$ with the following properties. Replacing $\theta+n\cdot\omega$ by $z+(n-k)\cdot\omega$ and restricting z in

$$\left\{z\in\mathbb{C}:\min_{\sigma=\pm1}\|z+\sigma\theta_s\|<\delta_s^{\frac{1}{10^4}}\right\}, \tag{3.12}$$

we see that $T_{\widetilde{\Omega}_k^s}$ becomes

$$M_s(z)=T(z)_{\widetilde{\Omega}_k^s-k}=(\cos 2\pi(z+n\cdot\omega)\delta_{n,n'}-E+\varepsilon\Delta)_{\widetilde{\Omega}_k^s-k}.$$

Then, $M_s(z)_{(\widetilde{\Omega}_k^s-k)\setminus(A_k^s-k)}$ is invertible and we can define the Schur complement

$$S_s(z) = M_s(z)_{A_k^s-k} - R_{A_k^s-k}M_s(z)R_{(\widetilde{\Omega}_k^s-k)\setminus(A_k^s-k)}(M_s(z)_{(\widetilde{\Omega}_k^s-k)\setminus(A_k^s-k)})^{-1}$$
$$\times R_{(\widetilde{\Omega}_k^s-k)\setminus(A_k^s-k)}M_s(z)R_{A_k^s-k}.$$

Moreover, if z belongs to the set defined by (3.12), then we have

$$\max_x \sum_y |S_s(z)(x,y)| < 4 + \sum_{l=0}^{s-1}\delta_l < 10 \tag{3.13}$$

and

$$\det S_s(z) \overset{\delta_{s-1}}{\sim} \|z-\theta_s\|\cdot\|z+\theta_s\|. \tag{3.14}$$

$(\mathbf{e})_s$ We say a finite set $\Lambda \subset \mathbb{Z}^d$ is s-**good** if and only if

$$\begin{cases} k' \in Q_{s'}, \quad \widetilde{\Omega}_{k'}^{s'} \subset \Lambda, \quad \widetilde{\Omega}_{k'}^{s'} \subset \Omega_k^{s'+1} \Rightarrow \widetilde{\Omega}_k^{s'+1} \subset \Lambda \quad \text{for } s' < s,\\ \{k \in P_s : \widetilde{\Omega}_k^s \subset \Lambda\} \cap Q_s = \emptyset. \end{cases} \tag{3.15}$$

Assume that Λ is s-**good**. Then,

$$\|T_\Lambda^{-1}\| < \delta_{s-1}^{-3} \sup_{\{k\in P_s:\widetilde{\Omega}_k^s\subset\Lambda\}} \|\theta + k\cdot\omega - \theta_s\|^{-1}\cdot\|\theta+k\cdot\omega+\theta_s\|^{-1} < \delta_s^{-3}, \tag{3.16}$$

$$|T_\Lambda^{-1}(x,y)| < \mathrm{e}^{-\gamma_s\|x-y\|_1} \quad \text{for } \|x-y\| > N_s^{c^3}, \tag{3.17}$$

where

$$\gamma_0 = \frac{1}{2}|\log\varepsilon|, \quad \gamma_s = \gamma_{s-1}(1-N_s^{\frac{1}{c}-1})^3.$$

Thus,

$$\gamma_s \searrow \gamma_\infty \geqslant \frac{1}{2}\gamma_0 = \frac{1}{4}|\log\varepsilon|.$$

$(\mathbf{f})_s$ We have

$$\left\{k \in \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s-1} l_i : \min_{\sigma=\pm1}\|\theta+k\cdot\omega+\sigma\theta_s\| < 10\delta_s^{\frac{1}{100}}\right\} \subset P_s. \tag{3.18}$$

The main theorem of this section is as follows.

Theorem 3.2. *Let $\omega \in \mathcal{R}_{\tau,\gamma}$. Then, there is some $\varepsilon_0(d,\tau,\gamma) > 0$ so that for $0 < \varepsilon \leqslant \varepsilon_0$, the statement $\mathcal{P}_s$ holds for all $s \geqslant 1$.*

The following three subsections are devoted to the proof of Theorem 3.2.

3.1 The initial step

Recalling (3.1)–(3.2) and $\cos 2\pi\theta_0 = E$, we have

$$|D_n| = 2|\sin\pi(\theta+n\cdot\omega+\theta_0)\sin\pi(\theta+n\cdot\omega-\theta_0)|$$
$$\geqslant 2\|\theta+n\cdot\omega+\theta_0\|\cdot\|\theta+n\cdot\omega-\theta_0\|.$$

Define $\delta_0 = \varepsilon^{1/10}$ and

$$P_0 = \mathbb{Z}^d, \quad Q_0 = \{k \in P_0 : \min(\|\theta+k\cdot\omega+\theta_0\|, \|\theta+k\cdot\omega-\theta_0\|) < \delta_0\}.$$

We say a finite set $\Lambda \subset \mathbb{Z}^d$ is 0-**good** if and only if

$$\Lambda \cap Q_0 = \emptyset.$$

Lemma 3.3. *If the finite set $\Lambda \subset \mathbb{Z}^d$ is* 0-**good***, then*

$$\|T_\Lambda^{-1}\| < 2\|D_\Lambda^{-1}\| < \delta_0^{-2}, \tag{3.19}$$

$$|T_\Lambda^{-1}(x,y)| < \mathrm{e}^{-\gamma_0\|x-y\|_1} \quad \text{for } \|x-y\| > 0, \tag{3.20}$$

where $\gamma_0 = 5|\log \delta_0| = \frac{1}{2}|\log \varepsilon|$.

Proof. Assuming Λ is 0-**good**, we have

$$\|D_\Lambda^{-1}\| < \frac{1}{2}\delta_0^{-2}, \quad \|\varepsilon D_\Lambda^{-1}\Delta_\Lambda\| < d\varepsilon\delta_0^{-2} < \frac{1}{2}\delta_0^7 < \frac{1}{2}.$$

Thus,

$$T_\Lambda^{-1} = (I + \varepsilon D_\Lambda^{-1}\Delta_\Lambda)^{-1} D_\Lambda^{-1}$$

and $(I + \varepsilon D_\Lambda^{-1}\Delta_\Lambda)^{-1}$ may be expanded in the Neumann series

$$(I + \varepsilon D_\Lambda^{-1}\Delta_\Lambda)^{-1} = \sum_{i=0}^{+\infty}(-\varepsilon D_\Lambda^{-1}\Delta_\Lambda)^i.$$

Hence,

$$\|T_\Lambda^{-1}\| < 2\|D_\Lambda^{-1}\| < \delta_0^{-2},$$

which implies (3.19).

In addition, if $\|x-y\|_1 > i$, then

$$((\varepsilon D_\Lambda^{-1}\Delta_\Lambda)^i D_\Lambda^{-1})(x,y) = 0.$$

Hence,

$$|T_\Lambda^{-1}(x,y)| = \left|\sum_{i \geqslant \|x-y\|_1}((\varepsilon D_\Lambda^{-1}\Delta_\Lambda)^i D_\Lambda^{-1})(x,y)\right| < \delta_0^{7\|x-y\|_1 - 2}.$$

In particular,

$$|T_\Lambda^{-1}(x,y)| < \mathrm{e}^{-\gamma_0\|x-y\|_1} \quad \text{for } \|x-y\| > 0$$

with $\gamma_0 = 5|\log \delta_0| = \frac{1}{2}|\log \varepsilon|$, which yields (3.20). □

3.2 Verification of $\mathcal{P}_1$

If $\Lambda \cap Q_0 \neq \emptyset$, then the Neumann series argument of the previous subsection does not work. Thus we use the resolvent identity argument to estimate T_Λ^{-1}, where Λ is 1-**good** (1-**good** will be specified later) but might intersect with Q_0 (not 0-**good**).

Firstly, we construct blocks Ω_k^1 ($k \in P_1$) to cover the singular point Q_0. Secondly, we get the bound estimate

$$\|T_{\widetilde{\Omega}_k^1}^{-1}\| < \delta_0^{-2}\|\theta + k\cdot\omega - \theta_1\|^{-1} \cdot \|\theta + k\cdot\omega + \theta_1\|^{-1},$$

where $\widetilde{\Omega}_k^1$ is an extension of Ω_k^1, and θ_1 is obtained by analyzing the root of the equation $\det T(z - k\cdot\omega)_{\widetilde{\Omega}_k^1} = 0$ about z. Finally, we combine the estimate of $T_{\widetilde{\Omega}_k^1}^{-1}$ to get that of T_Λ^{-1} by the resolvent identity assuming that Λ is 1-**good**.

Recall that

$$1 < c^{20} < \frac{1}{\tau}.$$

Let

$$N_1 = \left[\left|\log\frac{\gamma}{\delta_0}\right|^{\frac{1}{c^5\tau}}\right].$$

Define

$$Q_0^{\pm} = \{k \in \mathbb{Z}^d : \|\theta + k \cdot \omega \pm \theta_0\| < \delta_0\}, \quad Q_0 = Q_0^+ \cup Q_0^-,$$
$$\widetilde{Q}_0^{\pm} = \{k \in \mathbb{Z}^d : \|\theta + k \cdot \omega \pm \theta_0\| < \delta_0^{\frac{1}{100}}\}, \quad \widetilde{Q}_0 = \widetilde{Q}_0^+ \cup \widetilde{Q}_0^-.$$

The proof can be decomposed into three steps.

Step 1. The case $(\mathbf{C}1)_0$ occurs, i.e.,

$$\operatorname{dist}(\widetilde{Q}_0^-, Q_0^+) > 100N_1^c. \tag{3.21}$$

Remark 3.4. We have

$$\operatorname{dist}(\widetilde{Q}_0^-, Q_0^+) = \operatorname{dist}(\widetilde{Q}_0^+, Q_0^-).$$

Thus (3.21) also implies

$$\operatorname{dist}(\widetilde{Q}_0^+, Q_0^-) > 100N_1^c.$$

We refer to Appendix A for a detailed proof.

Assuming (3.21), we define

$$P_1 = Q_0 = \{k \in \mathbb{Z}^d : \min(\|\theta + k \cdot \omega + \theta_0\|, \|\theta + k \cdot \omega - \theta_0\|) < \delta_0\}. \tag{3.22}$$

Associate every $k \in P_1$ with an N_1-block $\Omega_k^1 := \Lambda_{N_1}(k)$ and an N_1^c-block $\widetilde{\Omega}_k^1 := \Lambda_{N_1^c}(k)$. Then, $\widetilde{\Omega}_k^1 - k \subset \mathbb{Z}^d$ is independent of $k \in P_1$ and symmetrical about the origin. If $k \neq k' \in P_1$,

$$\|k - k'\| \geqslant \min\left(100N_1^c, \left|\log \frac{\gamma}{2\delta_0}\right|^{\frac{1}{\tau}}\right) \geqslant 100N_1^c.$$

Thus

$$\operatorname{dist}(\widetilde{\Omega}_k^1, \widetilde{\Omega}_{k'}^1) > 10 \operatorname{diam} \widetilde{\Omega}_k^1 \quad \text{for } k \neq k' \in P_1.$$

For $k \in Q_0^-$, we consider

$$M_1(z) := T(z)_{\widetilde{\Omega}_k^1 - k} = (\cos 2\pi(z + n \cdot \omega)\delta_{n,n'} - E + \varepsilon\Delta)_{n \in \widetilde{\Omega}_k^1 - k}$$

defined in

$$\{z \in \mathbb{C} : |z - \theta_0| < \delta_0^{\frac{1}{10}}\}. \tag{3.23}$$

For $n \in (\widetilde{\Omega}_k^1 - k) \setminus \{0\}$, we have that for $0 < \delta_0 \ll 1$,

$$\begin{aligned}
\|z + n \cdot \omega - \theta_0\| &\geqslant \|n \cdot \omega\| - |z - \theta_0| \\
&\geqslant \gamma \mathrm{e}^{-N_1^{c\tau}} - \delta_0^{\frac{1}{10}} \\
&\geqslant \gamma \mathrm{e}^{-|\log \frac{\gamma}{\delta_0}|^{\frac{1}{c^4}}} - \delta_0^{\frac{1}{10}} \\
&> \delta_0^{\frac{1}{10^4}}.
\end{aligned}$$

For $n \in \widetilde{\Omega}_k^1 - k$, we have

$$\begin{aligned}
\|z + n \cdot \omega + \theta_0\| &\geqslant \|\theta + (n + k) \cdot \omega + \theta_0\| - |z - \theta_0| - \|\theta + k \cdot \omega - \theta_0\| \\
&\geqslant \delta_0^{\frac{1}{100}} - \delta_0^{\frac{1}{10}} - \delta_0 \\
&> \frac{1}{2}\delta_0^{\frac{1}{100}}.
\end{aligned}$$

Since $\delta_0 \gg \varepsilon$, by the Neumann series argument, we have

$$\|(M_1(z)_{(\widetilde{\Omega}_k^1 - k) \setminus \{0\}})^{-1}\| < 3\delta_0^{-\frac{1}{50}}.$$

Now we can apply the Schur complement lemma (see Lemma B.1 in the appendix) to provide desired estimates. By Lemma B.1, $M_1(z)^{-1}$ is controlled by the inverse of the Schur complement (of $(\widetilde{\Omega}_k^1-k)\backslash\{0\}$), i.e.,

$$\begin{aligned} S_1(z) &= M_1(z)_{\{0\}} - R_{\{0\}}M_1(z)R_{(\widetilde{\Omega}_k^1-k)\backslash\{0\}}(M_1(z)_{(\widetilde{\Omega}_k^1-k)\backslash\{0\}})^{-1}R_{(\widetilde{\Omega}_k^1-k)\backslash\{0\}}M_1(z)R_{\{0\}} \\ &= -2\sin\pi(z-\theta_0)\sin\pi(z+\theta_0)+r(z) \\ &= g(z)((z-\theta_0)+r_1(z)), \end{aligned}$$

where $g(z)$ and $r_1(z)$ are analytic functions in the set defined by (3.23), satisfying $|g(z)|\geqslant 2\|z+\theta_0\|>\delta_0^{\frac{1}{100}}$ and $|r_1(z)|<\varepsilon^2\delta_0^{-1}<\varepsilon$. Since

$$|r_1(z)|<|z-\theta_0| \quad \text{for } |z-\theta_0|=\delta_0^{\frac{1}{10}},$$

using the Rouché theorem implies that the equation

$$(z-\theta_0)+r_1(z)=0$$

has a unique root θ_1 in the set (3.23), which satisfies

$$|\theta_0-\theta_1|=|r_1(\theta_1)|<\varepsilon, \quad |(z-\theta_0)+r_1(z)|\sim|z-\theta_1|.$$

Moreover, θ_1 is the unique root of $\det M_1(z)=0$ in the set (3.23). Since $\|z+\theta_0\|>\frac{1}{2}\delta_0^{\frac{1}{100}}$ and $|\theta_0-\theta_1|<\varepsilon$, we get

$$\|z+\theta_1\|\sim\|z+\theta_0\|,$$

which shows for z being in the set (3.23) that

$$|S_1(z)|\sim\|z+\theta_1\|\cdot\|z-\theta_1\|, \tag{3.24}$$

$$\begin{aligned} \|M_1(z)^{-1}\| &< 4(1+\|(M_1(z)_{(\widetilde{\Omega}_k^1-k)\backslash\{0\}})^{-1}\|)^2(1+|S_1(z)|^{-1}) \\ &< \delta_0^{-2}\|z+\theta_1\|^{-1}\cdot\|z-\theta_1\|^{-1}, \end{aligned} \tag{3.25}$$

where in the first inequality we use Lemma B.1. Now, for $k\in Q_0^+$, we consider $M_1(z)$ in

$$\{z\in\mathbb{C}:|z+\theta_0|<\delta_0^{\frac{1}{10}}\}. \tag{3.26}$$

A similar argument shows that $\det M_1(z)=0$ has a unique root θ_1' in the set (3.26). We show $\theta_1+\theta_1'=0$. By Lemma C.1, $\det M_1(z)$ is an even function of z. Then, the uniqueness of the root implies $\theta_1'=-\theta_1$. Thus for z being in the set (3.26), both (3.24) and (3.25) hold as well. Finally, since $M_1(z)$ is 1-periodic, (3.24) and (3.25) remain valid for

$$z\in\left\{z\in\mathbb{C}:\min_{\sigma=\pm1}\|z+\sigma\theta_0\|<\delta_0^{\frac{1}{10}}\right\}. \tag{3.27}$$

From (3.22), we have that $\theta+k\cdot\omega$ belongs to the set in (3.27). Thus, for $k\in P_1$, we get

$$\begin{aligned} \|T_{\widetilde{\Omega}_k^1}^{-1}\| &= \|M_1(\theta+k\cdot\omega)^{-1}\| \\ &< \delta_0^{-2}\|\theta+k\cdot\omega-\theta_1\|^{-1}\cdot\|\theta+k\cdot\omega+\theta_1\|^{-1}. \end{aligned} \tag{3.28}$$

Step 2. The case $(\mathbf{C}2)_0$ occurs, i.e.,

$$\operatorname{dist}(\widetilde{Q}_0^-,Q_0^+)\leqslant 100N_1^c.$$

Then, there exist $i_0\in Q_0^+$ and $j_0\in\widetilde{Q}_0^-$ with $\|i_0-j_0\|\leqslant 100N_1^c$ such that

$$\|\theta+i_0\cdot\omega+\theta_0\|<\delta_0, \quad \|\theta+j_0\cdot\omega-\theta_0\|<\delta_0^{\frac{1}{100}}.$$

Set $l_0 = i_0 - j_0$. Then,

$$\|l_0\| = \operatorname{dist}(Q_0^+, \widetilde{Q}_0^-) = \operatorname{dist}(\widetilde{Q}_0^+, Q_0^-).$$

Define

$$O_1 = Q_0^- \cup (Q_0^+ - l_0).$$

For $k \in Q_0^+$, we have

$$\begin{aligned}\|\theta + (k - l_0)\cdot\omega - \theta_0\| &< \|\theta + k\cdot\omega + \theta_0\| + \|l_0\cdot\omega + 2\theta_0\| \\ &< \delta_0 + \delta_0 + \delta_0^{\frac{1}{100}} < 2\delta_0^{\frac{1}{100}}.\end{aligned}$$

Thus,

$$O_1 \subset \{o \in \mathbb{Z}^d : \|\theta + o\cdot\omega - \theta_0\| < 2\delta_0^{\frac{1}{100}}\}.$$

For every $o \in O_1$, define its mirror point

$$o^* = o + l_0.$$

Next, define

$$P_1 = \left\{\frac{1}{2}(o + o^*) : o \in O_1\right\} = \left\{o + \frac{l_0}{2} : o \in O_1\right\}. \tag{3.29}$$

Associate every $k \in P_1$ with a $100N_1^c$-block $\Omega_k^1 := \Lambda_{100N_1^c}(k)$ and an $N_1^{c^2}$-block $\widetilde{\Omega}_k^1 := \Lambda_{N_1^{c^2}}(k)$. Thus,

$$Q_0 \subset \bigcup_{k\in P_1} \Omega_k^1,$$

and $\widetilde{\Omega}_k^1 - k \subset \mathbb{Z}^d + \frac{l_0}{2}$ is independent of $k \in P_1$ and symmetrical about the origin. Notice that

$$\begin{aligned}&\min\left(\left\|\frac{l_0}{2}\cdot\omega + \theta_0\right\|, \left\|\frac{l_0}{2}\cdot\omega + \theta_0 - \frac{1}{2}\right\|\right) \\ &= \frac{1}{2}\|l_0\cdot\omega + 2\theta_0\| \\ &\leqslant \frac{1}{2}(\|\theta + i_0\cdot\omega + \theta_0\| + \|\theta + j_0\cdot\omega - \theta_0\|) < \delta_0^{\frac{1}{100}}.\end{aligned}$$

Since $\delta_0 \ll 1$, only one of

$$\left\|\frac{l_0}{2}\cdot\omega + \theta_0\right\| < \delta_0^{\frac{1}{100}} \quad \text{and} \quad \left\|\frac{l_0}{2}\cdot\omega + \theta_0 - \frac{1}{2}\right\| < \delta_0^{\frac{1}{100}}$$

holds. Firstly, we consider the case

$$\left\|\frac{l_0}{2}\cdot\omega + \theta_0\right\| < \delta_0^{\frac{1}{100}}. \tag{3.30}$$

Let $k \in P_1$. Since $k = \frac{1}{2}(o + o^*) = (o + \frac{l_0}{2})$ (for some $o \in O_1$), we have

$$\|\theta + k\cdot\omega\| \leqslant \|\theta + o\cdot\omega - \theta_0\| + \left\|\frac{l_0}{2}\cdot\omega + \theta_0\right\| < 3\delta_0^{\frac{1}{100}}. \tag{3.31}$$

Thus if $k \neq k' \in P_1$, we obtain

$$\|k - k'\| \geqslant \left|\log\frac{\gamma}{6\delta_0^{\frac{1}{100}}}\right|^{\frac{1}{\tau}} \sim N_1^{c^5} \gg 10N_1^{c^2},$$

which implies

$$\operatorname{dist}(\widetilde{\Omega}_k^1, \widetilde{\Omega}_{k'}^1) > 10\operatorname{diam}\widetilde{\Omega}_k^1 \quad \text{for } k \neq k' \in P_1.$$

Consider

$$M_1(z) := T(z)_{\widetilde{\Omega}_k^1 - k} = (\cos 2\pi(z + n\cdot\omega)\delta_{n,n'} - E + \varepsilon\Delta)_{n\in\widetilde{\Omega}_k^1 - k}$$

in

$$\{z \in \mathbb{C} : |z| < \delta_0^{\frac{1}{10^3}}\}. \tag{3.32}$$

For $n \neq \pm \frac{l_0}{2}$ and $n \in \widetilde{\Omega}_k^1 - k$, we have

$$\begin{aligned} \|n \cdot \omega \pm \theta_0\| &\geqslant \left\| \left(n \mp \frac{l_0}{2} \right) \cdot \omega \right\| - \left\| \frac{l_0}{2} \omega + \theta_0 \right\| \\ &> \gamma \mathrm{e}^{-(2N_1^{c^2})^\tau} - \delta_0^{\frac{1}{100}} > 2\delta_0^{\frac{1}{10^4}}. \end{aligned}$$

Thus for z being in the set (3.32) and $n \neq \pm \frac{l_0}{2}$, we have

$$\|z + n \cdot \omega \pm \theta_0\| \geqslant \|n \cdot \omega \pm \theta_0\| - |z| > \delta_0^{\frac{1}{10^4}}.$$

Hence,

$$|\cos 2\pi(z + n \cdot \omega) - E| \geqslant \delta_0^{2 \times \frac{1}{10^4}} \gg \varepsilon.$$

Using the Neumann series argument, we conclude that

$$\|(M_1(z)_{(\widetilde{\Omega}_k^1 - k) \setminus \{\pm \frac{l_0}{2}\}})^{-1}\| < \delta_0^{-3 \times \frac{1}{10^4}}. \tag{3.33}$$

Thus by Lemma B.1, $M_1(z)^{-1}$ is controlled by the inverse of the Schur complement of $(\widetilde{\Omega}_k^1 - k) \setminus \{\pm \frac{l_0}{2}\}$, i.e.,

$$\begin{aligned} S_1(z) = M_1(z)_{\{\pm \frac{l_0}{2}\}} &- R_{\{\pm \frac{l_0}{2}\}} M_1(z) R_{(\widetilde{\Omega}_k^1 - k) \setminus \{\pm \frac{l_0}{2}\}} \\ &\times (M_1(z)_{(\widetilde{\Omega}_k^1 - k) \setminus \{\pm \frac{l_0}{2}\}})^{-1} R_{(\widetilde{\Omega}_k^1 - k) \setminus \{\pm \frac{l_0}{2}\}} M_1(z) R_{\{\pm \frac{l_0}{2}\}}. \end{aligned}$$

Clearly,

$$\begin{aligned} \det S_1(z) &= \det(M_1(z)_{\{\pm \frac{l_0}{2}\}}) + O(\varepsilon^2 \delta_0^{-\frac{3}{10^4}}) \\ &= 4 \sin \pi \left(z + \frac{l_0}{2} \cdot \omega - \theta_0 \right) \sin \pi \left(z + \frac{l_0}{2} \cdot \omega + \theta_0 \right) \\ &\quad \times \sin \pi \left(z - \frac{l_0}{2} \cdot \omega - \theta_0 \right) \sin \pi \left(z - \frac{l_0}{2} \cdot \omega + \theta_0 \right) + O(\varepsilon^{1.5}). \end{aligned}$$

If $l_0 = 0$, then

$$\det S_1(z) = -2 \sin \pi(z - \theta_0) \sin \pi(z + \theta_0) + O(\varepsilon^{1.5}).$$

In this case, the argument is easier, and we omit the discussion. In the following, we deal with $l_0 \neq 0$. By (3.30) and (3.32), we have

$$\begin{aligned} \left\| z + \frac{l_0}{2} \cdot \omega - \theta_0 \right\| &\geqslant \|l_0 \cdot \omega\| - \left\| \frac{l_0}{2} \cdot \omega + \theta_0 \right\| - |z| \\ &> \gamma \mathrm{e}^{-(100N_1^c)^\tau} - \delta_0^{\frac{1}{100}} - \delta_0^{\frac{1}{10^3}} \\ &> \delta_0^{\frac{1}{10^4}} \end{aligned}$$

and

$$\begin{aligned} \left\| z - \frac{l_0}{2} \cdot \omega + \theta_0 \right\| &\geqslant \|l_0 \cdot \omega\| - \left\| \frac{l_0}{2} \cdot \omega + \theta_0 \right\| - |z| \\ &> \gamma \mathrm{e}^{-(100N_1^c)^\tau} - \delta_0^{\frac{1}{100}} - \delta_0^{\frac{1}{10^3}} \\ &> \delta_0^{\frac{1}{10^4}}. \end{aligned}$$

Let z_1 satisfy

$$z_1 \equiv \frac{l_0}{2} \cdot \omega + \theta_0 \ (\mathrm{mod}\ \mathbb{Z}), \quad |z_1| = \left\| \frac{l_0}{2} \cdot \omega + \theta_0 \right\| < \delta_0^{\frac{1}{100}}.$$

Then,

$$\begin{aligned} \det S_1(z) &\sim \left\| z + \frac{l_0}{2} \cdot \omega - \theta_0 \right\| \cdot \left\| z - \frac{l_0}{2} \cdot \omega + \theta_0 \right\| \cdot |(z - z_1)(z + z_1) + r_1(z)| \\ &\overset{\delta_0^{\frac{2}{10^4}}}{\sim} |(z - z_1)(z + z_1) + r_1(z)|, \end{aligned}$$

where $r_1(z)$ is an analytic function in the set (3.32) with $|r_1(z)| < \varepsilon \ll \delta_0^{\frac{1}{10^3}}$. Applying the Rouché theorem shows that the equation

$$(z - z_1)(z + z_1) + r_1(z) = 0$$

has exact two roots θ_1 and θ_1' in the set (3.32), which are perturbations of $\pm z_1$. Notice that

$$\{|z| < \delta_0^{\frac{1}{10^3}} : \det M_1(z) = 0\} = \{|z| < \delta_0^{\frac{1}{10^3}} : \det S_1(z) = 0\},$$

and $\det M_1(z)$ is an even function (see Lemma C.1) of z. Thus,

$$\theta_1' = -\theta_1.$$

Moreover, we have

$$|\theta_1 - z_1| \leqslant |r_1(\theta_1)|^{\frac{1}{2}} < \varepsilon^{\frac{1}{2}}, \quad |(z - z_1)(z + z_1) + r_1(z)| \sim |(z - \theta_1)(z + \theta_1)|.$$

Thus for z being in the set (3.32), we have

$$\det S_1(z) \overset{\delta_0}{\sim} \|z - \theta_1\| \cdot \|z + \theta_1\|, \tag{3.34}$$

which implies

$$\|S_1(z)^{-1}\| \leqslant C\delta_0^{-1} \|z - \theta_1\|^{-1} \cdot \|z + \theta_1\|^{-1}.$$

Recalling (3.33), by Lemma B.1, we get

$$\begin{aligned} \|M_1(z)^{-1}\| &< 4(1 + \|(M_1(z)_{(\widetilde{\Omega}_k^1 - k)\setminus\{0\}})^{-1}\|)^2(1 + \|S_1(z)^{-1}\|) \\ &< \delta_0^{-2} \|z + \theta_1\|^{-1} \cdot \|z - \theta_1\|^{-1}. \end{aligned} \tag{3.35}$$

Thus for (3.30), both (3.34) and (3.35) are established for z belonging to

$$\{z \in \mathbb{C} : \|z\| < \delta_0^{\frac{1}{10^3}}\}$$

since $M_1(z)$ is 1-periodic (in z). By (3.31), for $k \in P_1$, we also have

$$\begin{aligned} \|T_{\widetilde{\Omega}_k^1}^{-1}\| &= \|M_1(\theta + k \cdot \omega)^{-1}\| \\ &< \delta_0^{-2} \|\theta + k \cdot \omega - \theta_1\|^{-1} \cdot \|\theta + k \cdot \omega + \theta_1\|^{-1}. \end{aligned} \tag{3.36}$$

For the case

$$\left\| \frac{l_0}{2} \cdot \omega + \theta_0 - \frac{1}{2} \right\| < \delta_0^{\frac{1}{100}}, \tag{3.37}$$

we have that for $k \in P_1$,

$$\left\| \theta + k \cdot \omega - \frac{1}{2} \right\| < 3\delta_0^{\frac{1}{100}}. \tag{3.38}$$

Consider

$$M_1(z) := T(z)_{\widetilde{\Omega}_k^1 - k} = (\cos 2\pi(z + n \cdot \omega)\delta_{n,n'} - E + \varepsilon\Delta)_{n \in \widetilde{\Omega}_k^1 - k}$$

in

$$\left\{z\in\mathbb{C}:\left|z-\frac{1}{2}\right|<\delta_0^{\frac{1}{10^3}}\right\}. \tag{3.39}$$

By a similar argument as above, we get

$$\|(M_1(z)_{(\widetilde{\Omega}_k^1-k)\setminus\{\pm\frac{l_0}{2}\}})^{-1}\|<\delta_0^{-3\times\frac{1}{10^4}}.$$

Thus, $M_1(z)^{-1}$ is controlled by the inverse of the Schur complement of $(\widetilde{\Omega}_k^1-k)\setminus\{\pm\frac{l_0}{2}\}$:

$$\begin{aligned}S_1(z)&=M_1(z)_{\{\pm\frac{l_0}{2}\}}-R_{\{\pm\frac{l_0}{2}\}}M_1(z)R_{(\widetilde{\Omega}_k^1-k)\setminus\{\pm\frac{l_0}{2}\}}\\&\quad\times(M_1(z)_{(\widetilde{\Omega}_k^1-k)\setminus\{\pm\frac{l_0}{2}\}})^{-1}R_{(\widetilde{\Omega}_k^1-k)\setminus\{\pm\frac{l_0}{2}\}}M_1(z)R_{\{\pm\frac{l_0}{2}\}}.\end{aligned}$$

Direct computation shows

$$\begin{aligned}\det S_1(z)&=\det(M_1(z)_{\{\pm\frac{l_0}{2}\}})+O(\varepsilon^2\delta_0^{-\frac{3}{10^4}})\\&=4\sin\pi\left(z+\frac{l_0}{2}\cdot\omega-\theta_0\right)\sin\pi\left(z+\frac{l_0}{2}\cdot\omega+\theta_0\right)\\&\quad\times\sin\pi\left(z-\frac{l_0}{2}\cdot\omega-\theta_0\right)\sin\pi\left(z-\frac{l_0}{2}\cdot\omega+\theta_0\right)+O(\varepsilon^{1.5}).\end{aligned}$$

By (3.37) and (3.39), we have

$$\begin{aligned}\left\|z+\frac{l_0}{2}\cdot\omega-\theta_0\right\|&\geqslant\|l_0\cdot\omega\|-\left\|\frac{l_0}{2}\cdot\omega+\theta_0-\frac{1}{2}\right\|-\left|z-\frac{1}{2}\right|\\&>\gamma\mathrm{e}^{-(100N_1^c)^\tau}-\delta_0^{\frac{1}{100}}-\delta_0^{\frac{1}{10^3}}\\&>\delta_0^{\frac{1}{10^4}}\end{aligned}$$

and

$$\begin{aligned}\left\|z-\frac{l_0}{2}\cdot\omega+\theta_0\right\|&\geqslant\|l_0\cdot\omega\|-\left\|\frac{l_0}{2}\cdot\omega+\theta_0-\frac{1}{2}\right\|-\left|z-\frac{1}{2}\right|\\&>\gamma\mathrm{e}^{-(100N_1^c)^\tau}-\delta_0^{\frac{1}{100}}-\delta_0^{\frac{1}{10^3}}\\&>\delta_0^{\frac{1}{10^4}}.\end{aligned}$$

Let z_1 satisfy

$$z_1\equiv\frac{l_0}{2}\cdot\omega+\theta_0\ (\mathrm{mod}\ \mathbb{Z}),\quad\left|z_1-\frac{1}{2}\right|=\left\|\frac{l_0}{2}\cdot\omega+\theta_0-\frac{1}{2}\right\|<\delta_0^{\frac{1}{100}}.$$

Then,

$$\begin{aligned}\det S_1(z)&\sim\left\|z+\frac{l_0}{2}\cdot\omega-\theta_0\right\|\cdot\left\|z-\frac{l_0}{2}\cdot\omega+\theta_0\right\|\cdot|(z-z_1)(z-(1-z_1))+r_1(z)|\\&\overset{\delta_0^{\frac{2}{10^4}}}{\sim}|(z-z_1)(z-(1-z_1))+r_1(z)|,\end{aligned}$$

where $r_1(z)$ is an analytic function in the set (3.39) with $|r_1(z)|<\varepsilon\ll\delta_0^{\frac{1}{10^3}}$. Using again the Rouché theorem shows that the equation

$$(z-z_1)(z-(1-z_1))+r_1(z)=0$$

has exact two roots θ_1 and θ_1' in (3.39), which are perturbations of z_1 and $1-z_1$. Notice that

$$\left\{\left|z-\frac{1}{2}\right|<\delta_0^{\frac{1}{10^3}}:\det M_1(z)=0\right\}=\left\{\left|z-\frac{1}{2}\right|<\delta_0^{\frac{1}{10^3}}:\det S_1(z)=0\right\},$$

and $\det M_1(z)$ is a 1-periodic even function of z (see Lemma C.1). Thus,

$$\theta_1' = 1 - \theta_1.$$

Moreover,

$$|\theta_1 - z_1| \leqslant |r_1(\theta_1)|^{\frac{1}{2}} < \varepsilon^{\frac{1}{2}}, \quad |(z - z_1)(z - 1 + z_1) + r_1(z)| \sim |(z - \theta_1)(z - (1 - \theta_1))|.$$

Thus for z belonging to the set (3.39), we have

$$\det S_1(z) \overset{\delta_0}{\sim} \|z - \theta_1\| \cdot \|z - (1 - \theta_1)\| = \|z - \theta_1\| \cdot \|z + \theta_1\|$$

and

$$\|M_1(z)^{-1}\| < \delta_0^{-2} \|z - \theta_1\|^{-1} \cdot \|z + \theta_1\|^{-1}.$$

Thus for (3.37), both (3.34) and (3.35) hold for z being in

$$\left\{ z \in \mathbb{C} : \left\| z - \frac{1}{2} \right\| < \delta_0^{\frac{1}{10^3}} \right\}.$$

By (3.38), for $k \in P_1$, we obtain

$$\begin{aligned} \|T^{-1}_{\widetilde{\Omega}^1_k}\| &= \|M_1(\theta + k \cdot \omega)^{-1}\| \\ &< \delta_0^{-2} \|\theta + k \cdot \omega - \theta_1\|^{-1} \cdot \|\theta + k \cdot \omega + \theta_1\|^{-1}. \end{aligned} \tag{3.40}$$

For $k \in P_1$, we define $A^1_k \subset \Omega^1_k$ to be

$$A^1_k := \begin{cases} \{k\}, & \text{Case } (\mathbf{C}1)_0, \\ \{o\} \cup \{o^*\}, & \text{Case } (\mathbf{C}2)_0, \end{cases} \tag{3.41}$$

where $k = \frac{1}{2}(o + o^*)$ for some $o \in O_1$ (see (3.29)) in the case $(\mathbf{C}2)_0$. We have verified $(\mathbf{a})_1$–$(\mathbf{d})_1$ and $(\mathbf{f})_1$.

Step 3. Application of the resolvent identity. Now we verify $(\mathbf{e})_1$, which is based on the iterating resolvent identity.

Note that

$$\left| \log \frac{\gamma}{\delta_1} \right| = \left| \log \frac{\gamma}{\delta_0} \right|^{c^5}.$$

Recall that

$$Q_1^{\pm} = \{k \in P_1 : \|\theta + k \cdot \omega \pm \theta_1\| < \delta_1\}, \quad Q_1 = Q_1^+ \cup Q_1^-.$$

We say that a finite set $\Lambda \subset \mathbb{Z}^d$ is 1-**good** if and only if

$$\begin{cases} \Lambda \cap Q_0 \cap \Omega^1_k \neq \emptyset \Rightarrow \widetilde{\Omega}^1_k \subset \Lambda, \\ \{k \in P_1 : \widetilde{\Omega}^1_k \subset \Lambda\} \cap Q_1 = \emptyset. \end{cases} \tag{3.42}$$

Theorem 3.5. *If Λ is 1-**good**, then*

$$\|T_\Lambda^{-1}\| < \delta_0^{-3} \sup_{\{k \in P_1 : \widetilde{\Omega}^1_k \subset \Lambda\}} \|\theta + k \cdot \omega - \theta_1\|^{-1} \cdot \|\theta + k \cdot \omega + \theta_1\|^{-1}, \tag{3.43}$$

$$|T_\Lambda^{-1}(x, y)| < \mathrm{e}^{-\gamma_1 \|x - y\|_1} \quad \textit{for } \|x - y\| > N_1^{c^3}, \tag{3.44}$$

where $\gamma_1 = \gamma_0(1 - N_1^{\frac{1}{c} - 1})^3$.

Proof. Define

$$2\Omega^1_k := \Lambda_{\operatorname{diam} \Omega^1_k}(k).$$

We have the following lemma.

Lemma 3.6. *For $k \in P_1 \setminus Q_1$, we have*

$$|T^{-1}_{\widetilde{\Omega}^1_k}(x,y)| < \mathrm{e}^{-\widetilde{\gamma}_0\|x-y\|_1} \quad \text{for } x \in \partial^- \widetilde{\Omega}^1_k,\ y \in 2\Omega^1_k, \tag{3.45}$$

where $\widetilde{\gamma}_0 = \gamma_0(1 - N_1^{\frac{1}{c}-1})$.

Proof. From our construction, we have

$$Q_0 \subset \bigcup_{k\in P_1} A^1_k \subset \bigcup_{k\in P_1} \Omega^1_k.$$

Thus,

$$(\widetilde{\Omega}^1_k \setminus A^1_k) \cap Q_0 = \emptyset,$$

which shows that $\widetilde{\Omega}^1_k \setminus A^1_k$ is 0-**good**. As a result, by (3.20), one has

$$|T^{-1}_{\widetilde{\Omega}^1_k\setminus A^1_k}(x,w)| < \mathrm{e}^{-\gamma_0\|x-w\|_1} \quad \text{for } x \in \partial^-\widetilde{\Omega}^1_k,\ w \in (\widetilde{\Omega}^1_k \setminus A^1_k) \cap 2\Omega^1_k.$$

Since (3.40) and $k \notin Q_1$, we have

$$\|T^{-1}_{\widetilde{\Omega}^1_k}\| < \delta_0^{-2}\delta_1^{-2} < \delta_1^{-3}.$$

Using the resolvent identity implies

$$\begin{aligned}
|T^{-1}_{\widetilde{\Omega}^1_k}(x,y)| &= \left| T^{-1}_{\widetilde{\Omega}^1_k\setminus A^1_k}(x,y)\chi_{\widetilde{\Omega}^1_k\setminus A^1_k}(y) - \sum_{(w',w)\in\partial A^1_k} T^{-1}_{\widetilde{\Omega}^1_k\setminus A^1_k}(x,w)\Gamma(w,w')T^{-1}_{\widetilde{\Omega}^1_k}(w',y)\right| \\
&< 4d \sup_{w\in\partial^+A^1_k} \mathrm{e}^{-\gamma_0\|x-w\|_1}\|T^{-1}_{\widetilde{\Omega}^1_k}\| \\
&< \sup_{w\in\partial^+A^1_k} \mathrm{e}^{-\gamma_0(\|x-y\|_1-\|y-w\|_1)+C|\log\delta_1|} \\
&< \mathrm{e}^{-\gamma_0(1-C(\|x-y\|_1^{\frac{1}{c}-1}+\frac{|\log\delta_1|}{\|x-y\|_1}))\|x-y\|_1} \\
&< \mathrm{e}^{-\gamma_0(1-N_1^{\frac{1}{c}-1})\|x-y\|_1} \\
&= \mathrm{e}^{-\widetilde{\gamma}_0\|x-y\|_1}
\end{aligned}$$

since

$$N_1^c \lesssim \operatorname{diam}\widetilde{\Omega}^1_k \sim \|x-y\|_1, \quad \|y-w\|_1 \lesssim \operatorname{diam}\Omega^1_k \lesssim (\operatorname{diam}\widetilde{\Omega}^1_k)^{\frac{1}{c}}$$

and

$$|\log\delta_1| \sim |\log\delta_0|^{c^5} \sim N_1^{c^{10}\tau} < N_1^{\frac{1}{c}}. \tag{3.46}$$

This completes the proof. □

We can prove Theorem 3.5 now. First, we prove the estimate (3.43) by Schur's test. Define

$$\widetilde{P}_1 = \{k \in P_1 : \Lambda \cap \Omega^1_k \cap Q_0 \neq \emptyset\}, \quad \Lambda' = \Lambda \setminus \bigcup_{k\in\widetilde{P}_1} \Omega^1_k.$$

Then, $\Lambda' \cap Q_0 = \emptyset$, which shows that Λ' is 0-**good**, and (3.19)–(3.20) hold for Λ'. We have the following cases.

(1) Let $x \notin \bigcup_{k\in\widetilde{P}_1} 2\Omega^1_k$. Thus $N_1 \leqslant \operatorname{dist}(x, \partial^-_\Lambda\Lambda')$. For $y \in \Lambda$, the resolvent identity reads as

$$T^{-1}_\Lambda(x,y) = T^{-1}_{\Lambda'}(x,y)\chi_{\Lambda'}(y) - \sum_{(w,w')\in\partial_\Lambda\Lambda'} T^{-1}_{\Lambda'}(x,w)\Gamma(w,w')T^{-1}_\Lambda(w',y).$$

Since

$$\sum_{y\in\Lambda'} |T^{-1}_{\Lambda'}(x,y)\chi_{\Lambda'}(y)| \leqslant |T^{-1}_{\Lambda'}(x,x)| + \sum_{\|x-y\|_1>0} |T^{-1}_{\Lambda'}(x,y)\chi_{\Lambda'}(y)|$$

$$\leqslant \|T_{\Lambda'}^{-1}\| + \sum_{\|x-y\|_1>0} \mathrm{e}^{-\gamma_0\|x-y\|_1}$$
$$\leqslant 2\delta_0^{-2}$$

and

$$\sum_{w\in\partial_\Lambda^-\Lambda'} |T_{\Lambda'}^{-1}(x,w)| \leqslant \sum_{\|x-w\|_1\geqslant N_1} \mathrm{e}^{-\gamma_0\|x-w\|_1} < \mathrm{e}^{-\frac{1}{2}\gamma_0 N_1},$$

we get

$$\begin{aligned}\sum_{y\in\Lambda}|T_\Lambda^{-1}(x,y)| &\leqslant \sum_{y\in\Lambda'}|T_{\Lambda'}^{-1}(x,y)\chi_{\Lambda'}(y)| + \sum_{y\in\Lambda,(w,w')\in\partial_\Lambda\Lambda'}|T_{\Lambda'}^{-1}(x,w)\Gamma(w,w')T_\Lambda^{-1}(w',y)|\\ &\leqslant 2\delta_0^{-2} + 2d\sum_{w\in\partial_\Lambda^-\Lambda'}|T_{\Lambda'}^{-1}(x,w)|\cdot\sup_{w'\in\Lambda}\sum_{y\in\Lambda}|T_\Lambda^{-1}(w',y)|\\ &\leqslant 2\delta_0^{-2} + \frac{1}{10}\sup_{w'\in\Lambda}\sum_{y\in\Lambda}|T_\Lambda^{-1}(w',y)|.\end{aligned}$$

(2) Let $x\in 2\Omega_k^1$ for some $k\in\widetilde{P}_1$. Thus by (3.42), we have $\widetilde{\Omega}_k^1\subset\Lambda$ and $k\notin Q_1$. For $y\in\Lambda$, using the resolvent identity shows

$$T_\Lambda^{-1}(x,y) = T_{\widetilde{\Omega}_k^1}^{-1}(x,y)\chi_{\widetilde{\Omega}_k^1}(y) - \sum_{(w,w')\in\partial_\Lambda\widetilde{\Omega}_k^1} T_{\widetilde{\Omega}_k^1}^{-1}(x,w)\Gamma(w,w')T_\Lambda^{-1}(w',y).$$

By (3.40) and (3.45), since

$$N_1 < \operatorname{diam}\widetilde{\Omega}_k^1 \lesssim \operatorname{dist}(x,\partial_\Lambda^-\widetilde{\Omega}_k^1),$$

we get

$$\begin{aligned}\sum_{y\in\Lambda}|T_\Lambda^{-1}(x,y)| &\leqslant \sum_{y\in\Lambda}|T_{\widetilde{\Omega}_k^1}^{-1}(x,y)\chi_{\widetilde{\Omega}_k^1}(y)| + \sum_{y\in\Lambda,(w,w')\in\partial_\Lambda\widetilde{\Omega}_k^1}|T_{\widetilde{\Omega}_k^1}^{-1}(x,w)\Gamma(w,w')T_\Lambda^{-1}(w',y)|\\ &< \#\widetilde{\Omega}_k^1\cdot\|T_{\widetilde{\Omega}_k^1}^{-1}\| + CN_1^{c^2d}\mathrm{e}^{-\widetilde{\gamma}_0N_1}\sup_{w'\in\Lambda}\sum_{y\in\Lambda}|T_\Lambda^{-1}(w',y)|\\ &< CN_1^{c^2d}\delta_0^{-2}\|\theta+k\cdot\omega-\theta_1\|^{-1}\cdot\|\theta+k\cdot\omega+\theta_1\|^{-1} + \frac{1}{10}\sup_{w'\in\Lambda}\sum_{y\in\Lambda}|T_\Lambda^{-1}(w',y)|\\ &< \frac{1}{2}\delta_0^{-3}\|\theta+k\cdot\omega-\theta_1\|^{-1}\cdot\|\theta+k\cdot\omega+\theta_1\|^{-1} + \frac{1}{10}\sup_{w'\in\Lambda}\sum_{y\in\Lambda}|T_\Lambda^{-1}(w',y)|.\end{aligned}$$

Combining the estimates in the above two cases yields

$$\begin{aligned}\|T_\Lambda^{-1}\| &\leqslant \sup_{x\in\Lambda}\sum_{y\in\Lambda}|T_\Lambda^{-1}(x,y)|\\ &< \delta_0^{-3}\sup_{\{k\in P_1:\widetilde{\Omega}_k^1\subset\Lambda\}}\|\theta+k\cdot\omega-\theta_1\|^{-1}\cdot\|\theta+k\cdot\omega+\theta_1\|^{-1}.\end{aligned}\tag{3.47}$$

Now we prove the off-diagonal decay estimate (3.44). For every $w\in\Lambda$, define its block in Λ:

$$J_w = \begin{cases}\Lambda_{\frac{1}{2}N_1}(w)\cap\Lambda & \text{if } w\notin\bigcup_{k\in\widetilde{P}_1}2\Omega_k^1, \qquad \text{(i)}\\ \widetilde{\Omega}_k^1 & \text{if } w\in 2\Omega_k^1 \text{ for some } k\in\widetilde{P}_1. \qquad \text{(ii)}\end{cases}$$

Then,

$$\operatorname{diam} J_w \leqslant \max(\operatorname{diam}\Lambda_{\frac{1}{2}N_1}(w),\operatorname{diam}\widetilde{\Omega}_k^1) < 3N_1^{c^2}.$$

For (i), since

$$\mathrm{dist}(w, \Lambda \cap Q_0) \geqslant \mathrm{dist}\bigg(w, \bigcup_{k \in \widetilde{P}_1} \Omega_k^1\bigg) \geqslant N_1,$$

we have $J_w \cap Q_0 = \emptyset$. Thus, J_w is 0-**good**. Noticing that $\mathrm{dist}(w, \partial_\Lambda^- J_w) \geqslant \frac{1}{2} N_1$, from (3.20), we have

$$|T_{J_w}^{-1}(w, w')| < \mathrm{e}^{-\gamma_0 \|w - w'\|_1} \quad \text{for } w' \in \partial_\Lambda^- J_w.$$

For (ii), by (3.45), we have

$$|T_{J_w}(w, w')| < \mathrm{e}^{-\widetilde{\gamma}_0 \|w - w'\|_1} \quad \text{for } w' \in \partial_\Lambda^- J_w.$$

Let $\|x - y\| > N_1^{c^3}$. Using the resolvent identity shows

$$T_\Lambda^{-1}(x, y) = T_{J_x}^{-1}(x, y)\chi_{J_x}(y) - \sum_{(w, w') \in \partial_\Lambda J_x} T_{J_x}^{-1}(x, w)\Gamma(w, w')T_\Lambda^{-1}(w', y).$$

The first term on the right-hand side (RHS) of the above identity is zero because $y \notin J_x$ (since $\|x - y\| > N_1^{c^3} > 3N_1^{c^2}$). It follows that

$$\begin{aligned}
|T_\Lambda^{-1}(x, y)| &\leqslant C N_1^{c^2 d} \mathrm{e}^{-\min(\gamma_0(1 - 2N_1^{-1}), \widetilde{\gamma}_0(1 - N_1^{-1}))\|x - x_1\|_1} |T_\Lambda^{-1}(x_1, y)| \\
&\leqslant C N_1^{c^2 d} \mathrm{e}^{-\widetilde{\gamma}_0(1 - N_1^{-1})\|x - x_1\|_1} |T_\Lambda^{-1}(x_1, y)| \\
&< \mathrm{e}^{-\widetilde{\gamma}_0(1 - N_1^{-1} - \frac{C \log N_1}{N_1})\|x - x_1\|_1} |T_\Lambda^{-1}(x_1, y)| \\
&< \mathrm{e}^{-\gamma_0(1 - N_1^{\frac{1}{c} - 1})^2 \|x - x_1\|_1} |T_\Lambda^{-1}(x_1, y)| \\
&= \mathrm{e}^{-\gamma_0' \|x - x_1\|_1} |T_\Lambda^{-1}(x_1, y)|
\end{aligned}$$

for some $x_1 \in \partial_\Lambda^+ J_x$, where $\gamma_0' = \gamma_0(1 - N_1^{\frac{1}{c} - 1})^2$. Then, iterate and stop for some step L such that $\|x_L - y\| < 3N_1^{c^2}$. Recalling (3.46) and (3.47), we get

$$\begin{aligned}
|T_\Lambda^{-1}(x, y)| &\leqslant \mathrm{e}^{-\gamma_0' \|x - x_1\|_1} \cdots \mathrm{e}^{-\gamma_0' \|x_{L-1} - x_L\|_1} |T_\Lambda^{-1}(x_L, y)| \\
&\leqslant \mathrm{e}^{-\gamma_0'(\|x - y\|_1 - 3N_1^{c^2})} \|T_\Lambda^{-1}\| \\
&< \mathrm{e}^{-\gamma_0'(1 - 3N_1^{c^2 - c^3})\|x - y\|_1} \delta_1^{-3} \\
&< \mathrm{e}^{-\gamma_0'(1 - 3N_1^{c^2 - c^3} - 3\frac{|\log \delta_1|}{N_1^{c^3}})\|x - y\|_1} \\
&< \mathrm{e}^{-\gamma_0'(1 - N_1^{\frac{1}{c} - 1})\|x - y\|_1} \\
&= \mathrm{e}^{-\gamma_1 \|x - y\|_1}.
\end{aligned}$$

This completes the proof of Theorem 3.5. □

3.3 The proof of Theorem 3.2: From $\mathcal{P}_s$ to $\mathcal{P}_{s+1}$

Proof of Theorem 3.2. We have finished the proof of $\mathcal{P}_1$ in Subsection 3.2. Assume that $\mathcal{P}_s$ holds. In order to complete the proof of Theorem 3.2, it suffices to establish $\mathcal{P}_{s+1}$.

In the following, we try to prove that $\mathcal{P}_{s+1}$ holds. For this purpose, we establish $(\mathbf{a})_{s+1}$–$(\mathbf{f})_{s+1}$ assuming $(\mathbf{a})_s$–$(\mathbf{f})_s$. We divide the proof into three steps. Let

$$Q_s^\pm = \{k \in P_s : \|\theta + k \cdot \omega \pm \theta_s\| < \delta_s\}, \quad Q_s = Q_s^+ \cup Q_s^- \tag{3.48}$$

and

$$\widetilde{Q}_s^\pm = \{k \in P_s : \|\theta + k \cdot \omega \pm \theta_s\| < \delta_s^{\frac{1}{100}}\}, \quad \widetilde{Q}_s = \widetilde{Q}_s^+ \cup \widetilde{Q}_s^-. \tag{3.49}$$

Step 1. The case $(\mathbf{C1})_s$ occurs, i.e.,

$$\mathrm{dist}(\widetilde{Q}_s^-, Q_s^+) > 100 N_{s+1}^c. \tag{3.50}$$

Remark 3.7. We can prove that

$$\operatorname{dist}(\widetilde{Q}_s^-, Q_s^+) = \operatorname{dist}(\widetilde{Q}_s^+, Q_s^-).$$

Thus (3.50) also implies that

$$\operatorname{dist}(\widetilde{Q}_s^+, Q_s^-) > 100N_{s+1}^c. \tag{3.51}$$

By (3.18) and the definitions of $Q_s^\pm$ (see (3.48)) and $\widetilde{Q}_s^\pm$ (see (3.49)), we obtain

$$Q_s^\pm = \left\{k \in \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s-1} l_i : \|\theta + k\cdot\omega \pm \theta_s\| < \delta_s\right\}, \tag{3.52}$$

$$\widetilde{Q}_s^\pm = \left\{k \in \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s-1} l_i : \|\theta + k\cdot\omega \pm \theta_s\| < \delta_s^{\frac{1}{100}}\right\}.$$

Then, the proof is similar to that of Remark 3.4 and we omit the details.

Assuming (3.50), we define

$$P_{s+1} = Q_s, \quad l_s = 0. \tag{3.53}$$

By (3.8) and (3.9), we have

$$P_{s+1} \subset \left\{k \in \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s} l_i : \min_{\sigma=\pm 1}(\|\theta + k\cdot\omega + \sigma\theta_s\|) < \delta_s\right\}. \tag{3.54}$$

Thus from (3.51), we obtain that for $k, k' \in P_{s+1}$ with $k \neq k'$,

$$\|k - k'\| > \min\left(\left|\log\frac{\gamma}{2\delta_s}\right|^{\frac{1}{\tau}}, 100N_{s+1}^c\right) \geqslant 100N_{s+1}^c. \tag{3.55}$$

In the following, we associate every $k \in P_{s+1}$ with blocks Ω_k^{s+1} and $\widetilde{\Omega}_k^{s+1}$ so that

$$\Lambda_{N_{s+1}}(k) \subset \Omega_k^{s+1} \subset \Lambda_{N_{s+1}+50N_s^{c^2}}(k),$$
$$\Lambda_{N_{s+1}^c}(k) \subset \widetilde{\Omega}_k^{s+1} \subset \Lambda_{N_{s+1}^c+50N_s^{c^2}}(k)$$

and

$$\begin{cases} \Omega_k^{s+1} \cap \widetilde{\Omega}_{k'}^{s'} \neq \emptyset \ (s' < s+1) \Rightarrow \widetilde{\Omega}_{k'}^{s'} \subset \Omega_k^{s+1}, \\ \widetilde{\Omega}_k^{s+1} \cap \widetilde{\Omega}_{k'}^{s'} \neq \emptyset \ (s' < s+1) \Rightarrow \widetilde{\Omega}_{k'}^{s'} \subset \widetilde{\Omega}_k^{s+1}, \\ \operatorname{dist}(\widetilde{\Omega}_k^{s+1}, \widetilde{\Omega}_{k'}^{s+1}) > 10\operatorname{diam}\widetilde{\Omega}_k^{s+1} \quad \text{for } k \neq k' \in P_{s+1}. \end{cases} \tag{3.56}$$

In addition, the set

$$\widetilde{\Omega}_k^{s+1} - k \subset \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s} l_i$$

is independent of $k \in P_{s+1}$ and is symmetrical about the origin.

Such Ω_k^{s+1} and $\widetilde{\Omega}_k^{s+1}$ can be constructed by the following argument (where we only consider $\widetilde{\Omega}_k^{s+1}$ since Ω_k^{s+1} is discussed by a similar argument). Fixing $k_0 \in Q_s^+$, we start from

$$J_{0,0} = \Lambda_{N_{s+1}^c}(k_0).$$

Define

$$H_r = (k_0 - P_{s+1} + P_{s-r}) \cup (k_0 + P_{s+1} - P_{s-r}), \quad 0 \leqslant r \leqslant s-1.$$

Notice that by (3.54), we have $k_0 - P_{s+1} \in \mathbb{Z}^d$, and $P_{s-r} \subset \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s-r-1} l_i$ by (3.8) and (3.9). Thus,

$$H_{s-r} \subset \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s-r-1} l_i.$$

Inductively define

$$J_{r,0} \subsetneqq J_{r,1} \subsetneqq \cdots \subsetneqq J_{r,t_r} =: J_{r+1,0},$$

where

$$J_{r,t+1} = J_{r,t} \cup \Bigg(\bigcup_{\{h \in H_r : \Lambda_{2N_{s-r}^{c^2}}(h) \cap J_{r,t} \neq \emptyset\}} \Lambda_{2N_{s-r}^{c^2}}(h) \Bigg),$$

and t_r is the largest integer satisfying the $\subsetneqq$ relationship (the following argument shows that $t_r < 10$). Thus,

$$h \in H_r, \quad \Lambda_{2N_{s-r}^{c^2}}(h) \cap J_{r+1,0} \neq \emptyset \Rightarrow \Lambda_{2N_{s-r}^{c^2}}(h) \subset J_{r+1,0}. \tag{3.57}$$

For $\widetilde{k} \in k_0 - P_{s+1}$, we have that by (3.54),

$$\min(\|\widetilde{k} \cdot \omega\|, \|\widetilde{k} \cdot \omega + 2\theta_s\|) < 2\delta_s.$$

For $k' \in P_{s-r}$, we get by (3.8) and (3.9) that

$$\min_{\sigma = \pm 1} (\|\theta + k' \cdot \omega + \sigma \theta_{s-r-1}\|) < \delta_{s-r-1} \quad \text{if } (\mathbf{C}1)_{\mathrm{s-r}} \text{ holds}, \tag{3.58}$$

$$\|\theta + k' \cdot \omega\| < 3\delta_{s-r-1}^{\frac{1}{100}} \quad \text{or} \quad \|\theta + k' \cdot \omega + \frac{1}{2}\| < 3\delta_{s-r-1}^{\frac{1}{100}} \quad \text{if } (\mathbf{C}2)_{\mathrm{s-r}} \text{ holds}. \tag{3.59}$$

Thus for $h \in k_0 - P_{s+1} + P_{s-r}$, we obtain that for (3.58),

$$\min_{\sigma = \pm 1} (\|\theta + h \cdot \omega + \sigma \theta_{s-r-1}\|, \|\theta + h \cdot \omega + 2\theta_s + \sigma \theta_{s-r-1}\|) < 2\delta_{s-r-1},$$

and for (3.59),

$$\min \left(\|\theta + h \cdot \omega\|, \left\|\theta + h \cdot \omega + \frac{1}{2}\right\|, \|\theta + h \cdot \omega + 2\theta_s\|, \left\|\theta + h \cdot \omega + \frac{1}{2} + 2\theta_s\right\| \right) < 4\delta_{s-r-1}^{\frac{1}{100}}.$$

Notice that $k_0 + P_{s+1} - P_{s-r} = 2k_0 - (k_0 - P_{s+1} + P_{s-r})$ is symmetrical to $k_0 - P_{s+1} + P_{s-r}$ about k_0. Thus, if a set Λ ($\subset \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s-r-1} l_i$) contains 10 distinct elements of H_r, then

$$\operatorname{diam} \Lambda > \left| \log \frac{\gamma}{8\delta_{s-r-1}^{\frac{1}{100}}} \right|^{\frac{1}{\tau}} \gg 100 N_{s-r}^{c^2}. \tag{3.60}$$

We claim that $t_r < 10$. Otherwise, there exist distinct $h_t \in H_r$ ($1 \leqslant t \leqslant 10$) such that

$$\Lambda_{2N_{s-r}^{c^2}}(h_1) \cap J_{r,0} \neq \emptyset, \quad \Lambda_{2N_{s-r}^{c^2}}(h_t) \cap \Lambda_{2N_{s-r}^{c^2}}(h_{t+1}) \neq \emptyset.$$

In particular,

$$\operatorname{dist}(h_t, h_{t+1}) \leqslant 4N_{s-r}^{c^2}.$$

Thus,

$$h_t \in \Lambda_{40N_{s-r}^{c^2}}(0) + h_1, \quad 1 \leqslant t \leqslant 10.$$

This contradicts (3.60). Thus, we have shown

$$J_{r+1,0} = J_{r,t_r} \subset \Lambda_{40N_{s-r}^{c^2}}(J_{r,0}). \tag{3.61}$$

Since

$$\sum_{r=0}^{s-1} 40 N_{s-r}^{c^2} < 50 N_s^{c^2},$$

we find $J_{s,0}$ to satisfy

$$\Lambda_{N_{s+1}^c}(k_0) = J_{0,0} \subset J_{s,0} \subset \Lambda_{50N_s^{c^2}}(J_{0,0}) \subset \Lambda_{N_{s+1}^c + 50N_s^{c^2}}(k_0).$$

Now, for any $k \in P_{s+1}$, define

$$\widetilde{\Omega}_k^{s+1} = J_{s,0} + (k - k_0). \tag{3.62}$$

Using $k - k_0 \in \mathbb{Z}^d$ and $\widetilde{\Omega}_k^{s+1} \subset \mathbb{Z}^d$ yields

$$\Lambda_{N_{s+1}^c}(k) \subset \widetilde{\Omega}_k^{s+1} \subset \Lambda_{N_{s+1}^c + 50N_s^{c^2}}(k).$$

We can verify (3.56). Since (3.55) and $50N_s^{c^2} \ll N_{s+1}^c$, we get

$$\mathrm{dist}(\widetilde{\Omega}_k^{s+1}, \widetilde{\Omega}_{k'}^{s+1}) > 10\,\mathrm{diam}\,\widetilde{\Omega}_k^{s+1} \quad \text{for } k \neq k' \in P_{s+1}.$$

Assume that for some $k \in P_{s+1}$ and $k' \in P_{s'}$ $(1 \leqslant s' \leqslant s)$, $\widetilde{\Omega}_k^{s+1} \cap \widetilde{\Omega}_{k'}^{s'} \neq \emptyset$. Then,

$$(\widetilde{\Omega}_k^{s+1} + (k_0 - k)) \cap (\widetilde{\Omega}_{k'}^{s'} + (k_0 - k)) \neq \emptyset. \tag{3.63}$$

From

$$\Lambda_{N_{s'}^c}(k') \subset \widetilde{\Omega}_{k'}^{s'} \subset \Lambda_{N_{s'}^c + 50N_{s'-1}^{c2}}(k') \subset \Lambda_{1.5N_{s'}^{c2}}(k'),$$

$\widetilde{\Omega}_k^{s+1} + (k_0 - k) = J_{s,0}$ and (3.63), we obtain

$$J_{s,0} \cap \Lambda_{1.5N_{s'}^{c2}}(k' + k_0 - k) \neq \emptyset.$$

Recalling (3.61), we have

$$J_{s,0} \subset \Lambda_{50N_{s'-1}^{c2}}(J_{s-s'+1,0}).$$

Thus,

$$\Lambda_{50N_{s'-1}^{c2}}(J_{s-s'+1,0}) \cap \Lambda_{1.5N_{s'}^{c2}}(k' + k_0 - k) \neq \emptyset.$$

From $50N_{s'-1}^{c^2} \ll 0.5N_{s'}^{c^2}$, it follows that

$$J_{s-s'+1,0} \cap \Lambda_{2N_{s'}^{c2}}(k' + k_0 - k) \neq \emptyset.$$

Since $k' \in P_{s'}$, we have $k' + k_0 - k \in H_{s-s'}$, and by (3.57),

$$\Lambda_{2N_{s'}^{c2}}(k' + k_0 - k) \subset J_{s-s'+1,0} \subset J_{s,0}.$$

Hence,

$$\widetilde{\Omega}_{k'}^{s'} \subset \Lambda_{2N_{s'}^{c2}}(k') \subset J_{s,0} + (k - k_0) = \widetilde{\Omega}_k^{s+1}.$$

Next, we show that $\widetilde{\Omega}_k^{s+1} - k$ is independent of k. For this, recalling (3.62), from $l_i \in \mathbb{Z}^d$, $\widetilde{\Omega}_k^{s+1} \subset \mathbb{Z}^d$ and $k \in P_{s+1} \subset \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s} l_i$, we obtain that

$$\widetilde{\Omega}_k^{s+1} - k \subset \mathbb{Z}^d - \frac{1}{2}\sum_{i=0}^{s} l_i = \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s} l_i,$$

and

$$\widetilde{\Omega}_k^{s+1} - k = J_{s,0} + (k - k_0) - k = \widetilde{\Omega}_{k_0}^{s+1} - k_0$$

is independent of k. Finally, we prove the symmetry property of $\widetilde{\Omega}_k^{s+1}$. The definition of H_r implies that it is symmetrical about k_0, which implies all $J_{r,t}$ is symmetrical about k_0 as well. In particular, $\widetilde{\Omega}_{k_0}^{s+1} = J_{s,0}$ is symmetrical about k_0, i.e., $\widetilde{\Omega}_{k_0}^{s+1} - k_0$ is symmetrical about the origin. In summary, we have established $(\mathbf{a})_{s+1}$ and $(\mathbf{b})_{s+1}$ in the case $(\mathbf{C}1)_s$.

Now we turn to the proof of $(\mathbf{c})_{s+1}$. First, in this construction, we have that for every $k' \in Q_s$ $(= P_{s+1})$,

$$\widetilde{\Omega}_{k'}^{s} \subset \Omega_{k'}^{s+1}.$$

For every $k \in P_{s+1}$, define

$$A_k^{s+1} = A_k^s.$$

Then, $A_k^{s+1} \subset \Omega_k^s \subset \Omega_k^{s+1}$ and $\# A_k^{s+1} = \# A_k^s \leqslant 2^s$. It remains to show that $\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}$ is s-**good**, i.e.,

$$\begin{cases} l' \in Q_{s'}, \quad \widetilde{\Omega}_{l'}^{s'} \subset (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}), \quad \widetilde{\Omega}_{l'}^{s'} \subset \Omega_l^{s'+1} \Rightarrow \widetilde{\Omega}_l^{s'+1} \subset (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}) \quad \text{for } s' < s, \\ \{l \in P_s : \widetilde{\Omega}_l^s \subset (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1})\} \cap Q_s = \emptyset. \end{cases}$$

Assume that

$$l' \in Q_{s'}, \quad \widetilde{\Omega}_{l'}^{s'} \subset (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}), \quad \widetilde{\Omega}_{l'}^{s'} \subset \Omega_l^{s'+1}.$$

We have the following two cases. The first case is $s' \leqslant s-2$. In this case, since $\emptyset \neq \widetilde{\Omega}_{l'}^{s'} \subset \widetilde{\Omega}_l^{s'+1} \cap \widetilde{\Omega}_k^{s+1}$, we get by using (3.56) that $\widetilde{\Omega}_l^{s'+1} \subset \widetilde{\Omega}_k^{s+1}$. Assuming

$$\widetilde{\Omega}_l^{s'+1} \cap A_k^{s+1} \neq \emptyset, \tag{3.64}$$

we have $\widetilde{\Omega}_l^{s'+1} \cap \widetilde{\Omega}_k^s \neq \emptyset$. Thus from (3.10) (since $s'+1 < s$), one has $\widetilde{\Omega}_l^{s'+1} \subset \widetilde{\Omega}_k^s$, which implies $\widetilde{\Omega}_{l'}^{s'} \subset (\widetilde{\Omega}_k^s \setminus A_k^s)$. Since $(\widetilde{\Omega}_k^s \setminus A_k^s)$ is $(s-1)$-**good**, we get

$$\widetilde{\Omega}_l^{s'+1} \subset (\widetilde{\Omega}_k^s \setminus A_k^s) \subset (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}).$$

This contradicts (3.64). We then consider the case $s' = s-1$. From $\widetilde{\Omega}_{l'}^{s-1} \subset \Omega_l^s$ and $\widetilde{\Omega}_l^s \cap A_k^s \neq \emptyset$, we have $l = k$ and $\widetilde{\Omega}_{l'}^{s-1} \subset (\widetilde{\Omega}_k^s \setminus A_k^s)$. This contradicts

$$\{l \in P_{s-1} : \widetilde{\Omega}_l^{s-1} \subset (\widetilde{\Omega}_k^s \setminus A_k^s)\} \cap Q_{s-1} = \emptyset,$$

because $(\widetilde{\Omega}_k^s \setminus A_k^s)$ is $(s-1)$-**good**. Finally, if $l \in Q_s$ and $\widetilde{\Omega}_l^s \subset \widetilde{\Omega}_k^{s+1}$, then $l = k$ since k is the only element of Q_s such that $\widetilde{\Omega}_k^s \subset \widetilde{\Omega}_k^{s+1}$ by the separation property of Q_s. As a result, $\widetilde{\Omega}_l^s \not\subset (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1})$, which implies

$$\{l \in P_s : \widetilde{\Omega}_l^s \subset (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1})\} \cap Q_s = \emptyset.$$

Moreover, the set

$$A_k^{s+1} - k = A_k^s - k$$

is independent of $k \in P_{s+1}$ and symmetrical about the origin due to the induction assumptions on A_k^s of the s-th step. This finishes the proof of $(\mathbf{c})_{s+1}$ in the case $(\mathbf{C}1)_s$.

In the following, we try to prove $(\mathbf{d})_{s+1}$ and $(\mathbf{f})_{s+1}$ in the case $(\mathbf{C}1)_s$. For the case $k \in Q_s^-$, we consider the analytic matrix-valued function

$$M_{s+1}(z) := T(z)_{\widetilde{\Omega}_k^{s+1}-k} = (\cos 2\pi(z + n\cdot\omega)\delta_{n,n'} - E + \varepsilon\Delta)_{n \in \widetilde{\Omega}_k^{s+1}-k}$$

defined in

$$\{z \in \mathbb{C} : |z - \theta_s| < \delta_s^{\frac{1}{10}}\}. \tag{3.65}$$

If $k' \in P_s$ and $\widetilde{\Omega}_{k'}^s \subset (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1})$, then $0 \neq \|k' - k\| \leqslant 2N_{s+1}^c$. Thus,

$$\begin{aligned} \|\theta + k'\cdot\omega - \theta_s\| &\geqslant \|(k'-k)\cdot\omega\| - \|\theta + k\cdot\omega - \theta_s\| \\ &\geqslant \gamma \mathrm{e}^{-(2N_{s+1}^c)^\tau} - \delta_s \\ &\geqslant \gamma \mathrm{e}^{-2^\tau |\log \frac{\gamma}{\delta_s}|^{\frac{1}{c^4}}} - \delta_s \\ &> \delta_s^{\frac{1}{10^4}}. \end{aligned}$$

By (3.51), we have $k' \notin \widetilde{Q}_s^+$, and thus,

$$\|\theta + k'\cdot\omega + \theta_s\| > \delta_s^{\frac{1}{100}}.$$

From (3.16), we obtain

$$\|T^{-1}_{\widetilde{\Omega}^{s+1}_k \setminus A^{s+1}_k}\| < \delta^{-3}_{s-1} \sup_{\{k' \in P_s : \widetilde{\Omega}^s_{k'} \subset (\widetilde{\Omega}^{s+1}_k \setminus A^{s+1}_k)\}} \|\theta + k' \cdot \omega - \theta_s\|^{-1} \cdot \|\theta + k' \cdot \omega + \theta_s\|^{-1}$$
$$< \frac{1}{2}\delta_s^{-2\times\frac{1}{100}}. \tag{3.66}$$

One may restate (3.66) as

$$\|(M_{s+1}(\theta + k \cdot \omega)_{(\widetilde{\Omega}^{s+1}_k \setminus A^{s+1}_k)-k})^{-1}\| < \frac{1}{2}\delta_s^{-2\times\frac{1}{100}}.$$

Notice that

$$\|z - (\theta + k \cdot \omega)\| \leqslant |z - \theta_s| + \|\theta + k \cdot \omega - \theta_s\|$$
$$< \delta_s^{\frac{1}{10}} + \delta_s < 2\delta_s^{\frac{1}{10}} \ll \delta_s^{2\times\frac{1}{100}}. \tag{3.67}$$

Thus by the Neumann series argument, we can show

$$\|(M_{s+1}(z)_{(\widetilde{\Omega}^{s+1}_k \setminus A^{s+1}_k)-k})^{-1}\| < \delta_s^{-2\times\frac{1}{100}}. \tag{3.68}$$

We may then control $M_{s+1}(z)^{-1}$ by the inverse of

$$S_{s+1}(z) = M_{s+1}(z)_{A^{s+1}_k - k} - R_{A^{s+1}_k - k} M_{s+1}(z) R_{(\widetilde{\Omega}^{s+1}_k \setminus A^{s+1}_k)-k}$$
$$\times (M_{s+1}(z)_{(\widetilde{\Omega}^{s+1}_k \setminus A^{s+1}_k)-k})^{-1} R_{(\widetilde{\Omega}^{s+1}_k \setminus A^{s+1}_k)-k} M_{s+1}(z) R_{A^{s+1}_k - k}.$$

Our next aim is to analyze the function $\det S_{s+1}(z)$. Since $A^{s+1}_k - k = A^s_k - k \subset \Omega^s_k - k$ and $\mathrm{dist}(\Omega^s_k, \partial^+ \widetilde{\Omega}^s_k) > 1$, we obtain

$$R_{A^{s+1}_k - k} M_{s+1}(z) R_{(\widetilde{\Omega}^{s+1}_k \setminus A^{s+1}_k)-k} = R_{A^s_k - k} M_{s+1}(z) R_{(\widetilde{\Omega}^s_k \setminus A^s_k)-k}.$$

Thus,

$$S_{s+1}(z) = M_{s+1}(z)_{A^s_k - k} - R_{A^s_k - k} M_{s+1}(z) R_{(\widetilde{\Omega}^s_k \setminus A^s_k)-k}$$
$$\times (M_{s+1}(z)_{(\widetilde{\Omega}^{s+1}_k \setminus A^{s+1}_k)-k})^{-1} R_{(\widetilde{\Omega}^s_k \setminus A^s_k)-k} M_{s+1}(z) R_{A^s_k - k}.$$

Since $\widetilde{\Omega}^s_k \setminus A^s_k$ is $(s-1)$-**good**, by (3.16)–(3.17), we get

$$\|T^{-1}_{\widetilde{\Omega}^s_k \setminus A^s_k}\| < \delta^{-3}_{s-1},$$
$$|T^{-1}_{\widetilde{\Omega}^s_k \setminus A^s_k}(x, y)| < \mathrm{e}^{-\gamma_{s-1}\|x-y\|_1} \quad \text{for } \|x - y\| > N^{c^3}_{s-1}.$$

Equivalently,

$$\|(M_{s+1}(\theta + k \cdot \omega)_{(\widetilde{\Omega}^s_k \setminus A^s_k)-k})^{-1}\| < \delta^{-3}_{s-1}, \tag{3.69}$$
$$|(M_{s+1}(\theta + k \cdot \omega)_{(\widetilde{\Omega}^s_k \setminus A^s_k)-k})^{-1}(x, y)| < \mathrm{e}^{-\gamma_{s-1}\|x-y\|_1} \quad \text{for } \|x - y\| > N^{c^3}_{s-1}. \tag{3.70}$$

In the set defined by (3.65), we claim that

$$|(M_{s+1}(z)_{(\widetilde{\Omega}^s_k \setminus A^s_k)-k})^{-1}(x, y)| < \delta_s^{10} \quad \text{for } \|x - y\| > N^{c^4}_{s-1}. \tag{3.71}$$

Proof of the claim (i.e., (3.71)). Define

$$T_1 = M_{s+1}(\theta + k \cdot \omega)_{(\widetilde{\Omega}^s_k \setminus A^s_k)-k}, \quad T_2 = M_{s+1}(z)_{(\widetilde{\Omega}^s_k \setminus A^s_k)-k}.$$

Then, $D = T_1 - T_2$ is diagonal so that $\|D\| < 5\pi\delta_s^{\frac{1}{10}}$ by (3.67). Using the Neumann series expansion yields

$$T_2^{-1} = (I - T_1^{-1}D)^{-1}T_1^{-1} = \sum_{i=0}^{+\infty}(T_1^{-1}D)^i T_1^{-1}. \tag{3.72}$$

By (3.69) and (3.70), we have

$$|T_1^{-1}(x,y)| < \delta_{s-1}^{-3}\mathrm{e}^{-\gamma_{s-1}(\|x-y\|_1 - N_{s-1}^{c^3})}.$$

Thus for $\|x-y\| > N_{s-1}^{c^4}$ and $0 \leqslant i \leqslant 200$,

$$\begin{aligned}|((T_1^{-1}D)^iT_1^{-1})(x,y)| &\leqslant (4\pi\delta_s^{\frac{1}{10}})^i \sum_{w_1,\dots,w_i}|T_1(x,w_1)\cdots T_1(w_{i-1},w_i)T_1(w_i,y)|\\ &< (4\pi\delta_s^{\frac{1}{10}})^i CN_s^{c^2d}\delta_{s-1}^{-3(i+1)}\mathrm{e}^{-\gamma_{s-1}(\|x-y\|_1-(i+1)N_{s-1}^{c^3})}\\ &< \delta_s^{\frac{1}{20}(i-1)}\mathrm{e}^{-\gamma_{s-1}(N_{s-1}^{c^4}-(i+1)N_{s-1}^{c^3})}.\end{aligned}$$

From $2 < \gamma_{s-1}$, $201N_{s-1}^{c^3} < \frac{1}{2}N_{s-1}^{c^4}$ and $|\log\delta_s| \sim |\log\delta_{s-1}|^{c^5} \sim N_s^{c^{10}\tau} \sim N_{s-1}^{c^{15}\tau} < N_{s-1}^{c^3}$, we get

$$\mathrm{e}^{-\gamma_{s-1}(N_{s-1}^{c^4}-(i+1)N_{s-1}^{c^3})} < \mathrm{e}^{-N_{s-1}^{c^4}} < \delta_s^{20}.$$

Hence,

$$\sum_{i=0}^{200}|((T_1^{-1}D)^iT_1^{-1})(x,y)| < \frac{1}{2}\delta_s^{10}. \tag{3.73}$$

For $i > 200$,

$$|((T_1^{-1}D)^iT_1^{-1})(x,y)| < (4\pi\delta_s^{\frac{1}{10}})^i\delta_{s-1}^{-3(i+1)} < \delta_s^{\frac{1}{20}i} < \delta_s^{10}\delta_s^{\frac{1}{20}(i-200)}.$$

Thus,

$$\sum_{i>200}|((T_1^{-1}D)^iT_1^{-1})(x,y)| < \frac{1}{2}\delta_s^{10}. \tag{3.74}$$

Combining (3.72)–(3.74), we get

$$|T_2^{-1}(x,y)| < \delta_s^{10} \quad \text{for } \|x-y\| > N_{s-1}^{c^4}.$$

This completes the proof of (3.71). □

Define $X = (\widetilde{\Omega}_k^s \setminus A_k^s) - k$ and $Y = (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}) - k$. Let $x \in X$ satisfy $\mathrm{dist}(x, A_k^s - k) \leqslant 1$. By the resolvent identity, we have that for any $y \in Y$,

$$\begin{aligned}&(M_{s+1}(z)_Y)^{-1}(x,y) - \chi_X(y)(M_{s+1}(z)_X)^{-1}(x,y)\\ &\quad = -\sum_{(w,w')\in\partial_Y X}(M_{s+1}(z)_X)^{-1}(x,w)\Gamma(w,w')(M_{s+1}(z)_Y)^{-1}(w',y).\end{aligned} \tag{3.75}$$

From

$$\mathrm{dist}(x,w) \geqslant \mathrm{dist}(A_k^s - k, \partial^-\widetilde{\Omega}_k^s - k) - 2 > N_s > N_{s-1}^{c^4},$$

(3.68) and (3.71), the RHS of (3.75) is bounded by $CN_s^{c^2d}\delta_s^{-\frac{1}{50}}\delta_s^{10} < \delta_s^9$. It then follows that

$$\begin{aligned}&R_{A_k^s-k}M_{s+1}(z)R_X(M_{s+1}(z)_Y)^{-1}\\ &\quad = R_{A_k^s-k}M_{s+1}(z)R_X(M_{s+1}(z)_X)^{-1}R_X + O(\delta_s^9).\end{aligned}$$

As a result,

$$R_{A_k^s-k}M_{s+1}(z)R_X(M_{s+1}(z)_Y)^{-1}R_XM_{s+1}(z)R_{A_k^s-k}$$

$$
\begin{aligned}
&= R_{A_k^s-k}M_{s+1}(z)R_X(M_{s+1}(z)_X)^{-1}R_XM_{s+1}(z)R_{A_k^s-k}+O(\delta_s^9)\\
&= R_{A_k^s-k}M_s(z)R_X(M_s(z)_X)^{-1}R_XM_s(z)R_{A_k^s-k}+O(\delta_s^9)
\end{aligned}
$$

and

$$
\begin{aligned}
S_{s+1}(z) &= M_s(z)_{A_k^s-k}-R_{A_k^s-k}M_s(z)R_X(M_s(z)_X)^{-1}R_XM_s(z)R_{A_k^s-k}+O(\delta_s^9)\\
&= S_s(z)+O(\delta_s^9),
\end{aligned}
$$

which implies (3.13) for the $(s+1)$-th step. Recalling (3.65) and (3.12), we have that by (3.14),

$$
\det S_s(z) \overset{\delta_{s-1}}{\sim} \|z-\theta_s\|\cdot\|z+\theta_s\|.
$$

By Hadamard's inequality, we obtain

$$
\begin{aligned}
\det S_{s+1}(z) &= \det S_s(z)+O((2^s)^2 10^{2^s}\delta_s^9)\\
&= \det S_s(z)+O(\delta_s^8),
\end{aligned}
$$

where we use the fact that $\#(A_k^s-k)\leqslant 2^s$, (3.13) and $\log\log|\log\delta_s|\sim s$. Notice that

$$
\begin{aligned}
\|z+\theta_s\| &\geqslant \|\theta+k\cdot\omega+\theta_s\|-\|z-\theta_s\|-\|\theta+k\cdot\omega-\theta_s\|\\
&> \delta_s^{\frac{1}{100}}-\delta_s^{\frac{1}{10}}-\delta_1\\
&> \frac{1}{2}\delta_s^{\frac{1}{100}}.
\end{aligned}
$$

Then, we have

$$
\det S_{s+1}(z) \overset{\delta_s}{\sim} (z-\theta_s)+r_{s+1}(z),
$$

where $r_{s+1}(z)$ is an analytic function defined in (3.65) with $|r_{s+1}(z)|<\delta_s^7$. Finally, by the Rouché theorem, the equation

$$
(z-\theta_s)+r_{s+1}(z)=0
$$

has a unique root θ_{s+1} in the set defined by (3.65), which satisfies

$$
|\theta_{s+1}-\theta_s|=|r_{s+1}(\theta_{s+1})|<\delta_s^7, \quad |(z-\theta_s)+r_{s+1}(z)|\sim|z-\theta_{s+1}|.
$$

Moreover, θ_{s+1} is also the unique root of $\det M_{s+1}(z)=0$ in the set defined by (3.65). From $\|z+\theta_s\| > \frac{1}{2}\delta_s^{\frac{1}{100}}$ and $|\theta_{s+1}-\theta_s|<\delta_s^7$, we have

$$
\|z+\theta_s\|\sim\|z+\theta_{s+1}\|.
$$

Thus, if z belongs to the set defined by (3.65), we have

$$
\det S_{s+1}(z) \overset{\delta_s}{\sim} \|z-\theta_{s+1}\|\cdot\|z+\theta_{s+1}\|. \tag{3.76}
$$

Since $|\log\delta_{s+1}|\sim|\log\delta_s|^{c^5}$, we get $\delta_{s+1}^{\frac{1}{10^4}}<\frac{1}{2}\delta_s^{\frac{1}{10}}$. Recalling (3.65), we see that (3.76) remains valid for z satisfying

$$
\|z-\theta_{s+1}\|<\delta_{s+1}^{\frac{1}{10^4}}.
$$

For $k\in Q_s^+$, one considers

$$
M_{s+1}(z):=T(z)_{\widetilde{\Omega}_k^{s+1}-k}=(\cos 2\pi(z+n\cdot\omega)\delta_{n,n'}-E+\varepsilon\Delta)_{n\in\widetilde{\Omega}_k^{s+1}-k}
$$

for z being in

$$
\{z\in\mathbb{C}:|z+\theta_s|<\delta_s^{\frac{1}{10}}\}. \tag{3.77}
$$

The same argument shows that $\det M_{s+1}(z) = 0$ has a unique root θ'_{s+1} in the set defined by (3.77). Since $\det M_{s+1}(z)$ is an even function of z, we get $\theta'_{s+1} = -\theta_{s+1}$. Thus, if z belongs to the set defined by (3.77), we also have (3.76). In conclusion, (3.76) is established for z belonging to

$$\Big\{z \in \mathbb{C} : \min_{\sigma=\pm 1} \|z + \sigma\theta_{s+1}\| < \delta_{s+1}^{\frac{1}{10^4}}\Big\},$$

which proves (3.14) for the $(s+1)$-th step. Combining $l_s = 0$, (3.52)–(3.53) and the following

$$\|\theta + k\cdot\omega \pm \theta_{s+1}\| < 10\delta_{s+1}^{\frac{1}{100}}, \quad |\theta_{s+1} - \theta_s| < \delta_s^7 \Rightarrow \|\theta + k\cdot\omega \pm \theta_s\| < \delta_s,$$

we get

$$\Big\{k \in \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s} l_i : \min_{\sigma=\pm 1} \|\theta + k\cdot\omega + \sigma\theta_{s+1}\| < 10\delta_{s+1}^{\frac{1}{100}}\Big\} \subset P_{s+1},$$

which proves (3.18) at the $(s+1)$-th step. Finally, we want to estimate $T^{-1}_{\widetilde{\Omega}_k^{s+1}}$. For $k \in P_{s+1}$, by (3.54), we obtain

$$\theta + k\cdot\omega \in \Big\{z \in \mathbb{C} : \min_{\sigma=\pm 1} \|z + \sigma\theta_s\| < \delta_s^{\frac{1}{10}}\Big\},$$

which together with (3.76) implies

$$\begin{aligned}
&|\det(T_{A_k^{s+1}} - R_{A_k^{s+1}} T R_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}} T^{-1}_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}} R_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}} T R_{A_k^{s+1}})| \\
&\quad = |\det S_{s+1}(\theta + k\cdot\omega)| \\
&\quad \geqslant \frac{1}{C}\delta_s \|\theta + k\cdot\omega - \theta_{s+1}\| \cdot \|\theta + k\cdot\omega + \theta_{s+1}\|.
\end{aligned}$$

By Cramer's rule and Hadamard's inequality (which, combined with (3.13), aims to establish the upper bound on the numerator in Cramer's representation of Green's function), one has

$$\begin{aligned}
&\|(T_{A_k^{s+1}} - R_{A_k^{s+1}} T R_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}} T^{-1}_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}} R_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}} T R_{A_k^{s+1}})^{-1}\| \\
&\quad < C 2^s 10^{2^s} \delta_s^{-1} \|\theta + k\cdot\omega - \theta_{s+1}\|^{-1} \cdot \|\theta + k\cdot\omega + \theta_{s+1}\|^{-1}.
\end{aligned}$$

From the Schur complement argument (see Lemma B.1) and (3.66), we get

$$\begin{aligned}
\|T^{-1}_{\widetilde{\Omega}_k^{s+1}}\| &< 4(1 + \|T^{-1}_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}}\|)^2 \\
&\quad \times (1 + \|(T_{A_k^{s+1}} - R_{A_k^{s+1}} T R_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}} T^{-1}_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}} R_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}} T R_{A_k^{s+1}})^{-1}\|) \\
&< \delta_s^{-2} \|\theta + k\cdot\omega - \theta_{s+1}\|^{-1} \cdot \|\theta + k\cdot\omega + \theta_{s+1}\|^{-1}.
\end{aligned} \tag{3.78}$$

Step 2. The case $(\mathbf{C}2)_s$ occurs, i.e.,

$$\mathrm{dist}(\widetilde{Q}_s^-, Q_s^+) \leqslant 100 N_{s+1}^c.$$

Then, there exist $i_s \in Q_s^+$ and $j_s \in \widetilde{Q}_s^-$ with $\|i_s - j_s\| \leqslant 100N_{s+1}^c$ such that

$$\|\theta + i_s\cdot\omega + \theta_s\| < \delta_s, \quad \|\theta + j_s\cdot\omega - \theta_s\| < \delta_s^{\frac{1}{100}}.$$

Define

$$l_s = i_s - j_s.$$

Using (3.8) and (3.9) yields

$$Q_s^+, \widetilde{Q}_s^- \subset P_s \subset \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s-1} l_i.$$

Thus $i_s \equiv j_s \pmod{\mathbb{Z}^d}$ and $l_s \in \mathbb{Z}^d$. Define

$$O_{s+1} = Q_s^- \cup (Q_s^+ - l_s). \tag{3.79}$$

For every $o \in O_{s+1}$, define its mirror point

$$o^* = o + l_s.$$

Then, we have

$$O_{s+1} \subset \left\{ o \in \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s-1} l_i : \|\theta + o \cdot \omega - \theta_s\| < 2\delta_s^{\frac{1}{100}} \right\}$$

and

$$O_{s+1} + l_s \subset \left\{ o^* \in \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s-1} l_i : \|\theta + o^* \cdot \omega + \theta_s\| < 2\delta_s^{\frac{1}{100}} \right\}.$$

Then, by (3.18), we obtain

$$O_{s+1} \cup (O_{s+1} + l_s) \subset P_s. \tag{3.80}$$

Define

$$P_{s+1} = \left\{ \frac{1}{2}(o + o^*) : o \in O_{s+1} \right\} = \left\{ o + \frac{l_s}{2} : o \in O_{s+1} \right\}. \tag{3.81}$$

Notice that

$$\begin{aligned} &\min\left(\left\| \frac{l_s}{2} \cdot \omega + \theta_s \right\|, \left\| \frac{l_s}{2} \cdot \omega + \theta_s - \frac{1}{2} \right\| \right) \\ &= \frac{1}{2}\|l_s \cdot \omega + 2\theta_s\| \leqslant \frac{1}{2}(\|\theta + i_s \cdot \omega + \theta_s\| + \|\theta + j_s \cdot \omega - \theta_s\|) < \delta_s^{\frac{1}{100}}. \end{aligned}$$

Since $\delta_s \ll 1$, only one of the following

$$\left\| \frac{l_s}{2} \cdot \omega + \theta_s \right\| < \delta_s^{\frac{1}{100}}, \quad \left\| \frac{l_s}{2} \cdot \omega + \theta_s - \frac{1}{2} \right\| < \delta_s^{\frac{1}{100}}$$

occurs. First, we consider the case

$$\left\| \frac{l_s}{2} \cdot \omega + \theta_s \right\| < \delta_s^{\frac{1}{100}}. \tag{3.82}$$

Let $k \in P_{s+1}$. From $k = o + \frac{l_s}{2}$ for some $o \in O_{s+1}$ and (3.82), we get

$$\|\theta + k \cdot \omega\| \leqslant \|\theta + o \cdot \omega - \theta_s\| + \left\| \frac{l_s}{2} \cdot \omega + \theta_s \right\| < 3\delta_s^{\frac{1}{100}},$$

which implies

$$P_{s+1} \subset \left\{ k \in \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s} l_i : \|\theta + k \cdot \omega\| < 3\delta_s^{\frac{1}{100}} \right\}. \tag{3.83}$$

Moreover, if $k \neq k' \in P_{s+1}$, then

$$\|k - k'\| > \left| \log \frac{\gamma}{6\delta_s^{\frac{1}{100}}} \right| \sim N_{s+1}^{c^5} \gg 10 N_{s+1}^{c^2}.$$

Similar to the proof that appears in Step 1 (i.e., the $(\mathbf{C}1)_s$ case), we can associate $k \in P_{s+1}$ with the blocks Ω_k^{s+1} and $\widetilde{\Omega}_k^{s+1}$ which satisfy

$$\begin{aligned} &\Lambda_{100N_{s+1}^c}(k) \subset \Omega_k^{s+1} \subset \Lambda_{100N_{s+1}^c + 50N_s^{c^2}}(k), \\ &\Lambda_{N_{s+1}^{c^2}}(k) \subset \widetilde{\Omega}_k^{s+1} \subset \Lambda_{N_{s+1}^{c^2} + 50N_s^{c^2}}(k) \end{aligned}$$

and

$$\begin{cases} \Omega_k^{s+1} \cap \widetilde{\Omega}_{k'}^{s'} \neq \emptyset \ (s' < s+1) \Rightarrow \widetilde{\Omega}_{k'}^{s'} \subset \Omega_k^{s+1}, \\ \widetilde{\Omega}_k^{s+1} \cap \widetilde{\Omega}_{k'}^{s'} \neq \emptyset \ (s' < s+1) \Rightarrow \widetilde{\Omega}_{k'}^{s'} \subset \widetilde{\Omega}_k^{s+1}, \\ \operatorname{dist}(\widetilde{\Omega}_k^{s+1}, \widetilde{\Omega}_{k'}^{s+1}) > 10 \operatorname{diam} \widetilde{\Omega}_k^{s+1} \quad \text{for } k \neq k' \in P_{s+1}. \end{cases} \tag{3.84}$$

In addition, the set

$$\widetilde{\Omega}_k^{s+1} - k \subset \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s} l_i$$

is independent of $k \in P_{s+1}$ and symmetrical about the origin. Clearly, in this construction, for every $k' \in Q_s$, there exists a $k = k' - \frac{l_s}{2}$ or a $k' + \frac{l_s}{2} \in P_{s+1}$ such that

$$\widetilde{\Omega}_{k'}^{s} \subset \Omega_k^{s+1}.$$

For every $k \in P_{s+1}$, we have $o, o^* \in P_s$ by (3.80). Define

$$A_k^{s+1} = A_o^s \cup A_{o^*}^s,$$

where $o \in O_{s+1}$ and $k = o + o^*$ (see (3.81)). Then,

$$\begin{aligned}&A_k^{s+1} \subset \Omega_o^s \cup \Omega_{o^*}^s \subset \Omega_k^{s+1},\\ &\#A_k^{s+1} = \#A_o^s + \#A_{o^*}^s \leqslant 2^{s+1}.\end{aligned}$$

Now we verify that $(\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1})$ is s-**good**, i.e.,

$$\begin{cases} l' \in Q_{s'}, \quad \widetilde{\Omega}_{l'}^{s'} \subset (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}), \quad \widetilde{\Omega}_{l'}^{s'} \subset \Omega_l^{s'+1} \Rightarrow \widetilde{\Omega}_l^{s'+1} \subset (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}) \quad \text{for } s' < s,\\ \{l \in P_s : \widetilde{\Omega}_l^s \subset (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1})\} \cap Q_s = \emptyset.\end{cases}$$

For this purpose, assume that

$$l' \in Q_{s'}, \quad \widetilde{\Omega}_{l'}^{s'} \subset (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}), \quad \widetilde{\Omega}_{l'}^{s'} \subset \Omega_l^{s'+1}.$$

If $s' \leqslant s-2$, since $\emptyset \neq \widetilde{\Omega}_{l'}^{s'} \subset \widetilde{\Omega}_l^{s'+1} \cap \widetilde{\Omega}_k^{s+1}$, by (3.84), we have $\widetilde{\Omega}_l^{s'+1} \subset \widetilde{\Omega}_k^{s+1}$. If $\widetilde{\Omega}_l^{s'+1} \cap A_k^{s+1} \neq \emptyset$, then we have $\widetilde{\Omega}_l^{s'+1} \cap A_o^s \neq \emptyset$ or $\widetilde{\Omega}_l^{s'+1} \cap A_{o^*}^s \neq \emptyset$. Thus, by (3.10) ($s'+1 < s$), we get $\widetilde{\Omega}_{l'}^{s'+1} \subset \widetilde{\Omega}_o^s$ or $\widetilde{\Omega}_{l'}^{s'+1} \subset \widetilde{\Omega}_{o^*}^s$, which implies $\widetilde{\Omega}_{l'}^{s'} \subset (\widetilde{\Omega}_o^s \setminus A_o^s)$ or $\widetilde{\Omega}_{l'}^{s'} \subset (\widetilde{\Omega}_{o^*}^s \setminus A_{o^*}^s)$. Thus, we have either $\widetilde{\Omega}_{l'}^{s'+1} \subset (\widetilde{\Omega}_o^s \setminus A_o^s) \subset (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1})$ or $\widetilde{\Omega}_{l'}^{s'+1} \subset (\widetilde{\Omega}_{o^*}^s \setminus A_{o^*}^s) \subset (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1})$ since both $(\widetilde{\Omega}_o^s \setminus A_o^s)$ and $(\widetilde{\Omega}_{o^*}^s \setminus A_{o^*}^s)$ are $(s-1)$-**good**. This leads to a contradiction. If $s' = s-1$, $\widetilde{\Omega}_{l'}^{s-1} \subset \Omega_l^s$ and $\widetilde{\Omega}_l^s \cap A_k^{s+1} \neq \emptyset$, then either $l = o$ or $l = o^*$, and thus $\widetilde{\Omega}_{l'}^{s-1} \subset (\widetilde{\Omega}_o^s \setminus A_o^s)$ or $\widetilde{\Omega}_{l'}^{s-1} \subset (\widetilde{\Omega}_{o^*}^s \setminus A_{o^*}^s)$. This contradicts

$$\{l \in P_{s-1} : \widetilde{\Omega}_l^{s-1} \subset (\widetilde{\Omega}_o^s \setminus A_o^s)\} \cap Q_{s-1} = \{l \in P_{s-1} : \widetilde{\Omega}_l^{s-1} \subset (\widetilde{\Omega}_{o^*}^s \setminus A_{o^*}^s)\} \cap Q_{s-1} = \emptyset$$

since both $(\widetilde{\Omega}_o^s \setminus A_o^s)$ and $(\widetilde{\Omega}_{o^*}^s \setminus A_{o^*}^s)$ are $(s-1)$-**good**. Finally, if $l \in Q_s$ and $\widetilde{\Omega}_l^s \subset \widetilde{\Omega}_k^{s+1}$, then $l = o$ or $l = o^*$. Thus $\widetilde{\Omega}_l^s \nsubseteq (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1})$, which implies

$$\{l \in P_s : \widetilde{\Omega}_l^s \subset (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1})\} \cap Q_s = \emptyset.$$

Moreover, we have

$$\begin{aligned}A_k^{s+1} - k &= (A_o^s - k) \cup (A_{o^*}^s - k)\\ &= \left((A_o^s - o) - \frac{l_s}{2}\right) \cup \left((A_{o^*}^s - o^*) + \frac{l_s}{2}\right)\end{aligned}$$

is independent of $k \in P_{s+1}$ and symmetrical about the origin.

Now consider the analytic matrix-valued function

$$M_{s+1}(z) := T(z)_{\widetilde{\Omega}_k^{s+1} - k} = (\cos 2\pi(z + n\cdot\omega)\delta_{n,n'} - E + \varepsilon\Delta)_{n \in \widetilde{\Omega}_k^{s+1} - k}$$

defined in

$$\{z \in \mathbb{C} : |z| < \delta_s^{\frac{1}{10^3}}\}. \tag{3.85}$$

If $k' \in P_s$ and $\widetilde{\Omega}_{k'}^s \subset (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1})$, then $k' \neq o, o^*$ and $\|k' - o\|, \|k' - o^*\| \leqslant 4N_{s+1}^{c^2}$. Thus,

$$\|\theta + k'\cdot\omega - \theta_s\| \geqslant \|(k'-o)\cdot\omega\| - \|\theta + o\cdot\omega - \theta_s\|$$

$$
\begin{aligned}
&\geqslant \gamma e^{-(4N_{s+1}^{c^2})^{\tau}} - 2\delta_s^{\frac{1}{100}}\\
&\geqslant \gamma e^{-4^{\tau}|\log\frac{\gamma}{\delta_s}|^{\frac{1}{c}}} - 2\delta_s^{\frac{1}{100}}\\
&> \delta_s^{\frac{1}{10^4}}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\theta + k'\cdot\omega + \theta_s\| &\geqslant \|(k'-o^*)\cdot\omega\| - \|\theta + o^*\cdot\omega + \theta_s\|\\
&\geqslant \gamma e^{-(4N_{s+1}^{c^2})^{\tau}} - 2\delta_s^{\frac{1}{100}}\\
&\geqslant \gamma e^{-4^{\tau}|\log\frac{\gamma}{\delta_s}|^{\frac{1}{c}}} - 2\delta_s^{\frac{1}{100}}\\
&> \delta_s^{\frac{1}{10^4}}.
\end{aligned}
$$

By (3.16), we have

$$
\begin{aligned}
\|T^{-1}_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}}\| &< \delta_{s-1}^{-3} \sup_{\{k'\in P_s:\widetilde{\Omega}_{k'}^s\subset(\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1})\}} \|\theta + k'\cdot\omega - \theta_s\|^{-1}\cdot\|\theta + k'\cdot\omega + \theta_s\|^{-1}\\
&< \frac{1}{2}\delta_s^{-3\times\frac{1}{10^4}}.
\end{aligned} \tag{3.86}
$$

One may restate (3.86) as

$$
\|(M_{s+1}(\theta + k\cdot\omega)_{(\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1})-k})^{-1}\| < \frac{1}{2}\delta_s^{-3\times\frac{1}{10^4}}.
$$

Since

$$
\begin{aligned}
\|z - (\theta + k\cdot\omega)\| &\leqslant |z| + \|\theta + k\cdot\omega\|\\
&< \delta_s^{\frac{1}{10^3}} + 3\delta_s^{\frac{1}{100}} < 2\delta_s^{\frac{1}{10^3}} \ll \delta_s^{3\times\frac{1}{10^4}},
\end{aligned} \tag{3.87}
$$

using the Neumann series argument, we obtain

$$
\|(M_{s+1}(z)_{(\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1})-k})^{-1}\| < \delta_s^{-3\times\frac{1}{10^4}}. \tag{3.88}
$$

We may control $M_{s+1}(z)^{-1}$ by the inverse of

$$
\begin{aligned}
S_{s+1}(z) = M_{s+1}(z)_{A_k^{s+1}-k} &- R_{A_k^{s+1}-k}M_{s+1}(z)R_{(\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1})-k}\\
&\times (M_{s+1}(z)_{(\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1})-k})^{-1}R_{(\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1})-k}M_{s+1}(z)R_{A_k^{s+1}-k}.
\end{aligned}
$$

Our next aim is to analyze $\det S_{s+1}(z)$. Since

$$
A_k^{s+1} - k = (A_o^s - k)\cup(A_{o^*}^s - k),\quad A_o^s - k\subset \Omega_o^s - k,\quad A_{o^*}^s - k\subset\Omega_{o^*}^s - k
$$

and

$$
\mathrm{dist}(\Omega_o^s - k, \Omega_{o^*}^s - k) > 10\,\mathrm{diam}\,\widetilde{\Omega}_o^s,
$$

we have

$$
M_{s+1}(z)_{A_k^{s+1}-k} = M_{s+1}(z)_{A_o^s-k}\oplus M_{s+1}(z)_{A_{o^*}^s-k}.
$$

From $\mathrm{dist}(\Omega_o^s, \partial^+\widetilde{\Omega}_o^s) > 1$ and $\mathrm{dist}(\Omega_{o^*}^s, \partial^+\widetilde{\Omega}_{o^*}^s) > 1$, we have

$$
\begin{aligned}
R_{A_o^s-k}M_{s+1}(z)R_{(\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1})-k} &= R_{A_o^s-k}M_{s+1}(z)R_{(\widetilde{\Omega}_o^s\setminus A_o^s)-k},\\
R_{A_{o^*}^s-k}M_{s+1}(z)R_{(\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1})-k} &= R_{A_{o^*}^s-k}M_{s+1}(z)R_{(\widetilde{\Omega}_{o^*}^s\setminus A_{o^*}^s)-k}.
\end{aligned}
$$

Define

$$X = (\widetilde{\Omega}_o^s \setminus A_o^s) - k, \quad X^* = (\widetilde{\Omega}_{o^*}^s \setminus A_{o^*}^s) - k, \quad Y = (\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}) - k.$$

Then, direct computation yields

$$\begin{aligned} &S_{s+1}(z) \\ &\quad = M_{s+1}(z)_{A_o^s-k} \oplus M_{s+1}(z)_{A_{o^*}^s-k} - (R_{A_o^s-k} \oplus R_{A_{o^*}^s-k})M_{s+1}(z)R_Y M_{s+1}(z)_Y^{-1} R_Y M_{s+1}(z) R_{A_k^{s+1}-k} \\ &\quad = (M_{s+1}(z)_{A_o^s-k} - R_{A_o^s-k} M_{s+1}(z) R_X M_{s+1}(z)_Y^{-1} R_Y M_{s+1}(z) R_{A_k^{s+1}-k}) \\ &\qquad \oplus (M_{s+1}(z)_{A_{o^*}^s-k} - R_{A_{o^*}^s-k} M_{s+1}(z) R_{X^*} M_{s+1}(z)_Y^{-1} R_Y M_{s+1}(z) R_{A_k^{s+1}-k}). \end{aligned} \tag{3.89}$$

Since $\widetilde{\Omega}_o^s \setminus A_o^s$ is $(s-1)$-**good**, by (3.16)–(3.17), we have

$$\begin{aligned} &\|T^{-1}_{\widetilde{\Omega}_o^s \setminus A_o^s}\| < \delta_{s-1}^{-3}, \\ &|T^{-1}_{\widetilde{\Omega}_o^s \setminus A_o^s}(x,y)| < \mathrm{e}^{-\gamma_{s-1}\|x-y\|_1} \quad \text{for } \|x-y\| > N_{s-1}^{c^3}. \end{aligned}$$

In other words,

$$\|(M_{s+1}(\theta + k\cdot\omega)_X)^{-1}\| < \delta_{s-1}^{-3}, \tag{3.90}$$

$$|(M_{s+1}(\theta + k\cdot\omega)_X)^{-1}(x,y)| < \mathrm{e}^{-\gamma_{s-1}\|x-y\|_1} \quad \text{for } \|x-y\| > N_{s-1}^{c^3}. \tag{3.91}$$

From the approximation (3.87), we deduce by the same argument as (3.71) that

$$|(M_{s+1}(z)_{(\widetilde{\Omega}_k^s \setminus A_k^s)-k})^{-1}(x,y)| < \delta_s^{10} \quad \text{for } \|x-y\| > N_{s-1}^{c^4}. \tag{3.92}$$

Let $x \in X$ and $\mathrm{dist}(x, A_o^s - k) \leqslant 1$. By the resolvent identity, we have that for any $y \in Y$,

$$\begin{aligned} &(M_{s+1}(z)_Y)^{-1}(x,y) - \chi_X(y)(M_{s+1}(z)_X)^{-1}(x,y) \\ &\quad = -\sum_{(w,w')\in\partial_Y X} (M_{s+1}(z)_X)^{-1}(x,w)\Gamma(w,w')(M_{s+1}(z)_Y)^{-1}(w',y). \end{aligned} \tag{3.93}$$

From

$$\mathrm{dist}(x,w) \geqslant \mathrm{dist}(A_o^s - k, \partial^- \widetilde{\Omega}_o^s - k) - 2 > N_s > N_{s-1}^{c^4},$$

(3.88) and (3.92), the RHS of (3.93) is bounded by

$$C N_s^{c^2 d} \delta_s^{-\frac{3}{10^4}} \delta_s^{10} < \delta_s^9.$$

It follows that

$$R_{A_o^s-k} M_{s+1}(z) R_X (M_{s+1}(z)_Y)^{-1} = R_{A_o^s-k} M_{s+1}(z) R_X (M_{s+1}(z)_X)^{-1} R_X + O(\delta_s^9).$$

Similarly,

$$R_{A_{o^*}^s-k} M_{s+1}(z) R_{X^*} (M_{s+1}(z)_Y)^{-1} = R_{A_{o^*}^s-k} M_{s+1}(z) R_{X^*} (M_{s+1}(z)_{X^*})^{-1} R_{X^*} + O(\delta_s^9).$$

Recalling (3.89), we get

$$\begin{aligned} S_{s+1}(z) &= (M_{s+1}(z)_{A_o^s-k} - R_{A_o^s-k} M_{s+1}(z) R_X (M_{s+1}(z)_X)^{-1} R_{(\widetilde{\Omega}_o^s\setminus A_o^s)-k} M_{s+1}(z) R_{A_o^s-k}) \\ &\quad \oplus (M_{s+1}(z)_{A_{o^*}^s-k} - R_{A_{o^*}^s-k} M_{s+1}(z) R_{X^*} (M_{s+1}(z)_{X^*})^{-1} R_{X^*} M_{s+1}(z) R_{A_{o^*}^s-k}) + O(\delta_s^9) \\ &= S_s\left(z - \frac{l_s}{2}\cdot\omega\right) \oplus S_s\left(z + \frac{l_s}{2}\cdot\omega\right) + O(\delta_s^9). \end{aligned} \tag{3.94}$$

From (3.82) and (3.85), we have

$$\left\|z - \frac{l_s}{2}\cdot\omega - \theta_s\right\| \leqslant |z| + \left\|\frac{l_s}{2}\cdot\omega + \theta_s\right\| < \delta_s^{\frac{1}{10^3}} + \delta_s^{\frac{1}{100}} < \delta_s^{\frac{1}{10^4}}$$

and

$$\left\|z+\frac{l_s}{2}\cdot\omega+\theta_s\right\|<|z|+\left\|\frac{l_s}{2}\cdot\omega+\theta_s\right\|<\delta_s^{\frac{1}{10^3}}+\delta_s^{\frac{1}{100}}<\delta_s^{\frac{1}{10^4}}.$$

Thus, both $z-\frac{l_s}{2}\cdot\omega$ and $z+\frac{l_s}{2}\cdot\omega$ belong to the set defined by (3.12), which together with (3.14) implies

$$\det S_s\left(z-\frac{l_s}{2}\cdot\omega\right)\overset{\delta_{s-1}}{\sim}\left\|\left(z-\frac{l_s}{2}\cdot\omega\right)-\theta_s\right\|\cdot\left\|\left(z-\frac{l_s}{2}\cdot\omega\right)+\theta_s\right\|, \tag{3.95}$$

$$\det S_s\left(z+\frac{l_s}{2}\cdot\omega\right)\overset{\delta_{s-1}}{\sim}\left\|\left(z+\frac{l_s}{2}\cdot\omega\right)-\theta_s\right\|\cdot\left\|\left(z+\frac{l_s}{2}\cdot\omega\right)+\theta_s\right\|. \tag{3.96}$$

Moreover,

$$\begin{aligned}\det S_{s+1}(z)&=\det S_s\left(z-\frac{l_s}{2}\omega\right)\cdot\det S_s\left(z+\frac{l_s}{2}\omega\right)+O((2^{s+1})^2 10^{2^{s+1}}\delta_s^9)\\&=\det S_s\left(z-\frac{l_s}{2}\omega\right)\cdot\det S_s\left(z+\frac{l_s}{2}\omega\right)+O(\delta_s^8)\end{aligned} \tag{3.97}$$

due to $\#(A_k^{s+1}-k)\leqslant 2^{s+1}$, (3.13) and $\log\log|\log\delta_s|\sim s$. Notice that

$$\begin{aligned}\left\|z+\frac{l_s}{2}\cdot\omega-\theta_s\right\|&\geqslant\|l_s\cdot\omega\|-\left\|z-\frac{l_s}{2}\cdot\omega-\theta_s\right\|\\&>\gamma\mathrm{e}^{-(100N_s^c)^\tau}-\delta_s^{\frac{1}{10^4}}\\&>\delta_s^{\frac{1}{10^4}}\end{aligned} \tag{3.98}$$

and

$$\begin{aligned}\left\|z-\frac{l_s}{2}\cdot\omega+\theta_s\right\|&\geqslant\|l_s\cdot\omega\|-\left\|z+\frac{l_s}{2}\cdot\omega+\theta_s\right\|\\&>\gamma\mathrm{e}^{-(100N_s^c)^\tau}-\delta_s^{\frac{1}{10^4}}\\&>\delta_s^{\frac{1}{10^4}}.\end{aligned} \tag{3.99}$$

Let z_{s+1} satisfy

$$z_{s+1}\equiv\frac{l_s}{2}\cdot\omega+\theta_s\ (\mathrm{mod}\ \mathbb{Z}),\quad |z_{s+1}|=\left\|\frac{l_s}{2}\cdot\omega+\theta_s\right\|<\delta_s^{\frac{1}{100}}. \tag{3.100}$$

From (3.95)–(3.99), we get

$$\det S_{s+1}(z)\overset{\delta_s}{\sim}(z-z_{s+1})\cdot(z+z_{s+1})+r_{s+1}(z),$$

where $r_{s+1}(z)$ is an analytic function in the set defined by (3.85) with $|r_{s+1}(z)|<\delta_s^7$. By the Rouché theorem, the equation

$$(z-z_{s+1})(z+z_{s+1})+r_{s+1}(z)=0$$

has exactly two roots θ_{s+1} and θ'_{s+1} in the set defined by (3.85), which are perturbations of $\pm z_{s+1}$, respectively. Notice that

$$\{|z|<\delta_s^{\frac{1}{10^3}}:\det M_{s+1}(z)=0\}=\{|z|<\delta_s^{\frac{1}{10^3}}:\det S_{s+1}(z)=0\}$$

and $\det M_{s+1}(z)$ is an even function of z. Thus,

$$\theta'_{s+1}=-\theta_{s+1}.$$

Moreover, we get

$$|\theta_{s+1}-z_{s+1}|\leqslant|r_{s+1}(\theta_{s+1})|^{\frac{1}{2}}<\delta_s^3 \tag{3.101}$$

and

$$|(z - z_{s+1})(z + z_{s+1}) + r_{s+1}(z)| \sim |(z - \theta_{s+1})(z + \theta_{s+1})|.$$

Thus for z being in the set defined by (3.85), we have

$$\det S_{s+1}(z) \overset{\delta_s}{\sim} \|z - \theta_{s+1}\| \cdot \|z + \theta_{s+1}\|. \tag{3.102}$$

Since $\delta_{s+1}^{\frac{1}{10^4}} < \frac{1}{2}\delta_s^{\frac{1}{10^3}}$, by combining (3.100) and (3.101), we get

$$\left\{z \in \mathbb{C} : \min_{\sigma=\pm 1} |z + \sigma\theta_{s+1}| < \delta_{s+1}^{\frac{1}{10^4}}\right\} \subset \{z \in \mathbb{C} : |z| < \delta_s^{\frac{1}{10^3}}\}.$$

Hence, (3.102) also holds for z belonging to

$$\{z \in \mathbb{C} : \|z \pm \theta_{s+1}\| < \delta_{s+1}^{\frac{1}{10^4}}\},$$

which proves (3.14) for the $(s+1)$-th step.

Notice that

$$\|\theta + k \cdot \omega + \theta_{s+1}\| < 10\delta_{s+1}^{\frac{1}{100}}, \quad |\theta_{s+1} - z_{s+1}| < \delta_s^3 \Rightarrow \left\|\theta + k \cdot \omega + \frac{l_s}{2} + \theta_s\right\| < \delta_s.$$

Thus if

$$k \in \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s} l_i \quad \text{and} \quad \|\theta + k \cdot \omega + \theta_{s+1}\| < 10\delta_{s+1}^{\frac{1}{100}},$$

then

$$k + \frac{l_s}{2} \in \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s-1} l_i \quad \text{and} \quad \left\|\theta + \left(k + \frac{l_s}{2}\right) \cdot \omega + \theta_s\right\| < \delta_s.$$

Thus by (3.52), we have $k + \frac{l_s}{2} \in Q_s^+$. Recalling also (3.79) and (3.81), we have $k \in P_{s+1}$. Thus,

$$\left\{k \in \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s} l_i : \|\theta + k \cdot \omega + \theta_{s+1}\| < 10\delta_{s+1}^{\frac{1}{100}}\right\} \subset P_{s+1}.$$

Similarly,

$$\left\{k \in \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s} l_i : \|\theta + k \cdot \omega - \theta_{s+1}\| < 10\delta_{s+1}^{\frac{1}{100}}\right\} \subset P_{s+1}.$$

Hence, we prove (3.18) for the $(s+1)$-th step.

Finally, we estimate $T^{-1}_{\widetilde{\Omega}_k^{s+1}}$. For $k \in P_{s+1}$, by (3.83), we have

$$\theta + k \cdot \omega \in \{z \in \mathbb{C} : \|z\| < \delta_s^{\frac{1}{10^3}}\}.$$

Thus from (3.102), we obtain

$$\begin{aligned}
&|\det(T_{A_k^{s+1}} - R_{A_k^{s+1}} T R_{\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}} T^{-1}_{\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}} R_{\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}} T R_{A_k^{s+1}})| \\
&\quad = |\det S_{s+1}(\theta + k \cdot \omega)| \\
&\quad \geqslant \frac{1}{C}\delta_s \|\theta + k \cdot \omega - \theta_{s+1}\| \cdot \|\theta + k \cdot \omega + \theta_{s+1}\|.
\end{aligned}$$

Using Cramer's rule and Hadamard's inequality implies

$$\begin{aligned}
&\|(T_{A_k^{s+1}} - R_{A_k^{s+1}} T R_{\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}} T^{-1}_{\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}} R_{\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}} T R_{A_k^{s+1}})^{-1}\| \\
&\quad < C 2^{s+1} 10^{2^{s+1}} \delta_s^{-1} \|\theta + k \cdot \omega - \theta_{s+1}\|^{-1} \cdot \|\theta + k \cdot \omega + \theta_{s+1}\|^{-1}.
\end{aligned}$$

Recalling the Schur complement argument (see Lemma B.1) and (3.86), we get

$$\begin{aligned}\|T^{-1}_{\widetilde{\Omega}_k^{s+1}}\| &< 4(1+\|T^{-1}_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}}\|)^2\\ &\quad\times(1+\|(T_{A_k^{s+1}}-R_{A_k^{s+1}}TR_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}}T^{-1}_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}}R_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}}TR_{A_k^{s+1}})^{-1}\|)\\ &< \delta_s^{-2}\|\theta+k\cdot\omega-\theta_{s+1}\|^{-1}\cdot\|\theta+k\cdot\omega+\theta_{s+1}\|^{-1}. \qquad (3.103)\end{aligned}$$

For the case

$$\left\|\frac{l_s}{2}\cdot\omega+\theta_s-\frac{1}{2}\right\|<\delta_s^{\frac{1}{100}}, \qquad (3.104)$$

we have

$$P_{s+1}\subset\left\{k\in\mathbb{Z}^d+\frac{1}{2}\sum_{i=0}^{s}l_i:\left\|\theta+k\cdot\omega-\frac{1}{2}\right\|<3\delta_s^{\frac{1}{100}}\right\}. \qquad (3.105)$$

Thus we can consider

$$M_{s+1}(z):=T(z)_{\widetilde{\Omega}_k^1-k}=(\cos 2\pi(z+n\cdot\omega)\delta_{n,n'}-E+\varepsilon\Delta)_{n\in\widetilde{\Omega}_k^1-k}$$

in

$$\left\{z\in\mathbb{C}:\left|z-\frac{1}{2}\right|<\delta_s^{\frac{1}{10^3}}\right\}. \qquad (3.106)$$

By similar arguments as above, we obtain that both θ_{s+1} and $1-\theta_{s+1}$ belong to the set defined by (3.106). Moreover, all the corresponding conclusions in the case of (3.82) hold for the case (3.104). Recalling (3.78), we know that the estimate (3.103) holds for the case (3.104) as well.

Step 3. Application of the resolvent identity. Finally, we aim to establish $(\mathbf{e})_{s+1}$ by iterating the resolvent identity.

Recall that

$$\left|\log\frac{\gamma}{\delta_{s+1}}\right|=\left|\log\frac{\gamma}{\delta_s}\right|^{c^5}.$$

Define

$$Q_{s+1}=\left\{k\in P_{s+1}:\min_{\sigma=\pm1}\|\theta+k\cdot\omega+\sigma\theta_{s+1}\|<\delta_{s+1}\right\}.$$

Assume that the finite set $\Lambda\subset\mathbb{Z}^d$ is $(s+1)$-**good**, i.e.,

$$\begin{cases} k'\in Q_{s'},\quad \widetilde{\Omega}_{k'}^{s'}\subset\Lambda,\quad \widetilde{\Omega}_{k'}^{s'}\subset\Omega_k^{s'+1}\Rightarrow\widetilde{\Omega}_k^{s'+1}\subset\Lambda\quad\text{for } s'<s+1,\\ \{k\in P_{s+1}:\widetilde{\Omega}_k^{s+1}\subset\Lambda\}\cap Q_{s+1}=\emptyset.\end{cases} \qquad (3.107)$$

It remains to verify the implications (3.16) and (3.17) with s replaced by $s+1$.

For $k\in P_t$ $(1\leqslant t\leqslant s+1)$, denote by

$$2\Omega_k^t:=\Lambda_{\operatorname{diam}\Omega_k^t}(k)$$

the "double"-size block of Ω_k^t. Moreover, define

$$\widetilde{P}_t=\{k\in P_t:\exists\, k'\in Q_{t-1}\text{ s.t. }\widetilde{\Omega}_{k'}^{t-1}\subset\Lambda,\widetilde{\Omega}_{k'}^{t-1}\subset\Omega_k^t\},\quad 1\leqslant t\leqslant s+1. \qquad (3.108)$$

Lemma 3.8. *For $k\in P_{s+1}\setminus Q_{s+1}$, we have*

$$|T^{-1}_{\widetilde{\Omega}_k^{s+1}}(x,y)|<\mathrm{e}^{-\widetilde{\gamma}_s\|x-y\|_1}\quad\textit{for } x\in\partial^-\widetilde{\Omega}_k^{s+1}\textit{ and } y\in 2\Omega_k^{s+1}, \qquad (3.109)$$

where $\widetilde{\gamma}_s=\gamma_s(1-N_{s+1}^{\frac{1}{c}-1})$.

Proof. First, notice that

$$\mathrm{dist}(\partial^-\widetilde{\Omega}_k^{s+1}, 2\Omega_k^{s+1}) \gtrsim \mathrm{diam}\,\widetilde{\Omega}_k^{s+1} > N_{s+1} \gg N_s^{c^3}.$$

Since $\widetilde{\Omega}_k^{s+1} \setminus A_k^{s+1}$ is s-**good**, we have that by (3.17),

$$|T^{-1}_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}}(x,w)| < \mathrm{e}^{-\gamma_s\|x-w\|_1} \quad \text{for } x\in\partial^-\widetilde{\Omega}_k^{s+1},\ w\in(\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1})\cap 2\Omega_k^{s+1}.$$

From (3.103) and $k\notin Q_{s+1}$, we obtain

$$\|T^{-1}_{\widetilde{\Omega}_k^{s+1}}\| < \delta_s^{-2}\delta_{s+1}^{-2} < \delta_{s+1}^{-3}.$$

Using the resolvent identity implies (since $x\in\partial^-\widetilde{\Omega}_k^{s+1}$)

$$\begin{aligned}
|T^{-1}_{\widetilde{\Omega}_k^{s+1}}(x,y)| &= \Bigg|T^{-1}_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}}(x,y)\chi_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}}(y) - \sum_{(w',w)\in\partial A_k^{s+1}} T^{-1}_{\widetilde{\Omega}_k^{s+1}\setminus A_k^{s+1}}(x,w)\Gamma(w,w')T^{-1}_{\widetilde{\Omega}_k^{s+1}}(w',y)\Bigg| \\
&< \mathrm{e}^{-\gamma_s\|x-y\|_1} + 2d\cdot 2^{s+1}\sup_{w\in\partial^+A_k^{s+1}} \mathrm{e}^{-\gamma_s\|x-w\|_1}\|T^{-1}_{\widetilde{\Omega}_k^{s+1}}\| \\
&< \mathrm{e}^{-\gamma_s\|x-y\|_1} + \sup_{w\in\partial^+A_k^{s+1}} \mathrm{e}^{-\gamma_s(\|x-y\|_1-\|y-w\|_1)+C|\log\delta_{s+1}|} \\
&< \mathrm{e}^{-\gamma_s\|x-y\|_1} + \mathrm{e}^{-\gamma_s(1-C(\|x-y\|_1^{\frac{1}{c}-1}+\frac{|\log\delta_{s+1}|}{\|x-y\|_1}))\|x-y\|_1} \\
&< \mathrm{e}^{-\gamma_s(1-N_{s+1}^{\frac{1}{c}-1})\|x-y\|_1} \\
&= \mathrm{e}^{-\widetilde{\gamma}_s\|x-y\|_1},
\end{aligned}$$

since

$$N_{s+1}^c \lesssim \mathrm{diam}\,\widetilde{\Omega}_k^{s+1} \sim \|x-y\|_1, \quad \|y-w\|_1 \lesssim \mathrm{diam}\,\Omega_k^{s+1} \lesssim (\mathrm{diam}\,\widetilde{\Omega}_k^{s+1})^{\frac{1}{c}}$$

and

$$|\log\delta_{s+1}| \sim |\log\delta_s|^{c^5} \sim N_{s+1}^{c^{10}\tau} < N_{s+1}^{\frac{1}{c}}. \tag{3.110}$$

This proves the lemma. □

Next, we consider the general case and finish the proof of $(\mathbf{e})_{s+1}$. Define

$$\Lambda' = \Lambda\setminus \bigcup_{k\in\widetilde{P}_{s+1}} \Omega_k^{s+1}.$$

We claim that Λ' is s-**good**. For $s'\leqslant s-1$, assume $\widetilde{\Omega}_{l'}^{s'}\subset\Lambda'$, $\widetilde{\Omega}_{l'}^{s'}\subset\Omega_l^{s'+1}$ and $\widetilde{\Omega}_l^{s'+1}\cap(\bigcup_{k\in\widetilde{P}_{s+1}}\Omega_k^{s+1})\neq\emptyset$. Thus by (3.84), we obtain $\widetilde{\Omega}_l^{s'+1}\subset\bigcup_{k\in\widetilde{P}_{s+1}}\Omega_k^{s+1}$, which contradicts $\widetilde{\Omega}_{l'}^{s'}\subset\Lambda'$. If there exists a k' such that $k'\in Q_s$ and $\widetilde{\Omega}_{k'}^s\subset\Lambda'\subset\Lambda$, then by (3.107), there exists a $k\in P_{s+1}$ such that

$$\widetilde{\Omega}_{k'}^s \subset \Omega_k^{s+1}\subset\Lambda.$$

Hence, recalling (3.108), one has $k\in\widetilde{P}_{s+1}$ and

$$\widetilde{\Omega}_{k'}^s \subset \bigcup_{k\in\widetilde{P}_{s+1}}\Omega_k^{s+1}.$$

This contradicts $\widetilde{\Omega}_{k'}^s\subset\Lambda'$. We have proven the claim. As a result, the estimates (3.16) and (3.17) hold with Λ replaced by Λ'. We now can estimate T_Λ^{-1}. For this purpose, we have the following two cases.

(1) Assume that $x\notin\bigcup_{k\in\widetilde{P}_{s+1}}2\Omega_k^{s+1}$. Then, $N_s^{c^3}\ll N_{s+1}\leqslant \mathrm{dist}(x,\partial_\Lambda^-\Lambda')$. For $y\in\Lambda$, using the resolvent identity shows

$$T_\Lambda^{-1}(x,y) = T_{\Lambda'}^{-1}(x,y)\chi_{\Lambda'}(y) - \sum_{(w,w')\in\partial_\Lambda\Lambda'} T_{\Lambda'}^{-1}(x,w)\Gamma(w,w')T_\Lambda^{-1}(w',y).$$

Since

$$\begin{aligned}\sum_{y\in\Lambda'}|T_{\Lambda'}^{-1}(x,y)\chi_{\Lambda'}(y)| &\leqslant \sum_{\|x-y\|\leqslant N_s^{c^3}}|T_{\Lambda'}^{-1}(x,y)|+\sum_{\|x-y\|>N_s^{c^3}}|T_{\Lambda'}^{-1}(x,y)|\\ &\leqslant N_s^{c^3}\cdot\|T_{\Lambda'}^{-1}\|+\sum_{\|x-y\|>N_s^{c^3}}\mathrm{e}^{-\gamma_s\|x-y\|_1}\\ &\leqslant 2N_s^{c^3}\delta_{s-1}^{-3}\delta_s^{-2}\\ &<\frac{1}{2}\delta_s^{-3}\end{aligned}$$

and

$$\sum_{w\in\partial_\Lambda^-\Lambda'}|T_{\Lambda'}^{-1}(x,w)|\leqslant\sum_{\|x-w\|_1\geqslant N_{s+1}}\mathrm{e}^{-\gamma_s\|x-w\|_1}<\mathrm{e}^{-\frac{1}{2}\gamma_sN_{s+1}},$$

we get

$$\begin{aligned}\sum_{y\in\Lambda}|T_\Lambda^{-1}(x,y)| &\leqslant \sum_{y\in\Lambda'}|T_{\Lambda'}^{-1}(x,y)\chi_{\Lambda'}(y)|+\sum_{y\in\Lambda,(w,w')\in\partial_\Lambda\Lambda'}|T_{\Lambda'}^{-1}(x,w)\Gamma(w,w')T_\Lambda^{-1}(w',y)|\\ &\leqslant\frac{1}{2}\delta_s^{-3}+2d\sum_{w\in\partial_\Lambda^-\Lambda'}|T_{\Lambda'}^{-1}(x,w)|\cdot\sup_{w'\in\Lambda}\sum_{y\in\Lambda}|T_\Lambda^{-1}(w',y)|\\ &\leqslant\frac{1}{2}\delta_s^{-3}+\frac{1}{10}\sup_{w'\in\Lambda}\sum_{y\in\Lambda}|T_\Lambda^{-1}(w',y)|.\end{aligned}$$

(2) Assume that $x\in 2\Omega_k^{s+1}$ for some $k\in\widetilde{P}_{s+1}$. Then, by (3.107), we have $\widetilde{\Omega}_k^{s+1}\subset\Lambda$ and $k\notin Q_{s+1}$. For $y\in\Lambda$, using the resolvent identity shows

$$T_\Lambda^{-1}(x,y)=T_{\widetilde{\Omega}_k^{s+1}}^{-1}(x,y)\chi_{\widetilde{\Omega}_k^{s+1}}(y)-\sum_{(w,w')\in\partial_\Lambda\widetilde{\Omega}_k^{s+1}}T_{\widetilde{\Omega}_k^{s+1}}^{-1}(x,w)\Gamma(w,w')T_\Lambda^{-1}(w',y).$$

By (3.103), (3.109) and

$$N_{s+1}<\operatorname{diam}\widetilde{\Omega}_k^{s+1}\lesssim\operatorname{dist}(x,\partial_\Lambda^-\widetilde{\Omega}_k^{s+1}),$$

we have

$$\begin{aligned}\sum_{y\in\Lambda}|T_\Lambda^{-1}(x,y)| &\leqslant\sum_{y\in\Lambda}|T_{\widetilde{\Omega}_k^{s+1}}^{-1}(x,y)\chi_{\widetilde{\Omega}_k^{s+1}}(y)|+\sum_{y\in\Lambda,(w,w')\in\partial_\Lambda\widetilde{\Omega}_k^{s+1}}|T_{\widetilde{\Omega}_k^{s+1}}^{-1}(x,w)\Gamma(w,w')T_\Lambda^{-1}(w',y)|\\ &<\#\widetilde{\Omega}_k^{s+1}\cdot\|T_{\widetilde{\Omega}_k^{s+1}}^{-1}\|+CN_{s+1}^{c^2d}\mathrm{e}^{-\widetilde{\gamma}_sN_{s+1}}\sup_{w'\in\Lambda}\sum_{y\in\Lambda}|T_\Lambda^{-1}(w',y)|\\ &<CN_{s+1}^{c^2d}\delta_s^{-2}\|\theta+k\cdot\omega-\theta_{s+1}\|^{-1}\cdot\|\theta+k\cdot\omega+\theta_{s+1}\|^{-1}+\frac{1}{10}\sup_{w'\in\Lambda}\sum_{y\in\Lambda}|T_\Lambda^{-1}(w',y)|\\ &<\frac{1}{2}\delta_s^{-3}\|\theta+k\cdot\omega-\theta_{s+1}\|^{-1}\cdot\|\theta+k\cdot\omega+\theta_{s+1}\|^{-1}+\frac{1}{10}\sup_{w'\in\Lambda}\sum_{y\in\Lambda}|T_\Lambda^{-1}(w',y)|.\end{aligned}$$

Combining the above two cases, we obtain

$$\begin{aligned}\|T_\Lambda^{-1}\| &\leqslant\sup_{x\in\Lambda}\sum_{y\in\Lambda}|T_\Lambda^{-1}(x,y)|\\ &<\delta_s^{-3}\sup_{\{k\in P_{s+1}:\widetilde{\Omega}_k^{s+1}\subset\Lambda\}}\|\theta+k\cdot\omega-\theta_{s+1}\|^{-1}\cdot\|\theta+k\cdot\omega+\theta_{s+1}\|^{-1}.\end{aligned}\tag{3.111}$$

Finally, we turn to the off-diagonal decay estimates. From (3.11), (3.107) and (3.108), it follows that for $k'\in\widetilde{P}_t\cap Q_t$ ($1\leqslant t\leqslant s$), there exists a $k\in\widetilde{P}_{t+1}$ such that

$$\widetilde{\Omega}_{k'}^t\subset\Omega_k^{t+1}$$

and

$$\widetilde{P}_{s+1} \cap Q_{s+1} = \emptyset.$$

Moreover,

$$\bigcup_{1\leqslant t\leqslant s+1} \bigcup_{k\in\widetilde{P}_t} \widetilde{\Omega}_k^t \subset \Lambda.$$

Hence for any $w \in \Lambda$, if

$$w \in \bigcup_{k\in\widetilde{P}_1} 2\Omega_k^1,$$

then there exists a $t \in [1, s+1]$ such that

$$w \in \bigcup_{k\in\widetilde{P}_t\setminus Q_t} 2\Omega_k^t.$$

For every $w \in \Lambda$, define its block in Λ:

$$J_w = \begin{cases} \Lambda_{\frac{1}{2}N_1}(w) \cap \Lambda & \text{if } w \notin \bigcup_{k\in\widetilde{P}_1} 2\Omega_k^1, \quad \text{(i)} \\ \widetilde{\Omega}_k^t & \text{if } w \in 2\Omega_k^t \text{ for some } k \in \widetilde{P}_t \setminus Q_t. \quad \text{(ii)} \end{cases}$$

Then, $\operatorname{diam} J_w \leqslant \operatorname{diam} \widetilde{\Omega}_k^{s+1} < 3N_{s+1}^{c^2}$. For (i), we have $J_w \cap Q_0 = \emptyset$ and $\operatorname{dist}(w, \partial_\Lambda^- J_w) \geqslant \frac{1}{2}N_1$. Thus,

$$|T_{J_w}^{-1}(w, w')| < \mathrm{e}^{-\gamma_0\|w-w'\|_1} \quad \text{for } w' \in \partial_\Lambda^- J_w.$$

For (ii), by (3.109), we have

$$|T_{J_w}^{-1}(w, w')| < \mathrm{e}^{-\widetilde{\gamma}_{t-1}\|w-w'\|_1} \quad \text{for } w' \in \partial_\Lambda^- J_w.$$

Let $\|x-y\| > N_{s+1}^{c^3}$. The resolvent identity reads as

$$T_\Lambda^{-1}(x, y) = T_{J_x}^{-1}(x, y)\chi_{J_x}(y) - \sum_{(w,w')\in\partial_\Lambda J_x} T_{J_x}^{-1}(x, w)\Gamma(w, w')T_\Lambda^{-1}(w', y).$$

The first term on the RHS of the above identity is zero since $\|x-y\| > N_{s+1}^{c^3} > 3N_{s+1}^{c^2}$ (so that $y \notin J_x$). It follows that

$$\begin{aligned} |T_\Lambda^{-1}(x, y)| &\leqslant CN_{s+1}^{c^2 d}\mathrm{e}^{-\min_t(\gamma_0(1-2N_1^{-1}),\widetilde{\gamma}_{t-1}(1-N_t^{-1}))\|x-x_1\|_1}|T_\Lambda^{-1}(x_1, y)| \\ &\leqslant CN_{s+1}^{c^2 d}\mathrm{e}^{-\widetilde{\gamma}_s(1-N_{s+1}^{-1})\|x-x_1\|_1}|T_\Lambda^{-1}(x_1, y)| \\ &< \mathrm{e}^{-\widetilde{\gamma}_s(1-N_{s+1}^{-1}-\frac{C\log N_{s+1}}{N_{s+1}})\|x-x_1\|_1}|T_\Lambda^{-1}(x_1, y)| \\ &< \mathrm{e}^{-\gamma_s(1-N_{s+1}^{\frac{1}{c}-1})^2\|x-x_1\|_1}|T_\Lambda^{-1}(x_1, y)| \\ &= \mathrm{e}^{-\gamma_s'\|x-x_1\|_1}|T_\Lambda^{-1}(x_1, y)| \end{aligned}$$

for some $x_1 \in \partial_\Lambda^+ J_x$, where $\gamma_s' = \gamma_s(1-N_{s+1}^{\frac{1}{c}-1})^2$. Iterate the above procedure and stop it if for some L, $\|x_L - y\| < 3N_{s+1}^{c^2}$. Recalling (3.110) and (3.111), we get

$$\begin{aligned} |T_\Lambda^{-1}(x, y)| &\leqslant \mathrm{e}^{-\gamma_s'\|x-x_1\|_1} \cdots \mathrm{e}^{-\gamma_s'\|x_{L-1}-x_L\|_1}|T_\Lambda^{-1}(x_L, y)| \\ &\leqslant \mathrm{e}^{-\gamma_s'(\|x-y\|_1-3N_{s+1}^{c^2})}\|T_\Lambda^{-1}\| \\ &< \mathrm{e}^{-\gamma_s'(1-3N_{s+1}^{c^2-c^3})\|x-y\|_1}\delta_{s+1}^{-3} \\ &< \mathrm{e}^{-\gamma_s'(1-3N_{s+1}^{c^2-c^3}-3\frac{|\log\delta_{s+1}|}{N_{s+1}^{c^3}})\|x-y\|_1} \end{aligned}$$

$$< \mathrm{e}^{-\gamma_s'(1-N_{s+1}^{\frac{1}{c}-1})\|x-y\|_1}$$
$$= \mathrm{e}^{-\gamma_{s+1}\|x-y\|_1}.$$

This gives the off-diagonal decay estimates.

We have completed the proof of Theorem 3.2. □

4 Arithmetic Anderson localization

As an application of Green's function estimates of the previous section, we prove the arithmetic version of Anderson localization below.

Proof of Theorem 1.2. First, recall

$$\Theta_{\tau_1} = \{(\theta,\omega) \in \mathbb{T}\times\mathcal{R}_{\tau,\gamma} : \text{the relation } \|2\theta+n\cdot\omega\| \leqslant \mathrm{e}^{-\|n\|^{\tau_1}} \text{ holds for finitely many } n\in\mathbb{Z}^d\},$$

where $0<\tau_1<\tau$.

We prove that for $0<\varepsilon\leqslant\varepsilon_0$, $\omega\in\mathcal{R}_{\tau,\gamma}$ and $(\theta,\omega)\in\Theta_{\tau_1}$, $H(\theta)$ has the only pure point spectrum with exponentially decaying eigenfunctions. Let ε_0 be given by Theorem 3.2. Fix ω and θ so that $\omega\in\mathcal{R}_{\tau,\gamma}$ and $(\theta,\omega)\in\Theta_{\tau_1}$. Let $E\in[-2,2]$ be a generalized eigenvalue of $H(\theta)$ and $u=\{u(n)\}_{n\in\mathbb{Z}^d}\neq 0$ be the corresponding generalized eigenfunction satisfying $|u(n)|\leqslant(1+\|n\|)^d$. From Schnol's theorem, it suffices to show that u decays exponentially. For this purpose, note first that there exists (since $(\theta,\omega)\in\Theta_{\tau_1}$) some $\widetilde{s}\in\mathbb{N}$ such that

$$\|2\theta+n\cdot\omega\| > \mathrm{e}^{-\|n\|^{\tau_1}} \quad \text{for all } n \text{ satisfying } \|n\|\geqslant N_{\widetilde{s}}. \tag{4.1}$$

We claim that there exists an $s_0>0$ such that for $s\geqslant s_0$,

$$\Lambda_{2N_s^{c^4}} \cap \left(\bigcup_{k\in Q_s}\widetilde{\Omega}_k^s\right) \neq \emptyset. \tag{4.2}$$

Otherwise, there exists a subsequence $s_i\to+\infty$ (as $i\to\infty$) such that

$$\Lambda_{2N_{s_i}^{c^4}} \cap \left(\bigcup_{k\in Q_{s_i}}\widetilde{\Omega}_k^{s_i}\right) = \emptyset. \tag{4.3}$$

Then, we can enlarge $\Lambda_{N_{s_i}^{c^4}}$ to $\widetilde{\Lambda}_i$ satisfying

$$\Lambda_{N_{s_i}^{c^4}} \subset \widetilde{\Lambda}_i \subset \Lambda_{N_{s_i}^{c^4}+50N_{s_i}^{c^2}}$$

and

$$\widetilde{\Lambda}_i\cap\widetilde{\Omega}_k^{s'}\neq\emptyset \Rightarrow \widetilde{\Omega}_k^{s'}\subset\widetilde{\Lambda}_i \quad \text{for } s'\leqslant s \text{ and } k\in P_{s'}.$$

From (4.3), we have

$$\widetilde{\Lambda}_i \cap \left(\bigcup_{k\in Q_{s_i}}\widetilde{\Omega}_k^{s_i}\right) = \emptyset,$$

which shows that $\widetilde{\Lambda}_i$ is s_i-**good**. As a result, for $n\in\Lambda_{N_{s_i}}$, since $\mathrm{dist}(n,\partial^-\widetilde{\Lambda}_{N_{s_i}^{c^4}})\geqslant\frac{1}{2}N_{s_i}^{c^4}>N_{s_i}^{c^3}$, we have

$$\begin{aligned} |u(n)| &\leqslant \sum_{(n',n'')\in\partial\widetilde{\Lambda}_i} |T^{-1}_{\widetilde{\Lambda}_{N_{s_i}^{c^4}}}(n,n')u(n'')| \\ &\leqslant 2d\sum_{n'\in\partial^-\widetilde{\Lambda}_i}|T^{-1}_{\widetilde{\Lambda}_i}(n,n')|\cdot\sup_{n''\in\partial^+\widetilde{\Lambda}_i}|u(n'')| \\ &\leqslant CN_{s_i}^{2c^4d}\cdot\mathrm{e}^{-\frac{1}{2}\gamma_\infty N_{s_i}^{c^4}} \to 0. \end{aligned}$$

From $N_{s_i} \to +\infty$, it follows that $u(n) = 0, \forall\, n \in \mathbb{Z}^d$. This contradicts $u \neq 0$, and the claim is proved.

Next, define

$$U_s = \Lambda_{8N_{s+1}^{c^4}} \backslash \Lambda_{4N_s^{c^4}}, \quad U_s^* = \Lambda_{10N_{s+1}^{c^4}} \backslash \Lambda_{3N_s^{c^4}}.$$

We can also enlarge U_s^* to $\widetilde{U}_s^*$ so that

$$U_s^* \subset \widetilde{U}_s^* \subset \Lambda_{50N_s^{c^2}}(U_s^*)$$

and

$$\widetilde{U}_s^* \cap \widetilde{\Omega}_k^{s'} \neq \emptyset \Rightarrow \widetilde{\Omega}_k^{s'} \subset \widetilde{U}_s^* \quad \text{for } s' \leqslant s \text{ and } k \in P_{s'}.$$

Let n satisfy $\|n\| > \max(4N_{\widetilde{s}}^{c^4}, 4N_{s_0}^{c^4})$. Then, there exists some $s \geqslant \max(\widetilde{s}, s_0)$ such that

$$n \in U_s. \tag{4.4}$$

Without loss of generality, by (4.2), we may assume

$$\Lambda_{2N_s^{c^4}} \cap \widetilde{\Omega}_k^s \neq \emptyset$$

for some $k \in Q_s^+$. Then, for $k \neq k' \in Q_s^+$, we have

$$\|k - k'\| > \left|\log \frac{\gamma}{2\delta_s}\right|^{\frac{1}{\tau}} \gtrsim N_{s+1}^{c^5} \gg \operatorname{diam} \widetilde{U}_s^*.$$

Thus,

$$\widetilde{U}_s^* \cap \left(\bigcup_{l \in Q_s^+} \widetilde{\Omega}_l^s \right) = \emptyset.$$

Now, if there exists an $l \in Q_s^-$ such that

$$\widetilde{U}_s^* \cap \widetilde{\Omega}_l^s \neq \emptyset,$$

then

$$N_s < N_s^{c^4} - 100N_s^{c^2} \leqslant \|l\| - \|k\| \leqslant \|l + k\| \leqslant \|l\| + \|k\| < 11N_{s+1}^{c^4}.$$

Recalling

$$Q_s \subset P_s \subset \mathbb{Z}^d + \frac{1}{2}\sum_{i=0}^{s-1} l_i,$$

we have $l + k \in \mathbb{Z}^d$. Hence by (4.1),

$$\begin{aligned} \mathrm{e}^{-(11N_{s+1}^{c^4})^{\tau_1}} &< \|2\theta + (l+k)\cdot\omega\| \\ &\leqslant \|\theta + l\cdot\omega - \theta_s\| + \|\theta + k\cdot\omega + \theta_s\| < 2\delta_s. \end{aligned}$$

This contradicts

$$|\log \delta_s| \sim N_{s+1}^{c^5\tau} \gg N_{s+1}^{c^4\tau_1}.$$

We thus have shown

$$\widetilde{U}_s^* \cap \left(\bigcup_{l \in Q_s} \widetilde{\Omega}_l^s \right) = \emptyset.$$

This implies that $\widetilde{U}_s^*$ is s-**good**.

Finally, recalling (4.4), we have

$$\operatorname{dist}(n, \partial^- \widetilde{U}_s^*) \geqslant \min(10N_{s+1}^{c^4} - |n|, |n| - 3N_s^{c^4}) - 1 \geqslant \frac{1}{5}\|n\| > N_s^{c^3}.$$

Then,

$$|u(n)| \leqslant \sum_{(n',n'') \in \partial \widetilde{U}_s^*} |T_{\widetilde{U}_s^*}^{-1}(n,n') u(n'')|$$

$$
\begin{aligned}
&\leqslant 2d \sum_{n' \in \partial^- \widetilde{U}_s^*} |T^{-1}_{\widetilde{U}_s^*}(n,n')| \cdot \sup_{n'' \in \partial^+ \widetilde{U}_s^*} |u(n'')| \\
&\leqslant C N_{s+1}^{2c^4 d} \cdot \mathrm{e}^{-\frac{1}{5}\gamma_\infty \|n\|} \\
&\leqslant C \|n\|^{2c^5 d} \cdot \mathrm{e}^{-\frac{1}{5}\gamma_\infty \|n\|} \\
&< \mathrm{e}^{-\frac{1}{6}\gamma_\infty \|n\|},
\end{aligned}
$$

which yields the exponential decay u.

We complete the proof of Theorem 1.2. □

Remark 4.1. Assume that for some $E \in [-2,2]$, the inductive process stops at a finite stage (i.e., $Q_s = \emptyset$ for some $s < \infty$). Then, for $N > N_s^{c^5}$, we can enlarge Λ_N to $\widetilde{\Lambda}_N$ with

$$\Lambda_N \subset \widetilde{\Lambda}_N \subset \Lambda_{N+50N_s^{c^2}}$$

and

$$\widetilde{\Lambda}_N \cap \widetilde{\Omega}_k^{s'} \neq \emptyset \Rightarrow \widetilde{\Omega}_k^{s'} \subset \widetilde{\Lambda}_N \quad \text{for } s' \leqslant s \text{ and } k \in P_{s'}.$$

Thus $\widetilde{\Lambda}_N$ is s-**good**. For $n \in \Lambda_{N^{\frac{1}{2}}}$, since $\operatorname{dist}(n, \partial^- \widetilde{\Lambda}_N) > N_s^{c^3}$, we have

$$
\begin{aligned}
|u(n)| &\leqslant \sum_{(n',n'') \in \partial \widetilde{\Lambda}_N} |T^{-1}_{\widetilde{\Lambda}_N}(n,n') u(n'')| \\
&\leqslant 2d \sum_{n' \in \partial^- \widetilde{\Lambda}_N} |T^{-1}_{\widetilde{\Lambda}_N}(n,n')| \cdot \sup_{n'' \in \partial^+ \widetilde{\Lambda}_N} |u(n'')| \\
&\leqslant C N^{2d} \cdot \mathrm{e}^{-\frac{1}{2}\gamma_\infty N} \to 0.
\end{aligned}
$$

Hence, such an E is not a generalized eigenvalue of $H(\theta)$.

5 $(\frac{1}{2}-)$-Hölder continuity of the IDS

In this section, we apply our estimates to obtaining $(\frac{1}{2}-)$-Hölder continuity of the IDS.

Proof of Theorem 1.4. Let T be given by (3.1). Fix $\mu > 0$, $\theta \in \mathbb{T}$ and $E \in [-2,2]$. Let ε_0 be defined in Theorem 3.2 and assume $0 < \varepsilon \leqslant \varepsilon_0$. Fix

$$0 < \eta < \eta_0 = \min(\mathrm{e}^{-(\frac{4}{\mu})^{\frac{c}{c-1}}}, \mathrm{e}^{-|\log \delta_0|^c}). \tag{5.1}$$

Denote by $\{\xi_r : r = 1, \ldots, R\} \subset \operatorname{span}(\delta_n : n \in \Lambda_N)$ the ℓ^2-orthonormal eigenvectors of T_{Λ_N} with the eigenvalues belonging to $[-\eta, \eta]$. We aim to prove that for sufficiently large N (depending on η),

$$R \leqslant (\#\Lambda_N) \eta^{\frac{1}{2}-\mu}.$$

From (5.1), we can choose $s \geqslant 1$ such that

$$|\log \delta_{s-1}|^c \leqslant |\log \eta| < |\log \delta_s|^c.$$

Enlarge Λ_N to $\widetilde{\Lambda}_N$ so that

$$\Lambda_N \subset \widetilde{\Lambda}_N \subset \Lambda_{N+50N_s^{c^2}}$$

and

$$\widetilde{\Lambda}_N \cap \widetilde{\Omega}_k^{s'} \neq \emptyset \Rightarrow \widetilde{\Omega}_k^{s'} \subset \widetilde{\Lambda}_N \quad \text{for } s' \leqslant s \text{ and } k \in P_{s'}.$$

Furthermore, define

$$\mathcal{K} = \left\{ k \in P_s : \widetilde{\Omega}_k^s \subset \widetilde{\Lambda}_N, \min_{\sigma = \pm 1} (\|\theta + k \cdot \omega + \sigma \theta_s\|) < \eta^{\frac{1}{2} - \frac{\mu}{2}} \right\}$$

and

$$\widetilde{\Lambda}'_N = \widetilde{\Lambda}_N \setminus \bigcup_{k\in\mathcal{K}} \Omega_k^s.$$

Thus by (3.10), we obtain

$$k' \in Q_{s'}, \quad \widetilde{\Omega}_{k'}^{s'} \subset \widetilde{\Lambda}'_N, \quad \widetilde{\Omega}_{k'}^{s'} \subset \Omega_k^{s'+1} \Rightarrow \widetilde{\Omega}_k^{s'+1} \subset \widetilde{\Lambda}'_N \quad \text{for } s' < s.$$

Since

$$|\log\eta| < |\log\delta_s|^c \sim |\log\delta_{s-1}|^{c^6} \sim N_s^{c^{11}\tau} < N_s^{\frac{1}{c}},$$

we obtain that from the resolvent identity,

$$\begin{aligned}\|T_{\widetilde{\Lambda}'_N}^{-1}\| &< \delta_{s-1}^{-3} \sup_{\{k\in P_s:\widetilde{\Omega}_k^s\subset\widetilde{\Lambda}'_N\}} \|\theta + k\cdot\omega - \theta_s\|^{-1}\cdot\|\theta+k\cdot\omega+\theta_s\|^{-1}\\ &< \delta_{s-1}^{-3}\eta^{\mu-1} < \frac{1}{2}\eta^{-1},\end{aligned} \tag{5.2}$$

where the last inequality follows from (5.1).

By the uniform distribution of $\{n\cdot\omega\}_{n\in\mathbb{Z}^d}$ in $\mathbb{T}$, we have

$$\begin{aligned}\#(\widetilde{\Lambda}_N \setminus \widetilde{\Lambda}'_N) &\leqslant \#\Omega_k^s \cdot \#\mathcal{K}\\ &\leqslant CN_s^{cd}\cdot\#\left\{k\in\mathbb{Z}+\sum_{i=0}^{s-1} l_i : \|k\| \leqslant N + 50N_s^{c^2}, \min_{\sigma=\pm1}(\|\theta+k\cdot\omega+\sigma\theta_s\|) < \eta^{\frac{1}{2}-\frac{\mu}{2}}\right\}\\ &\leqslant CN_s^{cd}\cdot\eta^{\frac{1}{2}-\frac{\mu}{2}}(N+50N_s^{c^2})^d\\ &\leqslant CN_s^{cd}\cdot\eta^{\frac{1}{2}-\frac{\mu}{2}}\#\Lambda_N\end{aligned}$$

for sufficiently large N.

For a vector $\xi\in\mathbb{C}^\Lambda$ with $\Lambda\subset\mathbb{Z}^d$, we define $\|\xi\|$ to be the ℓ^2-norm. Assume that $\xi\in\{\xi_r : r\leqslant R\}$ is an eigenvector of T_{Λ_N}. Then,

$$\|T_{\Lambda_N}\xi\| = \|R_{\Lambda_N}T\xi\| \leqslant \eta.$$

Hence,

$$\eta \geqslant \|R_{\widetilde{\Lambda}'_N}T_{\Lambda_N}\xi\| = \|R_{\widetilde{\Lambda}'_N}TR_{\widetilde{\Lambda}'_N}\xi + R_{\widetilde{\Lambda}'_N}TR_{\Lambda_N\setminus\widetilde{\Lambda}'_N}\xi - R_{\widetilde{\Lambda}'_N\setminus\Lambda_N}T\xi\|. \tag{5.3}$$

Applying $T_{\widetilde{\Lambda}'_N}^{-1}$ to (5.3) and (5.2) implies

$$\|R_{\widetilde{\Lambda}'_N}\xi + T_{\widetilde{\Lambda}'_N}^{-1}(R_{\widetilde{\Lambda}'_N}TR_{\Lambda_N\setminus\widetilde{\Lambda}'_N}\xi - R_{\widetilde{\Lambda}'_N\setminus\Lambda_N}T\xi)\| < \frac{1}{2}. \tag{5.4}$$

Define

$$H = \mathrm{Range}(T_{\widetilde{\Lambda}'_N}^{-1}(R_{\widetilde{\Lambda}'_N}TR_{\Lambda_N\setminus\widetilde{\Lambda}'_N} - R_{\widetilde{\Lambda}'_N\setminus\Lambda_N}T)).$$

Then,

$$\begin{aligned}\dim H &\leqslant \mathrm{Rank}(T_{\widetilde{\Lambda}'_N}^{-1}(R_{\widetilde{\Lambda}'_N}TR_{\Lambda_N\setminus\widetilde{\Lambda}'_N} - R_{\widetilde{\Lambda}'_N\setminus\Lambda_N}T))\\ &\leqslant \#(\widetilde{\Lambda}_N\setminus\widetilde{\Lambda}'_N) + \#(\widetilde{\Lambda}_N\setminus\Lambda_N)\\ &\leqslant CN_s^{cd}\cdot\eta^{\frac{1}{2}-\frac{\mu}{2}}\#\Lambda_N + CN_s^{c^2d}N^{d-1}\\ &\leqslant CN_s^{cd}\cdot\eta^{\frac{1}{2}-\frac{\mu}{2}}\#\Lambda_N.\end{aligned}$$

Denote by P_H the orthogonal projection to H. Applying $I-P_H$ to (5.4), we get

$$\|R_{\widetilde{\Lambda}'_N}\xi - P_HR_{\widetilde{\Lambda}'_N}\xi\|^2 = \|R_{\widetilde{\Lambda}'_N}\xi\|^2 - \|P_HR_{\widetilde{\Lambda}'_N}\xi\|^2 \leqslant \frac{1}{4}.$$

Before concluding the proof, we need a useful lemma.

Lemma 5.1. *Let H be a Hilbert space, and H_1 and H_2 be its subspaces. Let $\{\xi_r : r = 1, \ldots, R\}$ be a set of orthonormal vectors. Then, we have*

$$\sum_{r=1}^{R} \|P_{H_1} P_{H_2} \xi_r\|^2 \leqslant \dim H_1.$$

Proof. Denote by $\langle \cdot, \cdot \rangle$ the inner product on H. Let $\{\phi_i\}$ be the orthonormal basis of H_1. By Parseval's equality and Bessel's inequality, we have

$$\begin{aligned}\sum_{r=1}^{R} \|P_{H_1} P_{H_2} \xi_r\|^2 &= \sum_{r=1}^{R} \sum_{i} |\langle \phi_i, P_{H_2} \xi_r \rangle|^2 \\ &= \sum_{i} \sum_{r=1}^{R} |\langle P_{H_2} \phi_i, \xi_r \rangle|^2 \\ &\leqslant \sum_{i} \|P_{H_2} \phi_i\|^2 \\ &\leqslant \sum_{i} \|\phi_i\|^2 \leqslant \dim H_1.\end{aligned}$$

This completes the proof. □

Finally, it follows from Lemma 5.1 that

$$\begin{aligned}R = \sum_{r=1}^{R} \|\xi_r\|^2 &= \sum_{r=1}^{R} \|R_{\widetilde{\Lambda}'_N} \xi_r\|^2 + \sum_{r=1}^{R} \|R_{\Lambda_N \setminus \widetilde{\Lambda}'_N} \xi_r\|^2 \\ &\leqslant \frac{1}{4} R + \sum_{r=1}^{R} (\|P_H R_{\widetilde{\Lambda}'_N} \xi_r\|^2 + \|R_{\Lambda_N \setminus \widetilde{\Lambda}'_N} \xi_r\|^2) \\ &\leqslant \frac{1}{4} R + \dim H + \#(\Lambda_N \setminus \widetilde{\Lambda}'_N) \\ &\leqslant \frac{1}{4} R + C N_s^{cd} \cdot \eta^{\frac{1}{2} - \frac{\mu}{2}} \# \Lambda_N.\end{aligned}$$

Hence, we get

$$R \leqslant C N_s^{cd} \cdot \eta^{\frac{1}{2} - \frac{\mu}{2}} \# \Lambda_N \leqslant \eta^{\frac{1}{2} - \mu} \# \Lambda_N.$$

We finish the proof of Theorem 1.4. □

Remark 5.2. In the above proof, if the inductive process stops at a finite stage (i.e., $Q_s = \emptyset$ for some s) and $|\log \delta_s|^c \leqslant |\log \eta|$, then $\widetilde{\Lambda}_N$ is s-**good** and

$$\|T_{\widetilde{\Lambda}_N}^{-1}\| < \delta_{s-1}^{-3} \delta_s^{-2} < \frac{1}{2} \eta^{-1},$$

which implies

$$R \leqslant \frac{4}{3} \#(\widetilde{\Lambda}_N \setminus \Lambda_N) \leqslant C N_s^{c^2 d} N^{-1} \# \Lambda_N.$$

Letting $N \to \infty$, we get $\mathcal{N}(E + \eta) - \mathcal{N}(E - \eta) = 0$, which means $E \notin \sigma(H(\theta))$.

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Appendix A

Proof of Remark 3.4. Let $i \in Q_0^+$ and $j \in \widetilde{Q}_0^-$ satisfy

$$\|\theta + i\cdot\omega + \theta_0\| < \delta_0, \quad \|\theta + j\cdot\omega - \theta_0\| < \delta_0^{\frac{1}{100}}.$$

Then, (1.4) implies that $1, \omega_1, \dots, \omega_d$ are rationally independent and $\{k\cdot\omega\}_{k\in\mathbb{Z}^d}$ is dense in $\mathbb{T}$. Thus, there exists a $k \in \mathbb{Z}^d$ such that $\|2\theta + k\cdot\omega\|$ is sufficiently small with

$$\begin{aligned}\|\theta + (k-j)\cdot\omega + \theta_0\| &\leqslant \|2\theta + k\cdot\omega\| + \|\theta + j\cdot\omega - \theta_0\| < \delta_0^{\frac{1}{100}},\\ \|\theta + (k-i)\cdot\omega - \theta_0\| &\leqslant \|2\theta + k\cdot\omega\| + \|\theta + i\cdot\omega + \theta_0\| < \delta_0.\end{aligned}$$

We then obtain $k - j \in \widetilde{Q}_0^+$ and $k - i \in Q_0^-$, which imply

$$\mathrm{dist}(\widetilde{Q}_0^+, Q_0^-) \leqslant \mathrm{dist}(\widetilde{Q}_0^-, Q_0^+).$$

A similar argument shows

$$\mathrm{dist}(\widetilde{Q}_0^+, Q_0^-) \geqslant \mathrm{dist}(\widetilde{Q}_0^-, Q_0^+).$$

We have shown $\mathrm{dist}(\widetilde{Q}_0^+, Q_0^-) = \mathrm{dist}(\widetilde{Q}_0^-, Q_0^+)$. □

Appendix B

Lemma B.1 (Schur complement lemma). *Let $A \in \mathbb{C}^{d_1\times d_1}$, $D \in \mathbb{C}^{d_2\times d_2}$, $B \in \mathbb{C}^{d_1\times d_2}$ and $D \in \mathbb{C}^{d_2\times d_1}$ be matrices and*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Furthermore, assume that A is invertible and $\|B\|, \|C\| \leqslant 1$. Then, we have

(1)

$$\det M = \det A \cdot \det S,$$

where

$$S = D - CA^{-1}B$$

is called the Schur complement of A;

(2) *M is invertible if and only if S is invertible, and*

$$\|S^{-1}\| \leqslant \|M^{-1}\| < 4(1 + \|A^{-1}\|)^2(1 + \|S^{-1}\|). \tag{B.1}$$

Proof. Direct computation shows

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix},$$

which implies (B.1). □

Appendix C

Lemma C.1. *Let $l \in \frac{1}{2}\mathbb{Z}^d$ and $\Lambda \subset \mathbb{Z}^d + l$ be a finite set which is symmetrical about the origin (i.e., $n \in \Lambda \Leftrightarrow -n \in \Lambda$). Then,*

$$\det T(z)_\Lambda = \det(\cos 2\pi(z + n \cdot \omega)\delta_{n,n'} - E + \varepsilon\Delta)_{n\in\Lambda}$$

is an even function of z.

Proof. Define the unitary map

$$U_\Lambda : \ell^2(\Lambda) \to \ell^2(\Lambda) \quad \text{with } (U\phi)(n) = \phi(-n).$$

Then,

$$U_\Lambda^{-1} T(z)_\Lambda U_\Lambda = (\cos 2\pi(z - n \cdot \omega)\delta_{n,n'} - E + \varepsilon\Delta)_{n\in\Lambda} = T(-z)_\Lambda,$$

which implies $\det T(z)_\Lambda = \det T(-z)_\Lambda$. □