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Time-inconsistent stochastic linear-quadratic control problem with indefinite control weight costs

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Abstract A time-inconsistent linear-quadratic optimal control problem for stochastic differential equations is studied. We introduce conditions where the control cost weighting matrix is possibly singular. Under such conditions, we obtain a family of closed-loop equilibrium strategies via multi-person differential games. This result extends Yong's work (2017) in the case of stochastic differential equations, where a unique closed-loop equilibrium strategy can be derived under standard conditions (namely, the control cost weighting matrix is uniformly positive definite, and the other weighting coefficients are positive semidefinite).

Keywords linear-quadratic optimal control problem, time-inconsistent cost functional, generalized Riccati equation, indefinite control weight cost, closed-loop equilibrium strategy

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1 Introduction

Linear-quadratic optimal control problems (LQ problems for short) are an important class of optimal control problems, which have been studied extensively due to their importance and wide applications. In the early stages of these studies, the uniform positive definiteness of the control weighting matrix and the positive semidefiniteness of the other weighting coefficients, which are referred to as *standard* conditions, have been taken for granted in the literature [4, 5, 9, 15]. Under standard conditions, one can obtain a unique closed-loop optimal control via a corresponding Riccati equation in the deterministic case [1] and the stochastic case [31]. Later on, a great deal of research is devoted to relaxing the standard conditions (especially the uniform positive definiteness of the control weighting matrix). In the deterministic case, there has been tremendous interest in studying the so-called singular LQ problems (see, e.g., [8]), in which the control weighting matrix is possibly singular. In the stochastic case, Chen et al. [7] found that stochastic LQ problems could be solvable even if the control weighting matrix is negative definite. The reason is that the control acting on the diffusion term could bring a positive effect, which compensates the negative effect of the control weighting matrix to some extent. This fundamental observation stimulated a series of studies devoted to the solvability of the indefinite stochastic LQ problems (see [20, 26] and the references therein).

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A key feature of the above-mentioned studies is time consistency. Roughly speaking, it means that the optimal control we choose now will still be optimal in the future. However, such a nice property may not hold for many practical control problems. The main reason for that is people's subjective time preferences, which will cause a time-inconsistent phenomenon (see, e.g., [11, 12, 16, 18, 19, 23]). When the time consistency property is broken, the corresponding control problem is called a time-inconsistent control problem. There have been extensive studies for time-inconsistent control problems (see, e.g., [6, 10, 11, 13, 14, 17, 28, 29] and the references therein). To treat time inconsistency, researchers introduced notions of the *equilibrium control* and *closed-loop equilibrium strategy* (see, e.g., [14, 29, 30]). We are more interested in the closed-loop equilibrium strategy, as the control is in a closed-loop form. In this aspect, Yong [30] constructed such a closed-loop equilibrium strategy for time-inconsistent stochastic LQ problems under standard conditions. The basic idea is to use *multi-person differential games* to obtain a *Nash equilibrium* in discrete time, which is a kind of approximate time-consistent strategy, and take a limit of it later.

In this paper, we consider the problem of seeking conditions weaker than standard ones to guarantee the existence of closed-loop equilibrium strategies. We use the methods of multi-person differential games as in [30] to study this problem. The reason is that such constructive methods can allow us to make use of properties of stochastic LQ problems rather than treating them directly, which seems to be tough. However, some new difficulties occur when we want to make this extension. We list these difficulties in the following and explain how to overcome them briefly.

(1) In the multi-person differential games, a family of stochastic LQ problems is introduced. Hence, to relax the standard conditions in the time-inconsistent setting, we have to relax the standard conditions for stochastic LQ problems first. As we know, the uniform convexity condition is a candidate for this. However, it seems not ideal for our purpose due to the following reason: for stochastic LQ problems, the uniform convexity condition is imposed on the coefficients of state equations and cost functionals implicitly, while for our problems, we would have a family of stochastic LQ problems defined inductively. It is quite unclear how weaker conditions are put to the original problem to make sure that all these induced stochastic LQ problems satisfy the uniform convexity condition. Hence, we need to find other weaker conditions for stochastic LQ problems, from which we could eventually obtain weaker conditions for our time-inconsistent stochastic LQ problems. This is not easy due to the complexity of this problem (see Subsection 3.2 for details). Fortunately, we do find new conditions, which are weaker than standard ones and different from the uniform convexity one, to guarantee the closed-loop solvability of stochastic LQ problems (see Lemma 3.11 for details). The proof is based on some properties of the Moore-Penrose inverse of a matrix and a delicate approximation. As a result, under the new conditions, we can obtain a family of closed-loop optimal strategies for stochastic LQ problems, and we finally obtain conditions weaker than the standard ones for our time-inconsistent stochastic LQ problems.

(2) When one takes a limit for the Nash equilibrium of multi-person differential games, one will meet some obstacles due to the singularity of the control weighting matrix. For example, in order to make sure that $(R_k + D^\top P_k D)^\dagger$ converges to $(R + D^\top PD)^\dagger$, it is necessary and sufficient that

$$\mathrm{Rank}(R_k + D^\top P_k D) = \mathrm{Rank}(R + D^\top PD)$$

as $R_k + D^\top P_k D$ approaches $R + D^\top PD$ (see [22] or Lemma 3.7). Notice that the variation of P_k may affect $\mathrm{Rank}(R_k + D^\top P_k D)$.

(3) In [30], the closed-loop equilibrium strategy is defined to be the limit of the Nash equilibrium for the multi-person differential games. Our definition is a little different (see Definition 2.1), which is motivated by [10, 17, 27]. It requires more delicate treatments.

• Roughly speaking, in Definition 2.1, since the term $\frac{1}{\varepsilon_j}$ will tend to ∞ as j tends to ∞, we have to obtain an estimate of the convergence rate of the Nash equilibrium. To do that, we cannot follow the idea in [30], where the convergence is established by the Arzela-Ascoli theorem without the information of the convergence rate. In [10], in order to handle the essential difficulty in the infinite-dimensional setting, a sharp estimate is given. We follow [10] to give a similar sharp estimate but with a new method. The advantage of our method is that the idea is intuitive, and the proof is easier in some sense. It is based on some delicate construction and subtle use of the coupling relationship between the equations.

• To prove that the strategy obtained by taking a limit is indeed a closed-loop equilibrium strategy as in Definition 2.1, we need a special treatment of our partitions of the time interval $[0,T]$ in the construction of the multi-person differential games, since the convergence rate is just the same order as our partition size. Therefore, we have to accelerate the convergence rate of some time intervals to reach our goal (see Subsection 4.3 for more details). We impose a special structure on our partitions to achieve this.

The rest of this paper is organized as follows. In Section 2, we formulate the problem and present our main result. Section 3 is devoted to collecting some preliminary results. In Subsection 3.1, we give some technical lemmas and recall some properties of the Moore-Penrose inverse of a matrix. In Subsection 3.2, we introduce the new conditions for stochastic LQ problems. In Section 4, we prove our main result. The section is divided into three subsections: in Subsection 4.1, we introduce the multi-person differential games and obtain the Nash equilibrium of it, which is a kind of approximate time-consistent strategy; in Subsection 4.2, the convergence of the Nash equilibrium is established with a sharp estimate of the convergence rate; in Subsection 4.3, we prove that the limiting strategy is a closed-loop equilibrium strategy.

2 Problem formulation and the main result

To begin with, we give some notations. Let $(\Omega,\mathcal{F},\mathbb{F},\mathrm{P})$ be a filtered complete probability space on which a standard one-dimensional Brownian motion $\{W(t)\}_{t\geqslant 0}$ is defined and $\mathbb{F}=\{\mathcal{F}_t\}_{t\geqslant 0}$ is the natural filtration generated by $\{W(t)\}_{t\geqslant 0}$.

For a matrix M, write $\mathcal{R}(M)$ for the range of M, $\mathrm{Rank}(M)$ for the rank of M, and $M^\dagger$ for the Moore-Penrose inverse of M. For $n\in\mathbb{N}$, denote by $\mathbb{S}^n$ the space of all the $n\times n$ symmetric matrices and by $\mathbb{S}^n_+$ the set of all the $n\times n$ positive semidefinite matrices. Denote by $|M|_2$ the spectral norm of a matrix $M\in\mathbb{R}^{n\times m}$, which equals the square root of the largest eigenvalue of $M^\top M$, and by $|M|_{\mathrm{Tr}}$ the trace norm of a matrix $M\in\mathbb{R}^{n\times m}$, which equals the sum of the square roots of the eigenvalues of $M^\top M$.

For any $k\in\mathbb{N}$, $t\in[0,T]$ and $r\in[1,\infty)$, denote by $L^r_{\mathcal{F}_t}(\Omega;\mathbb{R}^k)$ the Banach space of all the $\mathcal{F}_t$-measurable random variables $\xi:\Omega\to\mathbb{R}^k$ such that $\mathrm{E}|\xi|^r_{\mathbb{R}^k}<\infty$ with the canonical norm. Denote by $L^r_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^k))$ the Banach space of all the $\mathbb{R}^k$-valued $\mathbb{F}$-adapted, continuous stochastic processes $\phi(\cdot)$ with the norm

$$|\phi(\cdot)|_{L^r_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^k))}\triangleq\Big(\mathrm{E}\max_{\tau\in[t,T]}|\phi(\tau)|^r_{\mathbb{R}^k}\Big)^{1/r}.$$

Fix any $r_1,r_2,r_3,r_4\in[1,\infty)$. Put

$$L^{r_1}_{\mathbb{F}}(\Omega;L^{r_2}(t,T;\mathbb{R}^k))=\bigg\{\varphi:(t,T)\times\Omega\to\mathbb{R}^k\ \bigg|\ \varphi(\cdot)\text{ is }\mathbb{F}\text{-adapted and }\mathrm{E}\bigg(\int_t^T|\varphi(\tau)|^{r_2}_{\mathbb{R}^k}d\tau\bigg)^{\frac{r_1}{r_2}}<\infty\bigg\},$$
$$L^{r_2}_{\mathbb{F}}(t,T;L^{r_1}(\Omega;\mathbb{R}^k))=\bigg\{\varphi:(t,T)\times\Omega\to\mathbb{R}^k\ \bigg|\ \varphi(\cdot)\text{ is }\mathbb{F}\text{-adapted and }\int_t^T(\mathrm{E}|\varphi(\tau)|^{r_1}_{\mathbb{R}^k})^{\frac{r_2}{r_1}}d\tau<\infty\bigg\}.$$

Both $L^{r_1}_{\mathbb{F}}(\Omega;L^{r_2}(t,T;\mathbb{R}^k))$ and $L^{r_2}_{\mathbb{F}}(t,T;L^{r_1}(\Omega;\mathbb{R}^k))$ are Banach spaces with the canonical norms. For $q\in[1,\infty)$, we simply denote $L^q_{\mathbb{F}}(\Omega;L^q(t,T;\mathbb{R}^k))$ by $L^q_{\mathbb{F}}(t,T;\mathbb{R}^k)$.

Let $T>0$ and $(t,x)\in[0,T)\times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$. Consider the following control system:

$$\begin{cases}dX(s)=(A(s)X(s)+B(s)u(s))ds+(C(s)X(s)+D(s)u(s))dW(s), & s\in[t,T],\\ X(t)=x,\end{cases}\tag{2.1}$$

where $A(\cdot),C(\cdot)\in L^\infty(0,T;\mathbb{R}^{n\times n})$, $B(\cdot),D(\cdot)\in L^\infty(0,T;\mathbb{R}^{n\times m})$ and the control $u\in\mathcal{U}[t,T]\triangleq L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$.

For any $(t,x) \in [0,T) \times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ and $u \in \mathcal{U}[t,T]$, the equation (2.1) admits a unique solution $X(\cdot) \in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))$.

We introduce the following cost functional:

$$J(t,x;u(\cdot)) = \frac{1}{2}\mathrm{E}_t\bigg[\int_t^T (\langle Q(s,t)X(s),X(s)\rangle + \langle R(s,t)u(s),u(s)\rangle)ds + \langle G(t)X(T),X(T)\rangle\bigg], \tag{2.2}$$

where $\mathrm{E}_t = \mathrm{E}(\cdot \mid \mathcal{F}_t)$ is the conditional expectation with respect to $\mathcal{F}_t$, $Q(\cdot,\cdot) \in C([0,T]^2;\mathbb{S}^n_+)$, $R(\cdot,\cdot) \in C([0,T]^2;\mathbb{S}^m_+)$ and $G(\cdot) \in C([0,T];\mathbb{S}^n_+)$.

For any $u(\cdot) \in \mathcal{U}[t,T]$, the cost functional $J(t,x;u(\cdot))$ is well defined. Therefore, we can introduce the following problem.

Problem (TISLQ) For any $(t,x) \in [0,T) \times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, find a $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ such that

$$J(t,x;\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x;u(\cdot)). \tag{2.3}$$

Any $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ satisfying (2.3) is called a pre-committed optimal control for a time-inconsistent stochastic linear-quadratic optimal control problem (Problem (TISLQ) for short) at (t,x). Although $\bar{u}(\cdot)$ is optimal for the cost functional $J(t,x;\cdot)$, it might not be very useful in practice since the pre-committed optimal control $\bar{u}(\cdot)$ may no longer be optimal later (see an illustrative example in [30]).

To treat the time inconsistency, we give the following definition.

Definition 2.1. Let any $\{\varepsilon_j\}_{j=1}^\infty \subset (0,+\infty)$ such that $\lim_{j\to\infty}\varepsilon_j = 0$. A matrix-valued function $\Theta(\cdot) \in L^2(0,T;\mathbb{R}^{m\times n})$ is called a closed-loop equilibrium strategy for Problem (TISLQ), if for any $(t,x) \in [0,T) \times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ and $u(\cdot) \in \mathcal{U}[t,T]$,

$$\varliminf_{j\to\infty} \frac{J(t,\overline{X}(t);u^{\varepsilon_j}(\cdot)) - J(t,\overline{X}(t);\bar{u}(\cdot))}{\varepsilon_j} \geqslant 0, \quad \text{P-a.s.}, \tag{2.4}$$

where

$$\begin{cases} d\overline{X}(s) = (A(s)+B(s)\Theta(s))\overline{X}(s)ds + (C(s)+D(s)\Theta(s))\overline{X}(s)dW(s), & s\in[t,T],\\ \overline{X}(t) = x, \end{cases} \tag{2.5}$$
$$\bar{u}(s) = \Theta(s)\overline{X}(s), \quad u^{\varepsilon_j}(s) = u(s)I_{[t,t+\varepsilon_j)}(s) + \Theta(s)X^{\varepsilon_j}(s)I_{[t+\varepsilon_j,T]}(s),$$

and $X^{\varepsilon_j}(\cdot)$ is the solution to (2.1) corresponding to the control $u^{\varepsilon_j}(\cdot)$. Furthermore,

$$V(t,x) \overset{\triangle}{=} J(t,x;\bar{u}(\cdot)) \tag{2.6}$$

is called the equilibrium value function of Problem (TISLQ).

Remark 2.2. From Definition 2.1, for any time $t \in [0,T)$ and state $x \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, any deviation of the closed-loop equilibrium strategy will be worse. This is why we call $\Theta(\cdot)$ a "closed-loop equilibrium strategy". Consequently, we do not regret at any time $t \in [0,T)$ if we use the closed-loop equilibrium strategy $\Theta(\cdot)$.

Remark 2.3. Note that the limit in (2.4) is taken with $\{\varepsilon_j\}_{j=1}^\infty \searrow 0$ rather than $\varepsilon \in \mathbb{R}$. We take this definition as in [27] to avoid the difficulty rising from the uncountability property of $\varepsilon > 0$.

To study the closed-loop equilibrium strategy for Problem (TISLQ), let us assume the following further conditions.

Assumption 2.4. *For* $0 \leqslant t \leqslant \tau \leqslant s \leqslant T$, *it holds that*

$$Q(s,t) \leqslant Q(s,\tau), \quad R(s,t) \leqslant R(s,\tau), \quad G(t) \leqslant G(\tau).$$

Assumption 2.5. *For any* $(t,s) \in [0,T]\times[0,T]$,

$$\mathcal{R}(R(s,t)) \supset \mathcal{R}(B(s)^\top) \cup \mathcal{R}(D(s)^\top).$$

Assumption 2.6. *There exist* $0 \leqslant k \leqslant m$ *and* $\delta > 0$ *such that*

$$\begin{aligned}&\mathrm{Rank}(R(s,t)) = k, \quad \forall\,(s,t) \in [0,T]^2,\\ &\lambda_i(s,t) \geqslant \delta, \quad \textit{if}\ \lambda_i(s,t) > 0,\ i \in \{1,\dots,m\},\ \forall\,(s,t) \in [0,T]^2,\end{aligned}$$

where $\{\lambda_i(\cdot,\cdot)\}_{i=1}^m$ *are the eigenvalues of* $R(\cdot,\cdot)$.

Assumption 2.7. *There exists a constant* $\mathcal{C} > 0$ *such that*

$$|Q(s,t) - Q(s,\tau)|_2 + |R(s,t) - R(s,\tau)|_2 + |G(t) - G(\tau)|_2 \leqslant \mathcal{C}|t-\tau|, \quad \forall\, 0 \leqslant t \leqslant \tau \leqslant s \leqslant T.$$

Remark 2.8. Assumptions 2.5 and 2.6 are weaker than $R(\cdot,\cdot) \geqslant \delta I$ in [30], and we adopt them to guarantee the solvability of the generalized Riccati equation. We will explain it in detail later in Subsection 3.2. Under such weaker assumptions, we can see a richer structure of this problem.

Remark 2.9. Assumption 2.6 is also used to obtain the sharp estimate in Proposition 4.4. If it does not hold, the desired estimate may not hold. An example is given below. Let $0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T$ be a partition of $[0,T]$,

$$t_k \leqslant T/2 < t_l, \quad R(s,t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/(s-t+1) & 0 \\ 0 & 0 & \max(0, t-T/2) \end{bmatrix}, \quad D(s) = 0, \quad 0 \leqslant t \leqslant s \leqslant T. \tag{2.7}$$

Then we have

$$\begin{cases} \lambda_1(s,t) = 0, \\ \lambda_2(s,t) = 1/(s-t+1) \geqslant 1/(1+T), \\ \lambda_3(s,t) = \max(0, t-T/2), \end{cases} \qquad \mathrm{Rank}(R(s,t)) = \begin{cases} 1, & 0 \leqslant t \leqslant T/2, \\ 2, & T/2 < t \leqslant T. \end{cases}$$

We need to obtain an estimate as

$$|R(s,t_l)^\dagger - R(s,t_k)^\dagger|_2 \leqslant \mathcal{C}|t_l - t_k|, \tag{2.8}$$

where $\mathcal{C}$ is a uniform constant. However, from (2.7), we get

$$R(s,t_l)^\dagger - R(s,t_k)^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_l - t_k & 0 \\ 0 & 0 & 1/(t_l - T/2) \end{bmatrix}.$$

Since $1/(t_l - T/2)$ will tend to ∞ as $t_l - t_k$ tends to zero, the estimate (2.8) fails to hold. Inspired by the above example, to handle the singularity, we pose Assumption 2.6.

We call Problem (TISLQ) by *Standard Problem* (*TISLQ*), if Assumptions 2.5 and 2.6 are replaced by the stronger condition

$$R(\cdot,\cdot) \geqslant \delta I. \tag{2.9}$$

Standard Problem (TISLQ) has been studied in [30]. Here, we are concerned with the problem under our weaker conditions, and we call it *Singular Problem* (*TISLQ*).

The main result of this paper is as follows.

Theorem 2.10. *Let Assumptions* 2.4–2.7 *hold. Then for any* $\theta(\cdot) \in L^2(0,T;\mathbb{R}^{m\times n})$*, there is a closed-loop equilibrium strategy* $\Theta(\cdot)$ *given by*

$$\begin{aligned}\Theta(s) = &-(R(s,s) + D(s)^\top \Gamma(s,s) D(s))^\dagger [B(s)^\top \Gamma(s,s) + D(s)^\top \Gamma(s,s) C(s)] \\ &+ \theta(s) - (R(s,s) + D(s)^\top \Gamma(s,s) D(s))^\dagger (R(s,s) + D(s)^\top \Gamma(s,s) D(s))\theta(s),\end{aligned} \tag{2.10}$$

where $\Gamma(\cdot,\cdot)$ *solves*

$$\begin{cases}\Gamma_s(s,t)+\Gamma(s,t)(A(s)+B(s)\Theta(s))+(A(s)+B(s)\Theta(s))^\top\Gamma(s,t)+Q(s,t)\\ \quad +(C(s)+D(s)\Theta(s))^\top\Gamma(s,t)(C(s)+D(s)\Theta(s))+\Theta(s)^\top R(s,t)\Theta(s)=0, \quad 0\leqslant t\leqslant s\leqslant T,\\ \Gamma(T,t)=G(t), \quad 0\leqslant t\leqslant T.\end{cases} \tag{2.11}$$

The equilibrium value function is given by

$$V(t,x)=\frac{1}{2}x^\top\Gamma(t,t)x, \quad \forall\,(t,x)\in[0,T)\times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n). \tag{2.12}$$

3 Some preliminary results

This section is devoted to providing some preliminaries.

3.1 Technical lemmas

In this subsection, we first give some technical lemmas which will be used later. Next, we review some properties of the Moore-Penrose inverse of a matrix.

Lemma 3.1. *Suppose* $M_1, M_2 \in \mathcal{S}^n_+$. *We have*

$$\mathrm{Rank}(M_1+M_2)\geqslant \max\{\mathrm{Rank}(M_1),\mathrm{Rank}(M_2)\}. \tag{3.1}$$

Lemma 3.1 should be a well-known result. However, we failed to find an exact reference. For readers' convenience, we provide a proof.

Proof of Lemma 3.1. It suffices to show that $\mathrm{Rank}(M_1+M_2)\geqslant\mathrm{Rank}(M_1)$, which is equivalent to

$$\dim(\mathrm{Ker}(M_1+M_2))\leqslant\dim(\mathrm{Ker}(M_1)).$$

Observe that for any $v\in\mathrm{Ker}(M_1+M_2)$,

$$v^\top(M_1+M_2)v=0\Leftrightarrow v^\top M_1v=0,\ v^\top M_2v=0.$$

Consequently, $v\in\mathrm{Ker}(M_1)\cap\mathrm{Ker}(M_2)$ and $\dim(\mathrm{Ker}(M_1+M_2))\leqslant\dim(\mathrm{Ker}(M_1))$. □

Lemma 3.2 (See [21, Chapter 2, Subsection 2.2, Theorem 4]). *Suppose that* $Y_1\in L^p_{\mathcal{F}}(\Omega;\mathbb{R})$ *and* $Y_2\in L^q_{\mathcal{F}}(\Omega;\mathbb{R})$ *with* $p,q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\mathrm{E}(|Y_1Y_2|\mid\mathcal{B})\leqslant[\mathrm{E}(|Y_1|^p\mid\mathcal{B})]^{1/p}[\mathrm{E}(|Y_2|^q\mid\mathcal{B})]^{1/q}, \quad \mathrm{P}\text{-}a.s., \tag{3.2}$$

where $\mathcal{B}$ *is a sub-sigma-algebra of* $\mathcal{F}$.

Lemma 3.3 (See [30, Lemma 2.4]). *Let* $x\in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, $\widetilde{G}\in\mathbb{S}^n_+$, $\widetilde{Q}(\cdot)\in L^\infty(0,T;\mathbb{S}^n_+)$ *and* $X(\cdot)$ *solve*

$$\begin{cases}dX(s)=A(s)X(s)ds+C(s)X(s)dW(s), \quad s\in[t,T],\\ X(t)=x.\end{cases} \tag{3.3}$$

Then for any $\tau\in[0,t]$,

$$\mathrm{E}_\tau\bigg(\int_t^T\langle\widetilde{Q}(s)X(s),X(s)\rangle ds+\langle\widetilde{G}X(T),X(T)\rangle\bigg)=\mathrm{E}_\tau(\langle\Pi(t)x,x\rangle), \tag{3.4}$$

where $\Pi(\cdot)$ *solves*

$$\begin{cases}\dot{\Pi}+\Pi A+A^\top\Pi+C^\top\Pi C+\widetilde{Q}=0 \quad \textit{in } [t,T],\\ \Pi(T)=\widetilde{G}.\end{cases} \tag{3.5}$$

Lemma 3.4 (See [2, Theorem 1]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be complete separable metric spaces, and $\mathcal{E}$ be a closed σ-compact subset of $\mathcal{X}\times\mathcal{Y}$. Then $\pi_1(\mathcal{E})$ is a Borel set in $\mathcal{X}$ and there exists a Borel function $\varphi:\pi_1(\mathcal{E})\to\mathcal{Y}$ whose graph is contained in $\mathcal{E}$, where π_1 denotes the projection of $\mathcal{X}\times\mathcal{Y}$ on $\mathcal{X}$.*

Now, we recall some properties of the Moore-Penrose inverse of a matrix.

Lemma 3.5 (See [3, Theorem 4.3]). *Let $M_1, M_2\in\mathcal{S}_+^n$, and $\mathrm{Rank}(M_1)=\mathrm{Rank}(M_2)$. Then $M_1\geqslant M_2$ if and only if $M_1^\dagger\leqslant M_2^\dagger$.*

Lemma 3.6 (See [22, Theorem 3.4]). *Suppose that M_1 and M_2 are matrices of the same dimension, and $\mathrm{Rank}(M_1)=\mathrm{Rank}(M_2)$. Then*

$$|M_1^\dagger-M_2^\dagger|_2\leqslant\mathcal{C}|M_1^\dagger|_2|M_2^\dagger|_2|M_1-M_2|_2. \tag{3.6}$$

Lemma 3.7 (See [22, Corollary 3.5]). *Given a sequence of matrices $\{M_n\}_{n=1}^\infty\in\mathbb{R}^{n\times m}$ satisfying that $\lim_{n\to\infty}|M_n-M|_2=0$, then $\lim_{n\to\infty}|M_n^\dagger-M^\dagger|_2=0$ if and only if $\lim_{n\to\infty}\mathrm{Rank}(M_n)=\mathrm{Rank}(M)$.*

3.2 Singular stochastic linear-quadratic optimal control problem

Let $\widetilde{Q}(\cdot)\in C([0,T];\mathbb{S}_+^n)$, $\widetilde{R}(\cdot)\in C([0,T];\mathbb{S}_+^m)$ and $\widetilde{G}\in\mathbb{S}_+^n$. If $Q(s,t)=\widetilde{Q}(s)$, $R(s,t)=\widetilde{R}(s)$ and $G(t)=\widetilde{G}$ for $0\leqslant t\leqslant s\leqslant T$ in (2.2), then Problem (TISLQ) reduces to a classical stochastic linear-quadratic optimal control problem (Problem (SLQ) for short). Moreover, with (2.9), it further reduces to a Standard Problem (SLQ).

In this subsection, we first review the closed-loop solvability of Problem (SLQ). Next, we prove the solvability of the generalized Riccati equation under assumptions weaker than the ones for Standard Problem (SLQ). The method is based on some delicate approximations. We call Problem (SLQ) under our weaker assumptions by *Singular Problem* (*SLQ*). Finally, we give a comparison principle for the generalized Riccati equation.

We first recall the following definition (see [25]).

Definition 3.8. We call $\Theta(\cdot)\in L^2(t,T;\mathbb{R}^{m\times n})$ a *closed-loop optimal strategy* for Problem (SLQ) on $[t,T]$, if

$$J(t,x;\Theta(\cdot)\overline{X}(\cdot))\leqslant J(t,x;u(\cdot)),\quad\forall\,x\in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n),\quad\forall\,u(\cdot)\in\mathcal{U}[t,T], \tag{3.7}$$

where $\overline{X}(\cdot)$ is the solution to the following closed-loop system:

$$\begin{cases}d\overline{X}(s)=(A(s)+B(s)\Theta(s))\overline{X}(s)ds+(C(s)+D(s)\Theta(s))\overline{X}(s)dW(s),\quad s\in[t,T],\\ \overline{X}(t)=x.\end{cases} \tag{3.8}$$

If a closed-loop optimal strategy exists on $[t,T]$, Problem (SLQ) is said to be closed-loop solvable on $[t,T]$.

The following result is an immediate corollary of [25, Theorem 5.2].

Lemma 3.9. *Problem* (SLQ) *admits a closed-loop optimal strategy on $[t,T]$ if and only if the generalized Riccati equation*

$$\begin{cases}\dot{P}+PA+A^\top P+C^\top PC+\widetilde{Q}\\ \quad-(PB+C^\top PD)(\widetilde{R}+D^\top PD)^\dagger(B^\top P+D^\top PC)=0\quad\text{in }[t,T],\\ P(T)=\widetilde{G}\end{cases} \tag{3.9}$$

admits a unique solution satisfying

$$\begin{cases}\widetilde{R}+D^\top PD\geqslant0,\quad\mathcal{R}(\widetilde{R}+D^\top PD)\supseteq\mathcal{R}(B^\top P+D^\top PC),\\ (\widetilde{R}+D^\top PD)^\dagger(B^\top P+D^\top PC)\in L^2(t,T;\mathbb{R}^{m\times n}).\end{cases} \tag{3.10}$$

Moreover, for any $\theta(\cdot)\in L^2(t,T;\mathbb{R}^{m\times n})$,

$$\Theta\triangleq-(R+D^\top PD)^\dagger(B^\top P+D^\top PC)+\theta-(R+D^\top PD)^\dagger(R+D^\top PD)\theta \tag{3.11}$$

is a closed-loop optimal strategy of Problem (SLQ) *and the optimal cost functional is*

$$J(t,x;\Theta(\cdot)\overline{X}(\cdot))=\frac{1}{2}x^\top P(t)x. \tag{3.12}$$

Remark 3.10. Note that our cost functional $J(t,x;u(\cdot))$ in (2.2) takes the conditional expectation rather than the expectation. Although Lemma 3.9 is stated with the expectation in [25], it is still true for the conditional expectation. Another property we want to mention is that if the generalized Riccati equation is solvable, then it is uniquely solvable [25].

In Standard Problem (SLQ), $\widetilde{R}(\cdot)\geqslant\delta I$ and $P(\cdot)\geqslant 0$, and then $\widetilde{R}+D^\top PD\geqslant\delta I$. The conditions (3.10) are satisfied naturally, and we have a unique closed-loop optimal strategy as in (3.11). On the other hand, when the uniform convexity condition [24] holds, $\widetilde{R}+D^\top PD$ is also nonsingular. In this paper, we consider that $\widetilde{R}+D^\top PD$ is singular.

Lemma 3.11. *Let* $\widetilde{G}\in\mathbb{S}_+^n$, $\widetilde{Q}(\cdot)\in L^\infty(0,T;\mathbb{S}_+^n)$ *and* $\widetilde{R}(\cdot)\in L^\infty(0,T;\mathbb{S}_+^m)$, *and denote the eigenvalues of* $\widetilde{R}(\cdot)$ *by* $\{\lambda_i(\cdot)\}_{i=1}^m$. *Suppose that*

$$\mathcal{R}(\widetilde{R}(s))\supset\mathcal{R}(B(s)^\top)\cup\mathcal{R}(D(s)^\top), \tag{3.13}$$

and there exist $0\leqslant k\leqslant m$ *and* $\delta>0$ *such that*

$$\begin{cases}\mathrm{Rank}(\widetilde{R}(s))=k, & a.e.\ s\in[0,T],\\ \lambda_i(s)\geqslant\delta, & if\ \lambda_i(s)>0,\ i\in\{1,\dots,m\},\ a.e.\ s\in[0,T].\end{cases} \tag{3.14}$$

Then the generalized Riccati equation

$$\begin{cases}\dot{P}+PA+A^\top P+C^\top PC+\widetilde{Q}\\ \quad -(PB+C^\top PD)(\widetilde{R}+D^\top PD)^\dagger(B^\top P+D^\top PC)=0 \quad in\ [0,T],\\ P(T)=\widetilde{G}\end{cases} \tag{3.15}$$

is uniquely solvable, and the following holds:

$$\begin{cases}\widetilde{R}+D^\top PD\geqslant 0, \quad \mathcal{R}(\widetilde{R}+D^\top PD)\supseteq\mathcal{R}(B^\top P+D^\top PC),\\ (\widetilde{R}+D^\top PD)^\dagger(B^\top P+D^\top PC)\in L^2(0,T;\mathbb{R}^{m\times n}).\end{cases} \tag{3.16}$$

Proof. We first claim that there exist measurable matrix-valued functions $\widetilde{V}(\cdot)$ and $\widetilde{\Lambda}(\cdot)$ such that

$$\widetilde{\Lambda}(\cdot)=\begin{bmatrix}\lambda_{i_1(\cdot)}(\cdot) & & \\ & \ddots & \\ & & \lambda_{i_k(\cdot)}(\cdot)\end{bmatrix}\geqslant\delta I_k, \quad i_l(\cdot)\in\{1,2,\dots,m\}, \quad l=1,2,\dots,k$$

and

$$\widetilde{R}(\cdot)=\widetilde{V}(\cdot)\begin{bmatrix}\widetilde{\Lambda}(\cdot) & 0\\ 0 & 0\end{bmatrix}\widetilde{V}(\cdot)^\top \quad \text{with } \widetilde{V}(\cdot)\widetilde{V}(\cdot)^\top=\widetilde{V}(\cdot)^\top\widetilde{V}(\cdot)=I. \tag{3.17}$$

We prove this claim by Lemma 3.4. Let $\mathcal{X}=\mathbb{R}^{m\times m}$, $\mathcal{Y}=\mathbb{R}^{m\times m}\times\mathbb{R}^{m\times m}$ and

$$\mathcal{E}=\Bigg\{\left(\widetilde{R},\widetilde{V},\begin{bmatrix}\widetilde{\Lambda} & 0\\ 0 & 0\end{bmatrix}\right)\in\mathcal{X}\times\mathcal{Y}\ \Bigg|\ \widetilde{R}\in\mathbb{S}^m,\ \widetilde{V}\widetilde{V}^\top=\widetilde{V}^\top\widetilde{V}=I,$$
$$\widetilde{\Lambda}\text{ is a }(k\times k)\text{-dimensional diagonal matrix, and }\widetilde{R}=\widetilde{V}\begin{bmatrix}\widetilde{\Lambda} & 0\\ 0 & 0\end{bmatrix}\widetilde{V}^\top\Bigg\}.$$

By the continuity of matrix multiplication, we have that $\mathcal{E}$ is closed and therefore σ-compact. Applying Lemma 3.4, we have

- $\pi_1(\mathcal{E})$ is a Borel set in $\mathcal{X}$;
- there is a Borel function $\varphi(\cdot) \equiv (\varphi_1(\cdot), \varphi_2(\cdot)) : \pi_1(\mathcal{E}) \to \mathcal{Y}$ with its graph contained in $\mathcal{E}$.

It follows from (3.14) that $\widetilde{R}(s) \in \pi_1(\mathcal{E})$ almost everywhere. Therefore, $\widetilde{R}(\cdot) : [0,T] \to \pi_1(\mathcal{E})$ is measurable. Thus, the composition $(\varphi_1(\widetilde{R}(\cdot)), \varphi_2(\widetilde{R}(\cdot))) : [0,T] \to \mathcal{Y}$ is measurable. Choosing

$$\widetilde{V}(\cdot) = \varphi_1(\widetilde{R}(\cdot)), \quad \begin{bmatrix} \widetilde{\Lambda}(\cdot) & 0 \\ 0 & 0 \end{bmatrix} = \varphi_2(\widetilde{R}(\cdot)),$$

we prove the claim.

Let

$$\widetilde{R}_n = \widetilde{V} \begin{bmatrix} \widetilde{\Lambda} & 0 \\ 0 & nI_{m-k} \end{bmatrix} \widetilde{V}^\top, \quad n = 1, 2, \ldots$$

Without loss of generality, we may assume $\delta \leqslant 1$. Then we have $\widetilde{R}_n \geqslant \delta I$.

Consider a sequence of approximation problems:

$$\begin{cases} \dot{P}_n + P_n A + A^\top P_n + C^\top P_n C + \widetilde{Q} \\ \quad -(P_n B + C^\top P_n D)(\widetilde{R}_n + D^\top P_n D)^\dagger (B^\top P_n + D^\top P_n C) = 0 \quad \text{in } [0,T], \\ P(T) = \widetilde{G}. \end{cases} \tag{3.18}$$

Since for any $n \geqslant 1$,

$$\widetilde{R}_n(\cdot) \geqslant \delta I, \quad \widetilde{Q}(\cdot) \geqslant 0, \quad \widetilde{G} \geqslant 0,$$

by the results of Standard Problem (SLQ) (see, e.g., [31, Theorem 7.2]), we know that the equations (3.18) have unique solutions $P_n(\cdot) \in C([0,T]; \mathbb{S}^n_+)$. For $j = 1, \ldots, n$, let e_j be the j-th unit vector that contains a 1 in the j-th position and zeros everywhere else. Then we have

$$\begin{aligned} e_j^\top P_n(t) e_j = e_j^\top \widetilde{G} e_j - \int_t^T e_j^\top [(P_n B + C^\top P_n D)(\widetilde{R}_n + D^\top P_n D)^\dagger (B^\top P_n + D^\top P_n C) \\ - (P_n A + A^\top P_n + C^\top P_n C + \widetilde{Q})] e_j ds. \end{aligned}$$

It follows from $(\widetilde{R}_n + D^\top P_n D) \geqslant \delta I > 0$ that

$$\begin{aligned} & e_j^\top P_n(t) e_j \leqslant e_j^\top \widetilde{G} e_j + \int_t^T e_j^\top (P_n A + A^\top P_n + C^\top P_n C + \widetilde{Q}) e_j ds, \\ & \sum_{j=1}^n e_j^\top P_n(t) e_j \leqslant \mathcal{C} \bigg[|\widetilde{G}|_{\mathrm{Tr}} + \int_t^T (|P_n(s)|_{\mathrm{Tr}} (|A|_{L^\infty(0,T;\mathbb{R}^{n\times n})} + |C|^2_{L^\infty(0,T;\mathbb{R}^{n\times n})}) + |\widetilde{Q}|_{L^\infty(0,T;\mathbb{S}^n_+)}) ds \bigg] \end{aligned}$$

and

$$|P_n(t)|_{\mathrm{Tr}} \leqslant C \bigg[|\widetilde{G}|_{\mathrm{Tr}} + \int_t^T (|P_n(s)|_{\mathrm{Tr}} (|A|_{L^\infty(0,T;\mathbb{R}^{n\times n})} + |C|^2_{L^\infty(0,T;\mathbb{R}^{n\times n})}) + |\widetilde{Q}|_{L^\infty(0,T;\mathbb{S}^n_+)}) ds \bigg].$$

By Gronwall's inequality and the equivalence of matrix norms, we get

$$|P_n(\cdot)|_{C([0,T];\mathbb{S}^n_+)} \leqslant \mathrm{e}^{\mathcal{C}_1 T} \mathcal{C}_0,$$

where $\mathcal{C}_1 = \mathcal{C}_1(A, C)$ and $\mathcal{C}_0 = \mathcal{C}_0(\widetilde{G}, \widetilde{Q})$. Therefore, $P_n(\cdot)$ is uniformly bounded. On the other hand, by $P_n(\cdot) \geqslant 0$ and Lemma 3.1, we have $\mathrm{Rank}(R_n) = \mathrm{Rank}(R_n + D^\top P_n D)$. By Lemma 3.5, we find

$$0 < (\widetilde{R}_n + D^\top P_n D)^\dagger \leqslant \widetilde{R}_n^\dagger = \widetilde{R}_n^{-1} \leqslant \frac{1}{\delta} I.$$

Thus,

$$|(\widetilde{R}_n + D^\top P_n D)^\dagger|_2 \leqslant \mathcal{C}(\delta).$$

This, together with

$$
\begin{aligned}
P_n(t) - P_n(s) = -\int_t^s & [(P_n B + C^\top P_n D)(\widetilde{R}_n + D^\top P_n D)^\dagger (B^\top P_n + D^\top P_n C) \\
& - P_n A - A^\top P_n - C^\top P_n C - \widetilde{Q}] d\tau,
\end{aligned}
$$

implies that

$$
|P_n(t) - P_n(s)|_2 \leqslant \mathcal{C}_2 (s - t),
$$

where $\mathcal{C}_2 = \mathcal{C}_2(A, B, C, D, \widetilde{G}, \widetilde{Q}, \delta)$ is fixed. Consequently, we get the equicontinuity of $\{P_n(\cdot)\}_{n=1}^\infty$.

By the Arzela-Ascoli theorem, there exists a subsequence of $\{P_n(\cdot)\}_{n=1}^\infty$ converging to some $P(\cdot)$ in $C([0,T]; \mathbb{S}_+^n)$. On the other hand, by (3.13) and (3.17), we can rewrite $\widetilde{R} + D^\top P_n D$ as

$$
\widetilde{R}(s) + D^\top(s) P_n(s) D(s) \equiv \widetilde{V}(s) \begin{bmatrix} \widetilde{K}_n(s) & 0 \\ 0 & 0 \end{bmatrix} \widetilde{V}^\top(s), \tag{3.19}
$$

where $\widetilde{K}_n(\cdot)$ is an $\mathbb{S}^k$-valued measurable function. For $\widetilde{K}_n(\cdot)$, similar to the proof of (3.17), we can prove that there exist measurable matrix-valued functions $\widetilde{V}_n(\cdot)$ and $\widetilde{\Sigma}_n(\cdot)$ such that

$$
\widetilde{K}_n(\cdot) = \widetilde{V}_n(\cdot) \widetilde{\Sigma}_n(\cdot) \widetilde{V}_n(\cdot)^\top \quad \text{with } \widetilde{V}_n(\cdot)\widetilde{V}_n(\cdot)^\top = \widetilde{V}_n(\cdot)^\top \widetilde{V}_n(\cdot) = I_k,
$$

and $\widetilde{\Sigma}_n(s)$ is a $k \times k$ diagonal matrix almost everywhere. Then we have

$$
\begin{aligned}
(\widetilde{R}_n + D^\top P_n D)^\dagger &= \left(D^\top P_n D + \widetilde{V} \begin{bmatrix} \widetilde{\Lambda} & 0 \\ 0 & n I_{m-k} \end{bmatrix} \widetilde{V}^\top \right)^\dagger \\
&= \left(\widetilde{R} + D^\top P_n D + \widetilde{V} \begin{bmatrix} 0 & 0 \\ 0 & n I_{m-k} \end{bmatrix} \widetilde{V}^\top \right)^\dagger \\
&= \left(\widetilde{V} \begin{bmatrix} \widetilde{K}_n & 0 \\ 0 & 0 \end{bmatrix} \widetilde{V}^\top + \widetilde{V} \begin{bmatrix} 0 & 0 \\ 0 & n I_{m-k} \end{bmatrix} \widetilde{V}^\top \right)^\dagger \\
&= \left(\widetilde{V} \begin{bmatrix} \widetilde{V}_n & 0 \\ 0 & I_{m-k} \end{bmatrix} \begin{bmatrix} \widetilde{\Sigma}_n & 0 \\ 0 & n I_{m-k} \end{bmatrix} \begin{bmatrix} \widetilde{V}_n^\top & 0 \\ 0 & I_{m-k} \end{bmatrix} \widetilde{V}^\top \right)^\dagger \\
&= \widetilde{V} \left(\begin{bmatrix} \widetilde{V}_n & 0 \\ 0 & I_{m-k} \end{bmatrix} \begin{bmatrix} \widetilde{\Sigma}_n^\dagger & 0 \\ 0 & \frac{1}{n} I_{m-k} \end{bmatrix} \begin{bmatrix} \widetilde{V}_n^\top & 0 \\ 0 & I_{m-k} \end{bmatrix} \right) \widetilde{V}^\top \\
&= \widetilde{V} \begin{bmatrix} \widetilde{V}_n \widetilde{\Sigma}_n^\dagger \widetilde{V}_n^\top & 0 \\ 0 & 0 \end{bmatrix} \widetilde{V}^\top + \widetilde{V} \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{n} I_{m-k} \end{bmatrix} \widetilde{V}^\top \\
&= (\widetilde{R} + D^\top P_n D)^\dagger + \widetilde{V} \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{n} I_{m-k} \end{bmatrix} \widetilde{V}^\top.
\end{aligned} \tag{3.20}
$$

By Lemma 3.7 and (3.20), taking the limit in (3.18), we get a solution to the Riccati equation (3.15).

Furthermore, from (3.13), (3.14) and Lemmas 3.1 and 3.5, we find

$$
\begin{cases}
\mathcal{R}(\widetilde{R} + D^\top P D) = \mathcal{R}(\widetilde{R}) \supseteq \mathcal{R}(B^\top P + D^\top P C), \\
\widetilde{R} + D^\top P D \geqslant 0, \\
(\widetilde{R} + D^\top P D)^\dagger (B^\top P + D^\top P C) \in L^2(s, T; \mathbb{R}^{m \times n}).
\end{cases}
$$

This completes the proof. □

Remark 3.12. From the proof of Lemma 3.11, one can see that (3.13) is posed to guarantee the second one in the constraint conditions (3.16). On the other hand, (3.14) is used to get the well-posedness of (3.18) and the solvability of (3.15).

Lemma 3.13. *Suppose that the generalized Riccati equation* (3.9) *is uniquely solvable. Then*

$$P(\cdot) \leqslant \widehat{P}(\cdot) \quad in\ [0,T], \tag{3.21}$$

where $\widehat{P}(\cdot)$ *satisfies*

$$\begin{cases} \dot{\widehat{P}} + \widehat{P}A + A^\top \widehat{P} + C^\top \widehat{P} C + \widetilde{Q} = 0 \quad in\ [0,T], \\ \widehat{P}(T) = \widetilde{G}. \end{cases} \tag{3.22}$$

Proof. By Lemma 3.9, we have

$$J(t,x;\Theta(\cdot)\overline{X}(\cdot)) = \frac{1}{2} x^\top P(t) x \leqslant J(t,x;0). \tag{3.23}$$

By Lemma 3.3, we have

$$J(t,x;0) = \frac{1}{2} x^\top \widehat{P}(t) x, \tag{3.24}$$

where $\widehat{P}(\cdot)$ satisfies (3.22). The conclusion follows from (3.23) and (3.24) immediately. □

4 Proof of the main result

We are now in a position to prove the main result. The proof is so long that we divide it into three subsections.

4.1 Multi-person differential games

In this subsection, following the idea in [30], we introduce a multi-person differential game.

For $N \in \mathbb{N}$, let $\mathcal{D}$ be an N-partition of $[0,T]$, i.e., $\mathcal{D} \triangleq \{[t_k, t_{k+1}]\}_{k=0}^{N-1}$, where $0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T$. Define the mesh size of $\mathcal{D}$ by

$$\|\mathcal{D}\| = \max_{0 \leqslant k \leqslant N-1} \{(t_{k+1} - t_k)\}.$$

Consider an N-person differential game, in which the k-th player controls the system on $[t_k, t_{k+1})$. The main rules are as follows:

(i) each player plays optimally based on the assumption that the latter players play optimally;

(ii) the $(k+1)$-th player's initial state is the k-th player's final state;

(iii) the k-th player still discounts the cost functional in his/her own way over the time interval $[t_{k+1}, T]$.

For $0 \leqslant k \leqslant N-1$, we set

$$Q_k(s) = Q(s,t_k), \quad R_k(s) = R(s,t_k), \quad G_k = G(t_k), \quad \mathcal{U}[t_k,t_{k+1}] = L^2_{\mathbb{F}}(t_k,t_{k+1};\mathbb{R}^m).$$

Let us first consider the $(N-1)$-th player who is not affected by any other players. Therefore, he/she just needs to behave optimally.

The $(N-1)$-th player. The state equation is

$$\begin{cases} dX_{N-1}(s) = (AX_{N-1} + Bu_{N-1})ds + (CX_{N-1} + Du_{N-1})dW(s) \quad \text{in } [t_{N-1}, t_N), \\ X_{N-1}(t_{N-1}) = x_{N-1} \in L^2_{\mathcal{F}_{t_{N-1}}}(\Omega;\mathbb{R}^n), \end{cases}$$

and the cost functional is

$$\begin{aligned} &J^{\mathcal{D}}_{N-1}(x_{N-1};u_{N-1}(\cdot)) \\ &\quad = \frac{1}{2}\mathrm{E}_{t_{N-1}}\left[\int_{t_{N-1}}^{t_N} (\langle Q_{N-1}X_{N-1}, X_{N-1}\rangle + \langle R_{N-1}u_{N-1}, u_{N-1}\rangle)ds + \langle G_{N-1}X_{N-1}(t_N), X_{N-1}(t_N)\rangle\right]. \end{aligned}$$

This is a Singular Problem (SLQ). By Lemma 3.11, the generalized Riccati equation

$$\begin{cases} \dot{P}_{N-1} + P_{N-1}A + A^\top P_{N-1} + C^\top P_{N-1}C + Q_{N-1} \\ \quad -(P_{N-1}B + C^\top P_{N-1}D)(R_{N-1} + D^\top P_{N-1}D)^\dagger (B^\top P_{N-1} + D^\top P_{N-1}C) = 0 \quad \text{in } [t_{N-1}, t_N), \\ P_{N-1}(t_N) = G_{N-1} \end{cases}$$

admits a unique solution such that

$$\begin{cases} R_{N-1} + D^\top P_{N-1}D \geqslant 0 & \text{in } [t_{N-1}, t_N], \\ \mathcal{R}(R_{N-1} + D^\top P_{N-1}D) \supseteq \mathcal{R}(B^\top P_{N-1} + D^\top P_{N-1}C) & \text{in } [t_{N-1}, t_N], \\ (R_{N-1} + D^\top P_{N-1}D)^\dagger (B^\top P_{N-1} + D^\top P_{N-1}C) \in L^2(t_{N-1}, t_N; \mathbb{R}^{m\times n}). \end{cases}$$

Furthermore, for any $\theta \in L^2(t_{N-1}, t_N; \mathbb{R}^{m\times n})$,

$$\begin{aligned} \Theta_{N-1} = &-(R_{N-1} + D^\top P_{N-1}D)^\dagger (B^\top P_{N-1} + D^\top P_{N-1}C) + \theta \\ &- (R_{N-1} + D^\top P_{N-1}D)^\dagger (R_{N-1} + D^\top P_{N-1}D)\theta \quad \text{in } [t_{N-1}, t_N] \end{aligned}$$

is a closed-loop optimal strategy for the $(N-1)$-th Player.

The $(N-2)$-th player. The state equation is

$$\begin{cases} dX_{N-2}(s) = (AX_{N-2} + Bu_{N-2})ds + (CX_{N-2} + Du_{N-2})dW(s) \quad \text{in } [t_{N-2}, t_{N-1}), \\ X_{N-2}(t_{N-2}) = x_{N-2} \in L^2_{\mathcal{F}_{t_{N-2}}}(\Omega; \mathbb{R}^n). \end{cases}$$

The $(N-2)$-th player will assume that the $(N-1)$-th player behaves optimally, while he/she still discounts the cost functional in his/her own way on the time interval $[t_{N-1}, t_N]$. Consequently, the cost functional is

$$\begin{aligned} &J^{\mathcal{D}}_{N-2}(x_{N-2}; u_{N-2}(\cdot)) \\ &\quad = \frac{1}{2}\mathrm{E}_{t_{N-2}}\bigg[\int_{t_{N-2}}^{t_{N-1}} (\langle Q_{N-2}X_{N-2}, X_{N-2}\rangle + \langle R_{N-2}u_{N-2}, u_{N-2}\rangle)ds \\ &\qquad + \int_{t_{N-1}}^{t_N} (\langle Q_{N-2}\overline{X}_{N-1}, \overline{X}_{N-1}\rangle + \langle R_{N-2}\bar{u}_{N-1}, \bar{u}_{N-1}\rangle)ds + \langle G_{N-2}\overline{X}_{N-1}(t_N), \overline{X}_{N-1}(t_N)\rangle\bigg] \\ &\quad = \frac{1}{2}\mathrm{E}_{t_{N-2}}\bigg[\int_{t_{N-2}}^{t_{N-1}} (\langle Q_{N-2}X_{N-2}, X_{N-2}\rangle + \langle R_{N-2}u_{N-2}, u_{N-2}\rangle)ds \\ &\qquad + \int_{t_{N-1}}^{t_N} \langle (Q_{N-2} + \Theta_{N-1}^\top R_{N-2}\Theta_{N-1})\overline{X}_{N-1}, \overline{X}_{N-1}\rangle ds + \langle G_{N-2}\overline{X}_{N-1}(t_N), \overline{X}_{N-1}(t_N)\rangle\bigg]. \end{aligned}$$

By Lemma 3.3, we can rewrite the cost functional as

$$\begin{aligned} &J^{\mathcal{D}}_{N-2}(x_{N-2}; u_{N-2}(\cdot)) \\ &\quad = \frac{1}{2}\mathrm{E}_{t_{N-2}}\bigg[\int_{t_{N-2}}^{t_{N-1}} (\langle Q_{N-2}X_{N-2}, X_{N-2}\rangle + \langle R_{N-2}u_{N-2}, u_{N-2}\rangle)ds \\ &\qquad + \langle \Gamma_{N-2}(t_{N-1})X_{N-2}(t_{N-1}), X_{N-2}(t_{N-1})\rangle\bigg], \end{aligned} \tag{4.1}$$

where $\Gamma_{N-2}(\cdot)$ solves the following equation:

$$\begin{cases} \dot{\Gamma}_{N-2} + \Gamma_{N-2}(A + B\Theta_{N-1}) + (A + B\Theta_{N-1})^\top \Gamma_{N-2} + Q_{N-2} \\ \quad +(C + D\Theta_{N-1})^\top \Gamma_{N-2}(C + D\Theta_{N-1}) + \Theta_{N-1}^\top R_{N-2}\Theta_{N-1} = 0 \quad \text{in } [t_{N-1}, t_N), \\ \Gamma_{N-2}(t_N) = G_{N-2}. \end{cases}$$

Although the change of $\theta(\cdot)$ on $[t_{N-1}, t_N]$ does not affect the $(N-1)$-th player's cost functional $J^{\mathcal{D}}_{N-1}$, it does have an influence on the $(N-2)$-th player's choice by $\Gamma_{N-2}(t_{N-1})$ in (4.1). In other words, due to the singularity of Problem (SLQ), the $(N-1)$-th player has more choices of control to obtain optimality. Since we can fix $\theta(\cdot) \in L^2(0, T; \mathbb{R}^{m\times n})$ at the beginning, we omit θ in $J^{\mathcal{D}}_{N-2}$ for the simplicity of notations.

Similar to the case of the $(N-1)$-th player, the generalized Riccati equation

$$\begin{cases} \dot{P}_{N-2} + P_{N-2}A + A^\top P_{N-2} + C^\top P_{N-2}C + Q_{N-2} \\ \quad -(P_{N-2}B + C^\top P_{N-2}D)(R_{N-2} + D^\top P_{N-2}D)^\dagger(B^\top P_{N-2} + D^\top P_{N-2}C) = 0 \quad \text{in } [t_{N-2}, t_{N-1}), \\ P_{N-2}(t_{N-1}) = \Gamma_{N-2}(t_{N-1}) \end{cases}$$

admits a unique solution such that

$$\begin{cases} R_{N-2} + D^\top P_{N-2}D \geqslant 0 & \text{in } [t_{N-2}, t_{N-1}], \\ \mathcal{R}(R_{N-2} + D^\top P_{N-2}D) \supseteq \mathcal{R}(B^\top P_{N-2} + D^\top P_{N-2}C) & \text{in } [t_{N-2}, t_{N-1}], \\ (R_{N-2} + D^\top P_{N-2}D)^\dagger(B^\top P_{N-2} + D^\top P_{N-2}C) \in L^2(t_{N-2}, t_{N-1}; \mathbb{R}^{m\times n}). \end{cases}$$

For any $\theta \in L^2(t_{N-2}, t_{N-1}; \mathbb{R}^{m\times n})$,

$$\begin{aligned} \Theta_{N-2} = &-(R_{N-2} + D^\top P_{N-2}D)^\dagger(B^\top P_{N-2} + D^\top P_{N-2}C) + \theta \\ &- (R_{N-2} + D^\top P_{N-2}D)^\dagger(R_{N-2} + D^\top P_{N-2}D)\theta \quad \text{in } [t_{N-2}, t_{N-1}] \end{aligned}$$

is a closed-loop optimal strategy for the $(N-2)$-th player.

The $(N-3)$-th player. The state equation is

$$\begin{cases} dX_{N-3}(s) = (AX_{N-3} + Bu_{N-3})ds + (CX_{N-3} + Du_{N-3})dW(s) \quad \text{in } [t_{N-3}, t_{N-2}), \\ X_{N-3}(t_{N-3}) = x_{N-3} \in L^2_{\mathcal{F}_{t_{N-3}}}(\Omega; \mathbb{R}^n). \end{cases}$$

The $(N-3)$-th player will assume that the $(N-1)$-th and $(N-2)$-th players behave optimally, while he/she still discounts the cost functional in his/her own way on $[t_{N-2}, t_N]$. Consequently, the cost functional is

$$\begin{aligned} &J^{\mathcal{D}}_{N-3}(x_{N-3}; u_{N-3}(\cdot)) \\ &= \frac{1}{2}\mathrm{E}_{t_{N-3}}\bigg[\int_{t_{N-3}}^{t_{N-2}} (\langle Q_{N-3}X_{N-3}, X_{N-3}\rangle + \langle R_{N-3}u_{N-3}, u_{N-3}\rangle)ds \\ &\quad + \int_{t_{N-2}}^{t_{N-1}} (\langle Q_{N-3}\overline{X}_{N-2}, \overline{X}_{N-2}\rangle + \langle R_{N-3}\bar{u}_{N-2}, \bar{u}_{N-2}\rangle)ds \\ &\quad + \int_{t_{N-1}}^{t_N} (\langle Q_{N-3}\overline{X}_{N-1}, \overline{X}_{N-1}\rangle + \langle R_{N-3}\bar{u}_{N-1}, \bar{u}_{N-1}\rangle)ds + \langle G_{N-3}\overline{X}_{N-1}(t_N), \overline{X}_{N-1}(t_N)\rangle\bigg] \\ &= \frac{1}{2}\mathrm{E}_{t_{N-3}}\bigg[\int_{t_{N-3}}^{t_{N-2}} (\langle Q_{N-3}X_{N-3}, X_{N-3}\rangle + \langle R_{N-3}u_{N-3}, u_{N-3}\rangle)ds \\ &\quad + \int_{t_{N-2}}^{t_{N-1}} \langle (Q_{N-3} + \Theta_{N-2}^\top R_{N-3}\Theta_{N-2})\overline{X}_{N-2}, \overline{X}_{N-2}\rangle ds \\ &\quad + \int_{t_{N-1}}^{t_N} \langle (Q_{N-3} + \Theta_{N-1}^\top R_{N-3}\Theta_{N-1})\overline{X}_{N-1}, \overline{X}_{N-1}\rangle ds + \langle G_{N-3}\overline{X}_{N-1}(t_N), \overline{X}_{N-1}(t_N)\rangle\bigg] \\ &= \frac{1}{2}\mathrm{E}_{t_{N-3}}\bigg[\int_{t_{N-3}}^{t_{N-2}} (\langle Q_{N-3}X_{N-3}, X_{N-3}\rangle + \langle R_{N-3}u_{N-3}, u_{N-3}\rangle)ds \\ &\quad + \int_{t_{N-2}}^{t_N} \langle (Q_{N-3} + \Theta_{\mathcal{D}}^\top R_{N-3}\Theta_{\mathcal{D}})\overline{X}_{\mathcal{D}}, \overline{X}_{\mathcal{D}}\rangle ds + \langle G_{N-3}\overline{X}_{\mathcal{D}}(t_N), \overline{X}_{\mathcal{D}}(t_N)\rangle\bigg], \end{aligned}$$

where

$$\begin{cases} \Theta_{\mathcal{D}}(s) = \Theta_{N-2}(s)I_{(t_{N-2}, t_{N-1}]} + \Theta_{N-1}(s)I_{(t_{N-1}, t_N]}, \\ \overline{X}_{\mathcal{D}}(s) = \overline{X}_{N-2}(s)I_{(t_{N-2}, t_{N-1}]} + \overline{X}_{N-1}I_{(t_{N-1}, t_N]}, \end{cases} \qquad s \in [t_{N-2}, t_N].$$

By Lemma 3.3 again, we rewrite the cost functional as

$$\begin{aligned}&J_{N-3}^{\mathcal{D}}(x_{N-3};u_{N-3}(\cdot))\\&\quad=\frac{1}{2}\mathrm{E}_{t_{N-3}}\bigg[\int_{t_{N-3}}^{t_{N-2}}(\langle Q_{N-3}X_{N-3},X_{N-3}\rangle+\langle R_{N-3}u_{N-3},u_{N-3}\rangle)ds\\&\qquad+\langle\Gamma_{N-3}(t_{N-2})X_{N-3}(t_{N-2}),X_{N-3}(t_{N-2})\rangle\bigg],\end{aligned}$$

where Γ_{N-3} solves the following equation:

$$\begin{cases}\dot{\Gamma}_{N-3}+\Gamma_{N-3}(A+B\Theta_{\mathcal{D}})+(A+B\Theta_{\mathcal{D}})^\top\Gamma_{N-3}+Q_{N-3}\\\quad+(C+D\Theta_{\mathcal{D}})^\top\Gamma_{N-3}(C+D\Theta_{\mathcal{D}})+\Theta_{\mathcal{D}}^\top R_{N-3}\Theta_{\mathcal{D}}=0\quad\text{in }[t_{N-2},t_N),\\\Gamma_{N-3}(t_N)=G_{N-3}.\end{cases}$$

Similarly, the generalized Riccati equation

$$\begin{cases}\dot{P}_{N-3}+P_{N-3}A+A^\top P_{N-3}+C^\top P_{N-3}C+Q_{N-3}\\\quad-(P_{N-3}B+C^\top P_{N-3}D)(R_{N-3}+D^\top P_{N-3}D)^\dagger(B^\top P_{N-3}+D^\top P_{N-3}C)=0\quad\text{in }[t_{N-3},t_{N-2}),\\P_{N-3}(t_{N-2})=\Gamma_{N-3}(t_{N-2})\end{cases}$$

admits a unique solution such that

$$\begin{cases}R_{N-3}+D^\top P_{N-3}D\geqslant0&\text{in }[t_{N-3},t_{N-2}],\\\mathcal{R}(R_{N-3}+D^\top P_{N-3}D)\supseteq\mathcal{R}(B^\top P_{N-3}+D^\top P_{N-3}C)&\text{in }[t_{N-3},t_{N-2}],\\(R_{N-3}+D^\top P_{N-3}D)^\dagger(B^\top P_{N-3}+D^\top P_{N-3}C)\in L^2(t_{N-3},t_{N-2};\mathbb{R}^{m\times n}).\end{cases}$$

The closed-loop optimal strategy for the $(N-3)$-th player is

$$\begin{aligned}\Theta_{N-3}=&-(R_{N-3}+D^\top P_{N-3}D)^\dagger(B^\top P_{N-3}+D^\top P_{N-3}C)+\theta\\&-(R_{N-3}+D^\top P_{N-3}D)^\dagger(R_{N-3}+D^\top P_{N-3}D)\theta\quad\text{in }[t_{N-3},t_{N-2}]\end{aligned}$$

for any $\theta\in L^2(t_{N-3},t_{N-2};\mathbb{R}^{m\times n})$.

By induction, we can construct sequences of $\{P_k(\cdot)\}_{k=0}^{N-1}$ and $\{\Gamma_k(\cdot)\}_{k=0}^{N-2}$, where $P_k(\cdot)$ solves the generalized Riccati equation

$$\begin{cases}\dot{P}_k+P_kA+A^\top P_k+C^\top P_kC+Q_k\\\quad-(P_kB+C^\top P_kD)(R_k+D^\top P_kD)^\dagger(B^\top P_k+D^\top P_kC)=0\quad\text{in }[t_k,t_{k+1}),\\P_k(t_{k+1})=\Gamma_k(t_{k+1})\end{cases}\tag{4.2}$$

and satisfies

$$\begin{cases}R_k+D^\top P_kD\geqslant0&\text{in }[t_k,t_{k+1}],\\\mathcal{R}(R_k+D^\top P_kD)\supseteq\mathcal{R}(B^\top P_k+D^\top P_kC)&\text{in }[t_k,t_{k+1}],\\(R_k+D^\top P_kD)^\dagger(B^\top P_k+D^\top P_kC)\in L^2(t_k,t_{k+1};\mathbb{R}^{m\times n}),\end{cases}\tag{4.3}$$

and $\Gamma_k(\cdot)$ solves

$$\begin{cases}\dot{\Gamma}_k+\Gamma_k(A+B\Theta_{\mathcal{D}})+(A+B\Theta_{\mathcal{D}})^\top\Gamma_k+Q_k\\\quad+(C+D\Theta_{\mathcal{D}})^\top\Gamma_k(C+D\Theta_{\mathcal{D}})+\Theta_{\mathcal{D}}^\top R_k\Theta_{\mathcal{D}}=0\quad\text{in }[t_{k+1},t_N),\\\Gamma_k(t_N)=G_k\end{cases}\tag{4.4}$$

with

$$\begin{cases}\Theta_k=-(R_k+D^\top P_kD)^\dagger(B^\top P_k+D^\top P_kC)+\theta-(R_k+D^\top P_kD)^\dagger(R_k+D^\top P_kD)\theta,\\\Theta_{\mathcal{D}}(s)=\displaystyle\sum_{k=0}^{N-1}\Theta_k(s)I_{(t_k,t_{k+1}]}(s),\quad s\in[0,t_N],\\\Gamma_{N-1}(t_N)=G_{N-1},\end{cases}\tag{4.5}$$

where $\theta(\cdot) \in L^2(0,T;\mathbb{R}^{m\times n})$.

In summary, let $\theta(\cdot) \in L^2(0,T;\mathbb{R}^{m\times n})$ be fixed. For any $0 \leqslant k \leqslant N-1$ and $x \in L^2_{\mathcal{F}_{t_k}}(\Omega;\mathbb{R}^n)$, consider the following closed-loop system:

$$\begin{cases} d\overline{X}_{\mathcal{D}} = (A+B\Theta_{\mathcal{D}})\overline{X}_{\mathcal{D}}ds + (C+D\Theta_{\mathcal{D}})\overline{X}_{\mathcal{D}}dW(s) \quad \text{in } [t_k,t_N], \\ \overline{X}_{\mathcal{D}}(t_k) = x. \end{cases} \tag{4.6}$$

Put

$$\bar{u}_{\mathcal{D}}(s) = \Theta_{\mathcal{D}}(s)\overline{X}_{\mathcal{D}}(s), \quad s \in [t_k,t_N].$$

Then for any $k \leqslant j \leqslant N-1$, we have

$$\inf_{u_j(\cdot)\in\mathcal{U}[t_j,t_{j+1}]} J_j^{\mathcal{D}}(\overline{X}_{\mathcal{D}}(t_j);u_j(\cdot)) = J_j^{\mathcal{D}}(\overline{X}_{\mathcal{D}}(t_j);\bar{u}_{\mathcal{D}}(\cdot)\,|_{[t_j,t_{j+1}]}) = \frac{1}{2}\langle P_{\mathcal{D}}(t_j)\overline{X}_{\mathcal{D}}(t_j),\overline{X}_{\mathcal{D}}(t_j)\rangle. \tag{4.7}$$

4.2 Well-posedness of the equation (2.11)

In this subsection, we establish the well-posedness of the equation (2.11) by means of solutions to (4.2) and (4.4) obtained in the previous subsection.

Note that in (4.4), $\Gamma_0(\cdot)$ is only defined on $[t_1,T]$, while $P_{\mathcal{D}}$ is defined on $[0,T]$. Therefore, we can actually get a unique solution $\Gamma_0(\cdot)$ defined on $[0,T]$ by solving

$$\begin{cases} \dot{\Gamma}_0 + \Gamma_0(A+B\Theta_{\mathcal{D}}) + (A+B\Theta_{\mathcal{D}})^\top\Gamma_0 + Q_0 + (C+D\Theta_{\mathcal{D}})^\top\Gamma_0(C+D\Theta_{\mathcal{D}}) \\ \quad +\Theta_{\mathcal{D}}^\top R_0\Theta_{\mathcal{D}} = 0 \quad \text{in } [0,T), \\ \Gamma_0(t_N) = G_0, \end{cases} \tag{4.8}$$

which is the extension of the previous one defined on $[t_1,T]$ by (4.4) (so we use the same notation).

For a given N-partition $\mathcal{D}$ of $[0,T]$, define

$$\begin{cases} P_{\mathcal{D}}(s) \equiv \displaystyle\sum_{k=0}^{N-1} P_k(s)I_{(t_k,t_{k+1}]}(s), \quad s\in[0,T], \\ \Gamma_{\mathcal{D}}(s,\tau) \equiv \displaystyle\sum_{k=0}^{N-2} \Gamma_k(s)I_{(t_{k+1},t_{k+2}]}(\tau) + \Gamma_0(s)I_{[t_0,t_1]}(\tau), \quad 0\leqslant\tau\leqslant s\leqslant T. \end{cases} \tag{4.9}$$

In the rest of this subsection, we prove the convergence of $(P_{\mathcal{D}}(\cdot),\Gamma_{\mathcal{D}}(\cdot,\cdot))$ as $\|\mathcal{D}\|\to 0$ and give a sharp estimate of the convergence rate. First, we show the uniform boundedness of $\{\Gamma_k(\cdot)\}_{k=0}^{N-2}$ and $\{P_k(\cdot)\}_{k=0}^{N-1}$.

Let $\widehat{G},\widehat{Q}\in\mathbb{S}^n_+$ satisfy

$$G(T)\leqslant\widehat{G}, \quad Q(s,T)\leqslant\widehat{Q}, \quad \forall\, s\in[0,T].$$

For $0\leqslant k\leqslant N-1$, consider the following equations:

$$\begin{cases} \dot{\widehat{P}}_k + \widehat{P}_kA + A^\top\widehat{P}_k + C^\top\widehat{P}_kC + Q_k = 0 \quad \text{in } [t_k,t_{k+1}), \\ \widehat{P}_k(t_{k+1}) = \Gamma_k(t_{k+1}) \end{cases} \tag{4.10}$$

and

$$\begin{cases} \dot{\widehat{\Pi}} + \widehat{\Pi}A + A^\top\widehat{\Pi} + C^\top\widehat{\Pi}C + \widehat{Q} = 0 \quad \text{in } [t_0,t_N), \\ \widehat{\Pi}(t_N) = \widehat{G}. \end{cases} \tag{4.11}$$

Proposition 4.1. *Let Assumptions* 2.4–2.6 *hold. Then*

$$0\leqslant\Gamma_l(s)\leqslant P_k(s)\leqslant\widehat{P}_k(s)\leqslant\widehat{\Pi}(s), \quad -1\leqslant l<k\leqslant N-1 \tag{4.12}$$

with the convention that $\Gamma_{-1}(\cdot)=0$.

Proof. By Assumption 2.4, Lemma 3.3 and the equation (4.4), we have

$$\Gamma_m(s) \geqslant \Gamma_n(s), \quad s \in [t_{N-1}, t_N], \quad 0 \leqslant n \leqslant m \leqslant N-2.$$

Inductively, we get

$$\begin{cases} \Gamma_m(s) \geqslant \Gamma_n(s), \quad s \in [t_{N-2}, t_{N-1}], \quad 0 \leqslant n \leqslant m \leqslant N-3, \\ \qquad\qquad\qquad \vdots \\ \Gamma_m(s) \geqslant \Gamma_n(s), \quad s \in [t_1, t_2], \quad n = m = 0. \end{cases} \tag{4.13}$$

On the other hand, it follows from Lemmas 3.3 and 3.9 that for any $x \in \mathbb{R}^n$, it holds that

$$\begin{aligned} \langle P_{N-1}(s)x, x\rangle = \mathrm{E}\bigg(\int_s^{t_N} &\langle (Q_{N-1} + \Theta_{N-1}^\top R_{N-1}\Theta_{N-1})\overline{X}_{N-1}, \overline{X}_{N-1}\rangle dt \\ &+ \langle G_{N-1}\overline{X}_{N-1}(t_N), \overline{X}_{N-1}(t_N)\rangle \bigg), \quad s \in [t_{N-1}, t_N] \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \langle \Gamma_k(s)x, x\rangle = \mathrm{E}\bigg(\int_s^{t_N} &\langle (Q_k + \Theta_{N-1}^\top R_k\Theta_{N-1})\overline{X}_{N-1}, \overline{X}_{N-1}\rangle dt \\ &+ \langle G_k\overline{X}_{N-1}(t_N), \overline{X}_{N-1}(t_N)\rangle \bigg), \quad s \in [t_{N-1}, t_N], \quad 0 \leqslant k \leqslant N-2, \end{aligned} \tag{4.15}$$

where $\overline{X}_{N-1}(\cdot)$ satisfies

$$\begin{cases} d\overline{X}_{N-1} = (A + B\Theta_{N-1})\overline{X}_{N-1}dt + (C + D\Theta_{N-1})\overline{X}_{N-1}dW(t) \quad \text{in } [s, t_N], \\ \overline{X}_{N-1}(s) = x, \quad s \in [t_{N-1}, t_N]. \end{cases}$$

By Assumption 2.4, we have

$$P_{N-1}(s) \geqslant \Gamma_k(s), \quad s \in [t_{N-1}, t_N], \quad 0 \leqslant k \leqslant N-2.$$

From (4.13) and a similar representation to (4.14) and (4.15), we can deduce

$$0 \leqslant \Gamma_l(s) \leqslant P_k(s), \quad s \in [t_k, t_{k+1}), \quad 0 \leqslant l < k \leqslant N-1.$$

Next, it follows from Lemma 3.13 that

$$P_k(s) \leqslant \widehat{P}_k(s), \quad s \in [t_k, t_{k+1}), \quad 0 \leqslant k \leqslant N-1.$$

Finally, note that for any $0 \leqslant k \leqslant N-1$,

$$\widehat{P}_k(t_{k+1} - 0) = \Gamma_k(t_{k+1}) = \Gamma_k(t_{k+1} + 0) \leqslant P_{k+1}(t_{k+1} + 0) \leqslant \widehat{P}_k(t_{k+1} + 0). \tag{4.16}$$

By Lemma 3.3 and (4.16), inductively, we can obtain

$$\widehat{P}_k(s) \leqslant \widehat{\Pi}(s), \quad s \in [t_k, t_{k+1}], \quad 0 \leqslant k \leqslant N-1.$$

The proof is completed. □

Having the above preparation, we can establish the convergence result for $(P_{\mathcal{D}}(s), \Gamma_{\mathcal{D}}(s, \tau))$ by the Arzela-Ascoli theorem as in [30]. However, that method gives no clue to the convergence rate which is needed to prove that the strategy derived is indeed a closed-loop equilibrium strategy as in Definition 2.1. This will be explained in detail in Subsection 4.3. Here, we follow some ideas in [10] to give a sharp estimate. It is based on some delicate construction and subtle use of the coupling relationship between the equations. The advantage of this new method is that the idea is intuitive, and the proof is easier in some sense.

Now we give the construction. Let $\theta(\cdot) \in L^2(0,T;\mathbb{R}^{m\times n})$ be fixed. For $l,m \in \mathbb{N}$, let $\mathcal{D}_l = \{[t_k,t_{k+1}]\}_{k=0}^{l-1}$ and $\mathcal{D}_{l+m} = \{[\tilde{t}_k,\tilde{t}_{k+1}]\}_{k=0}^{m_l-1}$ be two partitions of $[0,T]$ such that $\tilde{t}_{m_k} = t_k$ for $k=0,\dots,l$, and $m_0 = 0$. Thus, $\mathcal{D}_{l+m}$ is a refinement of the partition $\mathcal{D}_l$. Then we can solve (4.2) and (4.4) to obtain $P_{\mathcal{D}_l}$ and $P_{\mathcal{D}_{l+m}}$ as in (4.9) under the partitions $\mathcal{D}_l$ and $\mathcal{D}_{l+m}$, respectively.

Given a partition $\mathcal{D}_{l+m}$, one can see from (4.2) and (4.4) that $P_{\mathcal{D}_{l+m}}$ is determined by $\{Q(\cdot,\tilde{t}_k), R(\cdot,\tilde{t}_k), G(\tilde{t}_k)\}_{0\leqslant k\leqslant m_l}$, and $P_{\mathcal{D}_{l+m}}$ will change if $\{Q(\cdot,\tilde{t}_k), R(\cdot,\tilde{t}_k), G(\tilde{t}_k)\}_{0\leqslant k\leqslant m_l}$ is altered. Generally speaking, $P_{\mathcal{D}_{l+m}}$ is different from $P_{\mathcal{D}_l}$. Here, we provide a way to alter $\{Q(\cdot,\tilde{t}_k), R(\cdot,\tilde{t}_k), G(\tilde{t}_k)\}_{0\leqslant k\leqslant m_l}$ such that the corresponding $P_{\mathcal{D}_{l+m}}$ equals $P_{\mathcal{D}_l}$. For the equations (4.2) and (4.4) with respect to the partition $\mathcal{D}_{l+m}$, we make the following alterations inductively and denote the varied $P_{\mathcal{D}_{l+m}}$ by $\overline{P}_{\mathcal{D}_{l+m}}$:

$$
\begin{cases}
\overline{P}_{m_N-1} \quad \text{in } [\tilde{t}_{m_N-1},\tilde{t}_{m_N}], \quad \overline{\Gamma}_{m_N-2} \quad \text{in } [\tilde{t}_{m_N-1},\tilde{t}_{m_N}],\\
\qquad \vdots\\
\overline{P}_{m_{N-1}+1} \quad \text{in } [\tilde{t}_{m_{N-1}+1},\tilde{t}_{m_{N-1}+2}], \quad \overline{\Gamma}_{m_{N-1}} \quad \text{in } [\tilde{t}_{m_{N-1}+1},\tilde{t}_{m_N}],\\
\overline{P}_{m_{N-1}} \quad \text{in } [\tilde{t}_{m_{N-1}},\tilde{t}_{m_{N-1}+1}]
\end{cases}
$$
with the same $(Q(\cdot,t_{N-1}), R(\cdot,t_{N-1}), G(t_{N-1}))$,

$$
\begin{cases}
\overline{\Gamma}_{m_{N-1}-1} \quad \text{in } [\tilde{t}_{m_{N-1}},\tilde{t}_{m_N}],\\
\overline{P}_{m_{N-1}-1} \quad \text{in } [\tilde{t}_{m_{N-1}-1},\tilde{t}_{m_{N-1}}], \quad \overline{\Gamma}_{m_{N-1}-2} \quad \text{in } [\tilde{t}_{m_{N-1}-1},\tilde{t}_{m_N}],\\
\qquad \vdots\\
\overline{P}_{m_{N-2}+1} \quad \text{in } [\tilde{t}_{m_{N-2}+1},\tilde{t}_{m_{N-2}+2}], \quad \overline{\Gamma}_{m_{N-2}} \quad \text{in } [\tilde{t}_{m_{N-2}+1},\tilde{t}_{m_N}],\\
\overline{P}_{m_{N-2}} \quad \text{in } [\tilde{t}_{m_{N-2}},\tilde{t}_{m_{N-2}+1}]
\end{cases}
$$
with the same $(Q(\cdot,t_{N-2}), R(\cdot,t_{N-2}), G(t_{N-2}))$,

$$\vdots$$

Then we have the following result.

Proposition 4.2. *For any $l,m \in \mathbb{N}$, we have*
$$
\overline{P}_{\mathcal{D}_{l+m}} = P_{\mathcal{D}_l}. \tag{4.17}
$$

Proof. For $k = m_{N-1}, m_{N-1}+1,\dots,m_N-1$, by Lemma 3.3, we can rewrite the equation (4.2) as
$$
\begin{cases}
\dot{\overline{P}}_k + \overline{P}_k(A+B\overline{\Theta}_{\mathcal{D}_{l+m}}) + (A+B\overline{\Theta}_{\mathcal{D}_{l+m}})^\top \overline{P}_k + Q(t_{N-1})\\
\quad +(C+D\overline{\Theta}_{\mathcal{D}_{l+m}})^\top \overline{P}_k(C+D\overline{\Theta}_{\mathcal{D}_{l+m}}) + \overline{\Theta}_{\mathcal{D}_{l+m}}^\top R(t_{N-1})\overline{\Theta}_{\mathcal{D}_{l+m}} = 0 \quad \text{in } [\tilde{t}_k,\tilde{t}_{k+1}),\\
\overline{P}_k(\tilde{t}_{k+1}) = \overline{\Gamma}_k(\tilde{t}_{k+1}).
\end{cases}
\tag{4.18}
$$

For $k = m_N - 1$, it follows from (4.18) and (4.4) that
$$
\overline{P}_{m_N-1}(s) = \overline{\Gamma}_l(s), \quad s\in[\tilde{t}_{m_N-1},\tilde{t}_{m_N}], \quad l = m_{N-1}, m_{N-1}+1,\dots,m_N-2.
$$
Inductively, we have
$$
\begin{aligned}
&\overline{P}_{m_N-2}(s) = \overline{\Gamma}_l(s), \quad s\in[\tilde{t}_{m_N-2},\tilde{t}_{m_N-1}], \quad l = m_{N-1}, m_{N-1}+1,\dots,m_N-3,\\
&\overline{P}_{m_N-3}(s) = \overline{\Gamma}_l(s), \quad s\in[\tilde{t}_{m_N-3},\tilde{t}_{m_N-2}], \quad l = m_{N-1}, m_{N-1}+1,\dots,m_N-4,\\
&\qquad \vdots\\
&\overline{P}_{m_{N-1}+1}(s) = \overline{\Gamma}_{m_{N-1}}(s), \quad s\in[\tilde{t}_{m_{N-1}+1},\tilde{t}_{m_{N-1}+2}].
\end{aligned}
$$
Particularly, we have
$$
\overline{\Gamma}_l(t_{l+1}) = \overline{P}_{l+1}(t_{l+1}), \quad l = m_{N-1}, m_{N-1}+1,\dots,m_N-2,
$$

which means that $\overline{P}_{\mathcal{D}_{l+m}}(\cdot)$ is continuous on

$$\bigcup_{m_{N-1}\leqslant l\leqslant m_N-1}[\tilde{t}_l,\tilde{t}_{l+1}]\equiv[t_{N-1},t_N].$$

Then from (4.2) and (4.4), we find

$$\begin{aligned}&\overline{P}_{\mathcal{D}_{l+m}}(s)=P_{\mathcal{D}_l}(s),\quad s\in[t_{N-1},t_N],\\&\overline{\Gamma}_{m_{N-1}-1}(s)=\Gamma_{N-2}(s),\quad s\in[\tilde{t}_{m_{N-1}},\tilde{t}_{m_N}].\end{aligned}$$

Similarly, we can handle the cases $k=m_0,\dots,m_{N-1}-1$. □

Remark 4.3. The reason why Proposition 4.2 holds is that on each interval $[t_k,t_{k+1}]$ $(0\leqslant k\leqslant N-1)$, the corresponding problem for $\overline{P}_{\mathcal{D}_{l+m}}$ is "time-consistent". Moreover, we can get from the proof that

$$\begin{cases}\overline{\Gamma}_{m_{N-1}-1}(\cdot)=\Gamma_{N-2}(\cdot),\quad [\tilde{t}_{m_{N-1}},\tilde{t}_{m_N}]=[t_{N-1},t_N],\\ \overline{\Gamma}_{m_{N-2}-1}(\cdot)=\Gamma_{N-3}(\cdot),\quad [\tilde{t}_{m_{N-2}},\tilde{t}_{m_N}]=[t_{N-2},t_N],\\ \quad\vdots\\ \overline{\Gamma}_{m_1-1}(\cdot)=\Gamma_0(\cdot),\quad [\tilde{t}_{m_1},\tilde{t}_{m_N}]=[t_1,t_N].\end{cases}\tag{4.19}$$

Proposition 4.4. *For any partitions $\mathcal{D}_l$ and $\mathcal{D}_{l+m}$, and the corresponding Riccati equations* (4.2) *and* (4.4)*, we have*

$$\begin{cases}|P_{\mathcal{D}_{l+m}}(s)-P_{\mathcal{D}_l}(s)|_2\leqslant\mathcal{C}\|\mathcal{D}_l\|,\quad s\in[0,T],\\ |\Gamma_{\mathcal{D}_{l+m}}(s,\tau)-\Gamma_{\mathcal{D}_l}(s,\tau)|_2\leqslant\mathcal{C}\|\mathcal{D}_l\|,\quad 0\leqslant\tau\leqslant s\leqslant T,\end{cases}\tag{4.20}$$

where $\mathcal{C}$ is a constant independent of the choice of partitions.

Proof. It follows from Proposition 4.2 that

$$P_{\mathcal{D}_l}(s)\equiv\overline{P}_{\mathcal{D}_{l+m}}(s),\quad s\in[0,T].$$

Then for any $k=m_0,m_0+1,\dots,m_N-1$ and $t\in[\tilde{t}_k,\tilde{t}_{k+1}]$, we have

$$\begin{aligned}|\overline{P}_k(t)-P_k(t)|_2&\leqslant\int_t^{\tilde{t}_{k+1}}[(2|A|_{L^\infty(0,T;\mathbb{R}^{n\times n})}+|C|^2_{L^\infty(0,T;\mathbb{R}^{n\times n})})|\overline{P}_k-P_k|_2\\&\quad+\mathcal{C}\|\mathcal{D}_l\|+\mathcal{C}|\overline{P}_k-P_k|_2]ds+|\overline{\Gamma}_k(\tilde{t}_{k+1})-\Gamma_k(\tilde{t}_{k+1})|_2\\&\leqslant\mathcal{C}\bigg(\|\mathcal{D}_l\|+\int_t^{\tilde{t}_{k+1}}|\overline{P}_k-P_k|_2ds\bigg)+|\overline{\Gamma}_k(\tilde{t}_{k+1})-\Gamma_k(\tilde{t}_{k+1})|_2.\end{aligned}\tag{4.21}$$

On the other hand, for $k=m_0,m_0+1,\dots,m_N-2$ and $t\in[\tilde{t}_{k+1},\tilde{t}_{m_N}]$, it holds that

$$\begin{aligned}&|\overline{\Gamma}_k(t)-\Gamma_k(t)|_2\\&\quad\leqslant\mathcal{C}\bigg\{\|\mathcal{D}_l\|+\int_t^T[(1+|\theta|_2+|\theta|_2^2)|\overline{\Gamma}_k-\Gamma_k|_2+(1+|\theta|_2+|\theta|_2^2)|\overline{P}_{\mathcal{D}_{l+m}}-P_{\mathcal{D}_{l+m}}|_2]ds\bigg\}.\end{aligned}\tag{4.22}$$

It follows from (4.22) and Gronwall's inequality that

$$|\overline{\Gamma}_k(\tilde{t}_{k+1})-\Gamma_k(\tilde{t}_{k+1})|_2\leqslant\mathcal{C}\bigg[\|\mathcal{D}_l\|+\int_{\tilde{t}_{k+1}}^T(1+|\theta|_2+|\theta|_2^2)|\overline{P}_{\mathcal{D}_{l+m}}-P_{\mathcal{D}_{l+m}}|_2ds\bigg].$$

This, together with (4.21), implies that for $t\in[\tilde{t}_k,\tilde{t}_{k+1}]$,

$$|\overline{P}_k(t)-P_k(t)|_2\leqslant\mathcal{C}\bigg[\|\mathcal{D}_l\|+\int_t^T(1+|\theta|_2+|\theta|_2^2)|\overline{P}_{\mathcal{D}_{l+m}}-P_{\mathcal{D}_{l+m}}|_2ds\bigg].$$

Hence, for $t \in [0,T]$, it holds that

$$|\overline{P}_{\mathcal{D}_{l+m}}(t) - P_{\mathcal{D}_{l+m}}(t)|_2 \leqslant \mathcal{C}\bigg[\|\mathcal{D}_l\| + \int_t^T (1+|\theta|_2+|\theta|_2^2)|\overline{P}_{\mathcal{D}_{l+m}} - P_{\mathcal{D}_{l+m}}|_2 ds\bigg].$$

This, together with Gronwall's inequality, implies that

$$|\overline{P}_{\mathcal{D}_{l+m}}(t) - P_{\mathcal{D}_{l+m}}(t)|_2 = |P_{\mathcal{D}_{l+m}}(t) - P_{\mathcal{D}_l}(t)|_2 \leqslant \mathcal{C}\|\mathcal{D}_l\|, \quad t \in [0,T]. \tag{4.23}$$

From (4.22) and (4.23), by Gronwall's inequality again, we get

$$|\overline{\Gamma}_k(t) - \Gamma_k(t)|_2 \leqslant \mathcal{C}\|\mathcal{D}_l\|, \quad t \in [\tilde{t}_{k+1}, \tilde{t}_{m_N}]. \tag{4.24}$$

On the other hand, from (4.19) and (4.9), we obtain

$$\begin{aligned}
&|\Gamma_{\mathcal{D}_l}(s,\tau) - \Gamma_{\mathcal{D}_{l+m}}(s,\tau)|_2 \\
&\quad = \bigg|\sum_{k=0}^{N-2} \overline{\Gamma}_{m_{k+1}-1}(s) I_{(\tilde{t}_{m_{k+1}},\tilde{t}_{m_{k+2}}]}(\tau) + \overline{\Gamma}_{m_1-1}(s) I_{(\tilde{t}_{m_0},\tilde{t}_{m_1}]}(\tau) \\
&\qquad - \sum_{k=m_0}^{m_N-2} \Gamma_k(s) I_{(\tilde{t}_{k+1},\tilde{t}_{k+2}]}(\tau) - \Gamma_{m_0}(s) I_{(\tilde{t}_0,\tilde{t}_1]}(\tau)\bigg|_2 \\
&\quad = \bigg|\sum_{k=0}^{N-2} \sum_{m_{k+1}\leqslant j+1<j+2\leqslant m_{k+2}} (\overline{\Gamma}_{m_{k+1}-1}(s) I_{(\tilde{t}_{m_{k+1}},\tilde{t}_{m_{k+2}}]}(\tau) - \Gamma_j(s) I_{((\tilde{t}_{j+1},\tilde{t}_{j+2}]}(\tau)) \\
&\qquad + \overline{\Gamma}_{m_1-1}(s) I_{(\tilde{t}_{m_0},\tilde{t}_{m_1}]}(\tau) - \sum_{k=m_0}^{\max(m_1-2,m_0)} \Gamma_k(s) I_{(\tilde{t}_{k+1},\tilde{t}_{k+2}]}(\tau) - \Gamma_{m_0}(s) I_{(\tilde{t}_0,\tilde{t}_1]}(\tau)\bigg|_2.
\end{aligned} \tag{4.25}$$

It follows from (4.4) that

$$\begin{aligned}
&|\Gamma_k(s) - \Gamma_{k-1}(s)|_2 \\
&\quad \leqslant |G_k - G_{k-1}|_2 + \mathcal{C}\int_s^T |\Gamma_k(r) - \Gamma_{k-1}(r)|_2(|A|_{L^\infty(0,T;\mathbb{R}^{n\times n})} + |B|_{L^\infty(0,T;\mathbb{R}^{n\times m})}|\Theta_{\mathcal{D}_{l+m}}(r)|_2 \\
&\qquad + |C|^2_{L^\infty(0,T;\mathbb{R}^{n\times n})} + |D|_{L^\infty(0,T;\mathbb{R}^{n\times m})}|\Theta_{\mathcal{D}_{l+m}}(r)|_2^2) dr \\
&\qquad + \mathcal{C}\int_s^T (|Q_k(r) - Q_{k-1}(r)|_2 + |R_k(r) - R_{k-1}(r)|_2 |\Theta_{\mathcal{D}_{l+m}}(r)|_2^2) dr.
\end{aligned}$$

By the uniform boundedness of $P_k(\cdot)$ and Gronwall's inequality, we find that for $m_0 \leqslant k \leqslant m_N - 2$,

$$|\Gamma_k(s) - \Gamma_{k-1}(s)|_2 \leqslant \mathcal{C}|\tilde{t}_k - \tilde{t}_{k-1}|, \quad s \in [\tilde{t}_{k+1}, \tilde{t}_{m_N}]. \tag{4.26}$$

Now we analyze the first term on the right-hand side of (4.25). For any $0 \leqslant k \leqslant N-2$, $m_{k+1}-1 \leqslant j \leqslant m_{k+2}-2$ and $s \in [\tilde{t}_{j+1}, \tilde{t}_{m_N}]$, by (4.24) and (4.26), we obtain

$$\begin{aligned}
|\overline{\Gamma}_{m_{k+1}-1}(s) - \Gamma_j(s)|_2 &\leqslant |\overline{\Gamma}_{m_{k+1}-1}(s) - \Gamma_{m_{k+1}-1}(s)|_2 + |\Gamma_{m_{k+1}-1}(s) - \Gamma_j(s)|_2 \\
&\leqslant \mathcal{C}\|\mathcal{D}_l\| + \sum_{i=m_{k+1}-1}^{j-1} |\Gamma_{i+1}(s) - \Gamma_i(s)|_2 \\
&\leqslant \mathcal{C}\|\mathcal{D}_l\| + \mathcal{C}\sum_{i=m_{k+1}-1}^{j-1} |\tilde{t}_{i+1} - \tilde{t}_i| \\
&\leqslant \mathcal{C}\|\mathcal{D}_l\| + \mathcal{C}|\tilde{t}_{m_{k+2}-2} - \tilde{t}_{m_{k+1}-1}| \\
&\leqslant \mathcal{C}\|\mathcal{D}_l\|.
\end{aligned} \tag{4.27}$$

From (4.27), we see that

$$\left|\sum_{k=0}^{N-2}\sum_{m_{k+1}\leqslant j+1<j+2\leqslant m_{k+2}}(\overline{\Gamma}_{m_{k+1}-1}(s)I_{(\tilde{t}_{m_{k+1}},\tilde{t}_{m_{k+2}}]}(\tau)-\Gamma_j(s)I_{((\tilde{t}_{j+1},\tilde{t}_{j+2}]}(\tau))\right|_2\leqslant\mathcal{C}\|\mathcal{D}_l\|. \tag{4.28}$$

With a similar technique to (4.27), for the second term on the right-hand side of (4.25), we can prove

$$\left|\overline{\Gamma}_{m_1-1}(s)I_{(\tilde{t}_{m_0},\tilde{t}_{m_1}]}(\tau)-\sum_{k=m_0}^{\max(m_1-2,m_0)}\Gamma_k(s)I_{(\tilde{t}_{k+1},\tilde{t}_{k+2}]}(\tau)-\Gamma_{m_0}(s)I_{(\tilde{t}_0,\tilde{t}_1]}(\tau)\right|_2\leqslant\mathcal{C}\|\mathcal{D}_l\|. \tag{4.29}$$

Combining (4.25), (4.28) and (4.29), we conclude that

$$|\Gamma_{\mathcal{D}_l}(s,\tau)-\Gamma_{\mathcal{D}_{l+m}}(s,\tau)|_2\leqslant\mathcal{C}\|\mathcal{D}_l\|,\quad 0\leqslant\tau\leqslant s\leqslant T.$$

This completes the proof. □

Now we give the main result of this subsection.

Theorem 4.5. *Let Assumptions* 2.4–2.7 *hold. Then for any* $\theta(\cdot)\in L^2(0,T;\mathbb{R}^{m\times n})$*, there exists a unique solution* $\Gamma(\cdot,\cdot)$ *to the equation* (2.11)*. Furthermore,*

$$\lim_{\|\mathcal{D}\|\to 0}(|\Gamma_{\mathcal{D}}(s,\tau)-\Gamma(s,\tau)|_2+|P_{\mathcal{D}}(s)-\Gamma(s,s)|_2)=0 \tag{4.30}$$

uniformly in $(s,\tau)\in\Delta$.

Proof. Let $\theta(\cdot)\in L^2(0,T;\mathbb{R}^{m\times n})$ be fixed. For any partition $\{\mathcal{D}_l\}_{l=1}^{\infty}$, $\mathcal{D}_{l+1}$ is a refinement of $\mathcal{D}_l$ and $\lim_{l\to\infty}\|\mathcal{D}_l\|=0$. By Proposition 4.4, for any $(s,\tau)\in\Delta$, $\{\Gamma_{\mathcal{D}_l}(s,\tau)\}_{l=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}^{n\times n}$, and we denote the limit by $\Gamma(s,\tau)$. Then by (4.20),

$$\lim_{l\to\infty}|\Gamma_{\mathcal{D}_l}(s,\tau)-\Gamma(s,\tau)|_2=0\quad\text{uniformly in }(s,\tau)\in\Delta.$$

On the other hand, for $s\in(t_{k+1},t_{k+2}]$ $(0\leqslant k\leqslant N-2)$, by the property (4.26) and Proposition 4.1, we have

$$\begin{aligned}|P_{\mathcal{D}_l}(s)-\Gamma_{\mathcal{D}_l}(s,s)|_2&=|P_{k+1}(s)-\Gamma_k(s)|_2\\&\leqslant|P_{k+1}(s)-P_{k+1}(t_{k+2})|_2+|P_{k+1}(t_{k+2})-\Gamma_k(s)|_2\\&\leqslant\mathcal{C}\|\mathcal{D}_l\|+|\Gamma_{k+1}(t_{k+2})-\Gamma_k(s)|_2\\&\leqslant\mathcal{C}\|\mathcal{D}_l\|+|\Gamma_{k+1}(t_{k+2})-\Gamma_k(t_{k+2})|_2+|\Gamma_k(t_{k+2})-\Gamma_k(s)|_2\\&\leqslant\mathcal{C}\|\mathcal{D}_l\|.\end{aligned} \tag{4.31}$$

For $s\in[t_0,t_1]$, we have

$$\begin{aligned}|P_{\mathcal{D}_l}(s)-\Gamma_{\mathcal{D}_l}(s,s)|_2&=|P_0(s)-\Gamma_0(s)|_2\\&\leqslant|P_0(s)-P_0(t_1)|_2+|P_0(t_1)-\Gamma_0(s)|_2\\&\leqslant|P_0(s)-P_0(t_1)|_2+|\Gamma_0(t_1)-\Gamma_0(s)|_2\\&\leqslant\mathcal{C}\|\mathcal{D}_l\|.\end{aligned} \tag{4.32}$$

It follows from (4.31), (4.32) and (4.20) that

$$\lim_{l\to\infty}|P_{\mathcal{D}_l}(s)-\Gamma(s,s)|_2=0\quad\text{uniformly for }s\in[0,T].$$

By (4.4) and (4.8), we have that for $1\leqslant k\leqslant N-2$,

$$\begin{aligned}\Gamma_k(s)-G_k=\int_s^T&[\Gamma_k(A+B\Theta_{\mathcal{D}})+(A+B\Theta_{\mathcal{D}})^\top\Gamma_k+Q_k+(C+D\Theta_{\mathcal{D}})^\top\Gamma_k(C+D\Theta_{\mathcal{D}})\\&+\Theta_{\mathcal{D}}^\top R_k\Theta_{\mathcal{D}}]dr,\quad s\in[t_{k+1},T],\end{aligned}$$

and for $k=0$,

$$\Gamma_0(s)-G_0=\int_s^T[\Gamma_0(A+B\Theta_{\mathcal{D}})+(A+B\Theta_{\mathcal{D}})^\top\Gamma_0+Q_0+(C+D\Theta_{\mathcal{D}})^\top\Gamma_0(C+D\Theta_{\mathcal{D}})\\ +\Theta_{\mathcal{D}}^\top R_0\Theta_{\mathcal{D}}]dr,\quad s\in[0,T].$$

These, together with (4.9), imply that for $\tau\in(t_{k+1},t_{k+2}]$ $(1\leqslant k\leqslant N-2)$,

$$\Gamma_{\mathcal{D}_l}(s,\tau)=G(t_k)+\int_s^T[\Gamma_{\mathcal{D}_l}(r,\tau)(A+B\Theta_{\mathcal{D}_l})+(A+B\Theta_{\mathcal{D}_l})^\top\Gamma_{\mathcal{D}_l}(r,\tau)+Q(r,t_k)\\ +(C+D\Theta_{\mathcal{D}_l})^\top\Gamma_{\mathcal{D}_l}(r,\tau)(C+D\Theta_{\mathcal{D}_l})+\Theta_{\mathcal{D}_l}^\top R(r,t_k)\Theta_{\mathcal{D}_l}]dr,\quad s\in[\tau,T],\tag{4.33}$$

and for $\tau\in[0,t_2]$,

$$\Gamma_{\mathcal{D}_l}(s,\tau)=G(t_0)+\int_s^T[\Gamma_{\mathcal{D}_l}(r,\tau)(A+B\Theta_{\mathcal{D}_l})+(A+B\Theta_{\mathcal{D}_l})^\top\Gamma_{\mathcal{D}_l}(r,\tau)+Q(r,t_0)\\ +(C-D\Theta_{\mathcal{D}_l})^\top\Gamma_{\mathcal{D}_l}(r,\tau)(C-D\Theta_{\mathcal{D}_l})+\Theta_{\mathcal{D}_l}^\top R(r,t_0)\Theta_{\mathcal{D}_l}]dr,\quad s\in[\tau,T].\tag{4.34}$$

By Lemmas 3.1 and 3.7, letting l tend to ∞ in (4.33) and (4.34), we see that the equation (2.11) has a solution.

Now we are going to prove the uniqueness of the solution. Suppose that there are two solutions $\Gamma_1(\cdot,\cdot)$ and $\Gamma_2(\cdot,\cdot)$. Let $\Lambda(\cdot,\cdot)=\Gamma_1(\cdot,\cdot)-\Gamma_2(\cdot,\cdot)$. Then

$$\begin{cases}\Lambda_s(s,t)+\Lambda(s,t)A+A^\top\Lambda(s,t)-\Lambda(s,t)B\Theta_1-(B\Theta_1)^\top\Lambda(s,t)\\ \quad+(C+D\Theta_2)^\top\Lambda(s,t)(C+D\Theta_1)+F(s,t)=0,\quad 0\leqslant t\leqslant s\leqslant T,\\ \Lambda(T,t)=0,\end{cases}\tag{4.35}$$

where

$$F(s,t)=[-\Gamma_2(s,t)B(\Theta_1-\Theta_2)+(\Theta_2-\Theta_1)^\top B^\top\Gamma_2(s,t)+(D\Theta_1-D\Theta_2)^\top\Gamma_1(s,t)(C-D\Theta_1)\\ +(C-D\Theta_2)^\top\Gamma_2(s,t)(D\Theta_1-D\Theta_2)+\Theta_1^\top R(s,t)(\Theta_1-\Theta_2)+(\Theta_1-\Theta_2)^\top R(s,t)\Theta_2],$$

and for $i=1,2$,

$$\Theta_i(s)=(R(s,s)+D(s)^\top\Gamma_i(s,s)D(s))^\dagger(B(s)^\top\Gamma_i(s,s)+D(s)^\top\Gamma_i(s,s)C(s))+\theta(s)\\ -(R(s,s)+D(s)^\top\Gamma_i(s,s)D(s))^\dagger(R(s,s)+D(s)^\top\Gamma_i(s,s)D(s))\theta(s).$$

Unambiguously, we rewrite the equation (4.35) in the column form with the same notation:

$$\begin{cases}\Lambda_s(s,t)=H(\Gamma_1(s,s),\Gamma_2(s,s))\Lambda(s,t)+F(s,t),\quad s\in[t,T],\\ \Lambda(T,t)=0.\end{cases}$$

Then we have

$$\Lambda(s,t)=-\int_s^T\Phi(T-s,T-\tau)F(\tau,t)d\tau,\tag{4.36}$$

where $\Phi(s,t)$ satisfies $\Phi(t,t)=I$ and

$$\Phi_s(s,t)=-H(\Gamma_1(T-s,T-s),\Gamma_2(T-s,T-s))\Phi(s,t),\quad s\in[t,T].$$

On the other hand, by Lemma 3.6, we have

$$|F(\tau,t)|_{\mathbb{R}^{n^2}}\leqslant\mathcal{C}|\Lambda(\tau,\tau)|_{\mathbb{R}^{n^2}}(1+|\theta(\tau)|_2+|\theta(\tau)|_2^2).$$

This, together with (4.36), implies that

$$|\Lambda(s,t)|_{\mathbb{R}^{n^2}}\leqslant\mathcal{C}\int_s^T|\Lambda(\tau,\tau)|_{\mathbb{R}^{n^2}}(1+|\theta(\tau)|_2+|\theta(\tau)|_2^2)d\tau.$$

Setting $s=t$, by Gronwall's inequality, we have $\Lambda(t,t)=0$ for $t\in[0,T]$. Thus, by the equations (4.35) and (4.36), we get

$$\begin{cases} F(s,t)=0, \\ \Lambda(s,t)=0, \end{cases} \qquad 0\leqslant t\leqslant s\leqslant T.$$

This completes the proof. □

Remark 4.6. By (4.20) and (4.30), we have

$$|P_{\mathcal{D}_l}(s)-\Gamma(s,s)|_2\leqslant \mathcal{C}\|\mathcal{D}_l\|, \quad s\in[0,T],$$

which gives a convergence rate.

4.3 Existence of a closed-loop equilibrium strategy

Let $\theta(\cdot)\in L^2(0,T;\mathbb{R}^{m\times n})$ be fixed. In this subsection, we are going to prove that $\Theta(\cdot)$ obtained in (2.10) is indeed a closed-loop equilibrium strategy. Some delicate treatments are needed and stated as follows.

Let $\{\varepsilon_j\}_{j=1}^\infty\subset(0,+\infty)$ such that $\lim_{j\to\infty}\varepsilon_j=0$. We need to prove that

$$\varliminf_{j\to\infty}\frac{J(t,\overline{X}(t);u_{\varepsilon_j}(\cdot))-J(t,\overline{X}(t);\bar{u}(\cdot))}{\varepsilon_j}\geqslant 0, \quad \text{P-a.s.} \tag{4.37}$$

The basic idea is to make use of (4.7). However, there are several difficulties.

(1) Note that $\frac{1}{\varepsilon_j}$ tends to ∞ as j tends to ∞. Therefore, we cannot use (4.7) directly.

(2) To apply (4.7), we first decompose $J(t,\overline{X}(t);u_{\varepsilon_j}(\cdot))-J(t,\overline{X}(t);\bar{u}(\cdot))$ into a nonnegative term and three remainder terms, and then prove that the remainder terms are of order $o(\varepsilon_j)$. To this end, we apply Proposition 4.4, which shows that the convergence rate is determined by the size of the partition. Therefore, we should relate our partition size to $\{\varepsilon_j\}_{j=1}^\infty$.

Theorem 4.7. *Let Assumptions* 2.4–2.7 *hold. Then* $\Theta(\cdot)$ *given by* (2.10) *is a closed-loop equilibrium strategy.*

Proof. Let $\theta(\cdot)\in L^2(0,T;\mathbb{R}^{m\times n})$ be fixed, and $\{\varepsilon_j\}_{j=1}^\infty\subset(0,+\infty)$ such that $\lim_{j\to\infty}\varepsilon_j=0$. Choose a sequence of partitions $\{\mathcal{D}_j\}_{j=1}^\infty$ of $[0,T]$ such that for $j\in\mathbb{N}$,

(1) $\mathcal{D}_j$ contains $[t,t+\varepsilon_j]=[t_{j_k},t_{j_{k+1}}]$, where t is the time in (4.37), and $t_{j_{N_j}}=T$;

(2) $\mathcal{D}_{j+1}$ is a refinement of $\mathcal{D}_j$;

(3) $\|\mathcal{D}_j\|=\varepsilon_j$;

(4) $\max_{j_{k+1}\leqslant l\leqslant j_{N_j}-1}(t_{l+1}-t_l)=\varepsilon_j^2$.

Then

$$\begin{aligned}
&\frac{1}{\varepsilon_j}(J(t,\overline{X}(t);u_{\varepsilon_j}(\cdot))-J(t,\overline{X}(t);\bar{u}(\cdot)))\\
&\quad=\frac{1}{\varepsilon_j}(J(t,\overline{X}(t);u_{\varepsilon_j}(\cdot))-J(t,\overline{X}(t);\tilde{u}_{j_k}(\cdot))+J(t,\overline{X}(t);\tilde{u}_{j_k}(\cdot))-J(t,\overline{X}(t);\bar{u}_{\mathcal{D}_j}(\cdot))\\
&\qquad+J(t,\overline{X}(t);\bar{u}_{\mathcal{D}_j}(\cdot))-J(t,\overline{X}(t);\tilde{u}_{\mathcal{D}_j}(\cdot))+J(t,\overline{X}(t);\tilde{u}_{\mathcal{D}_j}(\cdot))-J(t,\overline{X}(t);\bar{u}(\cdot))),
\end{aligned}$$

where

$$\begin{cases}
\bar{u}(\cdot)=\Theta(\cdot)\overline{X}(\cdot), \quad \overline{X}(\cdot)=X(\cdot;t,\overline{X}(t),\bar{u}(\cdot)),\\
\bar{u}_{\mathcal{D}_j}(\cdot)=\Theta_{\mathcal{D}_j}(\cdot)\overline{X}_{\mathcal{D}_j}(\cdot), \quad \overline{X}_{\mathcal{D}_j}(\cdot)=X(\cdot;t,\overline{X}(t),\bar{u}_{\mathcal{D}_j}(\cdot)),\\
u_{\varepsilon_j}(\cdot)=u(\cdot)1_{[t,t+\varepsilon_j)}(\cdot)+\Theta(\cdot)X_{\varepsilon_j}(\cdot)1_{[t+\varepsilon_j,T]}(\cdot), \quad X_{\varepsilon_j}(\cdot)=X(\cdot;t,\overline{X}(t),u_{\varepsilon_j}(\cdot)),\\
\tilde{u}_{j_k}(\cdot)=u(\cdot)1_{[t,t+\varepsilon_j)}(\cdot)+\Theta_{\mathcal{D}_j}(\cdot)\tilde{X}_{\mathcal{D}_j}(\cdot)1_{[t+\varepsilon_j,T]}(\cdot), \quad \tilde{X}_{\mathcal{D}_j}(\cdot)=X(\cdot;t,\overline{X}(t),\tilde{u}_{j_k}(\cdot)),\\
\tilde{u}_{\mathcal{D}_j}(\cdot)=\Theta(\cdot)\overline{X}(\cdot)I_{[t,t+\varepsilon_j)}(\cdot)+\Theta_{\mathcal{D}_j}(\cdot)\widehat{X}_{\mathcal{D}_j}(\cdot)I_{[t+\varepsilon_j,T]}(\cdot),\\
\widehat{X}_{\mathcal{D}_j}(\cdot)=\overline{X}(\cdot)I_{[t,t+\varepsilon_j)}(\cdot)+X(\cdot;t+\varepsilon_j,\overline{X}(t+\varepsilon_j),\tilde{u}_{\mathcal{D}_j}(\cdot))I_{[t+\varepsilon_j,T]}(\cdot).
\end{cases}$$

By (4.7), we have $J(t,\overline{X}(t);\tilde{u}_{j_k}(\cdot)) \geqslant J(t,\overline{X}(t);\bar{u}_{\mathcal{D}_j}(\cdot))$. It suffices to prove that

$$\begin{cases} \varliminf_{j\to\infty} F_j^1 \equiv \varliminf_{j\to\infty} \dfrac{1}{\varepsilon_j}(J(t,\overline{X}(t);u_{\varepsilon_j}(\cdot)) - J(t,\overline{X}(t);\tilde{u}_{j_k}(\cdot))) \geqslant 0, \\ \varliminf_{j\to\infty} F_j^2 \equiv \varliminf_{j\to\infty} \dfrac{1}{\varepsilon_j}(J(t,\overline{X}(t);\tilde{u}_{\mathcal{D}_j}(\cdot)) - J(t,\overline{X}(t);\bar{u}(\cdot))) \geqslant 0, \\ \varliminf_{j\to\infty} F_j^3 \equiv \varliminf_{j\to\infty} \dfrac{1}{\varepsilon_j}(J(t,\overline{X}(t);\bar{u}_{\mathcal{D}_j}(\cdot)) - J(t,\overline{X}(t);\tilde{u}_{\mathcal{D}_j}(\cdot))) \geqslant 0. \end{cases} \tag{4.38}$$

Before proceeding further, let us give a useful estimate. From (4.5) and Proposition 4.4, we have

$$|\Theta_{\mathcal{D}_j}|_2 \leqslant \mathcal{C}(1+|\theta|_2) \tag{4.39}$$

and

$$|\Theta_{\mathcal{D}_j} - \Theta_{\mathcal{D}_{j+m}}|_2 \leqslant \mathcal{C}\|\mathcal{D}_j\|(1+|\theta|_2). \tag{4.40}$$

By letting $j\to\infty$ in (4.39) and $m\to\infty$ in (4.40), we get

$$\begin{cases} |\Theta|_2 \leqslant \mathcal{C}(1+|\theta|_2), \\ |\Theta_{\mathcal{D}_j} - \Theta|_2 \leqslant \mathcal{C}\|\mathcal{D}_j\|(1+|\theta|_2). \end{cases} \tag{4.41}$$

Now we turn to the proof of (4.38). Let us first estimate $|F_j^2|$. It follows from (4.39), (4.41), Lemma 3.2 and our partition property (4) that

$$\begin{aligned} |F_j^2| \leqslant{}& \frac{1}{2\varepsilon_j}\mathrm{E}_t\bigg[\int_{t+\varepsilon_j}^T |\langle Q(t)(\widehat{X}_{\mathcal{D}_j}+\overline{X}), \widehat{X}_{\mathcal{D}_j}-\overline{X}\rangle + \langle R(t)(\Theta_{\mathcal{D}_j}\widehat{X}_{\mathcal{D}_j}+\Theta\overline{X}), \Theta_{\mathcal{D}_j}\widehat{X}_{\mathcal{D}_j}-\Theta\overline{X}\rangle| ds \\ &+ |\langle G(t)(\widehat{X}_{\mathcal{D}_j}(T)+\overline{X}(T)), \widehat{X}_{\mathcal{D}_j}(T)-\overline{X}(T)\rangle|\bigg] \\ \leqslant{}& \mathcal{C}\frac{1}{2\varepsilon_j}\bigg[\bigg(\mathrm{E}_t\int_{t+\varepsilon_j}^T |\widehat{X}_{\mathcal{D}_j}+\overline{X}|_{\mathbb{R}^n}^2 ds\bigg)^{1/2}\bigg(\mathrm{E}_t\int_{t+\varepsilon_j}^T |\widehat{X}_{\mathcal{D}_j}-\overline{X}|_{\mathbb{R}^n}^2 ds\bigg)^{1/2} \\ &+ \bigg(\mathrm{E}_t\int_{t+\varepsilon_j}^T |\Theta_{\mathcal{D}_j}\widehat{X}_{\mathcal{D}_j}+\Theta\overline{X}|_{\mathbb{R}^m}^2 ds\bigg)^{1/2}\bigg(\mathrm{E}_t\int_{t+\varepsilon_j}^T |\Theta_{\mathcal{D}_j}\widehat{X}_{\mathcal{D}_j}-\Theta\overline{X}|_{\mathbb{R}^m}^2 ds\bigg)^{1/2} \\ &+ (\mathrm{E}_t|\widehat{X}_{\mathcal{D}_j}(T)+\overline{X}(T)|_{\mathbb{R}^n}^2)^{1/2}(\mathrm{E}_t|\widehat{X}_{\mathcal{D}_j}(T)-\overline{X}(T)|_{\mathbb{R}^n}^2)^{1/2}\bigg] \\ \leqslant{}& \mathcal{C}\frac{1}{2\varepsilon_j}\bigg[\Big(\mathrm{E}_t\sup_{s\in[t,T]}|\widehat{X}_{\mathcal{D}_j}|_{\mathbb{R}^n}^2 + \mathrm{E}_t\sup_{s\in[t,T]}|\overline{X}|_{\mathbb{R}^n}^2\Big)^{1/2}\Big(\mathrm{E}_t\sup_{s\in[t+\epsilon_j,T]}|\widehat{X}_{\mathcal{D}_j}-\overline{X}|_{\mathbb{R}^n}^2\Big)^{1/2} \\ &+ \Big(\mathrm{E}_t\sup_{s\in[t,T]}|\widehat{X}_{\mathcal{D}_j}|_{\mathbb{R}^n}^2 + \mathrm{E}_t\sup_{s\in[t,T]}|\overline{X}|_{\mathbb{R}^n}^2\Big)^{1/2} \\ &\times \Big(\mathrm{E}_t\sup_{s\in[t+\epsilon_j,T]}|\widehat{X}_{\mathcal{D}_j}-\overline{X}|_{\mathbb{R}^n}^2 + \varepsilon_j^4\cdot\mathrm{E}_t\sup_{s\in[t,T]}|\overline{X}|_{\mathbb{R}^n}^2\Big)^{1/2}\bigg]. \end{aligned} \tag{4.42}$$

Hence, it suffices to estimate $\mathrm{E}_t\sup_{s\in[t,T]}|\overline{X}|_{\mathbb{R}^n}^2$, $\mathrm{E}_t\sup_{s\in[t+\epsilon_j,T]}|\widehat{X}_{\mathcal{D}_j}-\overline{X}|_{\mathbb{R}^n}^2$ and $\mathrm{E}_t\sup_{s\in[t,T]}|\widehat{X}_{\mathcal{D}_j}|_{\mathbb{R}^n}^2$.

First, we handle $\mathrm{E}_t\sup_{s\in[t,T]}|\overline{X}|_{\mathbb{R}^n}^2$ in the following way. For any $\mathcal{A}\in\mathcal{F}_t$, we have

$$\begin{cases} d1_{\mathcal{A}}\overline{X} = (A+B\Theta)1_{\mathcal{A}}\overline{X}ds + (C+D\Theta)1_{\mathcal{A}}\overline{X}dW(s) \quad \text{in } [t,T], \\ 1_{\mathcal{A}}\overline{X}(t) = 1_{\mathcal{A}}x, \end{cases}$$

where $x\in L_{\mathcal{F}_t}^2(\Omega;\mathbb{R}^n)$. Then

$$\mathrm{E}\Big(1_{\mathcal{A}}\cdot\sup_{s\in[t,T]}|\overline{X}(s)|_{\mathbb{R}^n}^2\Big) = \mathrm{E}\sup_{s\in[t,T]}|1_{\mathcal{A}}\overline{X}(s)|_{\mathbb{R}^n}^2 \leqslant \mathcal{C}\mathrm{E}|1_{\mathcal{A}}x|_{\mathbb{R}^n}^2 = \mathcal{C}\mathrm{E}(1_{\mathcal{A}}\cdot|x|_{\mathbb{R}^n}^2),$$

where $\mathcal{C}$ is a constant depending on $A+B\Theta$, $C+D\Theta$ and T, but independent of $\mathcal{A}$. Since x is $\mathcal{F}_t$-measurable, we have

$$\mathrm{E}_t \sup_{s\in[t,T]} |\overline{X}|^2_{\mathbb{R}^n} \leqslant \mathcal{C}|x|^2_{\mathbb{R}^n}, \quad \text{P-a.s.} \tag{4.43}$$

Similarly, from (4.39) and (4.41), we can deduce

$$\mathrm{E}_t \sup_{s\in[t,T]} |\widehat{X}_{\mathcal{D}_j}|^2_{\mathbb{R}^n} \leqslant \mathcal{C}|x|^2_{\mathbb{R}^n}, \quad \text{P-a.s.}, \quad \forall\, j \geqslant 1. \tag{4.44}$$

Next, we deal with the term $\mathrm{E}_t \sup_{s\in[t+\varepsilon_j,T]} |\widehat{X}_{\mathcal{D}_j} - \overline{X}|^2_{\mathbb{R}^n}$. By our partition property (4), similar to the derivation of (4.43), for any $\mathcal{A}\in\mathcal{F}_t$, we can obtain

$$\begin{aligned}\mathrm{E} \sup_{s\in[t+\varepsilon_j,T]} |1_{\mathcal{A}}(\widehat{X}_{\mathcal{D}_j} - \overline{X})|^2_{\mathbb{R}^n} &\leqslant \mathcal{C}\bigg[\varepsilon_j^4 \int_{t+\varepsilon_j}^T (1+|\theta|_2^2)ds \cdot \mathrm{E}\Big(1_{\mathcal{A}} \sup_{s\in[t,T]} |\widehat{X}_{\mathcal{D}_j}|^2_{\mathbb{R}^n}\Big)\bigg] \\ &\leqslant \mathcal{C}\varepsilon_j^4 \mathrm{E}(1_{\mathcal{A}}|x|^2_{\mathbb{R}^n}).\end{aligned}$$

Therefore,

$$\mathrm{E}_t \sup_{s\in[t+\varepsilon_j,T]} |\widehat{X}_{\mathcal{D}_j} - \overline{X}|^2_{\mathbb{R}^n} \leqslant \mathcal{C}\varepsilon_j^4|x|^2_{\mathbb{R}^n}, \quad \text{P-a.s.}, \quad \forall\, j\geqslant 1. \tag{4.45}$$

It follows from (4.42)–(4.45) that

$$|F_j^2| \leqslant \mathcal{C}\varepsilon_j|x|^2_{\mathbb{R}^n}, \quad \text{P-a.s.}, \quad \forall\, j\geqslant 1. \tag{4.46}$$

Next, we estimate F_j^3. By Lemma 3.3, we find

$$\begin{aligned}|F_j^3| \leqslant \frac{1}{2\varepsilon_j}\mathrm{E}_t\bigg| \int_t^{t+\varepsilon_j} &[\langle Q(t)(\overline{X}_{\mathcal{D}_j} + \overline{X}), \overline{X}_{\mathcal{D}_j} - \overline{X}\rangle + \langle R(t)(\Theta_{\mathcal{D}_j}\overline{X}_{\mathcal{D}_j} + \Theta\overline{X}), \Theta_{\mathcal{D}_j}\overline{X}_{\mathcal{D}_j} - \Theta\overline{X}\rangle]ds \\ &+ \langle \Gamma_{\mathcal{D}_j}(t+\varepsilon_j)(\overline{X}_{\mathcal{D}_j}(t+\varepsilon_j) + \overline{X}(t+\varepsilon_j)), \overline{X}_{\mathcal{D}_j}(t+\varepsilon_j) - \overline{X}(t+\varepsilon_j)\rangle\bigg|.\end{aligned}$$

Via an argument similar to (4.42)–(4.45), we have

$$\begin{aligned}|F_j^3| \leqslant \mathcal{C}\frac{1}{2\varepsilon_j}\bigg[&\Big(\mathrm{E}_t \sup_{s\in[t,t+\varepsilon_j]} |\overline{X}_{\mathcal{D}_j}|^2_{\mathbb{R}^n} + \mathrm{E}_t \sup_{s\in[t,t+\varepsilon_j]} |\overline{X}|^2_{\mathbb{R}^n}\Big)^{1/2}\Big(\mathrm{E}_t \sup_{s\in[t,t+\epsilon_j]} |\overline{X}_{\mathcal{D}_j} - \overline{X}|^2_{\mathbb{R}^n}\Big)^{1/2} \\ &+ \Big(\mathrm{E}_t \sup_{s\in[t,t+\varepsilon_j]} |\overline{X}_{\mathcal{D}_j}|^2_{\mathbb{R}^n} + \mathrm{E}_t \sup_{s\in[t,t+\varepsilon_j]} |\overline{X}|^2_{\mathbb{R}^n}\Big)^{1/2}\bigg(\mathrm{E}_t \sup_{s\in[t,t+\epsilon_j]} |\overline{X}_{\mathcal{D}_j} - \overline{X}|^2_{\mathbb{R}^n} \\ &+ \varepsilon_j^2 \int_t^{t+\varepsilon_j} (1+|\theta|_2^2)ds \cdot \mathrm{E}_t \sup_{s\in[t,t+\varepsilon]} |\overline{X}|^2_{\mathbb{R}^n}\bigg)^{1/2}\bigg]\end{aligned}$$

and

$$\begin{cases}\mathrm{E}_t \sup\limits_{s\in[t,T]} |\overline{X}_{\mathcal{D}_j}|^2_{\mathbb{R}^n} \leqslant \mathcal{C}|x|^2_{\mathbb{R}^n}, \\ \mathrm{E}_t \sup\limits_{s\in[t,t+\varepsilon_j]} |\overline{X}_{\mathcal{D}_j} - \overline{X}|^2_{\mathbb{R}^n} \leqslant \mathcal{C}\varepsilon_j^2 \displaystyle\int_t^{t+\varepsilon_j} (1+|\theta|_2^2)ds \cdot |x|^2_{\mathbb{R}^n},\end{cases} \quad \text{P-a.s.}, \quad \forall\, j\geqslant 1.$$

Consequently, we can get

$$|F_j^3| \leqslant \mathcal{C}\bigg(\int_t^{t+\varepsilon_j} (1+|\theta|_2^2)ds\bigg)^{1/2} \cdot |x|^2_{\mathbb{R}^n}, \quad \text{P-a.s.}, \quad \forall\, j\geqslant 1. \tag{4.47}$$

Finally, with a calculation similar to F_j^2, we can deduce for F_j^1 that

$$|F_j^1| \leqslant \mathcal{C}\varepsilon_j|x|^2_{\mathbb{R}^n}, \quad \text{P-a.s.}, \quad \forall\, j\geqslant 1. \tag{4.48}$$

Letting $j\to\infty$ in (4.46)–(4.48), we obtain (4.38). This completes the proof. $\square$

From the above arguments, we immediately get the following corollary.

Let Assumptions 2.4–2.7 hold. Then for any $\theta(\cdot) \in L^2(0,T;\mathbb{R}^{m\times n})$, there is a closed-loop equilibrium strategy $\Theta(\cdot)$ given by

$$\begin{aligned}\Theta(s) = &-(R(s,s)+D(s)^\top\Gamma(s,s)D(s))^\dagger[B(s)^\top\Gamma(s,s)+D(s)^\top\Gamma(s,s)C(s)]\\ &+\theta(s)-(R(s,s)+D(s)^\top\Gamma(s,s)D(s))^\dagger(R(s,s)+D(s)^\top\Gamma(s,s)D(s))\theta(s),\end{aligned}\tag{4.49}$$

where $\Gamma(\cdot,\cdot)$ solves

$$\begin{cases}\Gamma_s(s,t)+\Gamma(s,t)(A(s)+B(s)\Theta(s))+(A(s)+B(s)\Theta(s))^\top\Gamma(s,t)+Q(s,t)\\ \quad +(C(s)+D(s)\Theta(s))^\top\Gamma(s,t)(C(s)+D(s)\Theta(s))+\Theta(s)^\top R(s,t)\Theta(s)=0, \quad 0\leqslant t\leqslant s\leqslant T,\\ \Gamma(T,t)=G(t), \quad 0\leqslant t\leqslant T.\end{cases}\tag{4.50}$$

The equilibrium value function is given by

$$V(t,x)=\frac{1}{2}x^\top\Gamma(t,t)x, \quad \forall\,(t,x)\in[0,T)\times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n).\tag{4.51}$$

Corollary 4.8. *Let Assumptions* 2.4–2.7 *hold. Then the equilibrium value function* V *is given by* (2.12).

Proof. From (4.7), (4.38), (4.46) and (4.47), we have

$$\lim_{j\to\infty}|F_j^2+F_j^3| = \lim_{j\to\infty}|J(t,x;\bar{u}(\cdot))-J(t,x;\bar{u}_{\mathcal{D}_j}(\cdot))| = \lim_{j\to\infty}\left|V(t,x)-\frac{1}{2}\langle P_{\mathcal{D}_j}x,x\rangle\right| = 0, \quad \text{P-a.s.}$$

This, together with Theorem 4.5, implies that

$$V(t,x)=\frac{1}{2}x^\top\Gamma(t,t)x, \quad \text{P-a.s.}$$

We complete the proof. □

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