SCIENCE CHINA
Mathematics

• ARTICLES •

January 2024 Vol. 67 No. 1: 1–22
https://doi.org/10.1007/s11425-022-2085-3

On the inviscid limit of the compressible Navier-Stokes equations near Onsager's regularity in bounded domains

Robin Ming Chen[1], Zhilei Liang[2], Dehua Wang[1,*] & Runzhang Xu[3]

[1]*Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA;*
[2]*School of Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China;*
[3]*College of Mathematical Sciences, Harbin Engineering University, Harbin 150001, China*

Email: mingchen@pitt.edu, liangzl@swufe.edu.cn, dwang@math.pitt.edu, xurunzh@hrbeu.edu.cn

Received August 21, 2022; accepted January 6, 2023; published online June 19, 2023

Abstract The viscous dissipation limit of weak solutions is considered for the Navier-Stokes equations of compressible isentropic flows confined in a bounded domain. We establish a Kato-type criterion for the validity of the inviscid limit for the weak solutions of the Navier-Stokes equations in a function space with the regularity index close to Onsager's critical threshold. In particular, we prove that under such a regularity assumption, if the viscous energy dissipation rate vanishes in a boundary layer of thickness in the order of the viscosity, then the weak solutions of the Navier-Stokes equations converge to a weak admissible solution of the Euler equations. Our approach is based on the commutator estimates and a subtle foliation technique near the boundary of the domain.

Keywords inviscid limit, Navier-Stokes equations, Euler equations, weak solutions, bounded domain, Kato-type criterion, Onsager's regularity

MSC(2020) 35Q30, 35D30, 76B03, 76D05, 76N10

Citation: Chen R M, Liang Z L, Wang D H, et al. On the inviscid limit of the compressible Navier-Stokes equations near Onsager's regularity in bounded domains. Sci China Math, 2024, 67: 1–22, https://doi.org/10.1007/s11425-022-2085-3

1 Introduction

We consider the isentropic Navier-Stokes equations describing the viscous compressible fluids occupied in a smooth and bounded domain $\Omega \subset \mathbb{R}^3$:

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon \boldsymbol{u}^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon \boldsymbol{u}^\varepsilon) + \operatorname{div}(\rho^\varepsilon \boldsymbol{u}^\varepsilon \otimes \boldsymbol{u}^\varepsilon) + \nabla(\rho^\varepsilon)^\gamma = \varepsilon \operatorname{div} \mathbb{S}(\nabla \boldsymbol{u}^\varepsilon), \end{cases} \tag{1.1}$$

where ρ^ε and $\boldsymbol{u}^\varepsilon$ denote the density and the velocity, respectively; the constant $\gamma > 1$ is the adiabatic exponent, $\mathbb{S}(\nabla \boldsymbol{u}^\varepsilon)$ is the viscous stress tensor of the form

$$\mathbb{S}(\nabla \boldsymbol{u}^\varepsilon) = \mu(\nabla \boldsymbol{u}^\varepsilon + (\nabla \boldsymbol{u}^\varepsilon)^\top) + \lambda \mathbb{I} \operatorname{div} \boldsymbol{u}^\varepsilon, \tag{1.2}$$

* Corresponding author

© Science China Press 2023

μ and λ are two constants satisfying $\mu > 0$ and $2\mu + 3\lambda \geqslant 0$, $\mathbb{I}$ is the identity matrix, and $\varepsilon > 0$ is a scaling factor. At least formally, one expects that the solution $(\rho^\varepsilon, \boldsymbol{u}^\varepsilon)$ to (1.1) as ε vanishes satisfies the corresponding Euler equations, i.e.,

$$\begin{cases} \partial_t \rho^0 + \operatorname{div}(\rho^0 \boldsymbol{u}^0) = 0, \\ \partial_t(\rho^0 \boldsymbol{u}^0) + \operatorname{div}((\rho^0 \boldsymbol{u}^0) \otimes \boldsymbol{u}^0) + \nabla(\rho^0)^\gamma = 0. \end{cases} \tag{1.3}$$

However, the rigorous mathematical justification is very complicated since the boundary layers appear (see [41, 46, 50]) if the equations (1.1) and (1.3) are subject to the no-slip boundary condition:

$$\boldsymbol{u}^\varepsilon = 0 \quad \text{on } \partial\Omega \tag{1.4}$$

and the slip boundary condition:

$$\boldsymbol{u}^0 \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \tag{1.5}$$

with $\boldsymbol{n}$ denoting the outward normal vector of the boundary $\partial\Omega$, respectively.

It is known that the mismatch of the boundary conditions (1.4) and (1.5) leads to the phenomenon of boundary layers, and not much has been understood theoretically. Concerning the incompressible Navier-Stokes equations, we see that a well-known result was originally due to Kato [33], where he introduced a boundary layer corrector and showed that the Leray-Hopf solutions of the Navier-Stokes equations converge to the smooth solutions of Euler equations in the sense of the energy norms, under the condition that the energy dissipation holds in a boundary layer of width ε, i.e., a necessary and sufficient condition of the limit $\boldsymbol{u}^\varepsilon \to \boldsymbol{u}^0$ in $L^\infty(0, T; L^2(\Omega))$ is

$$\varepsilon \int_0^T \|\nabla \boldsymbol{u}^\varepsilon\|^2_{L^2(\Gamma_{c\varepsilon})} dt \to 0 \quad \text{as } \varepsilon \to 0, \tag{1.6}$$

where $\Gamma_{c\varepsilon}$ is a very thin boundary layer of width proportional to ε. Some refinements and improvements of Kato's result were subsequently obtained; see, for example, [3, 16, 20, 34, 51] and the references therein. In addition to the above known fact that the anomalous energy dissipation leads to the failure of the inviscid limit to the smooth Euler solutions, Onsager [42] conjectured that the spatial Hölder continuity $C^{1/3}$ of the velocity field should be the threshold of regularity to sustain non-vanishing energy dissipation in the limit

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_\Omega |\nabla \boldsymbol{u}^\varepsilon|^2 dx dt > 0.$$

We mention that there have been many results toward both directions on Onsager's conjecture (see [4, 5, 8, 12, 15, 22, 32, 47] and the references therein). Assuming that the boundary layer of thickness $O(\varepsilon^\beta)$ with $\beta = \frac{3}{4}+$, Drivas and Nguyen [19] established, among other regularity conditions, a Kato-type boundary criterion for the solutions with Onsager's regularity. Recently, Chen et al. [10] improved the result of [19] by relaxing the exponent β from $\frac{3}{4}$ to 1. This matches the classical Kato's boundary layer condition (1.6) for weak solutions of the Euler equations with $C^{1/3}$ regularity.

It is natural to expect that similar issues arise in compressible fluids. However, much less is known in this case. Questions about regularity, stability and uniqueness become much more delicate. Extending Kato's idea to compressible flows, Sueur [49] proved that given a strong C^1 solution $(\rho^0, \boldsymbol{u}^0)$ to the Euler equations (1.3) with ρ^0 being bounded above and away from zero, there exists a sequence of weak solutions $(\rho^\varepsilon, \boldsymbol{u}^\varepsilon)$ of the equations (1.1) with the no-slip condition (1.4) that converges to $(\rho^0, \boldsymbol{u}^0)$ in the relative energy norm, provided that the near boundary limit

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\Gamma_{c\varepsilon}} \left(|\mathbb{S}(\nabla \boldsymbol{u}^\varepsilon)|^2 + \frac{\rho^\varepsilon |\boldsymbol{u}^\varepsilon|^2}{|d_\Omega(x)|^2} + \frac{[\rho^\varepsilon(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{n})]^2}{|d_\Omega(x)|^2} \right) dx dt = 0 \tag{1.7}$$

holds true, and moreover,

$$\lim_{\varepsilon \to 0} (\|\rho^\varepsilon - \rho^0\|_{L^\infty(0,T;L^\gamma(\Omega))} + \|\sqrt{\rho^\varepsilon}|\boldsymbol{u}^\varepsilon| - \sqrt{\rho^0}|\boldsymbol{u}^0|\|_{L^\infty(0,T;L^2(\Omega))}) = 0 \tag{1.8}$$

if it is true initially, where $d_\Omega(x) := \mathrm{dist}(x, \partial\Omega)$ denotes the distance of the point $x \in \Omega$ to the boundary $\partial\Omega$, and $\boldsymbol{u}\cdot\boldsymbol{n}$ denotes the normal component of $\boldsymbol{u}$. By the use of the Hardy inequality, the condition (1.7) reduces to the original Kato's criterion (1.6) when the density function ρ^ε is assumed to be a constant. In the same strong Euler solution setting, (1.7) can be weakened to a condition that only includes the tangential or the normal component of the velocity in the integrand [55, 56]. In [2], Bardos and Nguyen studied the inviscid limit for weak solutions of compressible flows. They proved that any weak solution with finite energy of the Navier-Stokes equations converges to the dissipative solution of the Euler equations in the domains without boundaries, while in the presence of boundaries, they established various criteria for the validity of the inviscid limit for both the Navier-friction boundary conditions and no-slip boundary conditions, and particularly, they extended Kato-Sueur's criteria from the strong solution to the dissipative solutions assumed for the Euler equations. We note that Sueur [49] also proved the validity of (1.8) for the weak solutions $(\rho^\varepsilon, \boldsymbol{u}^\varepsilon)$ of the equations (1.1) with the Navier-friction boundary conditions (see [6, 24, 27, 30, 52, 54] and the references therein for more discussions and results related to the Navier-type boundary conditions).

We also mention the seminal works [44, 45] in which the validity of the Prandtl expansion is introduced in a boundary strip of width proportional to $\varepsilon^{1/2}$, but such an expansion at the level of the Sobolev regularity is invalid (see, for example, [29]). Related important discussions and results can also be found in the paper [37] about the ill-posedness of solutions to the linearized Prandtl operator around a non-monotone shear flow in Sobolev spaces, and the paper [31] on the failure of H^s continuous dependence of solutions to the nonlinear Prandtl operator (see also [36] and the references therein for more studies).

On the other hand, it is well known that many solutions of the Euler system can develop singularities in finite time. Understanding the vanishing viscosity limit outside the classical solution regime naturally promotes the need for a weak solution theory. The main concern of this paper is the inviscid limit of the weak solutions $(\rho^\varepsilon, \boldsymbol{u}^\varepsilon)$ of the Navier-Stokes equations (1.1)–(1.2) under the boundary condition (1.4).

Before stating our main results, we define the weak solutions of the equations (1.1) and (1.3) as follows.

Definition 1.1 (Weak solutions of the Navier-Stokes equation)**.** For any fixed $T \in (0, \infty)$, the pair of functions $(\rho^\varepsilon, \boldsymbol{u}^\varepsilon)$ is a weak solution to the problem (1.1)–(1.2) if the following properties hold true:

- $\rho^\varepsilon \geqslant 0$ a.e., and
$$\rho^\varepsilon,\ (\rho^\varepsilon)^\gamma,\ \rho^\varepsilon|\boldsymbol{u}^\varepsilon|^2 \in L^\infty(0,T;L^1(\Omega)), \quad \nabla\boldsymbol{u}^\varepsilon \in L^2(0,T;L^2(\Omega)); \tag{1.9}$$
- $(\rho^\varepsilon, \boldsymbol{u}^\varepsilon)$ satisfies the system (1.1) in the sense of distributions $\mathcal{D}'(0,T;\Omega)$;
- the energy inequality holds:
$$\frac{d}{dt}\mathbb{E}[\rho^\varepsilon, \boldsymbol{u}^\varepsilon] + \varepsilon\int_\Omega \mathbb{S}(\nabla\boldsymbol{u}^\varepsilon) : \nabla\boldsymbol{u}^\varepsilon dx \leqslant 0 \quad \text{in } \mathcal{D}'(0,T), \tag{1.10}$$

where
$$\mathbb{E}[\rho^\varepsilon, \boldsymbol{u}^\varepsilon](t) := \int_\Omega e(\rho^\varepsilon, \boldsymbol{u}^\varepsilon)dx \quad \text{with } e(\rho^\varepsilon, \boldsymbol{u}^\varepsilon) := \frac{1}{2}\rho^\varepsilon|\boldsymbol{u}^\varepsilon|^2 + \frac{(\rho^\varepsilon)^\gamma}{\gamma-1}. \tag{1.11}$$

Note that the global existence of the above weak solutions has been established by Feireisl et al. [25] with the γ-pressure law for $\gamma > 3/2$.

The notion for the weak solution of the Euler equation (1.3) that will be used in this paper is given as follows.

Definition 1.2 (Weak admissible solutions of the Euler equations)**.** For any fixed $T \in (0, \infty)$, the pair of functions $(\rho^0, \boldsymbol{u}^0)$ is a weak admissible solution to the problem (1.3) if the following properties hold true:

- $\rho^0 \geqslant 0$ a.e., and
$$\rho^0,\ (\rho^0)^\gamma,\ \rho^0|\boldsymbol{u}^0|^2 \in L^\infty(0,T;L^1(\Omega)); \tag{1.12}$$
- $(\rho^0, \boldsymbol{u}^0)$ satisfies the system (1.3) in the sense of distributions $\mathcal{D}'(0,T;\Omega)$;

- the energy inequality holds:

$$\frac{d}{dt}\mathbb{E}[\rho^0, \boldsymbol{u}^0] \leqslant 0 \quad \text{in } \mathcal{D}'(0,T), \tag{1.13}$$

where $\mathbb{E}$ is given in (1.11).

Beginning with the ground-breaking work of De Lellis and Székelyhidi [17, 18], it has been understood via the method of convex integration that the system (1.3) is desperately ill-posed and admits infinitely many weak admissible solutions for a very large set of initial data [11, 13, 14, 21, 28]. It is thus natural to ask whether the vanishing viscosity limit could be regarded as a sound selection principle to identify the physically relevant solutions of the Euler system. Our result here provides a sufficient condition to confirm the vanishing viscosity limit as an admissible weak solution of the Euler equations when the physical domain has a boundary. Whether or not such an inviscid limit is unique under this regularity frame, however, still remains to be a very challenging problem.

Now let us introduce some notation that will be frequently used in this paper. For some small $h > 0$, we define

$$\Omega^h := \{x \in \Omega, d_\Omega(x) > h\} \quad \text{and} \quad \Gamma_h := \Omega \backslash \Omega^h. \tag{1.14}$$

For $d \in \mathbb{N}$ and $Q \subset \mathbb{R}^d$, we define the Besov space $B_r^{\sigma,\infty}(Q)$ ($r \in [1,\infty)$) as the space of the measurable functions with the norm

$$\|f\|_{B_r^{\sigma,\infty}(Q)} := \|f\|_{L^r(Q)} + \sup_{\zeta \in \mathbb{R}^d} \frac{\|f(\cdot+\zeta) - f(\cdot)\|_{L^r(Q\cap(Q-\{\zeta\}))}}{|\zeta|^\sigma}. \tag{1.15}$$

Now we are ready to state our main result below.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^3$ be a C^3 smooth bounded domain, and $(\rho^\varepsilon, \boldsymbol{u}^\varepsilon)$ be a weak solution to the problem* (1.1)–(1.2) *under the boundary condition* (1.4). *Assume that*

$$\sigma \in \left(\frac{1}{3}, \frac{1}{2}\right], \quad \gamma \geqslant 2, \quad 0 \leqslant \rho^\varepsilon(x,t) \leqslant \widehat{\rho} < \infty, \tag{1.16}$$

where the positive constant $\widehat{\rho}$ is independent of ε. Assume in addition that for any interior domain $\widetilde{\Omega}_T \subset\subset \Omega^\varepsilon \times (\varepsilon, T-\varepsilon)$,

$$\rho^\varepsilon,\ \boldsymbol{u}^\varepsilon \text{ and } \rho^\varepsilon\boldsymbol{u}^\varepsilon \text{ are uniformly in } \varepsilon \text{ bounded in } B_3^{\sigma,\infty}(\widetilde{\Omega}_T), \tag{1.17}$$

and for the near boundary domain $\Gamma_{8\varepsilon}$,

$$\boldsymbol{u}^\varepsilon \text{ is uniformly in } \varepsilon \text{ bounded in } L^4(0,T;L^\infty(\Gamma_{8\varepsilon})). \tag{1.18}$$

Then a necessary and sufficient condition for vanishing of dissipation, i.e.,

$$\lim_{\varepsilon\to 0} \varepsilon \int_0^T \int_\Omega \mathbb{S}(\nabla \boldsymbol{u}^\varepsilon) : \nabla \boldsymbol{u}^\varepsilon dxdt = 0 \tag{1.19}$$

is

$$\lim_{\varepsilon\to 0} \varepsilon \int_0^T \int_{\Gamma_{8\varepsilon}} |\nabla \boldsymbol{u}^\varepsilon|^2 dxdt = 0. \tag{1.20}$$

In addition, upon to a subsequence, the solution $(\rho^\varepsilon, \boldsymbol{u}^\varepsilon)$ converges weakly to a weak admissible solution $(\rho^0, \boldsymbol{u}^0)$ of the Euler equations (1.3) *as ε tends to zero. Moreover, for any $p \in (1, \frac{9}{2}]$, the following strong convergence holds true*:

$$\rho^\varepsilon \to \rho^0, \quad \rho^\varepsilon\boldsymbol{u}^\varepsilon \to \rho^0\boldsymbol{u}^0 \quad \text{locally in } L^3(0,T;L^p(\Omega)). \tag{1.21}$$

We have the following remarks on the theorem.

Remark 1.4. In (1.16), we use the range of the exponent $\sigma \in (\frac{1}{3}, \frac{1}{2}]$ to emphasize that we are mainly interested in the weak solutions with Onsager's regularity $\sigma = \frac{1}{3}+$, while if $\sigma > \frac{1}{2}$, the result is still valid. In (1.18), we choose 8ε as the width of the boundary strip for simplicity of our calculations, but it is not necessarily optimal.

Remark 1.5. We see that the strong convergence (1.8) obtained in [49] is in the energy norm, whereas the strong convergence (1.21) is weaker in time due to the relaxed regularity at the level of the Euler equations and the fact that the density is allowed to touch zero. On the other hand, the enhanced regularity assumption (1.17) allows one to conclude the convergence of the physical quantities—density and momentum—of the system. Moreover, applying the embedding theorem on Besov spaces (see [40, Theorem 2]) and the standard Sobolev embedding to (1.17):

$$\rho^\varepsilon,\, \boldsymbol{u}^\varepsilon,\, \rho^\varepsilon \boldsymbol{u}^\varepsilon \in B_3^{\sigma,\infty}(\widetilde{\Omega}_T) \hookrightarrow W^{\frac{1}{3},\frac{12-\delta}{5-3\sigma}}(\widetilde{\Omega}_T) \hookrightarrow\hookrightarrow L^4(\widetilde{\Omega}_T),$$

where $0 < \delta < 9(\sigma - \frac{1}{3})$, one can further deduce that

$$\rho^\varepsilon \to \rho^0, \quad \boldsymbol{u}^\varepsilon \to \boldsymbol{u}^0, \quad \rho^\varepsilon \boldsymbol{u}^\varepsilon \to \rho^0 \boldsymbol{u}^0 \quad \text{in } L^4(\widetilde{\Omega}_T), \tag{1.22}$$

implying the strong convergence of velocity as well. Note that compared with (1.21), the convergence (1.22) is stronger in time, at the price of a loss in space. A further implication of (1.22) is the convergence of the energy density

$$e(\rho^\varepsilon, \boldsymbol{u}^\varepsilon) \to e(\rho^0, \boldsymbol{u}^0) \quad \text{in } L^1(\widetilde{\Omega}_T).$$

Remark 1.6. The regularity conditions (1.16)–(1.17) are in accordance with those assumptions in [22], where Feireisl et al. proved that the weak solutions to the compressible isentropic Euler system (1.3) conserve energy locally in time, i.e., $\frac{d}{dt}\mathbb{E}[\rho^0, \boldsymbol{u}^0] = 0$ in $\mathcal{D}'(0,T)$.

Remark 1.7. The restriction $\gamma \geqslant 2$ in (1.16) is mainly for the convenience of mathematical calculations due to the appearance of vacuum states. The proof of Theorem 1.3 can be generalized to the case where the pressure function $P = P(\rho^\varepsilon)$ satisfies

$$P(\rho^\varepsilon) \in C^2([0, \widehat{\rho}]) \quad \text{and} \quad P'(0) = 0.$$

In fact, the condition on the pressure can be relaxed to $P \in C^{1,\gamma-1}$ for $\gamma \geqslant 1$ with a stronger regularity condition for the density [1].

We now explain the main difficulties and the strategy of the proof for Theorem 1.3. The result in Theorem 1.3 asserts the vanishing dissipation of weak solutions that possess the Onsager-type regularity in the interior domain and satisfy the celebrated Kato-type condition near the boundary. Under the assumption of the existence of a strong Euler solution [33,49,56], a Kato-type boundary corrector can be defined so that the relative energy estimates can be employed. When weak Euler solutions are considered [10, 19], on the other hand, such a boundary corrector approach becomes difficult to apply, and new ideas involving domain separation and global mollification are introduced. The current work is mainly motivated by [10,19] on the inviscid limit of weak solutions for the incompressible Navier-Stokes equations, and can be regarded as a generalization to the compressible fluids. Compared with the incompressible situation, new difficulties arise when the density function is included. (1) First, the nonlinear coupling of ρ^ε and $\boldsymbol{u}^\varepsilon$ makes the commutator estimates more complicated, and therefore, certain extra restriction on the density seems necessary and some higher regularity conditions on the velocity field are needed. In order to successfully use the commutator estimates, we consider in the interior domain the Besov regularity $B_3^{\sigma,\infty}$ ($\sigma > 1/3$) on the density, the velocity and the momentum. Such regularity assumptions are consistent with those in the work of Feireisl et al. [22], where the energy is conserved (locally in time) for the compressible Euler equations. (2) Another major difficulty in studying the zero viscosity limit is the boundary layer effect. The mismatch of velocity boundary conditions requires a transition mechanism near the boundary so that the overall convergence can be guaranteed. Moreover, the appearance of the density (with possible vacuum) makes the control on $\boldsymbol{u}^\varepsilon$ much subtler near the boundary layer. Our approach is to adopt the boundary layer foliation technique developed in our previous paper [10]. We mention that the boundary foliation technique is mainly for the sake of the regularization procedure, which guarantees that there is enough room left for the mollification of ρ^ε and $\boldsymbol{u}^\varepsilon$. Note that other types of global mollification techniques, for example, the shifted convolution, have been used to treat a class of trace problems [7, 39]. This shifted convolution approach could allow one to resolve the regularity issue up to the boundary; however it does not seem to be able to provide a boundary layer with our desired thickness as in Kato's criterion. Instead, our idea of the boundary layer foliation introduces more mollification scales in the hope of generating more cancellations to obtain a refined estimate of the global viscous dissipation. More precisely, we divide the near-boundary region into several subregions and mollify ρ^ε and $\boldsymbol{u}^\varepsilon$ by using different mollifiers in each subregion. We remark that there are enormous works in the literature on the related vanishing viscosity limits with boundaries [27,38,43,52,53,55,57,58] or without boundaries [6, 9, 23, 26].

We prove Theorem 1.3 in several steps. Firstly, we regularize the momentum equations in each small subregion, and then apply a smooth partition of unity to integrate the separated areas together. We want to point out that the mollification procedure should be done for both the space and time variables so that the mollified velocity becomes a legitimate test function. Secondly, we integrate the energy inequality and utilize the mollified momentum equations to bound the global viscous dissipation in the form of (3.23). To prove (1.19), we only need to show that the error terms will shrink to zero as the viscosity ε tends to zero. It is worthwhile pointing out that in the calculation, we make repeated use of the fact that the partition of unity $\{\xi_n\}$ satisfies

$$\sum_{|m-n|\leqslant 1} \xi_m\xi_n = 1, \quad \nabla\bigg(\sum_{|m-n|\leqslant 1} \xi_m\xi_n\bigg) = 0.$$

Finally, we prove that as $\varepsilon \to 0$, the sequence of solutions of the Navier-Stokes equations converges to a weak solution of the Euler equations, and furthermore, some strong convergence in L^p Sobolev topology is obtained.

The rest of this paper is organized as follows. In Section 2, we introduce the boundary layer foliation technique and some basic properties and useful lemmas. In Section 3, we devote ourselves to proving Theorem 1.3. The detailed proof is divided into several subsections corresponding to the steps mentioned above.

2 Preliminaries

In this section, we introduce the boundary layer foliation technique and some basic properties and useful lemmas.

2.1 Foliation of the near boundary domain

In order to clearly state the near boundary assumptions, we follow the ideas in [10] and construct the boundary layer sequence as follows. Let σ be as in (1.16). Define the increasing sequence $\{\beta_n^*\}$ as follows:

$$\beta_0^* = 0 \quad \text{and} \quad \beta_n^* = \frac{1}{2(1-\sigma)}\bigg(1+\frac{1}{3}\beta_{n-1}^*\bigg) \quad n = 1, 2, 3, \ldots \tag{2.1}$$

It is easy to see that

$$\beta_\infty^* = \lim_{n\to\infty} \beta_n^* = \frac{3}{5-6\sigma} > 1, \tag{2.2}$$

and hence, for some finite number $N = N(\sigma) \geqslant 1$,

$$0 = \beta_0^* < \beta_1^* < \beta_2^* < \cdots < \beta_{N-1}^* \leqslant 1 < \beta_N^*. \tag{2.3}$$

In light of (2.1)–(2.3), for each $n = 1, 2, \ldots$, there is a β_n close to β_n^* satisfying

$$0 = \beta_0 < \beta_n < \frac{1}{2(1-\sigma)}\bigg(1+\frac{1}{3}\beta_{n-1}\bigg), \quad \text{if } n \geqslant 1 \tag{2.4}$$

and

$$\beta_0 < \beta_1\ (= \beta) < \beta_2 < \cdots < \beta_{N-1} < 1 = \beta_N. \tag{2.5}$$

Utilizing (2.4)–(2.5) and (1.14), we decompose the whole domain Ω as

$$V_1 = \Omega^{\varepsilon^{\beta_1}}, \quad V_n = \Omega^{\varepsilon^{\beta_n}} - \Omega^{\varepsilon^{\beta_{n-1}}+\varepsilon^{\beta_n}}, \quad 2 \leqslant n \leqslant N, \quad V_{N+1} = \Omega - \left(\bigcup_{n=1}^{N} V_n\right) =: \Gamma_\varepsilon. \tag{2.6}$$

If ε is small, then we see that

$$V_k \cap V_m = \emptyset, \quad \text{if } |k-m| \geqslant 2 \tag{2.7}$$

and

$$\operatorname{meas} V_n \leqslant C\varepsilon^{\beta_{n-1}}, \quad \operatorname{meas}(V_n \cap V_{n+1}) \leqslant C\varepsilon^{\beta_{n+1}}, \quad \operatorname{meas}(V_n \cap V_{n+1}^c) \leqslant C\varepsilon^{\beta_{n-1}}, \tag{2.8}$$

where V^c denotes the complement of the set V.

2.2 Properties of mollifiers and some useful lemmas

Let $\Omega_T = \Omega \times (0,T)$ and $\Omega_T^\delta = \Omega^\delta \times (\delta, T-\delta)$ with $\delta > 0$ a given small constant. Define the mollification of f as

$$\overline{f_\delta}(x,t) = \int_0^T \int_\Omega f(x-y,t-s)\eta_\delta(y,s)dyds, \quad (x,t) \in \Omega_T^\delta, \tag{2.9}$$

where η_δ is the standard mollifier of width δ. A direct computation shows

$$\|\overline{f_\delta} - f\|_{L^r(\Omega_T^\delta)} \leqslant C\delta^\sigma \|f\|_{B_r^{\sigma,\infty}(\Omega_T)}. \tag{2.10}$$

Notice that

$$\begin{aligned}
\partial_{x_i}\overline{f_\delta}(x,t) &= \int_0^T \int_\Omega \eta_\delta(y,s)\partial_{x_i} f(x-y,t-s)dyds \\
&= -\int_0^T \int_\Omega \eta_\delta(y,s)\partial_{y_i} f(x-y,t-s)dyds \\
&= -\int_0^T \int_\Omega \eta_\delta(y,s)\partial_{y_i}(f(x-y,t-s)-f(x,t))dyds \\
&= \delta^{-1}\int_0^T \int_\Omega \partial_{y_i}\eta(y,s)(f(x-\delta y,t-\delta s)-f(x,t))dyds,
\end{aligned}$$

which together with (1.15) provides that for $\partial = \partial_{x_i}$ or $\partial = \partial_t$,

$$\|\partial\overline{f_\delta}\|_{L^r(\Omega_T^\delta)} \leqslant \delta^{\sigma-1}\|f\|_{B_r^{\sigma,\infty}(\Omega_T)}, \quad \forall\, r \in [1,\infty). \tag{2.11}$$

The two lemmas below are the commutator estimates and the Hardy-type embedding inequality, respectively, which will be used in the proof of Theorem 1.3.

Lemma 2.1 (See [10, 15]). *Let the exponents $r, r_1, r_2 \in [1,\infty)$ satisfy $r_1^{-1} + r_2^{-1} = r^{-1}$. Then for functions $f \in L^{r_1}(\Omega_T)$ and $g \in L^{r_2}(\Omega_T)$, the inequality*

$$\|\overline{(f\otimes g)_\delta} - \overline{f_\delta}\otimes\overline{g_\delta}\|_{L^r(\Omega_T^\delta)} \leqslant \|\overline{f_\delta} - f\|_{L^{r_1}(\Omega_T)}\|\overline{g_\delta} - g\|_{L^{r_2}(\Omega_T)} \tag{2.12}$$

is fulfilled locally in Ω_T.

Lemma 2.2 (See [35]). *Let $p \in [1,\infty)$ and $f \in W_0^{1,p}(\Omega_T)$. There is some constant C which depends on p and Ω_T such that*

$$\left\|\frac{f}{d_{\Omega_T}(x,t)}\right\|_{L^p(\Omega_T)} \leqslant C\|\partial f\|_{L^p(\Omega_T)},$$

where $\partial = \partial_{x_i}$ or $\partial = \partial_t$ and $d_{\Omega_T}(x,t) = \operatorname{dist}((x,t), \partial\Omega_T)$.

3 The proof of the main result

In this section, we prove Theorem 1.3. The main steps of the proof have been explained in Section 1. Below we provide the details. In the proof, we drop the superscript ε and denote $(\rho^\varepsilon, \boldsymbol{u}^\varepsilon)$ by $(\rho, \boldsymbol{u})$ for convenience, and use $C > 0$ to denote the generic constant that may depend on T, $\widehat{\rho}$, N, γ, μ, λ and Ω but is uniform in ε.

We first prove the equivalence of (1.19) and (1.20). For that purpose, it suffices to show that (1.20) leads to (1.19).

3.1 The representation of the momentum equations

Set

$$\mathbb{V}_T^n = (V_n \cap V_{n+1}^c) \times (\varepsilon^{\beta_1}, T - \varepsilon^{\beta_1}) \quad \text{and} \quad \mathbb{V}_T^{n,n+1} = (V_n \cap V_{n+1}) \times (\varepsilon^{\beta_1}, T - \varepsilon^{\beta_1}).$$

In terms of (2.6), we mollify f as

$$\overline{f_n} := \overline{f}_{\varepsilon^{\beta_n}}(x,t) = \begin{cases} \displaystyle\int_0^T \int_\Omega \eta_{\varepsilon^{\beta_n}}(x-y, t-s) f(y,s) dyds, & (x,t) \in \mathbb{V}_T^n, \\ \displaystyle\int_0^T \int_\Omega \eta_{\varepsilon^{\beta_{n+1}}}(x-y, t-s) f(y,s) dyds, & (x,t) \in \mathbb{V}_T^{n,n+1} \end{cases} \tag{3.1}$$

for each $n \in \{1, 2, \dots, N\}$. Clearly, one has

$$\overline{f_n} = \overline{f_{n+1}} \quad \text{on } \mathbb{V}_T^{n,n+1}. \tag{3.2}$$

Moreover, by (2.10)–(2.11), for $\partial = \partial_{x_i}$ or $\partial = \partial_t$ we deduce that for $p \in [1, \infty)$,

$$\|\partial \overline{f_n}\|_{L^p} \leqslant \begin{cases} C\varepsilon^{\beta_n(\sigma-1)} \|f\|_{B_p^{\sigma,\infty}(\Omega_T)}, & (x,t) \in \mathbb{V}_T^n, \\ C\varepsilon^{\beta_{n+1}(\sigma-1)} \|f\|_{B_p^{\sigma,\infty}(\Omega_T)}, & (x,t) \in \mathbb{V}_T^{n,n+1} \end{cases} \tag{3.3}$$

and

$$\|\overline{f_n} - f\|_{L^p} \leqslant \begin{cases} C\varepsilon^{\beta_n \sigma} \|f\|_{B_p^{\sigma,\infty}(\Omega_T)}, & (x,t) \in \mathbb{V}_T^n, \\ C\varepsilon^{\beta_{n+1} \sigma} \|f\|_{B_p^{\sigma,\infty}(\Omega_T)}, & (x,t) \in \mathbb{V}_T^{n,n+1}. \end{cases} \tag{3.4}$$

In view of (3.1), if we mollify the momentum equation in (1.1), we deduce that

$$\partial_t \overline{(\rho \boldsymbol{u})_n} + \operatorname{div} \overline{(\rho \boldsymbol{u} \otimes \boldsymbol{u})_n} + \nabla \overline{(\rho^\gamma)_n} = \varepsilon \operatorname{div} \overline{\mathbb{S}_n} \quad \text{a.e. in } V_n \times (\varepsilon^{\beta_1}, T - \varepsilon^{\beta_1}). \tag{3.5}$$

Introduce the smooth partition of unity associated with the collection $\{V_n\}_{n=1}^N$ such that

$$\xi_n \in C_0^1(V_n), \quad 0 \leqslant \xi_n \leqslant 1 \quad \text{and} \quad \sum_{n=1}^N \xi_n = 1. \tag{3.6}$$

Multiplying (3.5) by ξ_n and summing up the resulting expressions from $n = 1$ to N, we get

$$\partial_t \bigg(\sum_{n=1}^N \xi_n \overline{(\rho \boldsymbol{u})_n} \bigg) + \sum_{n=1}^N \xi_n \operatorname{div} \overline{(\rho \boldsymbol{u} \otimes \boldsymbol{u})_n} + \sum_{n=1}^N \xi_n \nabla \overline{(\rho^\gamma)_n} = \varepsilon \sum_{n=1}^N \xi_n \operatorname{div} \overline{\mathbb{S}_n}. \tag{3.7}$$

Take the cut-off functions $\psi(t) \in C_0^1(0,T)$ and $\theta(x)$ satisfying

$$0 \leqslant \theta(x) \leqslant 1 \text{ in } \Omega, \quad \theta(x) = 1 \text{ if } x \in \Omega^{4\varepsilon}, \quad \theta(x) = 0 \text{ if } x \in \Gamma_{2\varepsilon}, \quad |\nabla \theta| \leqslant 4\varepsilon^{-1}. \tag{3.8}$$

If we multiply (3.7) by $\psi\theta(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n})$ and integrate it over Ω_T, we obtain

$$\int_0^T \psi \int_\Omega \theta \bigg(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n} \bigg) \bigg(\sum_{n=1}^N \xi_n \partial_t \overline{(\rho \boldsymbol{u})_n} + \sum_{n=1}^N \xi_n \operatorname{div} \overline{(\rho \boldsymbol{u} \otimes \boldsymbol{u})_n} \bigg)$$

$$
\begin{aligned}
&+\int_0^T \psi \int_\Omega \theta\Big(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n}\Big)\Big(\sum_{n=1}^N \xi_n \nabla \overline{(\rho^\gamma)_n}\Big) \\
&= \varepsilon \int_0^T \psi \int_\Omega \theta\Big(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n}\Big)\Big(\sum_{n=1}^N \xi_n \mathrm{div} \overline{\mathbb{S}_n}\Big),
\end{aligned} \tag{3.9}
$$

where and hereafter we omit the expressions of the integral variables for simplicity.

3.2 The representation of the energy inequality

Observe from (1.10) that

$$
\varepsilon \int_0^T \psi \int_\Omega \mathbb{S}(\nabla \boldsymbol{u}) : \nabla \boldsymbol{u} \leqslant \int_0^T \psi' \mathbb{E}(t), \quad \psi \in C_0^1(0,T),
$$

which together with (3.9) gives us the following:

$$
\begin{aligned}
&\varepsilon \int_0^T \psi \int_\Omega \mathbb{S}(\nabla \boldsymbol{u}^\varepsilon) : \nabla \boldsymbol{u}^\varepsilon \\
&\leqslant \int_0^T \psi' \mathbb{E}(t) + \int_0^T \psi \int_\Omega \theta\Big(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n^\varepsilon}\Big)\Big(\sum_{n=1}^N \xi_n \partial_t \overline{(\rho \boldsymbol{u})_n} + \sum_{n=1}^N \xi_n \mathrm{div} \overline{(\rho \boldsymbol{u} \otimes \boldsymbol{u})_n}\Big) \\
&\quad + \int_0^T \psi \int_\Omega \theta\Big(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n}\Big)\Big(\sum_{n=1}^N \xi_n \nabla \overline{(\rho^\gamma)_n}\Big) \\
&\quad - \varepsilon \int_0^T \psi \int_\Omega \theta\Big(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n}\Big)\Big(\sum_{n=1}^N \xi_n \mathrm{div} \overline{\mathbb{S}_n}\Big).
\end{aligned} \tag{3.10}
$$

We need to handle the terms on the right-hand side of (3.10). A direct computation shows that

$$
\begin{aligned}
&\int_0^T \psi \int_\Omega \theta\Big(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n}\Big)\Big(\sum_{n=1}^N \xi_n \partial_t \overline{(\rho \boldsymbol{u})_n}\Big) \\
&= \int_0^T \psi \int_\Omega \theta\Big(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n}\Big)\Big(\sum_{n=1}^N \xi_n \partial_t (\overline{\rho_n}\, \overline{\boldsymbol{u}_n})\Big) \\
&\quad + \int_0^T \psi \int_\Omega \theta\Big(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n}\Big)\Big(\sum_{n=1}^N \xi_n \partial_t (\overline{(\rho \boldsymbol{u})_n} - \overline{\rho_n}\, \overline{\boldsymbol{u}_n})\Big)
\end{aligned} \tag{3.11}
$$

and

$$
\begin{aligned}
&\int_0^T \psi \int_\Omega \theta\Big(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n}\Big)\Big(\sum_{n=1}^N \xi_n \mathrm{div} \overline{(\rho \boldsymbol{u} \otimes \boldsymbol{u})_n}\Big) \\
&= \int_0^T \psi \int_\Omega \theta\Big(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n}\Big)\Big(\sum_{n=1}^N \xi_n \mathrm{div} (\overline{(\rho \boldsymbol{u})_n} \otimes \overline{\boldsymbol{u}_n})\Big) \\
&\quad + \int_0^T \psi \int_\Omega \theta\Big(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n}\Big)\sum_{n=1}^N \xi_n \mathrm{div} (\overline{(\rho \boldsymbol{u} \otimes \boldsymbol{u})_n} - \overline{(\rho \boldsymbol{u})_n} \otimes \overline{\boldsymbol{u}_n}).
\end{aligned} \tag{3.12}
$$

It follows from (2.7) and (3.6) that

$$
\xi_k \xi_m = 0, \quad \text{if } |k-m| \geqslant 2. \tag{3.13}
$$

Using (3.13), we have the following computation:

$$
\int_0^T \psi \int_\Omega \theta\Big(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n}\Big)\partial_t\Big(\sum_{n=1}^N \xi_n \overline{\rho_n}\, \overline{\boldsymbol{u}_n}\Big)
$$

$$
\begin{aligned}
&= \sum_{m,n=1}^{N} \int_0^T \psi \int_\Omega \theta \xi_m \xi_n \overline{\boldsymbol{u}_m} \partial_t (\overline{\rho_n}\, \overline{\boldsymbol{u}_n}) \\
&= \sum_{|m-n|\leqslant 1} \int_0^T \psi \int_\Omega \theta \xi_m \xi_n \overline{\boldsymbol{u}_m} \partial_t (\overline{\rho_n}\, \overline{\boldsymbol{u}_n}) \\
&= \sum_{|m-n|\leqslant 1} \int_0^T \psi \int_\Omega \theta \xi_m \xi_n \left[\partial_t \left(\overline{\rho_{\widetilde{n}}} \frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2} \right) + \partial_t \overline{\rho_{\widetilde{n}}} \frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2} \right],
\end{aligned} \tag{3.14}
$$

where the last equality holds due to (3.2) and

$$
\widetilde{n} = \begin{cases} \max\{m,n\}, & |m-n| = 1, \\ n, & |m-n| = 0. \end{cases} \tag{3.15}
$$

A similar argument gives that

$$
\begin{aligned}
&\int_0^T \psi \int_\Omega \theta \left(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n} \right) \left(\sum_{n=1}^N \xi_n \operatorname{div}(\overline{(\rho \boldsymbol{u})_n} \otimes \overline{\boldsymbol{u}_n}) \right) \\
&\quad = \sum_{|m-n|\leqslant 1} \int_0^T \psi \int_\Omega \theta \xi_m \xi_n \left[\operatorname{div} \left(\overline{(\rho \boldsymbol{u})_{\widetilde{n}}} \frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2} \right) + \operatorname{div} \overline{(\rho \boldsymbol{u})_{\widetilde{n}}} \frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2} \right].
\end{aligned} \tag{3.16}
$$

Next, by (3.1), (3.2) and (3.15), we mollify the mass equation to obtain

$$
0 = \partial_t \overline{\rho_n} + \operatorname{div} \overline{(\rho \boldsymbol{u})_n} = \partial_t \overline{\rho_{\widetilde{n}}} + \operatorname{div} \overline{(\rho \boldsymbol{u})_{\widetilde{n}}} \quad \text{a.e. in } V_n \times (\varepsilon^{\beta_1}, T - \varepsilon^{\beta_1}),
$$

which together with (3.14) and (3.16) ensures that

$$
\begin{aligned}
&\int_0^T \psi \int_\Omega \theta \left(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n} \right) \partial_t \left(\sum_{n=1}^N \xi_n \overline{\rho_n}\, \overline{\boldsymbol{u}_n} \right) \\
&\quad + \int_0^T \psi \int_\Omega \theta \left(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n} \right) \sum_{n=1}^N \xi_n \operatorname{div}(\overline{(\rho \boldsymbol{u})_n} \otimes \overline{\boldsymbol{u}_n}) \\
&\quad = \sum_{|m-n|\leqslant 1} \int_0^T \psi \int_\Omega \theta \xi_m \xi_n \left[\partial_t \left(\overline{\rho_{\widetilde{n}}} \frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2} \right) + \operatorname{div} \left(\overline{(\rho \boldsymbol{u})_{\widetilde{n}}} \frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2} \right) \right] \\
&\quad = - \sum_{|m-n|\leqslant 1} \int_0^T \psi' \int_\Omega \theta \xi_m \xi_n \overline{\rho_{\widetilde{n}}} \frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2} \\
&\qquad - \sum_{|m-n|\leqslant 1} \int_0^T \psi \int_\Omega \nabla \theta \xi_m \xi_n \cdot \overline{(\rho \boldsymbol{u})_{\widetilde{n}}} \frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2} - \sum_{|m-n|\leqslant 1} \int_0^T \psi \int_\Omega \theta \nabla(\xi_m \xi_n) \cdot \overline{(\rho \boldsymbol{u})_{\widetilde{n}}} \frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2} \\
&\quad = - \sum_{|m-n|\leqslant 1} \int_0^T \psi' \int_\Omega \theta \xi_m \xi_n \overline{\rho_{\widetilde{n}}} \frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2} + I_1 + I_2,
\end{aligned} \tag{3.17}
$$

where I_1 and I_2 denote the last two terms, respectively. As a result of (3.17) and (3.11)–(3.12), the first integral in (3.9) satisfies

$$
\begin{aligned}
&\int_0^T \psi \int_\Omega \theta \left(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n} \right) \left(\sum_{n=1}^N \xi_n \partial_t \overline{(\rho \boldsymbol{u})_n} + \sum_{n=1}^N \xi_n \operatorname{div} \overline{(\rho \boldsymbol{u} \otimes \boldsymbol{u})_n} \right) \\
&\quad = - \sum_{|m-n|\leqslant 1} \int_0^T \psi' \int_\Omega \theta \xi_m \xi_n \overline{\rho_{\widetilde{n}}} \frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2} + I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{3.18}
$$

where

$$I_3=\int_0^T\psi\int_\Omega\theta\Big(\sum_{n=1}^N\xi_n\overline{\boldsymbol{u}_n}\Big)\Big(\sum_{n=1}^N\xi_n\partial_t(\overline{(\rho\boldsymbol{u})_n}-\overline{\rho_n}\,\overline{\boldsymbol{u}_n})\Big)$$

and

$$I_4=\int_0^T\psi\int_\Omega\theta\Big(\sum_{n=1}^N\xi_n\overline{\boldsymbol{u}_n}\Big)\sum_{n=1}^N\xi_n\mathrm{div}(\overline{(\rho\boldsymbol{u}\otimes\boldsymbol{u})_n}-\overline{(\rho\boldsymbol{u})_n}\otimes\overline{\boldsymbol{u}_n}).$$

For the second term in (3.9), by (3.13) one has

$$\int_0^T\psi\int_\Omega\theta\Big(\sum_{n=1}^N\xi_n\overline{\boldsymbol{u}_n}\Big)\Big(\sum_{n=1}^N\xi_n\nabla\overline{(\rho^\gamma)_n}\Big)=\sum_{|m-n|\leqslant1}\int_0^T\psi\int_\Omega\theta\xi_m\xi_n\overline{\boldsymbol{u}_n}\cdot\nabla\overline{(\rho^\gamma)_n}. \tag{3.19}$$

We mollify the mass equation and then multiply it by $\gamma(\overline{\rho_n})^{\gamma-1}$ to obtain

$$(\gamma-1)(\overline{\rho_n})^\gamma\mathrm{div}\overline{\boldsymbol{u}_n}=-\partial_t(\overline{\rho_n})^\gamma-\mathrm{div}((\overline{\rho_n})^\gamma\overline{\boldsymbol{u}_n})+\gamma(\overline{\rho_n})^{\gamma-1}\mathrm{div}(\overline{(\boldsymbol{u}\rho)_n}-\overline{\boldsymbol{u}_n}\,\overline{\rho_n}). \tag{3.20}$$

With (3.20), the right-hand side of (3.19) becomes

$$\begin{aligned}&\sum_{|m-n|\leqslant1}\int_0^T\psi\int_\Omega\theta\xi_m\xi_n\overline{\boldsymbol{u}_n}\cdot\nabla\overline{(\rho^\gamma)_n}\\&\quad=\sum_{|m-n|\leqslant1}\int_0^T\psi\int_\Omega\mathrm{div}(\theta\xi_m\xi_n\overline{\boldsymbol{u}_n})((\overline{\rho_n})^\gamma-\overline{(\rho^\gamma)_n})\\&\qquad-\sum_{|m-n|\leqslant1}\int_0^T\psi\int_\Omega\nabla(\theta\xi_m\xi_n)\overline{\boldsymbol{u}_n}(\overline{\rho_n})^\gamma\\&\qquad+\frac{1}{\gamma-1}\sum_{|m-n|\leqslant1}\int_0^T\psi\int_\Omega\theta\xi_m\xi_n\mathrm{div}((\overline{\rho_n})^\gamma\overline{\boldsymbol{u}_n})\\&\qquad-\frac{\gamma}{\gamma-1}\sum_{|m-n|\leqslant1}\int_0^T\psi\int_\Omega\theta\xi_m\xi_n(\overline{\rho_n})^{\gamma-1}\mathrm{div}(\overline{(\boldsymbol{u}\rho)_n}-\overline{\boldsymbol{u}_n}\,\overline{\rho_n})\\&\qquad+\frac{1}{\gamma-1}\sum_{|m-n|\leqslant1}\int_0^T\psi\int_\Omega\theta\xi_m\xi_n\partial_t(\overline{\rho_n})^\gamma\\&\quad=J_1+J_2+J_3+J_4+\frac{1}{\gamma-1}\sum_{|m-n|\leqslant1}\int_0^T\psi\int_\Omega\theta\xi_m\xi_n\partial_t(\overline{\rho_n})^\gamma,\end{aligned} \tag{3.21}$$

where J_i $(i=1,2,3,4)$ denote the first four terms, respectively. Hence, combining (3.19) with (3.21), we obtain

$$\begin{aligned}&\int_0^T\psi\int_\Omega\theta\Big(\sum_{n=1}^N\xi_n\overline{\boldsymbol{u}_n}\Big)\Big(\sum_{n=1}^N\xi_n\nabla\overline{(\rho^\gamma)_n}\Big)\\&\quad=\sum_{i=1}^4J_i+\frac{1}{\gamma-1}\sum_{|m-n|\leqslant1}\int_0^T\psi\int_\Omega\theta\xi_m\xi_n\partial_t(\overline{\rho_n})^\gamma\\&\quad=\sum_{i=1}^4J_i-\frac{1}{\gamma-1}\sum_{|m-n|\leqslant1}\int_0^T\psi'\int_\Omega\theta\xi_m\xi_n(\overline{\rho_n})^\gamma.\end{aligned} \tag{3.22}$$

Therefore, if we substitute (3.18) and (3.22) into (3.10), we obtain

$$\varepsilon\int_0^T\psi\int_\Omega\mathbb{S}(\nabla\boldsymbol{u}^\varepsilon):\nabla\boldsymbol{u}^\varepsilon\leqslant\int_0^T\psi'\mathbb{E}(t)-\sum_{|m-n|\leqslant1}\int_0^T\psi'\int_\Omega\theta\xi_m\xi_n\Big(\overline{\rho_{\tilde n}}\frac{|\overline{\boldsymbol{u}_{\tilde n}}|^2}{2}+\frac{(\overline{\rho_{\tilde n}})^\gamma}{\gamma-1}\Big)$$

$$+\sum_{i=1}^{4} I_i+\sum_{i=1}^{4} J_i-\varepsilon\int_0^T\psi\int_\Omega\theta\Big(\sum_{n=1}^{N}\xi_n\overline{\boldsymbol{u}_n}\Big)\Big(\sum_{n=1}^{N}\xi_n\mathrm{div}\overline{\mathbb{S}_n}\Big). \tag{3.23}$$

3.3 The proof of (1.19): The global inviscid limit

The main task in this subsection is to prove (1.19). Since $\mathbb{S}(\nabla\boldsymbol{u}):\nabla\boldsymbol{u}\geqslant 0$, it suffices to show that the terms on the right-hand side in (3.23) will shrink to zero as $\varepsilon\to 0$.

By the definition of $\mathbb{E}$, the first line on the right-hand side of (3.23) becomes

$$\begin{aligned}
&\int_0^T\psi'\mathbb{E}(t)-\sum_{|m-n|\leqslant 1}\int_0^T\psi'\int_\Omega\theta\xi_m\xi_n\bigg(\overline{\rho_{\widetilde{n}}}\frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2}+\frac{(\overline{\rho_{\widetilde{n}}})^\gamma}{\gamma-1}\bigg)\\
&\quad=\int_0^T\psi'\int_\Omega\rho\frac{|\boldsymbol{u}|^2}{2}-\int_0^T\psi'\int_\Omega\sum_{|m-n|\leqslant 1}\theta\xi_m\xi_n\overline{\rho_{\widetilde{n}}}\frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2}\\
&\quad\quad+\int_0^T\psi'\int_\Omega\frac{\rho^\gamma}{\gamma-1}-\int_0^T\psi'\int_\Omega\sum_{|m-n|\leqslant 1}\theta\xi_m\xi_n\frac{(\overline{\rho_{\widetilde{n}}})^\gamma}{\gamma-1}.
\end{aligned}\tag{3.24}$$

It follows from (2.7) and (3.6) that

$$\sum_{|m-n|\leqslant 1}\xi_m\xi_n=\sum_{m,n=1}^{N}\xi_m\xi_n=\bigg(\sum_{n=1}^{N}\xi_n\bigg)^2=1\quad\text{and}\quad\nabla\bigg(\sum_{|m-n|\leqslant 1}\xi_m\xi_n\bigg)=0, \tag{3.25}$$

and thus,

$$\begin{aligned}
&\bigg|\int_0^T\psi'\int_\Omega\rho\frac{|\boldsymbol{u}|^2}{2}-\int_0^T\psi'\int_\Omega\sum_{|m-n|\leqslant 1}\theta\xi_m\xi_n\overline{\rho_{\widetilde{n}}}\frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2}\bigg|\\
&\quad=\bigg|\int_0^T\psi'\Big(\int_{\Gamma_\varepsilon}+\int_{\Omega^\varepsilon}\Big)\rho\frac{|\boldsymbol{u}|^2}{2}-\int_0^T\psi'\int_\Omega\sum_{|m-n|\leqslant 1}\theta\xi_m\xi_n\overline{\rho_{\widetilde{n}}}\frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2}\bigg|\\
&\quad\leqslant\int_0^T|\psi'|\int_{\Gamma_\varepsilon}\rho\frac{|\boldsymbol{u}|^2}{2}+\sum_{|m-n|\leqslant 1}\int_0^T|\psi'|\int_{V_m\cap V_n}\bigg|\rho\frac{|\boldsymbol{u}|^2}{2}-\theta\overline{\rho_{\widetilde{n}}}\frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2}\bigg|.
\end{aligned}$$

The conditions (1.16) and (1.18) guarantee that

$$\lim_{\varepsilon\to 0}\int_0^T|\psi'|\int_{\Gamma_\varepsilon}\rho\frac{|\boldsymbol{u}|^2}{2}=0,$$

and by (3.8) and the mollification properties,

$$\begin{aligned}
&\int_0^T\int_{V_m\cap V_n}\bigg|\rho\frac{|\boldsymbol{u}|^2}{2}-\theta\overline{\rho_{\widetilde{n}}}\frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2}\bigg|\\
&\quad=\int_0^T\int_{V_m\cap V_n}\bigg|\rho\frac{|\boldsymbol{u}|^2}{2}-\overline{\rho_{\widetilde{n}}}\frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2}+(1-\theta)\overline{\rho_{\widetilde{n}}}\frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2}\bigg|\\
&\quad\leqslant\int_0^T\int_{V_m\cap V_n}\bigg|\rho\frac{|\boldsymbol{u}|^2}{2}-\overline{\rho_{\widetilde{n}}}\frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2}\bigg|+\int_0^T\int_{\Gamma_{4\varepsilon}}(1-\theta)\bigg|\rho\frac{|\boldsymbol{u}|^2}{2}\bigg|\\
&\quad\to 0\ (\text{as }\varepsilon\to 0).
\end{aligned}$$

Hence, the above estimates yield the following:

$$\lim_{\varepsilon\to 0}\bigg|\int_0^T\psi'\int_\Omega\rho\frac{|\boldsymbol{u}|^2}{2}-\int_0^T\psi'\int_\Omega\sum_{|m-n|\leqslant 1}\theta\xi_m\xi_n\overline{\rho_{\widetilde{n}}}\frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2}\bigg|=0. \tag{3.26}$$

By the similar argument, we have

$$\lim_{\varepsilon\to 0}\bigg|\int_0^T\psi'\int_\Omega\frac{\rho^\gamma}{\gamma-1}-\int_0^T\psi'\int_\Omega\sum_{|m-n|\leqslant 1}\theta\xi_m\xi_n\frac{(\overline{\rho_{\widetilde{n}}})^\gamma}{\gamma-1}\bigg|=0. \tag{3.27}$$

As a result of (3.26) and (3.27), it follows from (3.24) that

$$\lim_{\varepsilon\to 0}\bigg|\int_0^T\psi'\mathbb{E}(t)-\sum_{|m-n|\leqslant 1}\int_0^T\psi'\int_\Omega\theta\xi_m\xi_n\bigg(\overline{\rho_{\widetilde{n}}}\frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2}+\frac{(\overline{\rho_{\widetilde{n}}})^\gamma}{\gamma-1}\bigg)\bigg|=0. \tag{3.28}$$

The zero ε-limit of the quantities in the last line of (3.23) is due to the following claim.

Claim 3.1. Under the assumptions of Theorem 1.3, we have

$$\lim_{\varepsilon\to 0}\bigg|\sum_{i=1}^4 I_i+\sum_{i=1}^4 J_i-\varepsilon\int_0^T\psi\int_\Omega\theta\bigg(\sum_{n=1}^N\xi_n\overline{\boldsymbol{u}_n}\bigg)\bigg(\sum_{n=1}^N\xi_n\mathrm{div}\overline{\mathbb{S}_n}\bigg)\bigg|=0. \tag{3.29}$$

Assume that (3.29) holds. We can continue to prove (1.19) as follows. Choosing the test function ψ as the non-negative sequence of the form

$$\psi_n(t)=\begin{cases} nt, & 0\leqslant t\leqslant \dfrac{1}{n},\\ 1, & \dfrac{1}{n}\leqslant t\leqslant\dfrac{nT-1}{n},\\ n(T-t), & \dfrac{nT-1}{n}\leqslant t\leqslant T,\end{cases} \tag{3.30}$$

and making use of (3.28)–(3.29), we conclude from (3.23) that

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_\Omega\mathbb{S}(\nabla\boldsymbol{u}):\nabla\boldsymbol{u}=\lim_{n\to\infty}\lim_{\varepsilon\to 0}\varepsilon\int_0^T\psi_n\int_\Omega\mathbb{S}(\nabla\boldsymbol{u}^\varepsilon):\nabla\boldsymbol{u}^\varepsilon=0,$$

which is the desired property (1.19). We prove (3.29) next.

3.4 The proof of Claim 3.1

It follows from (2.10)–(2.11), (3.1), (3.15) and Lemma 2.1 that

$$\begin{aligned}\|\overline{(\rho\boldsymbol{u})_{\widetilde{n}}}-\overline{\rho_{\widetilde{n}}}\,\overline{\boldsymbol{u}_{\widetilde{n}}}\|_{L^{\frac32}}&\leqslant C\|\overline{\rho_{\widetilde{n}}}-\rho\|_{L^3}\|\overline{\boldsymbol{u}_{\widetilde{n}}}-\boldsymbol{u}^\varepsilon\|_{L^3}\\&\leqslant C\varepsilon^{2\sigma\beta_{\widetilde{n}}}\|\rho\|_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)}\|\boldsymbol{u}\|_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)},\end{aligned} \tag{3.31}$$

$$\begin{aligned}\|(\overline{(\rho\boldsymbol{u}\otimes\boldsymbol{u})_{\widetilde{n}}}-\overline{(\rho\boldsymbol{u})_{\widetilde{n}}}\otimes\overline{\boldsymbol{u}_{\widetilde{n}}})\|_{L^{\frac32}}&\leqslant\|\overline{(\rho\boldsymbol{u})_n}-\rho\boldsymbol{u}\|_{L^3}\|\overline{\boldsymbol{u}_n}-\boldsymbol{u}\|_{L^3}\\&\leqslant C\varepsilon^{2\sigma\beta_{\widetilde{n}}}\|\rho\boldsymbol{u}\|_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)}\|\boldsymbol{u}\|_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)},\end{aligned} \tag{3.32}$$

and

$$\|\partial_t\overline{\boldsymbol{u}_{\widetilde{n}}}\|_{L^3}+\|\nabla\overline{\boldsymbol{u}_{\widetilde{n}}}\|_{L^3}\leqslant C\varepsilon^{(\sigma-1)\beta_{\widetilde{n}}}\|\boldsymbol{u}\|_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)}. \tag{3.33}$$

By (1.17), (3.31), (3.33) and Lemma 2.2, we have the following estimate:

$$\begin{aligned}|I_3|&=\bigg|\int_0^T\psi\int_\Omega\theta\bigg(\sum_{n=1}^N\xi_n\overline{\boldsymbol{u}_n}\bigg)\bigg(\sum_{n=1}^N\xi_n\partial_t(\overline{(\rho\boldsymbol{u})_n}-\overline{\rho_n}\,\overline{\boldsymbol{u}_n})\bigg)\bigg|\\&=\bigg|\sum_{|m-n|\leqslant 1}\int_0^T\psi\int_\Omega\theta\xi_m\xi_n\overline{\boldsymbol{u}_m}\partial_t(\overline{(\rho\boldsymbol{u})_n}-\overline{\rho_n}\,\overline{\boldsymbol{u}_n})\bigg|\\&\leqslant\bigg|\sum_{|m-n|\leqslant 1}\int_0^T\int_\Omega(|\psi'||\overline{\boldsymbol{u}_{\widetilde{n}}}|+\psi|\partial_t\overline{\boldsymbol{u}_{\widetilde{n}}}|)(\overline{(\rho\boldsymbol{u})_{\widetilde{n}}}-\overline{\rho_{\widetilde{n}}}\,\overline{\boldsymbol{u}_{\widetilde{n}}})\bigg|\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\|\partial_t\overline{\boldsymbol{u}_{\widetilde{n}}}\|_{L^3}\|\overline{(\rho\boldsymbol{u})_{\widetilde{n}}}-\overline{\rho_n}\,\overline{\boldsymbol{u}_n}\|_{L^{\frac{3}{2}}}\\
&\leqslant C\varepsilon^{(3\sigma-1)\beta_{\widetilde{n}}}\|\boldsymbol{u}^\varepsilon\|^2_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)}\|\rho\|_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)}\\
&\leqslant C\varepsilon^{(3\sigma-1)\beta_{\widetilde{n}}}. \qquad (3.34)
\end{aligned}
$$

Using (3.15) once again, one has

$$
\begin{aligned}
|I_4| &= \left|\int_0^T\psi\int_\Omega\theta\Big(\sum_{n=1}^N\xi_n\overline{\boldsymbol{u}_n}\Big)\Big(\sum_{n=1}^N\xi_n\mathrm{div}(\overline{(\rho\boldsymbol{u}\otimes\boldsymbol{u})_n}-\overline{(\rho\boldsymbol{u})_n}\otimes\overline{\boldsymbol{u}_n})\Big)\right|\\
&= \left|\sum_{|m-n|\leqslant1}\int_0^T\psi\int_\Omega\nabla(\theta\xi_m\xi_n\overline{\boldsymbol{u}_{\widetilde{n}}})(\overline{(\rho\boldsymbol{u}\otimes\boldsymbol{u})_{\widetilde{n}}}-\overline{(\rho\boldsymbol{u})_{\widetilde{n}}}\otimes\overline{\boldsymbol{u}_{\widetilde{n}}})\right|\\
&\leqslant C\bigg[\varepsilon^{(3\sigma-1)\beta_{\widetilde{n}}}+\varepsilon^{1+(2\sigma-\frac{5}{3})}+\varepsilon\int_0^T\int_{\Gamma_{4\varepsilon}}|\nabla\boldsymbol{u}|^2\bigg]^{\frac{1}{2}}, \qquad (3.35)
\end{aligned}
$$

where for the last inequality, we have used the inequalities (3.36)–(3.38) as follows. First, from (1.17) and (3.32)–(3.33),

$$
\begin{aligned}
&\left|\int_0^T\psi\int_\Omega\theta\xi_m\xi_n\nabla\overline{\boldsymbol{u}_{\widetilde{n}}}(\overline{(\rho\boldsymbol{u}\otimes\boldsymbol{u})_{\widetilde{n}}}-\overline{(\rho\boldsymbol{u})_{\widetilde{n}}}\otimes\overline{\boldsymbol{u}_{\widetilde{n}}})\right|\\
&\quad\leqslant C\varepsilon^{(3\sigma-1)\beta_{\widetilde{n}}}\|\boldsymbol{u}\|^2_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)}\|\rho\boldsymbol{u}\|_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)}\\
&\quad\leqslant C\varepsilon^{(3\sigma-1)\beta_{\widetilde{n}}}. \qquad (3.36)
\end{aligned}
$$

Second, observing from (3.8) that $\nabla\theta=0$ if $x\notin\Omega^{2\varepsilon}\cap\Gamma_{4\varepsilon}\subset V_N$, and using (3.1), (2.10), the Hardy inequality and (1.17)–(1.18), one deduces that

$$
\begin{aligned}
&\left|\int_0^T\psi\int_\Omega\nabla\theta\xi_m\xi_n\overline{\boldsymbol{u}_{\widetilde{n}}}(\overline{(\rho\boldsymbol{u}\otimes\boldsymbol{u})_{\widetilde{n}}}-\overline{(\rho\boldsymbol{u})_{\widetilde{n}}}\otimes\overline{\boldsymbol{u}_{\widetilde{n}}})\right|\\
&\quad= \left|\int_0^T\psi\int_\Omega\nabla\theta\xi_N^2\overline{\boldsymbol{u}_N}(\overline{(\rho\boldsymbol{u}\otimes\boldsymbol{u})_N}-\overline{(\rho\boldsymbol{u})_N}\otimes\overline{\boldsymbol{u}_N})\right|\\
&\quad\leqslant C\bigg(\int_0^T\int_{\Gamma_{4\varepsilon}\cap\Omega^{2\varepsilon}}|\overline{\boldsymbol{u}_N}|^4\bigg)^{\frac{1}{2}}\bigg(\int_0^T\int_{\Gamma_{4\varepsilon}\cap\Omega^{2\varepsilon}}|\nabla\theta(\overline{\boldsymbol{u}_N}-\boldsymbol{u}+\boldsymbol{u})|^2\bigg)^{\frac{1}{2}}\\
&\quad\leqslant C\bigg(\varepsilon\int_0^T\|\boldsymbol{u}\|^4_{L^\infty(\Gamma_{8\varepsilon})}\bigg)^{\frac{1}{2}}\bigg(\int_0^T\varepsilon^{-\frac{5}{3}}\|\overline{\boldsymbol{u}_N}-\boldsymbol{u}^\varepsilon\|^2_{L^3}+\int_0^T\int_{\Gamma_{4\varepsilon}}|\boldsymbol{u}\cdot\nabla\theta|^2\bigg)^{\frac{1}{2}}\\
&\quad\leqslant C\varepsilon^{\frac{1}{2}}\bigg(\int_0^T\varepsilon^{(2\sigma-\frac{5}{3})}\|\boldsymbol{u}\|^2_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)}dt+\int_0^T\int_{\Gamma_{4\varepsilon}}|\nabla\boldsymbol{u}|^2\bigg)^{\frac{1}{2}}\\
&\quad\leqslant C\bigg[\varepsilon^{1+(2\sigma-\frac{5}{3})}+\varepsilon\int_0^T\int_{\Gamma_{4\varepsilon}}|\nabla\boldsymbol{u}|^2\bigg]^{\frac{1}{2}}. \qquad (3.37)
\end{aligned}
$$

Third, by virtue of (3.2), (3.13), (3.15) and (3.25), a direct computation shows

$$
\begin{aligned}
&\sum_{|m-n|\leqslant1}\int_0^T\psi\int_\Omega\theta\nabla(\xi_m\xi_n)\overline{\boldsymbol{u}_{\widetilde{n}}}(\overline{(\rho\boldsymbol{u}\otimes\boldsymbol{u})_{\widetilde{n}}}-\overline{(\rho\boldsymbol{u})_{\widetilde{n}}}\otimes\overline{\boldsymbol{u}_{\widetilde{n}}})\\
&\quad=\sum_{m,n=1}^N\int_0^T\psi\int_\Omega\theta\nabla(\xi_m\xi_n)\overline{\boldsymbol{u}_m}(\overline{(\rho\boldsymbol{u}\otimes\boldsymbol{u})_n}-\overline{(\rho\boldsymbol{u})_n}\otimes\overline{\boldsymbol{u}_n})\\
&\quad=\int_0^T\psi\int_\Omega\theta\nabla\Big(\sum_{m,n=1}^N\xi_m\xi_n\Big)\overline{\boldsymbol{u}_m}(\overline{(\rho\boldsymbol{u}\otimes\boldsymbol{u})_n}-\overline{(\rho\boldsymbol{u})_n}\otimes\overline{\boldsymbol{u}_n})\\
&\quad=\int_0^T\psi\int_\Omega\theta\nabla\Big(\sum_{n=1}^N\xi_n\Big)^2\overline{\boldsymbol{u}_n}(\overline{(\rho\boldsymbol{u}\otimes\boldsymbol{u})_n}-\overline{(\rho\boldsymbol{u})_n}\otimes\overline{\boldsymbol{u}_n})
\end{aligned}
$$

$$= 0. \tag{3.38}$$

The same deduction as (3.37)–(3.38) gives rise to the following:

$$\begin{aligned}
&|I_1 + I_2| \\
&= \bigg| \sum_{|m-n|\leqslant 1} \int_0^T \psi \int_\Omega \xi_m \xi_n \nabla\theta \cdot \overline{(\rho \boldsymbol{u})_{\widetilde{n}}} \frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2} + \sum_{|m-n|\leqslant 1} \int_0^T \psi \int_\Omega \theta \nabla(\xi_m\xi_n) \cdot \overline{(\rho \boldsymbol{u})_{\widetilde{n}}} \frac{|\overline{\boldsymbol{u}_{\widetilde{n}}}|^2}{2} \bigg| \\
&= \bigg| \int_0^T \psi \int_\Omega \xi_N^2 \nabla\theta \cdot \overline{(\rho^\varepsilon \boldsymbol{u})_N} \frac{|\overline{\boldsymbol{u}_N}|^2}{2} \bigg| \\
&\leqslant C \bigg(\int_0^T \int_{\Gamma_{4\varepsilon}\cap\Omega^{2\varepsilon}} |\overline{\boldsymbol{u}_N}|^4 \bigg)^{\frac{1}{2}} \bigg(\int_0^T \int_{\Gamma_{4\varepsilon}\cap\Omega^{2\varepsilon}} |\nabla\theta(\overline{\boldsymbol{u}_N} - \boldsymbol{u} + \boldsymbol{u})|^2 \bigg)^{\frac{1}{2}} \\
&\leqslant C \bigg[\varepsilon^{1+(2\sigma - \frac{5}{3})} + \varepsilon \int_0^T \int_{\Gamma_{4\varepsilon}} |\nabla \boldsymbol{u}|^2 \bigg]^{\frac{1}{2}}.
\end{aligned} \tag{3.39}$$

In terms of (1.16) and (1.20), from the inequalities (3.34)–(3.35) and (3.39) we obtain

$$\lim_{\varepsilon\to 0} \bigg| \sum_{i=1}^4 I_i \bigg| = 0. \tag{3.40}$$

Next, we prove

$$\lim_{\varepsilon\to 0} \bigg| \varepsilon \int_0^T \psi \int_\Omega \theta \bigg(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n} \bigg) \bigg(\sum_{n=1}^N \xi_n \operatorname{div} \overline{\mathbb{S}_n} \bigg) \bigg| = 0. \tag{3.41}$$

To this end, we use the integration by parts to obtain

$$\begin{aligned}
&\varepsilon \int_0^T \psi \int_\Omega \theta \bigg(\sum_{n=1}^N \xi_n \overline{\boldsymbol{u}_n} \bigg) \bigg(\sum_{n=1}^N \xi_n \operatorname{div} \overline{\mathbb{S}_n} \bigg) \\
&= -\varepsilon \sum_{|m-n|\leqslant 1} \int_0^T \int_\Omega \theta \xi_m \xi_n \nabla \overline{\boldsymbol{u}_m}\, \overline{\mathbb{S}_n} - \varepsilon \sum_{|m-n|\leqslant 1} \int_0^T \int_\Omega \nabla\theta \xi_n \xi_m \overline{\boldsymbol{u}_m} : \overline{\mathbb{S}_n} \\
&\quad - \varepsilon \sum_{|m-n|\leqslant 1} \int_0^T \int_\Omega \theta \nabla(\xi_m \xi_n) \overline{\boldsymbol{u}_m}\, \overline{\mathbb{S}_n},
\end{aligned} \tag{3.42}$$

and then we need to estimate each of the three terms in (3.42). For the first term, we note that from (2.4),

$$1 + \frac{1}{3}\beta_{\widetilde{n}-1} + 2\beta_{\widetilde{n}}(\sigma - 1) > 0,$$

and from (2.8),

$$\operatorname{meas}(V_n \cap V_m) \leqslant \begin{cases} C\varepsilon^{\beta_{\widetilde{n}}}, & |n-m| = 1, \\ C\varepsilon^{\beta_{\widetilde{n}-1}}, & |n-m| = 0. \end{cases}$$

Then using (1.2), (1.17) and (2.11) shows

$$\begin{aligned}
&\bigg| \varepsilon \sum_{|m-n|\leqslant 1} \int_0^T \int_\Omega \theta \xi_m \xi_n \nabla \overline{\boldsymbol{u}_m}\, \overline{\mathbb{S}_n} \bigg| \\
&\quad \leqslant C\varepsilon \int_0^T \|\nabla \overline{\boldsymbol{u}_{\widetilde{n}}}\|_{L^3(V_n\cap V_m)} \|\overline{\mathbb{S}_{\widetilde{n}}}\|_{L^3(V_n\cap V_m)} \|\xi_n\xi_m\|_{L^3(V_n\cap V_m)} \\
&\quad \leqslant C\varepsilon^{1+\frac{1}{3}\beta_{\widetilde{n}-1}} \int_0^T \|\nabla \boldsymbol{u}\|^2_{L^3(V_n\cap V_m)} \\
&\quad \leqslant C\varepsilon^{1+2\beta_{\widetilde{n}}(\sigma-1)+\frac{1}{3}\beta_{\widetilde{n}-1}} \|\boldsymbol{u}\|^2_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)}
\end{aligned}$$

$$\leqslant C\varepsilon^{1+2\beta_{\widetilde{n}}(\sigma-1)+\frac{1}{3}\beta_{\widetilde{n}-1}}. \tag{3.43}$$

Since $\sigma > \frac{1}{3}$, one has

$$1+2\beta_{\widetilde{n}}(\sigma-1)+\frac{1}{3}\beta_{\widetilde{n}-1} > 1+\beta_{\widetilde{n}}\bigg(\frac{1}{3}+2(\sigma-1)\bigg) \geqslant \frac{4}{3}+2(\sigma-1) = 2\bigg(\sigma-\frac{1}{3}\bigg) > 0.$$

Hence, we take the limit in (3.43) to conclude

$$\lim_{\varepsilon\to 0}\varepsilon\sum_{|m-n|\leqslant 1}\int_0^T\int_\Omega \theta\xi_m\xi_n\nabla\overline{\boldsymbol{u}_m}\,\overline{\mathbb{S}_n}=0. \tag{3.44}$$

For the second term, we note that (3.8) implies $\nabla\theta=0$ if $x\notin\Omega^{2\varepsilon}\cap\Gamma_{4\varepsilon}\subset V_N$. Then, by (2.10) and the Hardy inequality, one has

$$\begin{aligned}
&\bigg|\varepsilon\sum_{|m-n|\leqslant 1}\int_0^T\int_\Omega\nabla\theta\xi_n\xi_m\overline{\boldsymbol{u}_m}\,\overline{\mathbb{S}_n}\bigg|\\
&\quad=\varepsilon\int_0^T\int_{\Omega^{2\varepsilon}\cap\Gamma_{4\varepsilon}}\nabla\theta\xi_N^2\overline{\boldsymbol{u}_N}\,\overline{\mathbb{S}_N}\\
&\quad\leqslant\varepsilon\int_0^T\|\nabla\overline{\boldsymbol{u}_N}\|^2_{L^2(\Omega^{2\varepsilon}\cap\Gamma_{4\varepsilon})}+C\varepsilon\int_0^T(\||\boldsymbol{u}||\nabla\theta|\|^2_{L^2(\Gamma_{4\varepsilon})}+\|\nabla\theta\|^2_{L^6(\Gamma_{4\varepsilon})}\|\overline{\boldsymbol{u}_N}-\boldsymbol{u}\|^2_{L^3(V_N)})\\
&\quad\leqslant C\varepsilon\int_0^T\|\nabla\boldsymbol{u}\|^2_{L^2(\Gamma_{8\varepsilon})}+C\varepsilon\int_0^T\|\nabla\theta\|^2_{L^6(\Gamma_{4\varepsilon})}\|\overline{\boldsymbol{u}_N}-\boldsymbol{u}\|^2_{L^3(V_N)}\\
&\quad\leqslant C\varepsilon\int_0^T\|\nabla\boldsymbol{u}\|^2_{L^2(\Gamma_{8\varepsilon})}+C\varepsilon^{-\frac{2}{3}}\int_0^T\|\overline{\boldsymbol{u}_N}-\boldsymbol{u}\|^2_{L^3(V_N)}\\
&\quad\leqslant C\varepsilon\int_0^T\|\nabla\boldsymbol{u}\|^2_{L^2(\Gamma_{8\varepsilon})}+C\varepsilon^{-\frac{2}{3}+2\sigma}\|\boldsymbol{u}\|^2_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)},
\end{aligned}$$

which, along with (1.17) and (1.20), implies

$$\lim_{\varepsilon\to 0}\bigg|\varepsilon\sum_{|m-n|\leqslant 1}\int_0^T\int_\Omega\nabla\theta\xi_n\xi_m\overline{\boldsymbol{u}_m}\,\overline{\mathbb{S}_n}\bigg|=0. \tag{3.45}$$

For the third term, the same deduction as (3.38) provides that

$$\begin{aligned}
\sum_{|m-n|\leqslant 1}\int_0^T\psi\int_\Omega\theta\nabla(\xi_m\xi_n)\overline{\boldsymbol{u}_m}\,\overline{\mathbb{S}_n}&=\int_0^T\psi\int_\Omega\theta\sum_{|m-n|\leqslant 1}\nabla(\xi_m\xi_n)\overline{\boldsymbol{u}_m}\,\overline{\mathbb{S}_n}\\
&=\int_0^T\psi\int_\Omega\theta\nabla\bigg(\sum_{n=1}^N\xi_n\bigg)^2\overline{\boldsymbol{u}_m}\,\overline{\mathbb{S}_n}\\
&=0.
\end{aligned}\tag{3.46}$$

Therefore, (3.41) follows directly from (3.44)–(3.46).

It remains to estimate the error terms J_i $(i=1,2,3,4)$ in (3.29). We compute J_1 as follows:

$$\begin{aligned}
J_1&=\sum_{|m-n|\leqslant 1}\int_0^T\psi\int_\Omega\operatorname{div}(\theta\xi_m\xi_n\overline{\boldsymbol{u}_n})((\overline{\rho_n})^\gamma-\overline{(\rho^\gamma)_n})\\
&=\sum_{|m-n|\leqslant 1}\int_0^T\psi\int_\Omega(\nabla\theta\xi_m\xi_n\overline{\boldsymbol{u}_n}+\theta\xi_m\xi_n\nabla\overline{\boldsymbol{u}_n}+\theta\nabla(\xi_m\xi_n)\overline{\boldsymbol{u}_n})((\overline{\rho_n})^\gamma-\overline{(\rho^\gamma)_n}).
\end{aligned}\tag{3.47}$$

Similar to (3.38) and (3.46), one deduces that

$$\sum_{|m-n|\leqslant 1}\int_0^T\psi\int_\Omega\theta\nabla(\xi_m\xi_n)\overline{\boldsymbol{u}_n}((\overline{\rho_n})^\gamma-\overline{(\rho^\gamma)_n})=0.$$

Then from (3.47), we have

$$J_1 = \sum_{|m-n|\leqslant 1} \int_0^T \psi \int_\Omega (\nabla\theta\xi_m\xi_n\overline{\boldsymbol{u}_n} + \theta\xi_m\xi_n\nabla\overline{\boldsymbol{u}_n})((\overline{\rho_n})^\gamma - \overline{(\rho^\gamma)_n}). \tag{3.48}$$

To proceed, we need the following proposition, the proof of which will be provided in Appendix A.

Proposition 3.2. *Let $\overline{f_n}$ be the same as defined in* (3.1)*, and $\widetilde{n}$ be as in* (3.15)*. Then if $\gamma \geqslant 2$, we have*

$$\|(\overline{\rho_n})^\gamma - \overline{(\rho^\gamma)_n}\|_{L^{\frac{3}{2}}} \leqslant C\varepsilon^{2\sigma\beta_{\widetilde{n}}}\|\rho\|^2_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)}. \tag{3.49}$$

By (3.49), as well as (1.17)–(1.18), (2.10), (3.8) and (3.33), a careful computation shows

$$\begin{aligned}
&\bigg|\sum_{|m-n|\leqslant 1} \int_0^T \psi \int_\Omega \nabla\theta\xi_m\xi_n\overline{\boldsymbol{u}_n}((\overline{\rho_n})^\gamma - \overline{(\rho^\gamma)_n})\bigg| \\
&= \bigg|\int_0^T \psi \int_\Omega \nabla\theta\xi_N^2\overline{\boldsymbol{u}_N}((\overline{\rho_N})^\gamma - \overline{(\rho^\gamma)_N})\bigg| \\
&\leqslant C\bigg(\int_0^T\int_{\Gamma_{4\varepsilon}\cap\Omega^{2\varepsilon}} |\overline{(\rho^\gamma)_N}|^2\bigg)^{\frac{1}{2}}\bigg(\int_0^T\int_{\Gamma_{4\varepsilon}\cap\Omega^{2\varepsilon}} |\nabla\theta(\overline{\boldsymbol{u}_N} - \boldsymbol{u} + \boldsymbol{u})|^2\bigg)^{\frac{1}{2}} \\
&\leqslant C\varepsilon^{\frac{1}{2}}\bigg(\int_0^T\int_{\Gamma_{4\varepsilon}\cap\Omega^{2\varepsilon}} |\nabla\theta(\overline{\boldsymbol{u}_N} - \boldsymbol{u} + \boldsymbol{u})|^2\bigg)^{\frac{1}{2}} \\
&\leqslant C\varepsilon^{\frac{1}{2}}\bigg(\int_0^T \varepsilon^{-\frac{5}{3}}\|\overline{\boldsymbol{u}_N} - \boldsymbol{u}\|^2_{L^3} + \int_0^T\int_{\Gamma_{4\varepsilon}} |\boldsymbol{u}\cdot\nabla\theta|^2\bigg)^{\frac{1}{2}} \\
&\leqslant C\bigg(\varepsilon^{1+(2\sigma-\frac{5}{3})} + \varepsilon\int_0^T\int_{\Gamma_{4\varepsilon}} |\nabla\boldsymbol{u}|^2\bigg)^{\frac{1}{2}}
\end{aligned} \tag{3.50}$$

and

$$\begin{aligned}
&\sum_{|m-n|\leqslant 1}\bigg|\int_0^T \psi \int_\Omega \theta\xi_m\xi_n\nabla\overline{\boldsymbol{u}_n}((\overline{\rho_n})^\gamma - \overline{(\rho^\gamma)_n})\bigg| \\
&\leqslant C\|\nabla\overline{\boldsymbol{u}_n}\|_{L^3}\|(\overline{\rho_n})^\gamma - \overline{(\rho^\gamma)_n}\|_{L^{\frac{3}{2}}} \\
&\leqslant C\varepsilon^{(3\sigma-1)\beta_{\widetilde{n}}}\|\boldsymbol{u}\|_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)}\|\rho\|^2_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)} \\
&\leqslant C\varepsilon^{(3\sigma-1)\beta_{\widetilde{n}}}.
\end{aligned}$$

Substituting the above two inequalities into (3.48) and using (1.16) and (1.20) give rise to

$$\lim_{\varepsilon\to 0} J_1 = 0. \tag{3.51}$$

Next, we consider $J_2 + J_3$. From (3.25), one has

$$\sum_{|m-n|\leqslant 1} \int_0^T \psi \int_\Omega \theta\nabla(\xi_m\xi_n)\overline{\boldsymbol{u}_n}(\overline{\rho_n})^\gamma = 0,$$

and thus,

$$\begin{aligned}
&J_2 + J_3 \\
&= \frac{1}{\gamma-1}\sum_{|m-n|\leqslant 1} \int_0^T \psi \int_\Omega \theta\xi_m\xi_n \mathrm{div}((\overline{\rho_n})^\gamma\overline{\boldsymbol{u}_n}) - \sum_{|m-n|\leqslant 1} \int_0^T \psi \int_\Omega \nabla(\theta\xi_m\xi_n)\overline{\boldsymbol{u}_n}(\overline{\rho_n})^\gamma \\
&= -\frac{\gamma}{\gamma-1}\sum_{|m-n|\leqslant 1} \int_0^T \psi \int_\Omega \nabla\theta\xi_m\xi_n\overline{\boldsymbol{u}_n}(\overline{\rho_n})^\gamma.
\end{aligned} \tag{3.52}$$

Following the same steps as in (3.50), from (3.52) we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}|J_2+J_3| &= \frac{\gamma}{\gamma-1}\lim_{\varepsilon\to 0}\left|\sum_{|m-n|\leqslant 1}\int_0^T\psi\int_\Omega \nabla\theta\xi_m\xi_n\overline{\boldsymbol{u}_n}(\overline{\rho_n})^\gamma\right| \\ &\leqslant C\lim_{\varepsilon\to 0}\left[\varepsilon^{1+(2\sigma-\frac{5}{3})}+\varepsilon\int_0^T\int_{\Gamma_{4\varepsilon}}|\nabla\boldsymbol{u}|^2\right]^{\frac{1}{2}} \\ &= 0. \end{aligned}\tag{3.53}$$

Now we estimate J_4. It follows from (1.17)–(1.18), (2.12), (3.4) and (3.15) that

$$\begin{aligned}&\left|\sum_{|m-n|\leqslant 1}\int_0^T\psi\int_\Omega \theta\xi_m\xi_n\nabla(\overline{\rho_n})^{\gamma-1}(\overline{(\boldsymbol{u}\rho)_n}-\overline{\boldsymbol{u}_n}\,\overline{\rho_n})\right| \\ &\quad\leqslant C\|\nabla\overline{\rho_n}\|_{L^3}\|\overline{(\boldsymbol{u}\rho)_n}-\overline{\boldsymbol{u}_n}\,\overline{\rho_n}\|_{L^{\frac{3}{2}}} \\ &\quad\leqslant C\|\nabla\overline{\rho_n}\|_{L^3}\|\overline{\boldsymbol{u}_n}-\boldsymbol{u}\|_{L^3}\|\overline{\rho_n}-\rho\|_{L^3} \\ &\quad\leqslant C\varepsilon^{(3\sigma-1)\beta_{\tilde{n}}}\|\rho\|^2_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)}\|\boldsymbol{u}\|_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)} \\ &\quad\leqslant C\varepsilon^{(3\sigma-1)\beta_{\tilde{n}}}\end{aligned}$$

and

$$\begin{aligned}&\left|\sum_{|m-n|\leqslant 1}\int_0^T\psi\int_\Omega \nabla\theta\xi_m\xi_n(\overline{\rho_n})^{\gamma-1}(\overline{(\boldsymbol{u}\rho)_n}-\overline{\boldsymbol{u}_n}\,\overline{\rho_n})\right| \\ &\quad=\left|\int_0^T\psi\int_{\Omega^{2\varepsilon}\cap\Gamma_{4\varepsilon}}\nabla\theta\xi_N^2((\overline{\rho_N})^{\gamma-1}(\overline{(\boldsymbol{u}\rho)_N}-\overline{\boldsymbol{u}_N}\,\overline{\rho_N}))\right| \\ &\quad\leqslant C\|\nabla\theta\|_{L^3((0,T)\times(\Omega^{2\varepsilon}\cap\Gamma_{4\varepsilon}))}\|\overline{\boldsymbol{u}_N}-\boldsymbol{u}\|_{L^3((0,T)\times(\Omega^{2\varepsilon}\cap\Gamma_{4\varepsilon}))}\|\overline{\rho_N}-\rho\|_{L^3((0,T)\times(\Omega^{2\varepsilon}\cap\Gamma_{4\varepsilon}))} \\ &\quad\leqslant C\varepsilon^{2\sigma-\frac{2}{3}}.\end{aligned}$$

The above two inequalities and (1.16) ensure that

$$\begin{aligned}\lim_{\varepsilon\to 0}|J_4| &= \lim_{\varepsilon\to 0}\left|\sum_{|m-n|\leqslant 1}\int_0^T\psi\int_\Omega \theta\xi_m\xi_n\nabla(\overline{\rho_n})^{\gamma-1}(\overline{(\boldsymbol{u}\rho)_n}-\overline{\boldsymbol{u}_n}\,\overline{\rho_n})\right| \\ &\quad+\lim_{\varepsilon\to 0}\left|\sum_{|m-n|\leqslant 1}\int_0^T\psi\int_\Omega \nabla\theta\xi_m\xi_n(\overline{\rho_n})^{\gamma-1}(\overline{(\boldsymbol{u}\rho)_n}-\overline{\boldsymbol{u}_n}\,\overline{\rho_n})\right| \\ &= 0.\end{aligned}\tag{3.54}$$

Therefore, from (3.51) and (3.53)–(3.54), we obtain

$$\lim_{\varepsilon\to 0}\left|\sum_{i=1}^4 J_i\right|=0,$$

which together with (3.40) and (3.41) generates the desired property (3.29). The proof of Claim 3.1 is completed.

3.5 Convergence to a weak solution of Euler equations

Under the conditions (1.16)–(1.17), there exists a pair of functions $(\rho^0,\boldsymbol{u}^0)$ such that for any $\widetilde{\Omega}_T\subset\subset\Omega_T$,

$$\begin{aligned}&\rho^\varepsilon\rightharpoonup\rho^0\quad\text{weakly in } B_3^{\sigma,\infty}(\widetilde{\Omega}_T)\cap L^\infty(\Omega_T),\\ &\boldsymbol{u}^\varepsilon\rightharpoonup\boldsymbol{u}^0,\quad \rho^\varepsilon\boldsymbol{u}^\varepsilon\rightharpoonup\boldsymbol{M}\quad\text{weakly in } B_3^{\sigma,\infty}(\widetilde{\Omega}_T).\end{aligned}\tag{3.55}$$

It follows from (1.10) that

$$\sqrt{\varepsilon}\nabla \boldsymbol{u}^\varepsilon \in L^2(0,T;L^2(\Omega)).$$

This along with the conditions (1.16)–(1.17) enables us to deduce from the equations (1.1) that for any $\widetilde{\Omega} \subset\subset \Omega$ and $I \subset\subset [0,T]$,

$$\partial_t \rho^\varepsilon,\ \partial_t(\rho^\varepsilon \boldsymbol{u}^\varepsilon) \in L^2(I;W^{-1,\frac{3}{2}}(\widetilde{\Omega})). \tag{3.56}$$

Note that for any $\widetilde{\Omega} \subset\subset \Omega$ and $I \subset\subset [0,T]$,

$$B_3^{\sigma,\infty}(I\times\widetilde{\Omega}) \hookrightarrow L^3(I;B_3^{\sigma,\infty}(\widetilde{\Omega})). \tag{3.57}$$

In addition, the embedding theorem in the three-dimensional Besov spaces (see [40, Theorem 2]) together with the Sobolev embedding implies that if $\sigma > \frac{1}{3}$, then

$$B_3^{\sigma,\infty}(\widetilde{\Omega}) \hookrightarrow W^{\frac{1}{3},\frac{9-\delta}{4-3\sigma}}(\widetilde{\Omega}) \hookrightarrow\hookrightarrow L^{\frac{9}{2}}(\widetilde{\Omega}), \tag{3.58}$$

where $0<\delta<9(\sigma-\frac{1}{3})$. By means of (1.17) and (3.56)–(3.58), we obtain from the Aubin-Lions lemma (see [48]) that for all $p\in(1,\frac{9}{2}]$,

$$\rho^\varepsilon \to \rho^0 \quad \text{strongly in } L^3(I;L^p(\widetilde{\Omega})). \tag{3.59}$$

This and the weak convergence of $\boldsymbol{u}^\varepsilon$ in (3.55) imply that $\boldsymbol{M}=\rho^0\boldsymbol{u}^0$ by the uniqueness of the limit. Hence, similar to the deduction of (3.59), it follows that

$$\rho^\varepsilon\boldsymbol{u}^\varepsilon \to \boldsymbol{M}=\rho^0\boldsymbol{u}^0 \quad \text{strongly in } L^3(I;L^p(\widetilde{\Omega})). \tag{3.60}$$

Consequently, from (3.55) and (3.60), we have

$$\rho^\varepsilon\boldsymbol{u}^\varepsilon\otimes\boldsymbol{u}^\varepsilon \to \rho^0\boldsymbol{u}^0\otimes\boldsymbol{u}^0 \quad \text{in } \mathcal{D}'(\widetilde{\Omega}_T). \tag{3.61}$$

Next, the uniform bound on the density in (1.16) and the strong convergence in (3.59) guarantee that

$$(\rho^\varepsilon)^\gamma \to (\rho^0)^\gamma \quad \text{in } \mathcal{D}'(\widetilde{\Omega}_T). \tag{3.62}$$

Finally, it follows from (1.10) that as $\varepsilon\to 0$,

$$\varepsilon\left|\int_0^T\int_\Omega \mathbb{S}:\nabla\varphi\right| \leqslant C\varepsilon^{\frac{1}{2}}\left(\varepsilon\int_0^T\int_\Omega|\nabla\boldsymbol{u}^\varepsilon|^2\right)^{\frac{1}{2}} \to 0 \quad \text{for any } \varphi\in C_c^1(\Omega_T). \tag{3.63}$$

Therefore, having (3.59)–(3.63) obtained, we are able to test the equations (1.1) against smooth functions and then take the ε-limit to the resulting integral quantities to conclude that the limit function $(\rho^0,\boldsymbol{u}^0)$ is the solution of the equations (1.3) in the sense of distributions. Finally, the strong convergence (1.21) follows directly from (3.59)–(3.60). The proof of Theorem 1.3 is thus completed.

Acknowledgements Robin Ming Chen was supported by National Science Foundation of USA (Grant No. DMS-1907584). Zhilei Liang was supported by the Fundamental Research Funds for the Central Universities (Grant No. JBK 2202045). Dehua Wang was supported by National Science Foundation of USA (Grant Nos. DMS-1907519 and DMS-2219384). Runzhang Xu was supported by National Natural Science Foundation of China (Grant No. 12271122).

References

1 Akramov I, Debiec T, Skipper J, et al. Energy conservation for the compressible Euler and Navier-Stokes equations with vacuum. Anal PDE, 2020, 13: 789–811

2 Bardos C, Nguyen T T. Remarks on the inviscid limit for the compressible flows. In: Recent Advances in Partial Differential Equations and Applications. Contemporary Mathematics, vol. 666. Providence: Amer Math Soc, 2016, 55–67

3 Bardos C, Titi E. Mathematics and turbulence: Where do we stand? J Turbul, 2013, 14: 42–76

4 Bardos C, Titi E. Onsager's conjecture for the incompressible Euler equations in bounded domains. Arch Ration Mech Anal, 2018, 228: 197–207

5 Bardos C, Titi E, Wiedemann E. Onsager's conjecture with physical boundaries and an application to the vanishing viscosity limit. Comm Math Phys, 2019, 370: 291–310

6 Basarić D. Vanishing viscosity limit for the compressible Navier-Stokes system via measure-valued solutions. NoDEA Nonlinear Differential Equations Appl, 2020, 27: 57

7 Boyer F. Trace theorems and spatial continuity properties for the solutions of the transport equation. Differential Integral Equations, 2005, 18: 891–934

8 Buckmaster T, De Lellis C, Székelyhidi L Jr, et al. Onsager's conjecture for admissible weak solutions. Comm Pure Appl Math, 2019, 72: 229–274

9 Chen G-Q, Perepelitsa M. Vanishing viscosity limit of the Navier-Stokes equations to the Euler equations for compressible fluid flow. Comm Pure Appl Math, 2010, 63: 1469–1504

10 Chen R M, Liang Z L, Wang D H. A Kato-type criterion for vanishing viscosity near Onsager's critical regularity. Arch Ration Mech Anal, 2022, 246: 535–559

11 Chen R M, Vasseur A F, Yu C. Global ill-posedness for a dense set of initial data to the isentropic system of gas dynamics. Adv Math, 2021, 393: 108057

12 Cheskidov A, Constantin P, Friedlander S, et al. Energy conservation and Onsager's conjecture for the Euler equations. Nonlinearity, 2008, 21: 1233–1252

13 Chiodaroli E. A counterexample to well-posedness of entropy solutions to the compressible Euler system. J Hyperbolic Differ Equ, 2014, 11: 493–519

14 Chiodaroli E, De Lellis C, Kreml O. Global ill-posedness of the isentropic system of gas dynamics. Comm Pure Appl Math, 2015, 68: 1157–1190

15 Constantin P, E W, Titi E. Onsager's conjecture on the energy conservation for solutions of Euler's equation. Comm Math Phys, 1994, 165: 207–209

16 Constantin P, Kukavica I, Vicol V. On the inviscid limit of the Navier-Stokes equations. Proc Amer Math Soc, 2015, 143: 3075–3090

17 De Lellis C, Székelyhidi L Jr. The Euler equations as a differential inclusion. Ann of Math (2), 2009, 170: 1417–1436

18 De Lellis C, Székelyhidi L Jr. On admissibility criteria for weak solutions of the Euler equations. Arch Ration Mech Anal, 2010, 195: 225–260

19 Drivas T D, Nguyen H Q. Onsager's conjecture and anomalous dissipation on domains with boundary. SIAM J Math Anal, 2018, 50: 4785–4811

20 E W. Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equation. Acta Math Sin (Engl Ser), 2000, 16: 207–218

21 Feireisl E. Maximal dissipation and well-posedness for the compressible Euler system. J Math Fluid Mech, 2014, 16: 447–461

22 Feireisl E, Gwiazda P, Świerczewska-Gwiazda A, et al. Regularity and energy conservation for the compressible Euler equations. Arch Ration Mech Anal, 2017, 223: 1375–1395

23 Feireisl E, Hofmanová M. On convergence of approximate solutions to the compressible Euler system. Ann PDE, 2020, 6: 11

24 Feireisl E, Klingenberg C, Markfelder S. Euler system with a polytropic equation of state as a vanishing viscosity limit. J Math Fluid Mech, 2022, 24: 67

25 Feireisl E, Novotný A, Petzeltová H. On the existence of globally defined weak solutions to the Navier-Stokes equations. J Math Fluid Mech, 2001, 3: 358–392

26 Geng Y C, Li Y C, Zhu S G. Vanishing viscosity limit of the Navier-Stokes equations to the Euler equations for compressible fluid flow with vacuum. Arch Ration Mech Anal, 2019, 234: 727–775

27 Gie G-M, Kelliher J P. Boundary layer analysis of the Navier-Stokes equations with generalized Navier boundary conditions. J Differential Equations, 2012, 253: 1862–1892

28 Girl V, Kwon H. On non-uniqueness of continuous entropy solutions to the isentropic compressible Euler equations. Arch Ration Mech Anal, 2022, 245: 1213–1283

29 Grenier E, Nguyen T T. L^∞ instability of Prandtl layers. Ann PDE, 2019, 5: 18

30 Guo L, Li F C, Xie F. Asymptotic limits of the isentropic compressible viscous magnetohydrodynamic equations with Navier-slip boundary conditions. J Differential Equations, 2019, 267: 6910–6957

31 Guo Y, Nguyen T. A note on Prandtl boundary layers. Comm Pure Appl Math, 2011, 64: 1416–1438

32 Isett P. A proof of Onsager's conjecture. Ann of Math (2), 2018, 188: 871–963

33 Kato T. Remarks on zero viscosity limit for nonstationary Navier-Stokes flows with boundary. In: Seminar on Nonlinear Partial Differential Equations. Mathematical Sciences Research Institute Publications, vol. 2. New York: Springer, 1984, 85–98

34 Kelliher J P. Observations on the vanishing viscosity limit. Trans Amer Math Soc, 2017, 369: 2003–2027
35 Kufner A, John O, Fučík S. Function Spaces. Monographs and Textbooks on Mechanics of Solids and Fluids. Leyden: Noordhoff International Publishing, 1977
36 Kukavica I, Nguyen T T, Vicol V, et al. On the Euler+Prandtl expansion for the Navier-Stokes equations. J Math Fluid Mech, 2022, 24: 47
37 Liu C J, Yang T. Ill-posedness of the Prandtl equations in Sobolev spaces around a shear flow with general decay. J Math Pures Appl (9), 2017, 108: 150–162
38 Masmoudi N. Remarks about the inviscid limit of the Navier-Stokes system. Comm Math Phys, 2007, 270: 777–788
39 Mischler S. On the trace problem for the solutions of the Vlasov equation. Comm Partial Differential Equations, 2000, 25: 1415–1443
40 Muramatu T. On imbedding theorems for Besov spaces of functions defined in general regions. Publ Res Inst Math Sci, 1971, 7: 261–285
41 Oleinik O A, Samokhin V N. Mathematical Models in Boundary Layer Theory. Boca Raton: Chapman & Hall/CRC, 1999
42 Onsager L. Statistical hydrodynamics. Nuovo Cim, 1949, 6: 279–287
43 Paddick M. The strong inviscid limit of the isentropic compressible Navier-Stokes equations with Navier boundary conditions. Discrete Contin Dyn Syst, 2016, 36: 2673–2709
44 Sammartino M, Caflisch R E. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. Comm Math Phys, 1998, 192: 433–461
45 Sammartino M, Caflisch R E. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution. Comm Math Phys, 1998, 192: 463–491
46 Schlichting H. Boundary-Layer Theory. New York: McGraw-Hill, 1979
47 Shvydkoy R. Lectures on the Onsager conjecture. Discrete Contin Dyn Syst Ser S, 2010, 3: 473–496
48 Simon J. Compact sets in the space $L^p(0,T;B)$. Ann Mat Pura Appl (4), 1987, 146: 65–96
49 Sueur F. On the inviscid limit for the compressible Navier-Stokes system in an impermeable bounded domain. J Math Fluid Mech, 2014, 16: 163–178
50 von Kármán T, Tsien H S. Boundary layer in compressible fluids. J Aeronaut Sci, 1938, 5: 227–232
51 Wang X M. A Kato type theorem on zero viscosity limit of Navier-Stokes flows. Indiana Univ Math J, 2001, 50: 223–241
52 Wang Y, Xin Z P, Yong Y. Uniform regularity and vanishing viscosity limit for the compressible Navier-Stokes with general Navier-slip boundary conditions in 3-dimensional domains. SIAM J Math Anal, 2015, 47: 4123–4191
53 Wang Y G, Williams M. The inviscid limit and stability of characteristic boundary layers for the compressible Navier-Stokes equations with Navier-friction boundary conditions. Ann Inst Fourier (Grenoble), 2013, 62: 2257–2314
54 Wang Y G, Yin J R, Zhu S Y. Vanishing viscosity limit for incompressible Navier-Stokes equations with Navier boundary conditions for small slip length. J Math Phys, 2017, 58: 101507
55 Wang Y G, Zhu S Y. On the vanishing dissipation limit for the full Navier-Stokes-Fourier system with non-slip condition. J Math Fluid Mech, 2018, 20: 393–419
56 Wang Y G, Zhu S Y. On the inviscid limit for the compressible Navier-Stokes system with no-slip boundary condition. Quart Appl Math, 2018, 76: 499–514
57 Xiao Y L, Xin Z P. On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition. Comm Pure Appl Math, 2007, 60: 1027–1055
58 Xin Z P, Yanagisawa T. Zero-viscosity limit of the linearized Navier-Stokes equations for a compressible viscous fluid in the half-plane. Comm Pure Appl Math, 1999, 52: 479–541

Appendix A The proof of Proposition 3.2

The ideas of the proof follow exactly from [22]. From the Taylor expansion, we have that for fixed $x \in V_n \cap V_m$ and $t \in (0,T)$,

$$|(\overline{\rho_n})^\gamma - \rho^\gamma - \gamma\rho^{\gamma-1}(\overline{\rho_n} - \rho)|(x,t) \leqslant \gamma(\gamma-1)\rho^{\gamma-2}(\overline{\rho_n} - \rho)^2(x,t) \leqslant C(\overline{\rho_n} - \rho)^2(x,t), \tag{A.1}$$

where the last inequality holds for $\gamma \geqslant 2$. Similarly,

$$|\rho^\gamma(y,t) - \rho^\gamma(x,t) - \gamma\rho^{\gamma-1}(x,t)(\rho(y,t) - \rho(x,t))| \leqslant C|\rho(y,t) - \rho(x,t)|^2. \tag{A.2}$$

Noting that the absolute value function is convex, we mollify (A.2) (in y) and use the Jensen inequality to obtain

$$|\overline{(\rho^\gamma)_n} - \rho^\gamma - \gamma\rho^{\gamma-1}(\overline{\rho_n} - \rho)| \leqslant C\overline{(|\rho(y,t) - \rho(x,t)|^2)_n},$$

which together with (A.1) shows that

$$\begin{aligned}|(\overline{\rho_n})^\gamma - \overline{(\rho^\gamma)_n}| &\leqslant |(\overline{\rho_n})^\gamma - \rho^\gamma - \gamma\rho^{\gamma-1}(\overline{\rho_n} - \rho)| + |\overline{(\rho^\gamma)_n} - \rho^\gamma - \gamma\rho^{\gamma-1}(\overline{\rho_n} - \rho)| \\ &\leqslant C(\overline{\rho_n} - \rho)^2 + C\overline{(|\rho(y,t) - \rho(x,t)|^2)_n}.\end{aligned}$$

By (2.10), (3.15) and the properties of mollification, one deduces

$$\begin{aligned}&\|(\overline{\rho_n})^\gamma - \overline{(\rho^\gamma)_n}\|_{L^{\frac{3}{2}}(V_n\cap V_m)} \\ &\quad\leqslant C\|(\overline{\rho_n} - \rho)^2 + \overline{(|\rho(y,t) - \rho(x,t)|^2)_n}\|_{L^{\frac{3}{2}}(V_n\cap V_m)} \\ &\quad\leqslant C\|\overline{(|\rho(y,t) - \rho(x,t)|^2)_n}\|_{L^{\frac{3}{2}}(V_n\cap V_m)} \\ &\quad\leqslant C\||\rho(y,t) - \rho(x,t)|^2\|_{L^{\frac{3}{2}}(V_n\cap V_m)} \\ &\quad\leqslant C\varepsilon^{\sigma\beta_{\tilde{n}}}\|\rho\|^2_{B_3^{\sigma,\infty}(\widetilde{\Omega}_T)}.\end{aligned}$$

This is the desired inequality (3.49).