SCIENCE CHINA
Mathematics

• ARTICLES •

September 2023 Vol. 66 No. 9: 1993–2020
https://doi.org/10.1007/s11425-022-2025-5

On the steady Prandtl boundary layer expansions

Chen Gao[1,2,*] & Liqun Zhang[1,2,3]

[1]*Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing* 100190, *China;*
[2]*School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing* 100049, *China;*
[3]*Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing* 100190, *China*

Email: gaochen@amss.ac.cn, lqzhang@math.ac.cn

Received March 23, 2022; accepted August 29, 2022; published online March 16, 2023

Abstract In this paper, we consider the zero-viscosity limit of the 2D steady Navier-Stokes equations in $(0, L) \times \mathbb{R}^+$ with no-slip boundary conditions. By estimating the stream-function of the remainder, we justify the validity of the Prandtl boundary layer expansions. Specially, we show the global stability under the concavity condition of the Prandtl profile for an arbitrarily large constant L when the Euler flow is shear.

Keywords Navier-Stokes equations, Prandtl boundary layer, zero-viscosity limit, stream-function, estimates of the remainder

MSC(2020) 35Q30, 76D10

Citation: Gao C, Zhang L Q. On the steady Prandtl boundary layer expansions. Sci China Math, 2023, 66: 1993–2020, https://doi.org/10.1007/s11425-022-2025-5

1 Introduction

We consider the vanishing viscosity limit of steady Navier-Stokes equations

$$\begin{cases} U^\varepsilon U^\varepsilon_X + V^\varepsilon U^\varepsilon_Y - \varepsilon\Delta U^\varepsilon + P^\varepsilon_X = 0, \\ U^\varepsilon V^\varepsilon_X + V^\varepsilon V^\varepsilon_Y - \varepsilon\Delta V^\varepsilon + P^\varepsilon_Y = 0, \\ U^\varepsilon_X + V^\varepsilon_Y = 0, \\ U^\varepsilon\,|_{Y=0} = V^\varepsilon\,|_{Y=0} = 0 \end{cases} \tag{1.1}$$

in a two-dimensional domain $\Omega = \{(X, Y) : 0 < X < L, Y > 0\}$. A formal limit $\varepsilon \to 0$ should lead to the Euler flow $[U^0, V^0]$ inside Ω:

$$\begin{cases} U^0 U^0_X + V^0 U^0_Y + P^0_X = 0, \\ U^0 V^0_X + V^0 V^0_Y + P^0_Y = 0, \\ U^0_X + V^0_Y = 0, \\ V^0\,|_{Y=0} = 0. \end{cases} \tag{1.2}$$

* Corresponding author

Generically, there is a mismatch between the tangential velocities of the Euler flow $U_0(X,0)\neq 0$ and the prescribed Navier-Stokes flows $U^\varepsilon(X,0)=0$ on the boundary, because of the difference of boundary conditions imposed on the two systems.

Due to the mismatch on the boundary, Prandtl in 1904 proposed a thin fluid boundary layer of size $\sqrt{\varepsilon}$ to connect different velocities $U_0(X,0)$ and 0. We make use of the scaled boundary layer, or Prandtl's variables, i.e.,

$$x=X,\quad y=\frac{Y}{\sqrt{\varepsilon}}. \tag{1.3}$$

In these variables, we express the solution of the Navier-Stokes (NS) equations $[U^\varepsilon,V^\varepsilon]$ via $[u^\varepsilon,v^\varepsilon]$ as

$$[U^\varepsilon(X,Y),V^\varepsilon(X,Y)]=[u^\varepsilon(x,y),\sqrt{\varepsilon}v^\varepsilon(x,y)],$$

in which we note that the scaled normal velocity v^ε is $\frac{1}{\sqrt{\varepsilon}}$ of the original velocity V^ε. Similarly,

$$P^\varepsilon(X,Y)=p^\varepsilon(x,y).$$

In these new variables, the Navier-Stokes equations in (1.1) now read

$$\begin{cases}u^\varepsilon u^\varepsilon_x+v^\varepsilon u^\varepsilon_y+p^\varepsilon_x=u^\varepsilon_{yy}+\varepsilon u^\varepsilon_{xx},\\ \varepsilon[u^\varepsilon v^\varepsilon_x+v^\varepsilon v^\varepsilon_y]+p^\varepsilon_y=\varepsilon[v^\varepsilon_{yy}+\varepsilon v^\varepsilon_{xx}],\\ u^\varepsilon_x+v^\varepsilon_y=0.\end{cases} \tag{1.4}$$

Let $\varepsilon\to 0$. It leads to the Prandtl equations

$$\begin{cases}u^0_pu^0_{px}+v^0_pu^0_{py}-u^0_{pyy}+p^0_{px}=0,\\ p^0_{py}=0,\\ u^0_{px}+v^0_{py}=0.\end{cases} \tag{1.5}$$

Prandtl hypothesized that when viscosity ε is small, the Navier-Stokes flow can be approximately decomposed into the following two parts:

$$\begin{aligned}U^\varepsilon(X,Y)&\approx u^0_e(X,Y)-u^0_e(X,0)+u^0_p\bigg(X,\frac{Y}{\sqrt{\varepsilon}}\bigg),\\ V^\varepsilon(X,Y)&\approx v^0_e(X,Y)+\sqrt{\varepsilon}v^0_p\bigg(X,\frac{Y}{\sqrt{\varepsilon}}\bigg),\end{aligned} \tag{1.6}$$

in which (u^0_e,v^0_e) denotes the Euler flow.

We attempt to verify the Prandtl boundary layer expansion (1.6) under more general conditions. Now let us review the main problems in the boundary layer theory. Two important open problems in this area are the well-posedness of Prandtl equations and the justification of viscosity vanishing limits. The first problem is relatively well understood and proved in some cases. Sammartino and Caflisch [24] obtained their result for the analytic class. For the monotonic data, Oleinik and Samokhin [23] obtained the local existence of classical solutions of 2D Prandtl equations by using the Crocco transformation. Xin and Zhang [27] proved the global existence of weak solutions to this system for the favorable pressure, which by their regularity result are classical solutions. Lately, Alexandre et al. [1] and Masmoudi and Wong [22] independently proved, by the energy methods, the local well-posedness of Prandtl equations in the Sobolev space under the monotonic assumptions. Meanwhile, Liu et al. [20] generalized the result in the 3D case with special structures. There are some results of well-posedness in the Gevrey class [2,8]. On the other hand, Gérard-Varet and Dormy [5] established the linearized ill-posedness without the monotonicity condition in the Sobolev space for Prandtl equations. There are some relevant results in [3,9,19].

For the second problem, the verification of the viscosity vanishing limits is more difficult and remains a challenging problem in general. The problem in the analytic case was proved by Sammartino and

Caflisch [25]. Wang et al. [26] gave a new proof by the energy method. In 2014, Maekawa [21] proved the convergence under the assumption of the initial vorticity vanishing in the neighborhood of the boundary. By the energy method, Fei et al. [4] generalized this result to the 3D case. Gérard-Varet et al. [7] established the Gevrey stability for shear flows. There are some results of instability in the Sobolev space [10–12]. For the steady case, important progress was made by Guo and Nguyen [15] and Iyer [16] for Prandtl boundary layer expansions for the steady Navier-Stokes flows over a moving boundary. Especially, Guo and Iyer [14] recently proved the convergence result for no-slip boundary conditions in shear Euler flows in the case where the length of the region L is small. Meanwhile, Gérard-Varet and Maekawa [6] obtained stability of shear flows of Prandtl type in some Sobolev space for a narrow X-periodic domain. Those results are great inspirations to us.

In our first result, we assume that the outside Euler flow

$$[U^0, V^0] \equiv [u_e^0(X,Y), v_e^0(X,Y)]$$

satisfies the following hypotheses:

$$0 < c_0 \leqslant u_e^0 \leqslant C_0 < \infty, \tag{1.7}$$

$$\|v_e^0\|_{L^\infty} \ll 1, \tag{1.8}$$

$$\|\langle Y\rangle^k \nabla^m v_e^0\|_{L^\infty} < \infty \quad \text{for sufficiently large } k,\ m \geqslant 0, \tag{1.9}$$

$$\|\langle Y\rangle^k \nabla^m u_e^0\|_{L^\infty} < \infty \quad \text{for sufficiently large } k,\ m \geqslant 1. \tag{1.10}$$

Here, $\langle Y\rangle = Y+1$.

We consider the Prandtl equations with the positive data, i.e.,

$$\begin{cases} u_p^0 u_{px}^0 + v_p^0 u_{py}^0 - u_{pyy}^0 + p_{px}^0 = 0, \quad p_{py}^0 = 0, \quad u_{px}^0 + v_{py}^0 = 0, \quad (x,y) \in (0,L)\times\mathbb{R}_+, \\ u_p^0\,|_{x=0} = U_P^0(y), \quad u_p^0\,|_{y=0} = v_p^0\,|_{y=0} = 0, \quad u_p^0\,|_{y\uparrow\infty} = u_e^0\,|_{Y=0}. \end{cases} \tag{1.11}$$

U_P^0 is a prescribed smooth function such that

$$\begin{aligned} &U_P^0 > 0 \quad \text{for } y>0, \quad \partial_y U_P^0(0) > 0, \quad \partial_y^2 U_P^0 - u_e^0(x,0)u_{ex}^0(x,0) \sim y^2 \quad \text{near } y=0, \\ &\partial_y^m\{U_P^0 - u_e^0(x,0)\} \quad \text{decays fast for any } m \geqslant 0. \end{aligned} \tag{1.12}$$

In fact, under the above conditions on U_P^0, the equations in (1.11) admit a classical solution $[u_p^0, v_p^0]$. Now we state our first result.

Theorem 1.1. *Assume that the Euler flow* $[u_e^0, v_e^0]$ *satisfies* (1.7)–(1.10)*,* U_P^0 *is a smooth function satisfying* (1.12) *and high-order compatibility conditions, and* L *is a constant small enough. Then there exist* $C(L), \varepsilon_0(L) > 0$ *depending on* L *such that for* $0 < \varepsilon \leqslant \varepsilon_0$*, the equations in* (1.1) *admit a solution* $[U^\varepsilon, V^\varepsilon] \in W^{2,2}(\Omega)$ *satisfying*

$$\begin{aligned} &\|U^\varepsilon - u_e^0 + u_e^0\,|_{Y=0} - u_p^0\|_{L^\infty} \leqslant C\sqrt{\varepsilon}, \\ &\|V^\varepsilon - v_e^0\|_{L^\infty} \leqslant C\sqrt{\varepsilon} \end{aligned} \tag{1.13}$$

with the following boundary conditions:

$$\begin{aligned} &[U^\varepsilon, V^\varepsilon]\,|_{Y=0} = 0, \\ &[U^\varepsilon, V^\varepsilon]\,|_{X=0} = \left[u_e^0(0,Y) - u_e^0(0,0) + u_p^0\left(0, \frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}a_0, v_e^0(0,Y) + \sqrt{\varepsilon}b_0\right], \\ &[U^\varepsilon, V^\varepsilon]_{X=L} = \left[u_e^0(L,Y) - u_e^0(L,0) + u_p^0\left(L, \frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}a_L, v_e^0(L,Y) + \sqrt{\varepsilon}b_L\right]. \end{aligned} \tag{1.14}$$

Here,

$$\begin{aligned}
a_0(Y) &= u_e^1(0,Y) + u_b^1\left(0, \frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}u_e^2(0,Y) + \sqrt{\varepsilon}\hat{u}_b^2\left(0, \frac{Y}{\sqrt{\varepsilon}}\right),\\
a_L(Y) &= u_e^1(L,Y) + u_b^1\left(L, \frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}u_e^2(L,Y) + \sqrt{\varepsilon}\hat{u}_b^2\left(L, \frac{Y}{\sqrt{\varepsilon}}\right),\\
b_0(Y) &= v_b^0\left(0, \frac{Y}{\sqrt{\varepsilon}}\right) + v_e^1(0,Y) + \sqrt{\varepsilon}v_b^1\left(0, \frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}v_e^2(0,Y) + \varepsilon\hat{v}_b^2\left(0, \frac{Y}{\sqrt{\varepsilon}}\right),\\
b_L(Y) &= v_b^0\left(L, \frac{Y}{\sqrt{\varepsilon}}\right) + v_e^1(L,Y) + \sqrt{\varepsilon}v_b^1\left(L, \frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}v_e^2(L,Y) + \varepsilon\hat{v}_b^2\left(L, \frac{Y}{\sqrt{\varepsilon}}\right)
\end{aligned} \tag{1.15}$$

are smooth functions constructed in Proposition 2.7.

For the second result, we consider L is any given positive constant. We assume that the Euler flow $[U^0, V^0] \equiv [u_e^0(Y), 0]$ is a shear flow, i.e., it satisfies the following hypotheses:

$$\begin{aligned}
&0 < c_0 \leqslant u_e^0 \leqslant C_0 < \infty,\\
&\|\langle Y\rangle^k \nabla^m u_e^0\|_{L^\infty} < \infty \quad \text{for sufficiently large } k,\ m \geqslant 1.
\end{aligned} \tag{1.16}$$

Here, $\langle Y\rangle = Y + 1$.

We assume that $[u_p^0, v_p^0]$ is a smooth solution of Prandtl equations (1.11) satisfying the following hypotheses:

$$\begin{aligned}
&u_p^0 > 0, \quad -u_{pyy}^0 \geqslant 0 \quad \text{for } y > 0,\\
&u_{py}^0 > 0 \quad \text{for } y \geqslant 0,\\
&\nabla^m\{u_p^0 - u_e^0(0)\} \quad \text{decays fast for any } m \geqslant 0.
\end{aligned} \tag{1.17}$$

Because the Euler flow here is independent of x, by the classical result of Oleinik and Samokhin [23], for any given $L > 0$, this kind of solution exists. An important example is the famous Blasius self-similar solution.

Now we state our second result.

Theorem 1.2. *Assume that the Euler flow* $[u_e^0, v_e^0]$ *satisfies* (1.16)*, the Prandtl profile* $[u_p^0, v_p^0]$ *satisfies* (1.17)*, and* $L > 0$ *is any given constant. Then there exist* $C(L), \varepsilon_0(L) > 0$ *depending on* L *such that for* $0 < \varepsilon \leqslant \varepsilon_0$*, the equations in* (1.1) *admit a solution* $[U^\varepsilon, V^\varepsilon] \in W^{2,2}(\Omega)$ *satisfying*

$$\begin{aligned}
&\|U^\varepsilon - u_e^0 + u_e^0|_{Y=0} - u_p^0\|_{L^\infty} \leqslant C\sqrt{\varepsilon},\\
&\|V^\varepsilon\|_{L^\infty} \leqslant C\sqrt{\varepsilon}
\end{aligned} \tag{1.18}$$

with the boundary conditions

$$\begin{aligned}
&[U^\varepsilon, V^\varepsilon]|_{Y=0} = 0,\\
&[U^\varepsilon, V^\varepsilon]|_{X=0} = \left[u_e^0(Y) - u_e^0(0) + u_p^0\left(0, \frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}a_0, \sqrt{\varepsilon}b_0\right],\\
&[U^\varepsilon, V^\varepsilon]_{X=L} = \left[u_e^0(Y) - u_e^0(0) + u_p^0\left(L, \frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}a_L, \sqrt{\varepsilon}b_L\right].
\end{aligned} \tag{1.19}$$

Here,

$$\begin{aligned}
a_0(Y) &= u_e^1(0,Y) + u_b^1\left(0, \frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}u_e^2(0,Y) + \sqrt{\varepsilon}\hat{u}_b^2\left(0, \frac{Y}{\sqrt{\varepsilon}}\right),\\
a_L(Y) &= u_e^1(L,Y) + u_b^1\left(L, \frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}u_e^2(L,Y) + \sqrt{\varepsilon}\hat{u}_b^2\left(L, \frac{Y}{\sqrt{\varepsilon}}\right),\\
b_0(Y) &= v_b^0\left(0, \frac{Y}{\sqrt{\varepsilon}}\right) + v_e^1(0,Y) + \sqrt{\varepsilon}v_b^1\left(0, \frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}v_e^2(0,Y) + \varepsilon\hat{v}_b^2\left(0, \frac{Y}{\sqrt{\varepsilon}}\right),\\
b_L(Y) &= v_b^0\left(L, \frac{Y}{\sqrt{\varepsilon}}\right) + v_e^1(L,Y) + \sqrt{\varepsilon}v_b^1\left(L, \frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}v_e^2(L,Y) + \varepsilon\hat{v}_b^2\left(L, \frac{Y}{\sqrt{\varepsilon}}\right)
\end{aligned} \tag{1.20}$$

are smooth functions constructed in Proposition 2.7.

The theorems show if the expansion (1.6) is right on $\partial\Omega$, then it is right in Ω. In Theorem 1.1, the Euler profile is considered as the perturbation of the shear flow and the Prandtl profile is positive. In contrast to the main theorem in [14], the Euler flow is shear and the positive Prandtl profile should satisfy another condition. If $[u_p^0(x,y), v_p^0(x,y)]$ solves Prandtl equations, then so does $[\frac{\lambda^2}{\sigma}u_p^0(\sigma x, \lambda y), v_p^0(\sigma x, \lambda y)]$ for any $\lambda, \sigma > 0$. In their result, σ should be chosen small enough. The proof of Theorem 1.1 is different from the previous works. We can obtain the unified estimates no matter that the Euler flow is a shear flow or its perturbation. However, to show the expansion (1.6) if the Euler flow is the perturbation of the shear flow, Iyer [17] obtained another estimate based on the shear flow case in [11] with the moving boundary condition on $Y = 0$. It does not work with the no-slip boundary condition $U^\varepsilon\,|_{Y=0} = 0$.

Theorem 1.2 is the first global result in steady Prandtl expansions because there is no assumption on the smallness of the boundary layer profile $u_p^0 - u_e^0\,|_{Y=0}$ and L can be chosen for any large constant. In this situation, the concavity condition on the Prandtl profile $-u_{pyy}^0 \geqslant 0$ is needed to obtain a nice estimate. The positive and concave solution of Prandtl equations is always called the solution in the monotonic class. In the classic book of Oleinik and Samokhin [23], they showed the well-posedness of the Prandtl equation in this class by the Crocco transformation. Especially, the famous Blasius self-similar solution of Prandtl equations is covered in this result.

To prove the theorems, we first construct the approximate solutions $\boldsymbol{U}_s = [U_s, V_s]$ of Navier-Stokes equations. We write the remainder as $\boldsymbol{U} := \boldsymbol{U}^\varepsilon - \boldsymbol{U}_s$. So $\boldsymbol{U}$ satisfies the following linearized Navier-Stokes equations:

$$-\varepsilon\Delta\boldsymbol{U} + \boldsymbol{U}_s\cdot\nabla\boldsymbol{U} + \boldsymbol{U}\cdot\nabla\boldsymbol{U}_s + \nabla P = \boldsymbol{F}. \tag{1.21}$$

The key point is to show the linear stability of the above equation. The method of [14] is by taking the partial derivatives of the vorticity equations with respect to x, they found the Rayleigh term and bi-Laplacian terms enjoy good interaction properties. What we choose to estimate is the stream-function of $\boldsymbol{U}$ which has natural boundary conditions. We can estimate the second derivatives of the stream-function which can be dominated by $\boldsymbol{F}$, which essentially leads to the proof of the first theorem. To obtain the second result, we need to do more subtle calculations. We observe a lot of cancellation in the estimates, which plays a significant role in the linear stability analysis of the equation (1.21).

The rest of this paper is organized as follows. In Section 2, we construct the approximate solutions to the Navier-Stokes equations. In Section 3, we estimate the stream-function of the remainder. In Section 4, we prove the main theorems. The details of linearized Euler and Prandtl equations are placed in Appendix A.

2 Construction of the approximate solutions

We construct the approximate solutions in this section. We need high-order approximations, as compared with (1.6), in order to control the remainders. Precisely, we search for approximate solutions of the Navier-Stokes equations in the following forms:

$$\begin{aligned}
U^\varepsilon(X,Y) &\approx u_e^0(X,Y) + u_b^0\left(X, \frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}\left[u_e^1(X,Y) + u_b^1\left(X,\frac{Y}{\sqrt{\varepsilon}}\right)\right]\\
&\quad + \varepsilon\left[u_e^2(X,Y) + u_b^2\left(X,\frac{Y}{\sqrt{\varepsilon}}\right)\right],\\
V^\varepsilon(X,Y) &\approx v_e^0(X,Y) + \sqrt{\varepsilon}\left[v_b^0\left(X,\frac{Y}{\sqrt{\varepsilon}}\right) + v_e^1(X,Y)\right] + \varepsilon\left[v_b^1\left(X,\frac{Y}{\sqrt{\varepsilon}}\right) + v_e^2(X,Y)\right]\\
&\quad + \varepsilon^{\frac{3}{2}}v_b^2\left(X,\frac{Y}{\sqrt{\varepsilon}}\right),\\
P^\varepsilon(X,Y) &\approx p_e^0(X,Y) + p_b^0\left(X,\frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}\left[p_e^1(X,Y) + p_b^1\left(X,\frac{Y}{\sqrt{\varepsilon}}\right)\right]\\
&\quad + \varepsilon\left[p_e^2(X,Y) + p_b^2\left(X,\frac{Y}{\sqrt{\varepsilon}}\right)\right] + \varepsilon^{\frac{3}{2}}p_b^3\left(X,\frac{Y}{\sqrt{\varepsilon}}\right),
\end{aligned} \tag{2.1}$$

in which $[u_e^j, v_e^j, p_e^j]$ and $[u_b^j, v_b^j, p_b^j]$ with $j = 0, 1, 2$ denote the Euler profiles and boundary layer profiles, respectively. Here, we note that these profile solutions also depend on ε. The Euler flows are always evaluated at (X, Y), whereas the boundary layer profiles are at $(X, \frac{Y}{\sqrt{\varepsilon}})$.

For convenience, we introduce some notation here. We write

$$\begin{aligned}
&\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2_{X,Y}},\\
&\langle \cdot, \cdot \rangle_{Y=0} = \langle \cdot, \cdot \rangle_{L^2_X(Y=0)},\\
&\| \cdot \| = \| \cdot \|_{L^2_{X,Y}}
\end{aligned}$$

and

$$\| \cdot \|_\infty = \| \cdot \|_{L^\infty_{X,Y}} = \| \cdot \|_{L^\infty_{x,y}}.$$

The notation $a \lesssim b$ means that there exists a positive constant C_0, s.t. $a \leqslant C_0 b$, where C_0 is independent of $\sqrt{\varepsilon}$ and L when $L \leqslant 1$, and C_0 is independent of $\sqrt{\varepsilon}$ but dependent on L when $L \geqslant 1$. The notation $a \lesssim_L b$ means that there exists a positive constant $C_0(L)$ such that $a \leqslant C_0(L) b$, where $C_0(L)$ is independent of $\sqrt{\varepsilon}$ but dependent on L. We let $a = O(b)$ denote $|a| \lesssim b$.

2.1 The zeroth-order profiles

Recall the Euler flow $[u_e^0, v_e^0]$. Let ψ be the stream-function of $[u_e^0, v_e^0]$:

$$\psi(X, Y) := \int_0^Y u_e^0(X, Y') dY', \quad \psi_Y = u_e^0, \quad \psi_X = -v_e^0. \tag{2.2}$$

Then the Euler equations in (1.2) are equivalent to

$$\Delta \psi = F_e(\psi). \tag{2.3}$$

From the assumptions in (1.7)–(1.10), we can know that F_e together with sufficiently many derivatives is bounded and decays in its argument.

For Prandtl equations, there is a famous result due to Oleinik and Samokhin [23].

Proposition 2.1 (See [23]). *Assume that the boundary data $U_P^0 \in C^\infty$ satisfies* (1.12)*. Then for some $L > 0$, the equations in* (1.11) *have a solution $[u_p^0, v_p^0]$ such that for some $y_0, m_0 > 0$,*

$$\begin{aligned}
&\sup_{(0,L)\times(0,\infty)} |u_p^0, \partial_y u_p^0, \partial_{yy} u_p^0, \partial_x u_p^0| \lesssim 1,\\
&\inf_{(0,L)\times(0,y_0)} \partial_y u_p^0 > m_0 > 0,\\
&u_p^0 > 0 \quad \text{for } y > 0.
\end{aligned} \tag{2.4}$$

Follow the proof of Oleinik and Samokhin in [23], and it is easy to have the following lemma.

Lemma 2.2. *If $U_P^0 \in C^\infty$ satisfies* (1.12) *and high-order parabolic compatibility conditions, then*

$$\| \langle y \rangle^M \nabla^k (u_p^0(x, y) - u_e^0(x, 0)) \|_\infty \lesssim 1 \quad \text{for } 0 \leqslant k \leqslant K, \tag{2.5}$$

where K and M are constants.

By using the Crocco transformation in [23], we easily see the following proposition.

Proposition 2.3. *Assume that $u_e^0|_{Y=0}$ is independent of x, U_P^0 is the smooth function satisfying* (1.12)*, and*

$$\begin{aligned}
&M_1 \left(1 - \frac{U_P^0}{u_e^0(0)}\right) \sqrt{-\ln \mu \left(1 - \frac{U_P^0}{u_e^0(0)}\right)} \leqslant \frac{\partial_y U_P^0}{u_e^0(0)} \leqslant M_2 \left(1 - \frac{U_P^0}{u_e^0(0)}\right) \sqrt{-\ln \mu \left(1 - \frac{U_P^0}{u_e^0(0)}\right)},\\
&- M_3 \sqrt{-\ln \mu \left(1 - \frac{U_P^0}{u_e^0(0)}\right)} \leqslant \frac{\partial_y^2 U_P^0}{\partial_y U_P^0} \leqslant -M_4 \sqrt{-\ln \mu \left(1 - \frac{U_P^0}{u_e^0(0)}\right)},
\end{aligned}$$

$$\left|\frac{\partial_y^3 U_P^0 \partial_y U_P^0 - (\partial_y^2 U_P^0)^2}{(\partial_y U_P^0)^2}\right| \leqslant M_5,$$

where μ and M_i are positive constants and $0 < \mu < 1$; moreover, U_P^0 satisfies the high-order parabolic compatibility conditions and the high-order derivatives of U_P^0 decay fast enough, and then for any $L > 0$, (1.11) *admits a smooth solution $[u_p^0, v_p^0]$ satisfying* (1.17) *and*

$$\|\langle y\rangle^M \nabla^k (u_p^0(x,y) - u_e^0(0))\|_\infty \lesssim 1 \quad \textit{for } 0 \leqslant k \leqslant K, \tag{2.6}$$

where K and M are large constants.

Notice that

$$u_p^0(x,y) - u_e^0(0) \sim \exp(-\alpha y^2),$$

and the Blasius self-similar solution is in this class. We can also deal with the case

$$u_p^0(x,y) - u_e^0(0) \sim \exp(-\alpha y).$$

After we solve the Prandtl equations (1.5), we let

$$u_b^0(x,y) = u_p^0(x,y) - u_e^0(x,0), \quad v_b^0(x,y) = \int_y^\infty u_{bx}^0(x,y')dy'$$

and $p_b^0 = 0$.

2.2 The high-order profiles

By the method of asymptotic matching expansions, we can deduce the equations of $[u_e^j, v_e^j]$ and $[u_b^j, v_b^j]$ ($j = 1, 2$). The first-order Euler profile $[u_e^1, v_e^1, p_e^1]$ solves the linearized Euler equations around $[u_e^0, v_e^0]$:

$$\begin{cases} u_e^0 u_{eX}^1 + u_{eX}^0 u_e^1 + v_e^0 u_{eY}^1 + u_{eY}^0 v_e^1 + p_{eX}^1 = 0, \\ u_e^0 v_{eX}^1 + v_{eX}^0 u_e^1 + v_e^0 v_{eY}^1 + v_{eY}^0 v_e^1 + p_{eY}^1 = 0, \\ \partial_X u_e^1 + \partial_Y v_e^1 = 0, \\ v_e^1 |_{Y=0} = -v_b^0 |_{y=0}. \end{cases} \tag{2.7}$$

Following the idea of Iyer [17], we introduce new independent variables by

$$\theta(X,Y) = X, \quad \psi(X,Y) = \int_0^Y u_e^0(X,Y')dY'. \tag{2.8}$$

Let ψ^1 be the stream-function of $[u_e^1, v_e^1]$:

$$\psi^1(X,Y) := \int_0^Y u_e^1(X,Y')dY' - \int_0^X v_e^1(X',0)dX', \quad \psi_Y^1 = u_e^1, \quad \psi_X^1 = -v_e^1.$$

Then the first-order Euler layer equations in (2.7) are equivalent to

$$\partial_\theta[\Delta_{XY}\psi^1 - F_e'(\psi)\psi^1] = 0, \tag{2.9}$$

which is reduced to finding a solution of the following equations:

$$\begin{cases} \Delta\psi^1 - F_e'(\psi)\psi^1 = 0, \\ \psi^1 |_{X=0} = \psi_0^1(Y), \quad \psi^1 |_{X=L} = \psi_L^1(Y), \\ \psi^1 |_{Y=0} = \int_0^X v_b^0(X',0)dX', \quad \psi^1 |_{Y\to\infty} = 0. \end{cases} \tag{2.10}$$

It is a standard elliptic problem, and we have the following result.

Lemma 2.4. *If v_b^0 is a smooth function, and for any $L>0$, $\psi_0^1(Y)$ and $\psi_L^1(Y)$ satisfy the compatibility conditions on the corner, then* (2.10) *admits a smooth solution satisfying the following estimate*:

$$\|\langle Y\rangle^M\nabla^k\psi^1\|\lesssim 1\quad \text{for } 1\leqslant k\leqslant K,\quad \text{where } K \text{ and } M \text{ are large constants.} \tag{2.11}$$

We will prove Lemma 2.4 in Appendix A.

Remark 2.5. The boundary conditions of ψ^1 in (2.10) imply the following boundary conditions of $[u_e^1,v_e^1]$:

$$\begin{aligned}&u_e^1\,|_{X=0}=\partial_Y\psi_0^1(Y),\quad u_e^1\,|_{X=L}=\partial_Y\psi_L^1(Y),\\&v_e^1\,|_{Y=0}=-v_b^0(X,0),\quad [u_e^1,v_e^1]\,|_{Y\to\infty}=0.\end{aligned} \tag{2.12}$$

So we actually construct a solution $[u_e^1,v_e^1]$ to the equations in (2.7) with the boundary condition (2.12).

Next, we need to solve the first-order boundary layer profile. For simplicity, we introduce the following notation:

$$\begin{aligned}&u_p^k:=u_b^k+\sum_{j=0}^k\frac{y^j}{j!}\partial_Y^ju_e^{k-j}\,|_{Y=0},\quad u_e^{(k)}:=\sum_{j=0}^k\frac{y^j}{j!}\partial_Y^ju_e^{k-j}\,|_{Y=0},\\&v_p^k:=v_b^k-v_b^k\,|_{y=0}+\sum_{j=0}^k\frac{y^{j+1}}{(j+1)!}\partial_Y^{j+1}v_e^{k-j}\,|_{Y=0},\quad v_e^{(k)}:=\sum_{j=0}^k\frac{y^{j+1}}{(j+1)!}\partial_Y^{j+1}v_e^{k-j}\,|_{Y=0}.\end{aligned} \tag{2.13}$$

$[u_b^1,v_b^1,p_b^1]$ satisfies the linearized Prandtl equations around $[u_p^0,v_p^0]$:

$$\begin{cases}u_p^0\partial_xu_b^1+u_b^1\partial_xu_p^0+\partial_yu_p^0[v_b^1-v_b^1\,|_{y=0}]+v_p^0\partial_yu_b^1-\partial_{yy}u_b^1+\partial_xp_b^1=f^{(1)},\\\partial_yp_b^1=0,\\\partial_xu_b^1+\partial_yv_b^1=0,\\u_b^1\,|_{y=0}=-u_e^1\,|_{Y=0},\quad [u_b^1,v_b^1]\,|_{y\to\infty}=0,\end{cases} \tag{2.14}$$

where

$$f^{(1)}=-\left\{u_b^0u_{ex}^{(1)}+u_{bx}^0u_e^{(1)}+v_b^0\partial_yu_e^{(1)}+u_{by}^0v_e^{(1)}\right\}. \tag{2.15}$$

We see that $f^{(1)}$ decays fast when $y\to\infty$ from Lemma 2.2. Since the above equations are linear parabolic-type equations, we add a boundary condition on $u_b^1\,|_{x=0}$:

$$\begin{cases}u_p^0\partial_xu_b^1+u_b^1\partial_xu_p^0+v_p^0\partial_yu_b^1+[v_b^1-v_b^1\,|_{y=0}]\partial_yu_p^0-\partial_{yy}u_b^1+\partial_xp_b^1=f^{(1)},\\\partial_yp_b^1=0,\\\partial_xu_b^1+\partial_yv_b^1=0,\\u_b^1\,|_{x=0}=U_B^1,\quad u_b^1\,|_{y=0}=-u_e^1\,|_{Y=0},\quad [u_b^1,v_b^1]\,|_{y\to\infty}=0.\end{cases} \tag{2.16}$$

We can also discuss the compatibility conditions like Prandtl equations. In our case, v_p^0 is different from that in [13], because $v_p^0\sim yv_{eY}^0(x,0)$ as y goes to ∞, and we still have the following lemma.

Lemma 2.6. *If $f^{(1)}$ and its derivatives are bounded and decay rapidly, and they satisfy the parabolic compatibility conditions, then* (2.16) *admits a unique solution $[u_b^1,v_b^1]$ and*

$$\|\langle y\rangle^M\nabla^ku_b^1\|_\infty+\|\langle y\rangle^M\nabla^kv_b^1\|_\infty\lesssim 1\quad \text{for } 0\leqslant k\leqslant K, \tag{2.17}$$

where K and M are large constants.

We will give the proof of Lemma 2.6 in Appendix A.

The second-order Euler profile $[u_e^2,v_e^2,p_e^2]$ satisfies the linearized Euler equations around $[u_e^0,v_e^0]$ with the force terms, i.e.,

$$\begin{cases}u_e^0u_{eX}^2+u_{eX}^0u_e^2+v_e^0u_{eY}^2+u_{eY}^0v_e^2+p_{eX}^2=F^{(2)},\\u_e^0v_{eX}^2+v_{eX}^0u_e^2+v_e^0v_{eY}^2+v_{eY}^0v_e^2+p_{eY}^2=G^{(2)},\\\partial_Xu_e^2+\partial_Yv_e^2=0,\\v_e^2\,|_{Y=0}=-v_b^1\,|_{y=0},\end{cases} \tag{2.18}$$

where

$$\begin{aligned} F^{(2)} &= -(u_e^1 u_{ex}^1 + v_e^1 u_{eY}^1) + \Delta u_e^0, \\ G^{(2)} &= -(u_e^1 v_{ex}^1 + v_e^1 v_{eY}^1) + \Delta v_e^0. \end{aligned} \tag{2.19}$$

We can treat the above equations as the equation of the first-order Euler flow, i.e.,

$$\partial_\theta \left[\Delta_{XY}\psi^2 - F_e'(\psi)\psi^2 - \frac{F_e''(\psi)}{2}(\psi^1)^2 \right] = \frac{\Delta_{XY}^2 \psi}{u}. \tag{2.20}$$

Let

$$H(\theta, \psi) = \int_0^\theta \frac{\Delta_{XY}^2 \psi(\theta', Y(\theta', \psi))}{u(\theta', \psi)} d\theta'.$$

Noticing that $\psi \sim Y$ when $Y \to \infty$, we see that H is of fast decay as $\psi \to \infty$ because of (1.10) and (1.9). We can find a solution of the following equations:

$$\begin{cases} \Delta_{XY}\psi^2 - F_e'(\psi)\psi^2 - \dfrac{F_e''(\psi)}{2}(\psi^1)^2 = H(\theta(X,Y), \psi(X,Y)), \\ \psi^2 \mid_{X=0} = \psi_0^2(Y), \quad \psi^2 \mid_{X=L} = \psi_L^2(Y), \\ \psi^2 \mid_{Y=0} = \displaystyle\int_0^X v_b^1(X', 0) dX', \quad \psi^2 \mid_{Y\to\infty} = 0 \end{cases} \tag{2.21}$$

with suitable $\psi_0^2(Y)$ and $\psi_L^2(Y)$, and we have estimates of the second-order Euler flow

$$\|\langle Y\rangle^M \nabla^k \psi^2\| \lesssim 1 \quad \text{for } 1 \leqslant k \leqslant K, \quad \text{where } K \text{ and } M \text{ are large constants.} \tag{2.22}$$

The second-order boundary layer profile $[u_b^2, v_b^2, p_b^2]$ is similar to the first-order boundary layer profile, and we need to solve the following equations:

$$\begin{cases} u_p^0 \partial_x u_b^2 + u_b^2 \partial_x u_p^0 + v_p^0 \partial_y u_b^2 + [v_b^2 - v_b^2 \mid_{y=0}] \partial_y u_p^0 - \partial_{yy} u_b^2 + \partial_x p_b^2 = f^{(2)}, \\ \partial_y p_b^2 = g^{(2)}, \\ \partial_x u_b^2 + \partial_y v_b^2 = 0, \\ u_b^2 \mid_{x=0} = U_B^2, \quad u_b^2 \mid_{y=0} = -u_e^2 \mid_{Y=0}, \quad [u_b^2, v_b^2] \mid_{y\to\infty} = 0, \end{cases} \tag{2.23}$$

where

$$\begin{aligned} f^{(2)} &= -\{u_b^0 u_{ex}^{(2)} + u_{bx}^0 u_e^{(2)} + v_b^0 u_{ey}^{(2)} + u_{by}^0 v_e^{(2)} + u_p^1 u_{bx}^1 + u_b^1 u_{ex}^{(1)} + v_p^1 u_{by}^1 + v_b^1 u_{ey}^{(1)} - u_{bxx}^0\}, \\ g^{(2)} &= -\{u_b^0 v_{px}^0 + u_e^{(0)} v_{bx}^0 + v_b^0 v_{py}^0 + (v_e^{(0)} + v_e^1 \mid_{Y=0}) v_{by}^0 - v_{byy}^0\}. \end{aligned} \tag{2.24}$$

We can see that $f^{(2)}$ and $g^{(2)}$ decay fast when $y \to \infty$ from Lemmas 2.2 and 2.6. We can solve

$$p_b^2(x, y) = -\int_y^\infty g^{(2)}(x, y') dy'.$$

By using the same argument of Lemma 2.6, we have

$$\|\langle y\rangle^M \nabla^k u_b^2\|_\infty + \|\langle y\rangle^M \nabla^k v_b^2\|_\infty \lesssim 1 \quad \text{for } 0 \leqslant k \leqslant K, \tag{2.25}$$

where K and M are large constants.

After that, p_b^3 is solved by

$$p_b^3 = \int_y^\infty \left\{ \sum_{j=0}^1 [u_b^{1-j} v_{px}^j + u_e^{(1-j)} v_{bx}^j + v_b^{1-j} v_{py}^j + (v_e^{(1-j)} + v_e^{2-j} \mid_{Y=0}) v_{by}^j] - v_{byy}^1 \right\} dy'. \tag{2.26}$$

We can conclude the following proposition for the approximate profiles.

Proposition 2.7. *Under the assumptions of Theorem* 1.1 *or Theorem* 1.2*, then the equations* (2.7)*,* (2.14)*,* (2.18) *and* (2.23) *admit smooth solutions* $[u_e^j, v_e^j]$ *and* $[u_b^j, v_b^j]$ *for* $j=1,2$*, and the following estimates hold*:

$$\begin{aligned} &\|\langle y\rangle^M \nabla^k u_b^j\|_\infty + \|\langle y\rangle^M \nabla^k v_b^j\|_\infty \lesssim 1 \quad \text{for } 0 \leqslant k \leqslant K \text{ and } j=0,1,2, \\ &\|\langle Y\rangle^M \nabla^k u_e^j\|_\infty + \|\langle Y\rangle^M \nabla^k v_e^j\|_\infty \lesssim 1 \quad \text{for } 0 \leqslant k \leqslant K \text{ and } j=1,2, \end{aligned} \tag{2.27}$$

where K *and* M *are sufficiently large constants,* $\langle y\rangle = y+1$ *and* $\langle Y\rangle = Y+1$*.*

Notice that $v_b^2|_{y=0} \neq 0$. We need to match the boundary conditions at $y=0$ and also $v_b^2|_{y\to\infty}=0$. Then we can modify $[\hat u_b^2, \hat v_b^2]$ in the following way:

$$\begin{aligned} \hat u_b^2(x,y) &:= \chi(\sqrt{\varepsilon}y)u_b^2(x,y) - \sqrt{\varepsilon}\chi'(\sqrt{\varepsilon}y)\int_0^y u_b^2(x,y')dy', \\ \hat v_b^2(x,y) &:= \chi(\sqrt{\varepsilon}y)(v_b^2(x,y) - v_b^2(x,0)), \end{aligned} \tag{2.28}$$

where χ is a cut-off function satisfying $\chi|_{[0,1]}=1$ and $\chi|_{[2,\infty)}=0$.

Let $[U_s, V_s, P_s]$ be

$$\begin{aligned} U_s(X,Y) &= u_e^0(X,Y) + u_b^0\left(X,\frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}\left[u_e^1(X,Y) + u_b^1\left(X,\frac{Y}{\sqrt{\varepsilon}}\right)\right] \\ &\quad + \varepsilon\left[u_e^2(X,Y) + \hat u_b^2\left(X,\frac{Y}{\sqrt{\varepsilon}}\right)\right], \\ V_s(X,Y) &= v_e^0(X,Y) + \sqrt{\varepsilon}\left[v_b^0\left(X,\frac{Y}{\sqrt{\varepsilon}}\right) + v_e^1(X,Y)\right] + \varepsilon\left[v_b^1\left(X,\frac{Y}{\sqrt{\varepsilon}}\right) + v_e^2(X,Y)\right] \\ &\quad + \varepsilon^{\frac{3}{2}}\hat v_b^2\left(X,\frac{Y}{\sqrt{\varepsilon}}\right), \\ P_s(X,Y) &= p_e^0(X,Y) + p_b^0\left(X,\frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}\left[p_e^1(X,Y) + p_b^1\left(X,\frac{Y}{\sqrt{\varepsilon}}\right)\right] \\ &\quad + \varepsilon\left[p_e^2(X,Y) + p_b^2\left(X,\frac{Y}{\sqrt{\varepsilon}}\right)\right] + \varepsilon^{\frac{3}{2}}p_b^3\left(X,\frac{Y}{\sqrt{\varepsilon}}\right). \end{aligned} \tag{2.29}$$

Then the errors

$$\begin{aligned} R_1 &:= U_sU_{sX} + V_sU_{sY} - \varepsilon\Delta U_s + P_{sX}, \\ R_2 &:= U_sV_{sX} + V_sV_{sY} - \varepsilon\Delta V_s + P_{sY} \end{aligned} \tag{2.30}$$

satisfy

$$\|R_1\| + \|R_2\| \lesssim \varepsilon^{\frac{3}{2}}. \tag{2.31}$$

Remark 2.8. We obtain what $[U_s, V_s]$ look like when ε is small, i.e.,

$$\begin{aligned} U_s(X,Y) &= u_e^0(X,Y) + u_b^0\left(X,\frac{Y}{\sqrt{\varepsilon}}\right) + O(\sqrt{\varepsilon}), \\ V_s(X,Y) &= v_e^0(X,Y) + \sqrt{\varepsilon}\left(v_b^0\left(X,\frac{Y}{\sqrt{\varepsilon}}\right) + v_e^1(X,Y)\right) + O(\varepsilon). \end{aligned}$$

When $\frac{Y}{\sqrt{\varepsilon}} \leqslant 1$,

$$u_e^0(X,Y) + u_b^0\left(X,\frac{Y}{\sqrt{\varepsilon}}\right) = u_e^0(X,Y) - u_e^0(X,0) + u_p^0\left(X,\frac{Y}{\sqrt{\varepsilon}}\right) \gtrsim -Y + \frac{Y}{\sqrt{\varepsilon}} \gtrsim \frac{Y}{\sqrt{\varepsilon}};$$

when $1 \leqslant \frac{Y}{\sqrt{\varepsilon}} \leqslant \varepsilon^{-\frac{1}{4}}$,

$$u_e^0(X,Y) + u_b^0\left(X,\frac{Y}{\sqrt{\varepsilon}}\right) = u_e^0(X,Y) - u_e^0(X,0) + u_p^0\left(X,\frac{Y}{\sqrt{\varepsilon}}\right) \gtrsim -Y + 1 \gtrsim 1;$$

when $\frac{Y}{\sqrt{\varepsilon}} \gtrsim \varepsilon^{-\frac{1}{4}}$,

$$u_e^0(X,Y) + u_b^0\left(X, \frac{Y}{\sqrt{\varepsilon}}\right) \gtrsim 1.$$

So $U_s \sim \frac{Y}{\sqrt{\varepsilon}}$, when $Y \leqslant \sqrt{\varepsilon}$, and $U_s \sim 1$, when $Y \geqslant \sqrt{\varepsilon}$.

One can easily see that for $i, j \geqslant 0$,

$$\begin{aligned}
&\|\partial_X^j U_s\|_\infty \lesssim 1, \quad \|\partial_X^j V_s\|_\infty \lesssim 1,\\
&\sqrt{\varepsilon}^i \|Y^j \partial_Y^{i+j} U_s\|_\infty \lesssim \sqrt{\varepsilon}^i \|Y^j \partial_Y^{i+j} u_e^0\|_\infty + \sqrt{\varepsilon}^i \left\| \frac{y^j}{\sqrt{\varepsilon}^i} \partial_y^{i+j} u_b^0 \right\|_\infty + O(\sqrt{\varepsilon}) \lesssim 1,\\
&\sqrt{\varepsilon}^i \|Y^j \partial_Y^{i+j+1} V_s\|_\infty \lesssim \sqrt{\varepsilon}^i \|Y^j \partial_Y^{i+j+1} u_e^0\|_\infty + \sqrt{\varepsilon}^i \left\| \frac{y^j}{\sqrt{\varepsilon}^i} \partial_y^{i+j+1} u_b^0 \right\|_\infty + O(\sqrt{\varepsilon}) \lesssim 1.
\end{aligned} \tag{2.32}$$

In the shear flow Euler case,

$$U_{sX}(X,Y) \sim u_{bX}^0\left(X, \frac{Y}{\sqrt{\varepsilon}}\right), \quad V_s(X,Y) \sim \sqrt{\varepsilon}\left(v_b^0\left(X, \frac{Y}{\sqrt{\varepsilon}}\right) + v_e^1(X,Y)\right).$$

3 Estimates of the remainder

Now we begin to estimate the remainder. Let

$$U^\varepsilon = U_s + U, \quad V^\varepsilon = V_s + V. \tag{3.1}$$

Then

$$\begin{cases} U_s U_X + U_{sX} U + V_s U_Y + U_{sY} V - \varepsilon \Delta U + P_X = -\{R_1 + U U_X + V U_Y\},\\ U_s V_X + V_{sX} U + V_s V_Y + V_{sY} V - \varepsilon \Delta V + P_Y = -\{R_2 + U V_X + V V_Y\},\\ U_X + V_Y = 0. \end{cases} \tag{3.2}$$

We consider the linearized equations

$$\begin{cases} U_s U_X + U_{sX} U + V_s U_Y + U_{sY} V - \varepsilon \Delta U + P_X = F_1,\\ U_s V_X + V_{sX} U + V_s V_Y + V_{sY} V - \varepsilon \Delta V + P_Y = F_2,\\ U_X + V_Y = 0. \end{cases} \tag{3.3}$$

Our critical estimate is the following proposition.

Proposition 3.1. *Under the assumptions in Theorem* 1.1 *or Theorem* 1.2, *if*

$$[U,V] \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$$

satisfies (3.3), *then*

$$\|\sqrt{\varepsilon} U_X, \sqrt{\varepsilon} U_Y, \sqrt{\varepsilon} V_X, \sqrt{\varepsilon} V_Y, U, V\| \leqslant C(\|F_1\| + \|F_2\|). \tag{3.4}$$

Let Φ be the stream-function of U and V, i.e., $\Phi_X = -V$ and $\Phi_Y = U$. We can solve the stream-function in the following way:

$$\Phi(X,Y) = \int_0^Y U(X,Y')dY'. \tag{3.5}$$

According to the boundary conditions $[U,V]_\Omega = 0$ and $U_X + V_Y = 0$, we have

$$\begin{aligned} &\Phi|_{X=0} = \Phi|_{X=L} = \Phi|_{Y=0} = 0,\\ &\Phi_X|_{X=0} = \Phi_X|_{X=L} = \Phi_Y|_{Y=0} = 0. \end{aligned} \tag{3.6}$$

If $U, V \in L^2(\Omega)$, then $\Phi_Y, \Phi_X \in L^2(\Omega)$.

Since $[U, V]$ satisfies (3.3), we can deduce the equation of the stream-function

$$\begin{cases} U_s\Delta\Phi_X - \Phi_X\Delta U_s - \varepsilon\Delta^2\Phi + V_s\Delta\Phi_Y - \Phi_Y\Delta V_s = \partial_Y F_1 - \partial_X F_2, \\ \Phi\,|_{X=0} = \Phi\,|_{X=L} = \Phi\,|_{Y=0} = \Phi_X\,|_{X=0} = \Phi_X\,|_{X=L} = \Phi_Y\,|_{Y=0} = 0. \end{cases} \tag{3.7}$$

It is the fourth-order elliptic equation for Φ, and the boundary conditions are about Φ and its derivatives.

Let $G = \frac{\Phi}{U_s}$, and G and Φ satisfy

$$\begin{cases} \partial_{XX}[U_s^2 G_X] + \partial_{XY}[U_s^2 G_Y] - \varepsilon\Delta^2\Phi + R[\Phi] = \partial_Y F_1 - \partial_X F_2, \\ G\,|_{X=0} = G\,|_{X=L} = G\,|_{Y=0} = G_X\,|_{X=0} = G_X\,|_{X=L} = 0, \end{cases} \tag{3.8}$$

where $R[\Phi] = V_s\Delta\Phi_Y - U_{sX}\Delta\Phi - \Phi_Y\Delta V_s + \Phi\Delta U_{sX}$. We define two norms of G:

$$\begin{aligned} \|G\|_{\mathbb{X}}^2 &:= \|U_s G_X\|^2 + \|U_s G_Y\|^2 + \varepsilon\{\|\sqrt{U}_s G_{XX}\sqrt{\omega}\|^2 + 2\|\sqrt{U}_s G_{XY}\sqrt{\omega}\|^2 + \|\sqrt{U}_s G_{YY}\sqrt{\omega}\|^2\}, \\ \|G\|_{\mathbb{Y}}^2 &:= \varepsilon\{\|\sqrt{U}_s G_{XX}\|^2 + 2\|\sqrt{U}_s G_{XY}\|^2 + \|\sqrt{U}_s G_{YY}\|^2\}, \end{aligned} \tag{3.9}$$

where $\omega = L - X$.

We start with a Hardy-type inequality. Following the method of Guo and Iyer [14], we have the following lemma.

Lemma 3.2. *If $H \in W^{1,2}(0,\infty)$ and $0 < \xi \leqslant 1$, then*

$$\|H\|_{L_Y^2}^2 \leqslant C\xi\varepsilon\|\sqrt{U_s}H_Y\|_{L_Y^2}^2 + \frac{C}{\xi^2}\|U_s H\|_{L_Y^2}^2.$$

Proof. Let χ be a smooth cut-off function supported in $[0,2]$ and

$$\chi\,|_{[0,1]} = 1, \quad \int H^2 dY \lesssim \int H^2\chi^2\left(\frac{Y}{\xi\sqrt{\varepsilon}}\right)dY + \int H^2\left(1 - \chi\left(\frac{Y}{\xi\sqrt{\varepsilon}}\right)\right)^2 dY.$$

Recall the leading profile of U_s. $U_s \sim \frac{Y}{\sqrt{\varepsilon}}$, if $Y \leqslant \sqrt{\varepsilon}$, and $U_s \sim 1$, if $Y \geqslant \sqrt{\varepsilon}$. So when $\frac{Y}{\sqrt{\varepsilon}} \leqslant 1$,

$$1 - \chi\left(\frac{Y}{\xi\sqrt{\varepsilon}}\right) \lesssim \frac{Y}{\xi\sqrt{\varepsilon}} \lesssim \frac{U_s}{\xi},$$

and when $\frac{Y}{\sqrt{\varepsilon}} \geqslant 1$,

$$1 - \chi\left(\frac{Y}{\xi\sqrt{\varepsilon}}\right) \lesssim 1 \lesssim \frac{U_s}{\xi}.$$

We have

$$\int H^2\left(1 - \chi\left(\frac{Y}{\xi\sqrt{\varepsilon}}\right)\right)^2 dY \lesssim \frac{1}{\xi^2}\int U_s^2 H^2 dY$$

and

$$\begin{aligned} \int H^2\chi^2\left(\frac{Y}{\xi\sqrt{\varepsilon}}\right)dY &= -2\int YHH_Y\chi^2\left(\frac{Y}{\xi\sqrt{\varepsilon}}\right)dY - 2\int\frac{Y}{\xi\sqrt{\varepsilon}}H^2\chi'\left(\frac{Y}{\xi\sqrt{\varepsilon}}\right)\chi\left(\frac{Y}{\xi\sqrt{\varepsilon}}\right)dY \\ &\lesssim \int Y^2H_Y^2\chi^2\left(\frac{Y}{\xi\sqrt{\varepsilon}}\right)dY + \int\left(\frac{Y}{\xi\sqrt{\varepsilon}}\right)^2 H^2\left|\chi'\left(\frac{Y}{\xi\sqrt{\varepsilon}}\right)\right|\chi\left(\frac{Y}{\xi\sqrt{\varepsilon}}\right)dY \\ &\lesssim \xi\varepsilon\int U_s H_Y^2 dY + \frac{1}{\xi^2}\int U_s^2H^2 dY. \end{aligned}$$

The proof is completed. □

The next lemma is a basic elliptic estimate.

Lemma 3.3. *Let G be the solution of the equation* (3.8) *and $L > 0$. Then*

$$\|G\|_{\mathbb{Y}}^2 \lesssim \|G\|_{\mathbb{X}}^2 + |\langle\partial_Y F_1 - \partial_X F_2, G\rangle|. \tag{3.10}$$

Proof. Take the inner product of $(3.8)_1$ and $-G$.

The first term is

$$\langle \partial_{XX}[U_s^2 G_X], -G\rangle = \langle \partial_X[U_s^2 G_X], G_X\rangle = \langle U_s U_{sX} G_X, G_X\rangle = O(\|U_s G_X\|^2) = O(\|G\|_{\mathbb{X}}^2). \tag{3.11}$$

The second term is

$$\langle \partial_{XY}[U_s^2 G_Y], -G\rangle = \langle \partial_X[U_s^2 G_Y], G_Y\rangle = \langle U_s U_{sX} G_Y, G_Y\rangle = O(\|U_s G_Y\|^2) = O(\|G\|_{\mathbb{X}}^2). \tag{3.12}$$

The bi-Laplacian term is

$$\langle -\varepsilon\Delta^2\Phi, -G\omega\rangle = \varepsilon\langle \Phi_{XXXX} + 2\Phi_{XXYY} + \Phi_{YYYY}, G\omega\rangle.$$

It holds that

$$\begin{aligned}
\varepsilon\langle \Phi_{XXXX}, G\rangle &= -\varepsilon\langle \Phi_{XXX}, G_X\rangle \\
&= \varepsilon\langle \Phi_{XX}, G_{XX}\rangle \\
&= \varepsilon\langle U_s G_{XX} + 2U_{sX} G_X + U_{sXX} G, G_{XX}\rangle \\
&= \varepsilon\langle U_s G_{XX}, G_{XX}\rangle - \varepsilon\langle 2U_{sXX} G_X + U_{sXXX} G, G_X\rangle, \\
&= \varepsilon\langle U_s G_{XX}, G_{XX}\rangle + O(\varepsilon\|G_X\|^2).
\end{aligned}$$

Let $0 < \xi \leqslant 1$ be chosen later, and by Lemma 3.2,

$$\begin{aligned}
\|G_X\|^2 &\lesssim \frac{1}{\xi^2}\|U_s G_X\|^2 + \xi\varepsilon\|\sqrt{U_s} G_{XY}\|^2 \\
&\lesssim \frac{1}{\xi^2}\|G\|_{\mathbb{X}}^2 + \xi\|G\|_{\mathbb{Y}}^2.
\end{aligned}$$

Then

$$\varepsilon\langle \Phi_{XXXX}, G\rangle = \varepsilon\langle U_s G_{XX}, G_{XX}\rangle + O\left(\frac{\varepsilon}{\xi^2}\|G\|_{\mathbb{X}}^2 + \xi\varepsilon\|G\|_{\mathbb{Y}}^2\right). \tag{3.13}$$

Next,

$$\begin{aligned}
\langle -2\varepsilon\Phi_{XXYY}, -G\rangle &= -\langle 2\varepsilon\Phi_{XXY}, G_Y\rangle \\
&= \langle 2\varepsilon\Phi_{XY}, G_{XY}\rangle \\
&= 2\varepsilon\langle U_s G_{XY} + U_{sX} G_Y + U_{sY} G_X + U_{sXY} G, G_{XY}\rangle, \\
&= 2\varepsilon\langle U_s G_{XY}, G_{XY}\rangle - \varepsilon\langle U_{sXX} G_Y, G_Y\rangle - \varepsilon\langle U_{sYY} G_X, G_X\rangle \\
&\quad - 2\varepsilon\langle U_{sXY} G_Y, G_X\rangle - 2\varepsilon\langle U_{sXYY} G, G_X\rangle \\
&= 2\varepsilon\langle U_s G_{XY}, G_{XY}\rangle + O(\|G_X\|^2 + \|G_Y\|^2),
\end{aligned}$$

and by Lemma 3.2,

$$\begin{aligned}
\|G_Y\|^2 &\lesssim \frac{1}{\xi^2}\|U_s G_Y\|^2 + \xi\varepsilon\|\sqrt{U_s} G_{YY}\|^2 \\
&\lesssim \frac{1}{\xi^2}\|G\|_{\mathbb{X}}^2 + \xi\|G\|_{\mathbb{Y}}^2.
\end{aligned}$$

Then

$$\langle -2\varepsilon\Phi_{XXYY}, -G\rangle = 2\varepsilon\langle U_s G_{XY}, G_{XY}\rangle + O\left(\frac{1}{\xi^2}\|G\|_{\mathbb{X}}^2 + \xi\|G\|_{\mathbb{Y}}^2\right) \tag{3.14}$$

and

$$\langle -\varepsilon\Phi_{YYYY}, -G\rangle = -\varepsilon\langle \Phi_{YYY}, G_Y\rangle$$

$$
\begin{aligned}
&= \varepsilon\langle \Phi_{YY}, G_Y\rangle\,|_{Y=0} + \varepsilon\langle \Phi_{YY}, G_{YY}\rangle \\
&= \varepsilon\langle U_sG_{YY} + 2U_{sY}G_Y + U_{sYY}G, G_Y\rangle\,|_{Y=0} \\
&\quad + \varepsilon\langle U_sG_{YY} + 2U_{sY}G_Y + U_{sYY}G, G_{YY}\rangle \\
&= 2\varepsilon\langle U_{sY}G_Y, G_Y\rangle\,|_{Y=0} + \varepsilon\langle U_sG_{YY}, G_{YY}\rangle + \varepsilon\langle 2U_{sY}G_Y + U_{sYY}G, G_{YY}\rangle \\
&= \varepsilon\langle U_{sY}G_Y, G_Y\rangle\,|_{Y=0} + \varepsilon\langle U_sG_{YY}, G_{YY}\rangle - \varepsilon\langle 2U_{sYY}G_Y + U_{sYYY}G, G_Y\rangle \\
&= \varepsilon\langle U_{sY}G_Y, G_Y\rangle\,|_{Y=0} + \varepsilon\langle U_sG_{YY}, G_{YY}\rangle \\
&\quad + O\bigg(\|G_Y\|^2 + \varepsilon\|YU_{sYYY}\|_\infty\bigg\|\frac{G}{Y}\bigg\|\|G_Y\|\bigg) \\
&= \varepsilon\langle U_{sY}G_Y, G_Y\rangle\,|_{Y=0} + \varepsilon\langle U_sG_{YY}, G_{YY}\rangle + O(\|G_Y\|^2) \\
&= \varepsilon\langle U_{sY}G_Y, G_Y\rangle\,|_{Y=0} + \varepsilon\langle U_sG_{YY}, G_{YY}\rangle + O\bigg(\frac{1}{\xi^2}\|G\|_{\mathbb{X}}^2 + \xi\|G\|_{\mathbb{Y}}^2\bigg).
\end{aligned} \tag{3.15}
$$

Since $\varepsilon\langle U_{sY}G_Y, G_Y\rangle\,|_{Y=0}$ is positive, $\varepsilon\langle U_sG_{YY}, G_{YY}\rangle$ is what we want. According to the fact that

$$
\begin{aligned}
&\|V_sU_{sY}\|_\infty \lesssim \|v_e^0U_{sY}\|_\infty + 1 \lesssim \bigg\|\frac{v_e^0}{Y}\bigg\|_\infty \|YU_{sY}\|_\infty + 1 \lesssim 1, \\
&\|\Phi_X\| = \|U_sG_X + U_{sX}G\| \lesssim \|G_X\|, \\
&\|\Phi_Y\| = \|U_sG_Y + U_{sY}G\| \lesssim \|G_Y\| + \|U_{sY}Y\|_\infty\bigg\|\frac{G}{Y}\bigg\| \lesssim \|G_Y\|,
\end{aligned}
$$

the $R[\Phi]$ term can be estimated as

$$
\begin{aligned}
\langle V_s\Phi_{XXY}, -G\rangle &= \langle V_s\Phi_{XY}, G_X\rangle + \langle V_{sX}\Phi_{XY}, G\rangle \\
&= \langle V_s(U_sG_{XY} + U_{sX}G_Y + U_{sY}G_X + U_{sXY}G), G_X\rangle \\
&\quad - \langle V_{sX}\Phi_X, G_Y\rangle - \langle V_{sXY}\Phi_X, G\rangle \\
&= -\frac{1}{2}\langle (V_sU_s)_YG_X, G_X\rangle + \langle V_s(U_{sX}G_Y + U_{sY}G_X + U_{sXY}G), G_X\rangle \\
&\quad - \langle V_{sX}\Phi_X, G_Y\rangle - \langle V_{sXY}\Phi_X, G\rangle \\
&= O(\|G_X\|^2 + \|G_Y\|^2) = O\bigg(\frac{1}{\xi^2}\|G\|_{\mathbb{X}}^2 + \xi\|G\|_{\mathbb{Y}}^2\bigg),
\end{aligned} \tag{3.16}
$$

$$
\begin{aligned}
\langle V_s\Phi_{YYY}, -G\rangle &= \langle V_s\Phi_{YY}, G_Y\rangle + \langle V_{sY}\Phi_{YY}, G\rangle \\
&= \langle V_s(U_sG_{YY} + 2U_{sY}G_Y + U_{sYY}G), G_Y\rangle \\
&\quad - \langle V_{sY}\Phi_Y, G_Y\rangle - \langle V_{sYY}\Phi_Y, G\rangle \\
&= -\frac{1}{2}\langle (V_sU_s)_YG_Y, G_Y\rangle + \langle V_s(2U_{sY}G_Y + U_{sYY}G), G_Y\rangle \\
&\quad - \langle V_{sY}\Phi_Y, G_Y\omega\rangle - \langle V_{sYY}\Phi_Y, G\rangle \\
&= O(\|G_Y\|^2) = O\bigg(\frac{1}{\xi^2}\|G\|_{\mathbb{X}}^2 + \xi\|G\|_{\mathbb{Y}}^2\bigg),
\end{aligned} \tag{3.17}
$$

$$
\begin{aligned}
\langle -U_{sX}\Delta\Phi, -G\rangle &= -\langle U_{sX}\Phi_X, G_X\rangle - \langle U_{sXX}\Phi_X, G\rangle - \langle U_{sX}\Phi_Y, G_Y\rangle - \langle U_{sXY}\Phi_Y, G\rangle \\
&= O(\|G_X\|^2 + \|G_Y\|^2) = O\bigg(\frac{1}{\xi^2}\|G\|_{\mathbb{X}}^2 + \xi\|G\|_{\mathbb{Y}}^2\bigg),
\end{aligned} \tag{3.18}
$$

$$
\langle -\Phi_Y\Delta V_s + \Phi\Delta U_{sX}, -G\rangle = O(\|G_X\|^2 + \|G_Y\|^2) = O\bigg(\frac{1}{\xi^2}\|G\|_{\mathbb{X}}^2 + \xi\|G\|_{\mathbb{Y}}^2\bigg). \tag{3.19}
$$

Collecting (3.11)–(3.19) and choosing ξ small enough, we can obtain the inequality (3.10). □

The following two lemmas show the non-trivial estimates of G.

Lemma 3.4. *Let G be the solution of the equation* (3.8) *and $0 < L, \|v_e^0\|_\infty \ll 1$. Then*

$$
\|G\|_{\mathbb{X}}^2 \lesssim (\sqrt{\varepsilon} + L + \|v_e^0\|_\infty)\|G\|_{\mathbb{Y}}^2 + |\langle \partial_YF_1 - \partial_XF_2, G\omega\rangle|. \tag{3.20}
$$

Proof. Take the inner product of $(3.8)_1$ with $-G\omega$, where $\omega = L - x$.

Because $|U_{sX}| \lesssim U_s$ and $\omega \leqslant L$, the first term is

$$\begin{aligned}\langle \partial_{XX}[U_s^2 G_X], -G\omega\rangle &= -\langle \partial_X[U_s^2 G_X], G\rangle + \langle \partial_X[U_s^2 G_X], G_X\omega\rangle \\ &= \frac{3}{2}\langle U_s^2 G_X, G_X\rangle + \langle U_s U_{sX} G_X, G_X\omega\rangle \\ &= \frac{3}{2}\langle U_s^2 G_X, G_X\rangle + O(L\|U_s G_X\|^2). \end{aligned} \tag{3.21}$$

The second term is

$$\begin{aligned}\langle \partial_{XY}[U_s^2 G_Y], -G\omega\rangle &= \langle \partial_X[U_s^2 G_Y], G_Y\omega\rangle \\ &= \langle U_s^2 G_{XY}, G_Y\omega\rangle + \langle 2U_s U_{sX} G_Y, G_Y\omega\rangle \\ &= \frac{1}{2}\langle U_s^2 G_Y, G_Y\rangle + \langle U_s U_{sX} G_Y, G_Y\omega\rangle \\ &= \frac{1}{2}\langle U_s^2 G_Y, G_Y\rangle + O(L\|U_s G_Y\|^2). \end{aligned} \tag{3.22}$$

The bi-Laplacian term is

$$\langle -\varepsilon\Delta^2\Phi, -G\omega\rangle = \varepsilon\langle \Phi_{XXXX} + 2\Phi_{XXYY} + \Phi_{YYYY}, G\omega\rangle.$$

It holds that

$$\begin{aligned}\varepsilon\langle \Phi_{XXXX}, G\omega\rangle &= -\varepsilon\langle \Phi_{XXX}, G_X\omega\rangle + \varepsilon\langle \Phi_{XXX}, G\rangle \\ &= \varepsilon\langle \Phi_{XX}, G_{XX}\omega\rangle - 2\varepsilon\langle \Phi_{XX}, G_X\rangle \\ &= \varepsilon\langle U_s G_{XX} + 2U_{sX} G_X + U_{sXX} G, G_{XX}\omega\rangle \\ &\quad - 2\varepsilon\langle U_s G_{XX} + 2U_{sX} G_X + U_{sXX} G, G_X\rangle \\ &= \varepsilon\langle U_s G_{XX}, G_{XX}\omega\rangle - \varepsilon\langle 2U_{sX} G_X + U_{sXX} G, G_X\rangle \\ &\quad - \varepsilon\langle 2U_{sXX} G_X + U_{sXXX} G, G_X\omega\rangle \\ &= \varepsilon\langle U_s G_{XX}, G_{XX}\omega\rangle + O(\varepsilon\|G_X\|) \\ &= \varepsilon\langle U_s G_{XX}, G_{XX}\omega\rangle + O\left(\frac{\varepsilon}{\xi^2}\|G\|_{\mathbb{X}}^2 + \xi\varepsilon\|G\|_{\mathbb{Y}}^2\right). \end{aligned} \tag{3.23}$$

Next,

$$\begin{aligned}2\varepsilon\langle \Phi_{XXYY}, G\omega\rangle &= -2\varepsilon\langle \Phi_{XXY}, G_Y\omega\rangle \\ &= 2\varepsilon\langle \Phi_{XY}, G_{XY}\omega\rangle - 2\varepsilon\langle \Phi_{XY}, G_Y\rangle \\ &= 2\varepsilon\langle U_s G_{XY} + U_{sX} G_Y + U_{sY} G_X + U_{sXY} G, G_{XY}\omega\rangle \\ &\quad - 2\varepsilon\langle U_s G_{XY} + U_{sX} G_Y + U_{sY} G_X + U_{sXY} G, G_Y\rangle. \end{aligned}$$

$2\varepsilon\langle U_s G_{XY}, G_{XY}\omega\rangle$ is a positive term,

$$\begin{aligned}&2\varepsilon\langle U_{sX} G_Y + U_{sY} G_X + U_{sXY} G, G_{XY}\omega\rangle \\ &\quad = -\varepsilon\langle U_{sXX} G_Y, G_Y\omega\rangle + \varepsilon\langle U_{sX} G_Y, G_Y\rangle - \varepsilon\langle U_{sYY} G_X, G_X\omega\rangle \\ &\qquad - 2\varepsilon\langle U_{sXY} G_Y, G_X\omega\rangle - 2\varepsilon\langle U_{sXYY} G, G_Y\omega\rangle \\ &\quad = O\left(\varepsilon\|G_Y\|^2 + \varepsilon\|U_{sYY}\|_\infty\|G_X\sqrt{\omega}\|^2 + \varepsilon\|U_{sXY}\|_\infty\|G_X\|\|G_Y\| + \varepsilon\|YU_{sXYY}\|_\infty\left\|\frac{G}{Y}\right\|\|G\|\right) \\ &\quad = O(\|G_X\sqrt{\omega}\|^2 + \sqrt{\varepsilon}\|G_X\|\|G_Y\| + \sqrt{\varepsilon}\|G_Y\|^2),\end{aligned}$$

and by Lemma 3.2,

$$\|G_Y\|^2 \lesssim \xi\varepsilon\|\sqrt{U_s}G_{YY}\|^2 + \frac{1}{\xi^2}\|U_sG_Y\|^2 \lesssim \frac{1}{\xi^2}\|G\|_{\mathbb{X}}^2 + \xi\|G\|_{\mathbb{Y}}^2,$$
$$\|G_X\|^2 \lesssim \xi\varepsilon\|\sqrt{U_s}G_{XY}\|^2 + \frac{1}{\xi^2}\|U_sG_X\|^2 \lesssim \frac{1}{\xi^2}\|G\|_{\mathbb{X}}^2 + \xi\|G\|_{\mathbb{Y}}^2,$$
$$\|G_X\sqrt{\omega}\|^2 \lesssim \xi\varepsilon\|\sqrt{U_s}G_{XY}\sqrt{\omega}\|^2 + \frac{1}{\xi^2}\|U_sG_X\sqrt{\omega}\|^2 \lesssim \left(\xi + \frac{L}{\xi^2}\right)\|G\|_{\mathbb{X}}^2.$$

So we have

$$2\varepsilon\langle U_{sX}G_Y + U_{sY}G_X + U_{sXY}G, G_{XY}\omega\rangle = O\left(\left(\frac{\sqrt{\varepsilon}+L}{\xi^2} + \xi\right)\|G\|_{\mathbb{X}}^2 + \sqrt{\varepsilon}\xi\|G\|_{\mathbb{Y}}^2\right)$$

and

$$\begin{aligned}
&-2\varepsilon\langle U_sG_{XY} + U_{sX}G_Y + U_{sY}G_X + U_{sXY}G, G_Y\rangle\\
&\quad= -\varepsilon\langle U_{sX}G_Y, G_Y\rangle - 2\varepsilon\langle U_{sY}G_Y, G_X\rangle - 2\varepsilon\langle U_{sXY}G, G_Y\rangle\\
&\quad= O\left(\varepsilon\|G_Y\|^2 + \varepsilon\|U_{sY}\|_\infty\|G_X\|\|G_Y\| + \varepsilon\|U_{sXY}Y\|_\infty\left\|\frac{G}{Y}\right\|\|G_Y\|\right)\\
&\quad= O(\varepsilon\|G_Y\|^2 + \sqrt{\varepsilon}\|G_X\|\|G_Y\|)\\
&\quad= O\left(\frac{\sqrt{\varepsilon}}{\xi^2}\|G\|_{\mathbb{X}}^2 + \sqrt{\varepsilon}\xi\|G\|_{\mathbb{Y}}^2\right).
\end{aligned}$$

Therefore,

$$2\varepsilon\langle\Phi_{XXYY}, G\omega\rangle = 2\varepsilon\langle U_sG_{XY}, G_{XY}\omega\rangle + O\left(\left(\frac{\sqrt{\varepsilon}+L}{\xi^2} + \xi\right)\|G\|_{\mathbb{X}}^2 + \sqrt{\varepsilon}\xi\|G\|_{\mathbb{Y}}^2\right). \tag{3.24}$$

Integrating by parts, we have

$$\begin{aligned}
\varepsilon\langle\Phi_{YYYY}, G\omega\rangle &= -\varepsilon\langle\Phi_{YYY}, G_Y\omega\rangle\\
&= \varepsilon\langle\Phi_{YY}, G_Y\omega\rangle\,|_{Y=0} + \varepsilon\langle\Phi_{YY}, G_{YY}\omega\rangle\\
&= \varepsilon\langle U_sG_{YY} + 2U_{sY}G_Y + U_{sYY}G, G_Y\omega\rangle\,|_{Y=0}\\
&\quad + \varepsilon\langle U_sG_{YY} + 2U_{sY}G_Y + U_{sYY}G, G_{YY}\omega\rangle\\
&= 2\varepsilon\langle U_{sY}G_Y, G_Y\omega\rangle\,|_{Y=0} + \varepsilon\langle U_sG_{YY}, G_{YY}\omega\rangle + \varepsilon\langle 2U_{sY}G_Y + U_{sYY}G, G_{YY}\omega\rangle\\
&= \varepsilon\langle U_{sY}G_Y, G_Y\omega\rangle\,|_{Y=0} + \varepsilon\langle U_sG_{YY}, G_{YY}\omega\rangle - \varepsilon\langle 2U_{sYY}G_Y + U_{sYYY}G, G_Y\omega\rangle.
\end{aligned}$$

Because $U_{sY}\,|_{Y=0} > 0$, the first two terms above are positive and

$$\begin{aligned}
-\varepsilon\langle 2U_{sYY}G_Y + U_{sYYY}G, G_Y\omega\rangle &= O\left(\varepsilon\|U_{sYY}\|_\infty\|G_Y\sqrt{\omega}\|^2 + \varepsilon\|U_{sYYY}Y\|_\infty\left\|\frac{G}{Y}\sqrt{\omega}\right\|\|G_Y\sqrt{\omega}\|\right)\\
&= O\left(\left(\xi + \frac{L}{\xi^2}\right)\|G\|_{\mathbb{X}}^2\right).
\end{aligned}$$

Then we obtain

$$\varepsilon\langle\Phi_{YYYY}, G\omega\rangle = \varepsilon\langle U_{sY}G_Y, G_Y\omega\rangle\,|_{Y=0} + \varepsilon\langle U_sG_{YY}, G_{YY}\omega\rangle + O\left(\left(\xi + \frac{L}{\xi^2}\right)\|G\|_{\mathbb{X}}^2\right). \tag{3.25}$$

Finally, we deal with the $R[\Phi]$ term. Since

$$\begin{aligned}
\langle V_s\Phi_{XXY}, -G\omega\rangle &= \langle V_s\phi_{XY}, G_X\omega\rangle + \langle V_{sX}\Phi_{XY}, G\omega\rangle - \langle V_s\Phi_{XY}, G\rangle\\
&= \langle V_s(U_sG_{XY} + U_{sX}G_Y + U_{sY}G_X + U_{sXY}G), G_X\omega\rangle\\
&\quad - \langle V_{sX}\Phi_X, G_Y\omega\rangle - \langle V_{sXY}\Phi_X, G\omega\rangle + \langle V_s\Phi_X, G_Y\rangle + \langle V_{sY}\Phi_X, G\rangle\\
&= -\frac{1}{2}\langle (V_sU_s)_YG_X, G_X\omega\rangle + \langle V_s(U_{sX}G_Y + U_{sY}G_X + U_{sXY}G), G_X\omega\rangle\\
&\quad - \langle V_{sX}\Phi_X, G_Y\omega\rangle - \langle V_{sXY}\Phi_X, G\omega\rangle + \langle V_s\Phi_X, G_Y\rangle + \langle V_{sY}\Phi_X, G\rangle,
\end{aligned}$$

noticing that

$$\|V_sU_{sY}\|_\infty \lesssim \frac{\|v_e^0u_{by}^0\|_\infty}{\sqrt{\varepsilon}}+1 \lesssim \left\|\frac{v_e^0}{Y}\right\|_\infty \|yu_{by}^0\|_\infty+1\lesssim 1,$$
$$\|V_s\|_\infty \lesssim \|v_e^0\|_\infty+\sqrt{\varepsilon},$$

we obtain

$$\begin{aligned}\langle V_s\Phi_{XXY},-G\omega\rangle &= O(\|G_X\sqrt{\omega}\|^2+\|G_Y\sqrt{\omega}\|^2+\sqrt{\varepsilon}\|G_X\|\|G_Y\|+\|v_e^0\|_\infty\|G_X\|\|G_Y\|)\\ &= O\bigg(\bigg(\frac{\sqrt{\varepsilon}+L+\|v_e^0\|_\infty}{\xi^2}+\xi\bigg)\|G\|_{\mathbb{X}}^2+\xi(\sqrt{\varepsilon}+\|v_e^0\|_\infty)\|G\|_{\mathbb{Y}}^2\bigg). \end{aligned}\tag{3.26}$$

Similarly,

$$\begin{aligned}\langle V_s\Phi_{YYY},-G\omega\rangle &= \langle V_s\Phi_{YY},G_Y\omega\rangle+\langle V_{sY}\Phi_{YY},G\omega\rangle\\ &= \langle V_s(U_sG_{YY}+2U_{sY}G_Y+U_{sYY}G),G_Y\omega\rangle\\ &\quad -\langle V_{sY}\Phi_Y,G_Y\omega\rangle-\langle V_{sYY}\Phi_Y,G\omega\rangle\\ &= -\frac{1}{2}\langle (V_sU_s)_YG_Y,G_Y\omega\rangle+\langle V_s(2U_{sY}G_Y+U_{sYY}G),G_Y\omega\rangle\\ &\quad -\langle V_{sY}\Phi_Y,G_Y\omega\rangle-\langle V_{sYY}\Phi_Y,G\omega\rangle\\ &= O(\|G_Y\sqrt{\omega}\|^2+\|\Phi_Y\sqrt{\omega}\|^2)\\ &= O\bigg(\bigg(\xi+\frac{L}{\xi^2}\bigg)\|G\|_{\mathbb{X}}^2\bigg), \end{aligned}\tag{3.27}$$

$$\begin{aligned}\langle -U_{sX}\Delta\Phi,-G\omega\rangle &= -\langle U_{sX}\Phi_X,G_X\omega\rangle-\langle U_{sXX}\Phi_X,G\omega\rangle+\langle U_{sX}\Phi_X,G\rangle\\ &\quad -\langle U_{sX}\Phi_Y,G_Y\omega\rangle-\langle U_{sXY}\Phi_Y,G\omega\rangle\\ &= O(\|G_X\sqrt{\omega}\|^2+\|G_Y\sqrt{\omega}\|^2)\\ &= O\bigg(\bigg(\xi+\frac{L}{\xi^2}\bigg)\|G\|_{\mathbb{X}}^2\bigg), \end{aligned}\tag{3.28}$$

$$\langle -\Phi_Y\Delta V_s+\Phi\Delta U_{sX},-G\omega\rangle = O(\|G_X\sqrt{\omega}\|^2+\|G_Y\sqrt{\omega}\|^2)=O\bigg(\bigg(\xi+\frac{L}{\xi^2}\bigg)\|G\|_{\mathbb{X}}^2\bigg).\tag{3.29}$$

Collecting (3.21)–(3.29), we get

$$\|G\|_{\mathbb{X}}^2 \lesssim \bigg(\frac{\sqrt{\varepsilon}+L+\|v_e^0\|_\infty}{\xi^2}+\xi\bigg)\|G\|_{\mathbb{X}}^2+\xi(\sqrt{\varepsilon}+\|v_e^0\|_\infty)\|G\|_{\mathbb{Y}}^2+|\langle \partial_YF_1-\partial_XF_2,G\omega\rangle|.$$

Finally, we choose

$$\xi=(\sqrt{\varepsilon}+L+\|v_e^0\|_\infty)^{\frac{1}{4}}$$

to be a small constant, and then we finish the proof. □

Remark 3.5. The weight $L-X$ is inspired by Iyer [17]. He used the weight $1-x$ to overcome the difficulties of the non-shear Euler flow. However, in this paper, the weight $L-x$ leads to an estimate similar to the positive estimate in [15, 17].

Lemma 3.6. *Let G be the solution of the equation* (3.8) *with $[u_e^0(Y),0]$ satisfying* (1.16)*, and u_p^0 satisfy* (1.17)*. Then*

$$\|G\|_{\mathbb{X}}^2 \lesssim \sqrt{\varepsilon}\|G\|_{\mathbb{Y}}^2+\langle \partial_YF_1-\partial_XF_2,G\omega\rangle.\tag{3.30}$$

Proof. Take the inner product of $(3.8)_1$ with $-G\omega$, where $\omega=L-x$.

The first term is

$$\langle \partial_{XX}[U_s^2G_X],-G\omega\rangle = -\langle \partial_X[U_s^2G_X],G\rangle+\langle \partial_X[U_s^2G_X],G_X\omega\rangle$$

$$= \frac{3}{2}\langle U_s^2 G_X, G_X\rangle + \langle U_s U_{sX} G_X, G_X\omega\rangle. \tag{3.31}$$

The second term is

$$\begin{aligned}\langle \partial_{XY}[U_s^2 G_Y], -G\omega\rangle &= \langle \partial_X[U_s^2 G_Y], G_Y\omega\rangle \\ &= \langle U_s^2 G_{XY}, G_Y\omega\rangle + \langle 2U_s U_{sX} G_Y, G_Y\omega\rangle \\ &= \frac{1}{2}\langle U_s^2 G_Y, G_Y\rangle + \langle U_s U_{sX} G_Y, G_Y\omega\rangle. \end{aligned} \tag{3.32}$$

The bi-Laplacian term is

$$\langle -\varepsilon\Delta^2\Phi, -G\omega\rangle = \varepsilon\langle \Phi_{XXXX} + 2\Phi_{XXYY} + \Phi_{YYYY}, G\omega\rangle.$$

It holds that

$$\begin{aligned}\varepsilon\langle \Phi_{XXXX}, G\omega\rangle &= -\varepsilon\langle \Phi_{XXX}, G_X\omega\rangle + \varepsilon\langle \Phi_{XXX}, G\rangle \\ &= \varepsilon\langle \Phi_{XX}, G_{XX}\omega\rangle - 2\varepsilon\langle \Phi_{XX}, G_X\rangle \\ &= \varepsilon\langle U_s G_{XX} + 2U_{sX} G_X + U_{sXX} G, G_{XX}\omega\rangle \\ &\quad - 2\varepsilon\langle U_s G_{XX} + 2U_{sX} G_X + U_{sXX} G, G_X\rangle \\ &= \varepsilon\langle U_s G_{XX}, G_{XX}\omega\rangle - \varepsilon\langle 2U_{sX} G_X + U_{sXX} G, G_X\rangle \\ &\quad - \varepsilon\langle 2U_{sXX} G_X + U_{sXXX} G, G_X\omega\rangle \\ &= \varepsilon\langle U_s G_{XX}, G_{XX}\omega\rangle + O(\varepsilon\|G_X\|^2) \\ &= \varepsilon\langle U_s G_{XX}, G_{XX}\omega\rangle + O(\varepsilon\|G\|_{\mathbb{X}}^2 + \varepsilon\|G\|_{\mathbb{Y}}^2). \end{aligned} \tag{3.33}$$

Next,

$$\begin{aligned}2\varepsilon\langle \Phi_{XXYY}, G\omega\rangle &= -2\varepsilon\langle \Phi_{XXY}, G_Y\omega\rangle \\ &= 2\varepsilon\langle \Phi_{XY}, G_{XY}\omega\rangle - 2\varepsilon\langle \Phi_{XY}, G_Y\rangle \\ &= 2\varepsilon\langle U_s G_{XY} + U_{sX} G_Y + U_{sY} G_X + U_{sXY} G, G_{XY}\omega\rangle \\ &\quad - 2\varepsilon\langle U_s G_{XY} + U_{sX} G_Y + U_{sY} G_X + U_{sXY} G, G_Y\rangle \\ &= 2\varepsilon\langle U_s G_{XY}, G_{XY}\omega\rangle - \varepsilon\langle U_{sXX} G_Y, G_Y\omega\rangle + \varepsilon\langle U_{sX} G_Y, G_Y\rangle \\ &\quad - \varepsilon\langle U_{sYY} G_X, G_X\omega\rangle - 2\varepsilon\langle U_{sXY} G_Y, G_X\omega\rangle - 2\varepsilon\langle U_{sXYY} G, G_Y\omega\rangle \\ &\quad - \varepsilon\langle U_{sX} G_Y, G_Y\rangle - 2\varepsilon\langle U_{sY} G_Y, G_X\rangle - 2\varepsilon\langle U_{sXY} G, G_Y\rangle \\ &= 2\varepsilon\langle U_s G_{XY}, G_{XY}\omega\rangle - \varepsilon\langle U_{sYY} G_X, G_X\omega\rangle + O(\sqrt{\varepsilon}\|G_X\|^2 + \sqrt{\varepsilon}\|G_Y\|^2) \\ &= 2\varepsilon\langle U_s G_{XY}, G_{XY}\omega\rangle - \varepsilon\langle U_{sYY} G_X, G_X\omega\rangle + O(\sqrt{\varepsilon}\|G\|_{\mathbb{X}}^2 + \sqrt{\varepsilon}\|G\|_{\mathbb{Y}}^2) \end{aligned} \tag{3.34}$$

and

$$\begin{aligned}\varepsilon\langle \Phi_{YYYY}, G\omega\rangle &= -\varepsilon\langle \Phi_{YYY}, G_Y\omega\rangle \\ &= \varepsilon\langle \Phi_{YY}, G_Y\omega\rangle\,|_{Y=0} + \varepsilon\langle \Phi_{YY}, G_{YY}\omega\rangle \\ &= \varepsilon\langle U_s G_{YY} + 2U_{sY} G_Y + U_{sYY} G, G_Y\omega\rangle\,|_{Y=0} \\ &\quad + \varepsilon\langle U_s G_{YY} + 2U_{sY} G_Y + U_{sYY} G, G_{YY}\omega\rangle \\ &= 2\varepsilon\langle U_{sY} G_Y, G_Y\omega\rangle\,|_{Y=0} + \varepsilon\langle U_s G_{YY}, G_{YY}\omega\rangle + \varepsilon\langle 2U_{sY} G_Y + U_{sYY} G, G_{YY}\omega\rangle \\ &= \varepsilon\langle U_{sY} G_Y, G_Y\omega\rangle\,|_{Y=0} + \varepsilon\langle U_s G_{YY}, G_{YY}\omega\rangle - \varepsilon\langle 2U_{sYY} G_Y + U_{sYYY} G, G_Y\omega\rangle \\ &= \varepsilon\langle U_{sY} G_Y, G_Y\omega\rangle\,|_{Y=0} + \varepsilon\langle U_s G_{YY}, G_{YY}\omega\rangle \\ &\quad - 2\varepsilon\langle U_{sYY} G_Y, G_Y\omega\rangle + \frac{\varepsilon}{2}\langle U_{sYYYY} G, G\omega\rangle, \end{aligned} \tag{3.35}$$

because $U_{sY}\,|_{Y=0} > 0$, and the first term above is positive. Now, we begin to deal with the $R[\Phi]$ term, i.e.,

$$\langle V_s \Phi_{XXY}, -G\omega\rangle = \langle V_s \Phi_{XY}, G_X\omega\rangle + \langle V_{sX} \Phi_{XY}, G\omega\rangle - \langle V_s \Phi_{XY}, G\rangle$$

$$
\begin{aligned}
&= \langle V_s(U_sG_{XY} + U_{sX}G_Y + U_{sY}G_X + U_{sXY}G), G_X\omega\rangle \\
&\quad - \langle V_{sX}\Phi_X, G_Y\omega\rangle - \langle V_{sXY}\Phi_X, G\omega\rangle + \langle V_s\Phi_X, G_Y\rangle + \langle V_{sY}\Phi_X, G\rangle \\
&= \langle V_sU_sG_{XY}, G_X\omega\rangle + \langle V_sU_{sY}G_X, G_X\omega\rangle + O(\sqrt{\varepsilon}\|G_X\|^2 + \sqrt{\varepsilon}\|G_Y\|^2) \\
&= \frac{1}{2}\langle (V_sU_{sY} - V_{sY}U_s)G_X, G_X\omega\rangle + O(\sqrt{\varepsilon}\|G\|_{\mathbb{X}}^2 + \sqrt{\varepsilon}\|G\|_{\mathbb{Y}}^2),
\end{aligned} \tag{3.36}
$$

$$
\begin{aligned}
\langle V_s\Phi_{YYY}, -G\omega\rangle &= \langle V_s\Phi_{YY}, G_Y\omega\rangle + \langle V_{sY}\Phi_{YY}, G\omega\rangle \\
&= \langle V_s(U_sG_{YY} + 2U_{sY}G_Y + U_{sYY}G), G_Y\omega\rangle \\
&\quad - \langle V_{sY}\Phi_Y, G_Y\omega\rangle - \langle V_{sYY}\Phi_Y, G\omega\rangle \\
&= -\frac{1}{2}\langle (V_sU_s)_YG_Y, G_Y\omega\rangle + \langle 2V_sU_{sY}G_Y, G_Y\omega\rangle - \frac{1}{2}\langle (V_sU_{sYY})_YG, G\omega\rangle \\
&\quad - \langle V_{sY}U_sG_Y, G_Y\omega\rangle - \langle V_{sY}U_{sY}G, G_Y\omega\rangle \\
&\quad - \langle V_{sYY}U_sG_Y, G\omega\rangle - \langle V_{sYY}U_{sY}G, G\omega\rangle \\
&= -\frac{1}{2}\langle (V_sU_s)_YG_Y, G_Y\omega\rangle + \langle 2V_sU_{sY}G_Y, G_Y\omega\rangle - \frac{1}{2}\langle (V_sU_{sYY})_YG, G\omega\rangle \\
&\quad - \langle V_{sY}U_sG_Y, G_Y\omega\rangle + \frac{1}{2}\langle (V_{sY}U_{sY})_YG, G\omega\rangle \\
&\quad + \frac{1}{2}\langle (V_{sYY}U_s)_YG, G\omega\rangle - \langle V_{sYY}U_{sY}G, G\omega\rangle \\
&= \frac{3}{2}\langle (V_sU_{sY} - V_{sY}U_s)G_Y, G_Y\omega\rangle + \frac{1}{2}\langle (-V_sU_{sYYY} + V_{sYYY}U_s)G, G\omega\rangle,
\end{aligned} \tag{3.37}
$$

$$
\begin{aligned}
\langle -U_{sX}\Delta\Phi, -G\omega\rangle &= -\langle U_{sX}\Phi_X, G_X\omega\rangle - \langle U_{sXX}\Phi_X, G\omega\rangle + \langle U_{sX}\Phi_X, G\rangle \\
&\quad - \langle U_{sX}\Phi_Y, G_Y\omega\rangle - \langle U_{sXY}\Phi_Y, G\omega\rangle \\
&= -\langle U_{sX}\Phi_X, G_X\omega\rangle - \langle U_{sX}\Phi_Y, G_Y\omega\rangle \\
&\quad - \langle U_{sXY}\Phi_Y, G\omega\rangle + O(\sqrt{\varepsilon}\|G_X\|^2 + \sqrt{\varepsilon}\|G_Y\|^2) \\
&= -\langle U_{sX}U_sG_X, G_X\omega\rangle - \langle U_{sX}U_sG_Y, G_Y\omega\rangle - \langle U_{sX}U_{sY}G, G_Y\omega\rangle \\
&\quad - \langle U_{sXY}U_sG_Y, G\omega\rangle - \langle U_{sXY}U_{sY}G, G\omega\rangle + O(\sqrt{\varepsilon}\|G_X\|^2 + \sqrt{\varepsilon}\|G_Y\|^2) \\
&= -\langle U_{sX}U_sG_X, G_X\omega\rangle - \langle U_{sX}U_sG_Y, G_Y\omega\rangle \\
&\quad + \frac{1}{2}\langle (U_{sX}U_{sY})_YG, G\omega\rangle + \frac{1}{2}\langle (U_{sXY}U_s)_YG, G\omega\rangle \\
&\quad - \langle U_{sXY}U_{sY}G, G\omega\rangle + O(\sqrt{\varepsilon}\|G_X\|^2 + \sqrt{\varepsilon}\|G_Y\|^2) \\
&= -\langle U_{sX}U_sG_X, G_X\omega\rangle - \langle U_{sX}U_sG_Y, G_Y\omega\rangle \\
&\quad + \frac{1}{2}\langle (U_{sX}U_{sYY} + U_{sXYY}U_s)G, G\omega\rangle + O(\sqrt{\varepsilon}\|G\|_{\mathbb{X}}^2 + \sqrt{\varepsilon}\|G\|_{\mathbb{Y}}^2),
\end{aligned} \tag{3.38}
$$

and

$$
\begin{aligned}
\langle -\Phi_Y\Delta V_s + \Phi\Delta U_{sX}, -G\omega\rangle &= \langle \Phi_YV_{sYY}, G\omega\rangle - \langle \Phi U_{sXYY}, G\omega\rangle \\
&\quad + \langle \Phi_YV_{sXX}, G\omega\rangle - \langle \Phi U_{sXXX}, G\omega\rangle \\
&= \langle \Phi_YV_{sYY}, G\omega\rangle - \langle \Phi U_{sYY}, G\omega\rangle + O(\sqrt{\varepsilon}\|G_X\|^2 + \sqrt{\varepsilon}\|G_Y\|^2) \\
&= \langle U_sV_{sYY}G_Y, G\omega\rangle + \langle U_{sY}V_{sYY}G, G\omega\rangle \\
&\quad - \langle U_sU_{sXYY}, G\omega\rangle + O(\sqrt{\varepsilon}\|G_X\|^2 + \sqrt{\varepsilon}\|G_Y\|^2) \\
&= -\frac{1}{2}\langle (U_sV_{sYY})_YG, G\omega\rangle + \langle U_{sY}V_{sYY}G, G\omega\rangle \\
&\quad - \langle U_sU_{sXYY}, G\omega\rangle + O(\sqrt{\varepsilon}\|G_X\|^2 + \sqrt{\varepsilon}\|G_Y\|^2) \\
&= \frac{1}{2}\langle (U_{sY}V_{sYY} - U_sU_{sXYY})G, G\omega\rangle + O(\sqrt{\varepsilon}\|G\|_{\mathbb{X}}^2 + \sqrt{\varepsilon}\|G\|_{\mathbb{Y}}^2).
\end{aligned} \tag{3.39}
$$

Collecting (3.31)–(3.39), we have

$$
\langle \partial_{XX}[U_s^2G_X] + \partial_{XY}[U_s^2G_Y] - \varepsilon\Delta^2\Phi + R[\Phi], -G\omega\rangle
$$

$$
\begin{aligned}
&= \frac{3}{2}\|U_sG_X\|^2 + \frac{1}{2}\|U_sG_Y\|^2 + \varepsilon\langle U_{sY}G_Y, G_Y\omega\rangle\,|_{Y=0} \\
&\quad + \varepsilon\{\|\sqrt{U}_sG_{XX}\sqrt{\omega}\|^2 + 2\|\sqrt{U}_sG_{XY}\sqrt{\omega}\|^2 + \|\sqrt{U}_sG_{YY}\sqrt{\omega}\|^2\} \\
&\quad + \left\langle\left(-\varepsilon U_{sYY} + \frac{1}{2}V_sU_{sY} - \frac{1}{2}V_{sY}U_s\right)G_X, G_X\omega\right\rangle \\
&\quad + \left\langle\left(-2\varepsilon U_{sYY} + \frac{3}{2}V_sU_{sY} - \frac{3}{2}V_{sY}U_s\right)G_Y, G_Y\omega\right\rangle \\
&\quad + \frac{1}{2}\langle(\varepsilon U_{sYYYY} - U_sU_{sXYY} - V_sU_{sYYY} + U_{sX}U_{sYY} + U_{sY}V_{sYY})G, G\omega\rangle \\
&\quad + O(\sqrt{\varepsilon}\|G\|_{\mathbb{X}}^2 + \sqrt{\varepsilon}\|G\|_{\mathbb{Y}}^2).
\end{aligned} \tag{3.40}
$$

Notice that

$$
\begin{aligned}
-\varepsilon U_{sYY} + V_sU_{sY} - V_{sY}U_s &= -\varepsilon U_{sYY} + V_sU_{sY} + U_sU_{sX} \\
&= -u^0_{pyy} + v^0_pu^0_{py} + u^0_pu^0_{px} + O(\sqrt{\varepsilon}) = O(\sqrt{\varepsilon})
\end{aligned}
$$

and by the assumption (1.17),

$$
-\varepsilon U_{sYY} = -u^0_{pyy} + O(\sqrt{\varepsilon}) \geqslant O(\sqrt{\varepsilon}).
$$

Then

$$
\begin{aligned}
&\left\langle\left(-\varepsilon U_{sYY} + \frac{1}{2}V_sU_{sY} - \frac{1}{2}V_{sY}U_s\right)G_X, G_X\omega\right\rangle + \left\langle\left(-2\varepsilon U_{sYY} + \frac{3}{2}V_sU_{sY} - \frac{3}{2}V_{sY}U_s\right)G_Y, G_Y\omega\right\rangle \\
&\quad \geqslant O(\sqrt{\varepsilon}\|G_X\|^2 + \sqrt{\varepsilon}\|G_Y\|^2).
\end{aligned} \tag{3.41}
$$

Notice that

$$
\begin{aligned}
&Y^2(\varepsilon U_{sYYYY} - U_sU_{sXYY} - V_sU_{sYYY} + U_{sX}U_{sYY} + U_{sY}V_{sYY}) \\
&\quad = Y^2(\varepsilon U_{sYY} - U_sU_{sX} - V_sU_{sY})_{YY} \\
&\quad = y^2(u^0_{pyy} - u^0_pu^0_{px} - v^0_pu^0_{py})_{yy} + O(\sqrt{\varepsilon}) = O(\sqrt{\varepsilon}).
\end{aligned}
$$

So

$$
\begin{aligned}
&\frac{1}{2}\langle(\varepsilon U_{sYYYY} - U_sU_{sXYY} - V_sU_{sYYY} + U_{sX}U_{sYY} + U_{sY}V_{sYY})G, G\omega\rangle \\
&\quad = O\left(\sqrt{\varepsilon}\left\|\frac{G}{Y}\right\|^2\right) = O(\sqrt{\varepsilon}\|G_Y\|^2) = O(\sqrt{\varepsilon}\|G\|_{\mathbb{X}}^2 + \sqrt{\varepsilon}\|G\|_{\mathbb{Y}}^2).
\end{aligned} \tag{3.42}
$$

Combining (3.40)–(3.42), we obtain

$$
\begin{aligned}
&\langle\partial_{XX}[U_s^2G_X] + \partial_{XY}[U_s^2G_Y] - \varepsilon\Delta^2\Phi + R[\Phi], -G\omega\rangle \\
&\quad \geqslant \frac{3}{2}\|U_sG_X\|^2 + \frac{1}{2}\|U_sG_Y\|^2 + \varepsilon\langle U_{sY}G_Y, G_Y\omega\rangle\,|_{Y=0} \\
&\quad\quad + \varepsilon\{\|\sqrt{U}_sG_{XX}\sqrt{\omega}\|^2 + 2\|\sqrt{U}_sG_{XY}\sqrt{\omega}\|^2 + \|\sqrt{U}_sG_{YY}\sqrt{\omega}\|^2\} \\
&\quad\quad + O(\sqrt{\varepsilon}\|G\|_{\mathbb{X}}^2 + \sqrt{\varepsilon}\|G\|_{\mathbb{Y}}^2).
\end{aligned} \tag{3.43}
$$

So we complete the proof. □

Proof of Proposition 3.1. Under the assumptions of Theorem 1.1, by Lemmas 3.3 and 3.4, we have

$$
\begin{aligned}
&\|G\|_{\mathbb{Y}}^2 \lesssim \|G\|_{\mathbb{X}}^2 + |\langle\partial_YF_1 - \partial_XF_2, G\rangle|, \\
&\|G\|_{\mathbb{X}}^2 \lesssim (\sqrt{\varepsilon} + L + \|v^0_e\|_\infty)\|G\|_{\mathbb{Y}}^2 + |\langle\partial_YF_1 - \partial_XF_2, G\omega\rangle|,
\end{aligned}
$$

and similarly, under the assumptions of Theorem 1.2, by Lemmas 3.3 and 3.6, we have

$$
\begin{aligned}
&\|G\|_{\mathbb{Y}}^2 \lesssim \|G\|_{\mathbb{X}}^2 + |\langle\partial_YF_1 - \partial_XF_2, G\rangle|, \\
&\|G\|_{\mathbb{X}}^2 \lesssim \sqrt{\varepsilon}\|G\|_{\mathbb{Y}}^2 + |\langle\partial_YF_1 - \partial_XF_2, G\omega\rangle|.
\end{aligned}
$$

Let $\delta = \sqrt{\varepsilon} + L + \|v_e^0\|_\infty$ in the first case, $\delta = \sqrt{\varepsilon}$ in the second case, and δ be small. Then we have

$$\begin{aligned}\|G\|_{\mathbb{X}}^2 + \|G\|_{\mathbb{Y}}^2 &\lesssim \|G\|_{\mathbb{X}}^2 + |\langle \partial_Y F_1 - \partial_X F_2, G\rangle| \\ &\lesssim \delta\|G\|_{\mathbb{Y}}^2 + |\langle \partial_Y F_1 - \partial_X F_2, G\omega\rangle| + |\langle \partial_Y F_1 - \partial_X F_2, G\rangle| \\ &\lesssim (\|F_1\| + \|F_2\|)(\|G_Y\| + \|G_X\|) \\ &\lesssim (\|F_1\| + \|F_2\|)(\|G\|_{\mathbb{X}} + \|G\|_{\mathbb{Y}}).\end{aligned}$$

It is easy to see that

$$\|\sqrt{\varepsilon}\Phi_{XX}, \sqrt{\varepsilon}\Phi_{XY}, \sqrt{\varepsilon}\Phi_{YY}, \Phi_X, \Phi_Y\| \lesssim \|G\|_{\mathbb{X}} + \|G\|_{\mathbb{Y}} \lesssim \|F_1\| + \|F_2\|.$$

Then we complete the proof. □

4 The proof of the main theorems

Proofs of Theorems 1.1 *and* 1.2. Let $\boldsymbol{U}_s = [U_s, V_s]$, $\boldsymbol{U} = [U, V]$ and $\boldsymbol{R} = [R_1, R_2]$. Now we write the Navier-Stokes equation in the following form:

$$\begin{cases} -\varepsilon\Delta\boldsymbol{U} + \boldsymbol{U}_s\cdot\nabla\boldsymbol{U} + \boldsymbol{U}\cdot\nabla\boldsymbol{U}_s + \boldsymbol{U}\cdot\nabla\boldsymbol{U} + \nabla P = -\boldsymbol{R}, \\ \nabla\cdot\boldsymbol{U} = 0, \quad \boldsymbol{U}|_\Omega = 0. \end{cases} \tag{4.1}$$

We use the method of the contraction mapping. Define

$$\|\boldsymbol{U}\|_{\mathbb{Z}} := \|\boldsymbol{U}\| + \sqrt{\varepsilon}\|\nabla\boldsymbol{U}\| + \varepsilon^{\frac{3}{2}}\|\nabla^2\boldsymbol{U}\|.$$

We set $\mathcal{T}: W^{2,2}(\Omega) \to W^{2,2}(\Omega)$ as $\mathcal{T}(\boldsymbol{U}) = \boldsymbol{W}$, where $\boldsymbol{W}$ is given by

$$\begin{cases} -\varepsilon\Delta\boldsymbol{W} + \boldsymbol{U}_s\cdot\nabla\boldsymbol{W} + \boldsymbol{W}\cdot\nabla\boldsymbol{U}_s + \nabla P = -\boldsymbol{R} - \boldsymbol{U}\cdot\nabla\boldsymbol{U}, \\ \nabla\cdot\boldsymbol{W} = 0, \quad \boldsymbol{W}|_\Omega = 0. \end{cases} \tag{4.2}$$

Let

$$B = \{\boldsymbol{U} \in W^{2,2}(\Omega) : \|\boldsymbol{U}\|_{\mathbb{Z}} \leqslant C_0(L)\varepsilon^{\frac{3}{2}}\},$$

where C_0 is chosen later. Next, we prove that $\mathcal{T}$ is a contractive mapping in B, if $\|\boldsymbol{R}\| \leqslant C_1\varepsilon^{\frac{3}{2}}$. We write $\boldsymbol{F} = -\boldsymbol{R} - \boldsymbol{U}\cdot\nabla\boldsymbol{U}$, and from Proposition 3.1,

$$\|\boldsymbol{W}\| + \sqrt{\varepsilon}\|\nabla\boldsymbol{W}\| \lesssim \|\boldsymbol{F}\|.$$

Due to the $W^{2,2}$ estimate of Stokes equations in a convex polygon in [18],

$$\varepsilon\|\nabla^2\boldsymbol{W}\| \lesssim_L \|\boldsymbol{F}\| + \|\nabla\boldsymbol{W}\| + \frac{1}{\sqrt{\varepsilon}}\|\boldsymbol{W}\| \lesssim_L \frac{1}{\sqrt{\varepsilon}}\|\boldsymbol{F}\|.$$

So we get

$$\|\boldsymbol{W}\|_{\mathbb{Z}} \leqslant C_2(L)\|\boldsymbol{F}\|.$$

It is easy to see that

$$\|\boldsymbol{U}\cdot\nabla\boldsymbol{U}\| \lesssim_L \|\boldsymbol{U}\|_{L^\infty}\|\nabla\boldsymbol{U}\| \lesssim_L \|\boldsymbol{U}\|^{\frac{1}{4}}\|\nabla\boldsymbol{U}\|^{\frac{3}{2}}\|\nabla^2\boldsymbol{U}\|^{\frac{1}{4}} \lesssim_L \varepsilon^{-\frac{9}{8}}\|\boldsymbol{U}\|_{\mathbb{Z}}^2.$$

It implies that

$$\|\boldsymbol{W}\|_{\mathbb{Z}} \leqslant C_2(L)\|\boldsymbol{F}\| + C_3(L)\varepsilon^{-\frac{9}{8}}\|\boldsymbol{U}\|_{\mathbb{Z}}^2 \leqslant (C_1C_2 + C_3C_0^2\varepsilon^{\frac{3}{8}})\varepsilon^{\frac{3}{2}}.$$

Select

$$C_0(L) = C_1(L)C_2(L) + 1, \quad \mathcal{T}(B) \subset B,$$

when ε is small enough. If $\boldsymbol{U}_1, \boldsymbol{U}_2 \in B$,

$$\begin{aligned}
\|\mathcal{T}(\boldsymbol{U}_1 - \boldsymbol{U}_2)\|_{\mathbb{Z}} &\leqslant C_2(L)\|\boldsymbol{U}_1 \cdot \nabla \boldsymbol{U}_1 - \boldsymbol{U}_2 \cdot \nabla \boldsymbol{U}_2\| \\
&\leqslant C_2(L)\|(\boldsymbol{U}_1 - \boldsymbol{U}_2) \cdot \nabla \boldsymbol{U}_1\| + \|\boldsymbol{U}_2 \cdot \nabla(\boldsymbol{U}_1 - \boldsymbol{U}_2)\| \\
&\leqslant C_2(L)\|(\boldsymbol{U}_1 - \boldsymbol{U}_2)\|_\infty \|\nabla \boldsymbol{U}_1\| + \|\boldsymbol{U}_2\|_\infty \|\nabla(\boldsymbol{U}_1 - \boldsymbol{U}_2)\| \\
&\leqslant C_3(L)\varepsilon^{-\frac{9}{8}}(\|\boldsymbol{U}_1\|_{\mathbb{Z}} + \|\boldsymbol{U}_2\|_{\mathbb{Z}})\|\boldsymbol{U}_1 - \boldsymbol{U}_2\|_{\mathbb{Z}} \\
&\leqslant 2C_1(L)C_3(L)\varepsilon^{\frac{3}{8}}\|\boldsymbol{U}_1 - \boldsymbol{U}_2\|_{\mathbb{Z}},
\end{aligned}$$

so $\mathcal{T}$ is a contraction mapping on B when ε is small enough, and we can conclude that the equation (4.1) admits a unique solution and

$$\|\boldsymbol{U}\|_{L^\infty} \lesssim_L \varepsilon^{-\frac{5}{8}}\|\boldsymbol{U}\|_{\mathbb{Z}} \lesssim_L \varepsilon^{\frac{7}{8}}.$$

So we have

$$\begin{aligned}
&\left|U^\varepsilon(X,Y) - u_e^0(X,Y) - u_b^0\left(X, \frac{Y}{\sqrt{\varepsilon}}\right)\right| \\
&\quad = \left|\sqrt{\varepsilon}u_e^1(X,Y) + \sqrt{\varepsilon}u_b^1\left(X, \frac{Y}{\sqrt{\varepsilon}}\right) + \varepsilon u_e^2(X,Y) + \varepsilon \hat{u}_b^2\left(X, \frac{Y}{\sqrt{\varepsilon}}\right) + U(X,Y)\right| \\
&\quad \lesssim_L \sqrt{\varepsilon}, \\
&|V^\varepsilon(X,Y) - v_e^0(X,Y)| \\
&\quad = \left|\sqrt{\varepsilon}v_b^0\left(X, \frac{Y}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}v_e^1(X,Y) + \varepsilon v_b^1\left(X, \frac{Y}{\sqrt{\varepsilon}}\right) + \varepsilon v_e^2(X,Y) + \varepsilon^{\frac{3}{2}}\hat{v}_b^2\left(X, \frac{Y}{\sqrt{\varepsilon}}\right) + V(X,Y)\right| \\
&\quad \lesssim_L \sqrt{\varepsilon},
\end{aligned}$$

which ends the proof. □

Acknowledgements Liqun Zhang was supported by National Natural Science Foundation of China (Grant Nos. 11471320 and 11631008).

References

1 Alexandre R, Wang Y-G, Xu C-J, et al. Well-posedness of the Prandtl equation in Sobolev spaces. J Amer Math Soc, 2015, 28: 745–784

2 Dietert H, Gérard-Varet D. Well-posedness of the Prandtl equations without any structural assumption. Ann PDE, 2019, 5: 8

3 E W N, Engquist B. Blowup of solutions of the unsteady Prandtl's equation. Comm Pure Appl Math, 1997, 50: 1287–1293

4 Fei M W, Tao T, Zhang Z F. On the zero-viscosity limit of the Navier-Stokes equations in $\mathbb{R}^3_+$ without analyticity. J Math Pures Appl (9), 2018, 112: 170–229

5 Gérard-Varet D, Dormy E. On the ill-posedness of the Prandtl equation. J Amer Math Soc, 2010, 23: 591–609

6 Gérard-Varet D, Maekawa Y. Sobolev stability of Prandtl expansions for the steady Navier-Stokes equations. Arch Ration Mech Anal, 2019, 233: 1319–1382

7 Gérard-Varet D, Maekawa Y, Masmoudi N. Gevrey stability of Prandtl expansions for 2-dimensional Navier-Stokes flows. Duke Math J, 2018, 167: 2531–2631

8 Gérard-Varet D, Masmoudi N. Well-posedness for the Prandtl system without analyticity or monotonicity. Ann Sci Éc Norm Supér (4), 2015, 48: 1273–1325

9 Gérard-Varet D, Nguyen T. Remarks on the ill-posedness of the Prandtl equation. Asymptot Anal, 2012, 77: 71–88

10 Grenier E, Guo Y, Nguyen T. Spectral instability of characteristic boundary layer flows. Duke Math J, 2016, 165: 3085–3146

11 Grenier E, Nguyen T. On nonlinear instability of Prandtl's boundary layers: The case of Rayleigh's stable shear flows. arXiv:1706.01282, 2017

12 Grenier E, Nguyen T. L^∞ instability of Prandtl layers. Ann PDE, 2019, 5: 18
13 Guo Y, Iyer S. Regularity and expansion for steady Prandtl equations. Comm Math Phys, 2021, 382: 1403–1447
14 Guo Y, Iyer S. Validity of steady Prandtl layer expansions. Comm Pure Appl Math, 2023, in press
15 Guo Y, Nguyen T. Prandtl boundary layer expansions of steady Navier-Stokes flows over a moving plate. Ann PDE, 2017, 3: 10
16 Iyer S. Steady Prandtl boundary layer expansions over a rotating disk. Arch Ration Mech Anal, 2017, 224: 421–469
17 Iyer S. Steady Prandtl layers over a moving boundary: Nonshear Euler flows. SIAM J Math Anal, 2019, 51: 1657–1695
18 Kellogg R B, Osborn J E. A regularity result for the Stokes problem in a convex polygon. J Funct Anal, 1976, 21: 397–431
19 Liu C-J, Wang Y-G, Yang T. On the ill-posedness of the Prandtl equations in three space dimensions. Arch Ration Mech Anal, 2016, 220: 83–108
20 Liu C-J, Wang Y-G, Yang T. Global existence of weak solutions to the three-dimensional Prandtl equations with a special structure. Discrete Contin Dyn Syst Ser S, 2016, 9: 2011–2029
21 Maekawa Y. On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half plane. Comm Pure Appl Math, 2014, 67: 1045–1128
22 Masmoudi N, Wong T K. Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods. Comm Pure Appl Math, 2015, 68: 1683–1741
23 Oleinik O A, Samokhin V N. Mathematical Models in Boundary Layer Theory. Applied Mathematics and Mathematical Computation, vol. 15. Boca Raton: Chapman & Hall/CRC, 1999
24 Sammartino M, Caflisch R E. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. Comm Math Phys, 1998, 192: 433–461
25 Sammartino M, Caflisch R E. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution. Comm Math Phys, 1998, 192: 463–491
26 Wang C, Wang Y X, Zhang Z F. Zero-viscosity limit of the Navier-Stokes equations in the analytic setting. Arch Ration Mech Anal, 2017, 224: 555–595
27 Xin Z P, Zhang L Q. On the global existence of solutions to the Prandtl's system. Adv Math, 2004, 181: 88–133

Appendix A

In this appendix, we prove Lemmas 2.4 and 2.6. We write the equations (2.10) and (2.21) in the uniform way, i.e.,

$$\begin{cases}\Delta\psi^i - F_e'(\psi)\psi^i = F^i,\\ \psi^i\,|_{X=0} = \psi_0^i(Y), \quad \psi^i\,|_{X=L} = \psi_L^i(Y),\\ \psi^i\,|_{Y=0} = \displaystyle\int_0^X v_b^{i-1}(X',0)dX', \quad \psi^i\,|_{Y\to\infty} = 0.\end{cases} \tag{A.1}$$

Here, ψ is the stream-function of $[u_e^0, v_e^0]$, $\psi_Y = u_e^0$, $\psi_X = -v_e^0$ and

$$\Delta\psi = F_e(\psi). \tag{A.2}$$

We homogenize the system (A.1). Let

$$\tilde{\psi} = \psi^i - \frac{L-x}{L}\psi_0^i(Y) - \frac{x}{L}\left[\psi_L^i(Y) - \chi(Y)\int_0^L v_b^{i-1}(X',0)dX'\right] - \chi(Y)\int_0^X v_b^{i-1}(X',0)dX',$$

where $\chi(Y)$ is a nonnegative smooth cut-off function, $\chi\,|_{[0,1]} = 1$ and $\chi\,|_{[2,\infty]} = 0$. $\tilde{\psi}$ satisfies

$$\begin{cases}\Delta\tilde{\psi} - F_e'(\psi)\tilde{\psi} = \tilde{F},\\ \tilde{\psi}\,|_{\partial\Omega} = 0.\end{cases} \tag{A.3}$$

Notice that

$$\Delta\psi = F_e(\psi),$$

so

$$0 < c_0 \leqslant u_e^0 = \psi_Y \leqslant C_0 < \infty$$

satisfies

$$\Delta u_e^0 = F_e'(\psi) u_e^0.$$

Let $w = \frac{\tilde{\psi}}{u_e^0}$. Then

$$u_e^0 \Delta w + 2\nabla u_e^0 \cdot \nabla w = \tilde{F}.$$

By the above equation times u_e^0, we show that the equation (A.3) is equivalent to

$$\begin{cases} \partial_X[(u_e^0)^2 w_X] + \partial_Y[(u_e^0)^2 w_Y] = u_e^0 \tilde{F}, \\ w\,|_{\partial\Omega} = 0. \end{cases} \tag{A.4}$$

We can easily get *a priori* estimates of the equation (A.4). Multiplying the equation (A.4) by w and integrating in Ω, we get

$$\|u_e^0 w_X\|^2 + \|u_e^0 w_Y\|^2 = -\langle w, u_e^0 \tilde{F}\rangle \lesssim \|w\| \|F\| \lesssim \|w_X\| \|F\|.$$

So we have

$$\|\nabla w\| \lesssim \|\tilde{F}\|. \tag{A.5}$$

The inequality (A.5) actually shows the existence of the solution to the equation (A.4). Moreover, if $\tilde{F}$ is a smooth function decaying fast when $Y \to \infty$, then

$$\|\langle Y\rangle^M \nabla^k w\| \lesssim 1 \quad \text{for } 1 \leqslant k \leqslant K, \quad \text{where } K \text{ and } M \text{ are large constants.} \tag{A.6}$$

So Lemma 2.4 is right.

Next, we discuss the boundary layer profile $[u_b^i, v_b^i]$ $(i = 1, 2)$. For convenience, we write

$$[\bar{u}, \bar{v}] := [u_p^0, v_p^0],$$

and we homogenize the system (2.16) as in [13]:

$$\begin{aligned} &u(x,y) = u_b^1(x,y) + u_e^1(x,0)\eta(y), \\ &v(x,y) = v_b^1(x,y) - v_b^1(x,0) + u_{eX}^1(x,0) I_\eta(y), \\ &I_\eta(y) := \int_y^\infty \eta(y') dy'. \end{aligned} \tag{A.7}$$

Here, we select η to be a C^∞ function satisfying the following:

$$\eta(0) = 1, \quad \int_0^\infty \eta dy = 0, \quad \eta \text{ decays fast as } y \to \infty. \tag{A.8}$$

Due to (2.16), the homogenized unknowns $[u, v]$ satisfy the system

$$\begin{cases} \bar{u}\partial_x u + u\partial_x \bar{u} + \bar{v}\partial_y u + v\partial_y \bar{u} - \partial_{yy} u + p_x = f^{(1)} + F =: h, \\ p_y = 0, \\ \partial_x u + \partial_y v = 0, \\ u\,|_{x=0} = U_B^1 + u_e^1(0,0)\eta(y) =: u_0(y), \quad [u,v]\,|_{y=0} = 0, \quad u\,|_{y\to\infty} = 0, \end{cases} \tag{A.9}$$

where

$$F = \bar{u} u_{eX}^1(x,0)\eta + \bar{u}_x u_e^1(x,0)\eta + \bar{v} u_e^1(x,0)\eta' + \bar{u}_y u_{eX}^1(x,0) I_\eta - u_e^1(x,0)\eta''. \tag{A.10}$$

Noticing that p is independent of y, we evaluate the equation as $y \to \infty$, and we have $p_x = 0$. We still use the stream-function of $[u, v]$:

$$\phi(x,y) := \int_0^y u(x,y') dy', \quad \partial_y \phi = u, \quad \partial_x \phi = -v. \tag{A.11}$$

Then ϕ satisfies

$$\begin{cases} \bar{u}\phi_{xy} + \bar{u}_x\phi_y + \bar{v}\phi_{yy} - \phi_x\bar{u}_y - \phi_{yyy} = h, \\ \Phi\,|_{x=0} = \int_0^y u_0(y')dy', \quad \Phi\,|_{y=0} = \phi_y\,|_{y=0} = 0, \quad \phi_y\,|_{y\to\infty} = 0. \end{cases} \tag{A.12}$$

In order to give the *a priori* estimate of (A.12), we define $g = \frac{\phi}{\bar{u}}$. Recall $\bar{u} \sim y$ when $y \leqslant 1$, $\bar{u} \sim 1$ when $y \geqslant 1$, and $\phi\,|_{y=0} = \phi_y\,|_{y=0} = 0$, so g is well defined. g satisfies

$$\begin{cases} \partial_x[\bar{u}^2 g_y] - \partial_y^3[\bar{u}g] + \bar{v}\partial_y^2[\bar{u}g] - \bar{u}\bar{v}_{yy}g = h, \\ G\,|_{x=0} = \dfrac{\int_0^y u_0(y')dy'}{\bar{u}}, \quad G\,|_{y=0} = 0, \quad g_y\,|_{y\to\infty} = 0. \end{cases} \tag{A.13}$$

Now we define the norms of g:

$$\begin{aligned} \|g\|_{\Xi_0} &:= \sup_{0\leqslant x_0\leqslant L} \|\bar{u}g_y\rho\|_{L_y^2(x=x_0)} + \|\sqrt{\bar{u}}g_{yy}\rho\|_{L_{x,y}^2}, \\ \|g\|_{\Xi_1} &:= \sup_{0\leqslant x_0\leqslant L} \left\|\bar{u}g_{xy}\frac{\rho}{\langle y\rangle}\right\|_{L_y^2(x=x_0)} + \left\|\sqrt{\bar{u}}g_{xyy}\frac{\rho}{\langle y\rangle}\right\|_{L_{x,y}^2}, \end{aligned} \tag{A.14}$$

where $\rho = \langle y\rangle^N$ for a large constant N. Next, let us prove the following *a priori* estimate of g.

Lemma A.1. *Suppose that g is a smooth solution of* (A.13)*, and $L > 0$ is small enough. Then*

$$\|g\|_{\Xi_0}^2 \lesssim \|\bar{u}g_y\rho\|_{L_y^2(x=0)}^2 + \|h\rho\|_{L_{x,y}^2}^2, \tag{A.15}$$

$$\|g_x\|_{\Xi_1}^2 \lesssim \left\|\bar{u}g_{xy}\frac{\rho}{\langle y\rangle}\right\|_{L_y^2(x=0)}^2 + \|g\|_{\Xi_0}^2 + \left\|h_x\frac{\rho}{\langle y\rangle}\right\|_{L_{x,y}^2}^2. \tag{A.16}$$

Proof. Multiplying the equation (A.13) by $g_y\rho^2$ and integrating in $(0, x_0)\times(0,\infty)$, we have

$$\begin{aligned} \int_0^{x_0}\int_0^\infty [\bar{u}^2 g_y]_x g_y\rho^2 dxdy &= \int_0^{x_0}\int_0^\infty \bar{u}^2 g_{xy}g_y\rho^2 dxdy + \int_0^{x_0}\int_0^\infty 2\bar{u}\bar{u}_x g_y^2\rho^2 dxdy \\ &= \frac{1}{2}\|\bar{u}g_y\rho\|_{L_y^2(x=x_0)}^2 - \frac{1}{2}\|\bar{u}g_y\rho\|_{L_y^2(x=0)}^2 + \int_0^{x_0}\int_0^\infty \bar{u}\bar{u}_x g_y^2\rho^2 dxdy. \end{aligned}$$

We can dominate $\|g_y\|$ by $\|g\|_{\Xi_0}$. Let $0 < \xi \leqslant 1$ be a constant being chosen later. $\chi(y)$ is a smooth cut-off function satisfying $\chi\,|_{[0,1]} = 1$ and $\chi\,|_{[2,\infty]} = 0$. Then

$$\|g_y\rho\|_{L_{x,y}^2} \lesssim \left\|g_y\left[1 - \chi\left(\frac{y}{\xi}\right)\right]\rho\right\|_{L_{x,y}^2} + \left\|g_y\chi\left(\frac{y}{\xi}\right)\rho\right\|_{L_{x,y}^2}.$$

When $y \leqslant 1$, $1 - \chi(\frac{y}{\xi}) \lesssim \frac{y}{\xi} \lesssim \frac{\bar{u}}{\xi}$; when $y > 1$, $1 - \chi(\frac{y}{\xi}) \lesssim \bar{u} \lesssim \frac{\bar{u}}{\xi}$. So

$$\left\|g_y\left[1 - \chi\left(\frac{y}{\xi}\right)\right]\rho\right\|_{L_{x,y}^2} \lesssim \frac{1}{\xi^2}\|\bar{u}g_y\rho\|_{L_{x,y}^2}^2 \lesssim \frac{L}{\xi^2}\|g\|_{\Xi_0}^2$$

and

$$\begin{aligned} \left\|g_y\chi\left(\frac{y}{\xi}\right)\rho\right\|_{L_{x,y}^2}^2 &= -\int_0^{x_0}\int_0^\infty 2yg_yg_{yy}\chi^2\left(\frac{y}{\xi}\right)\rho^2 dxdy - \int_0^{x_0}\int_0^\infty \frac{2}{\xi}yg_y^2\chi\left(\frac{y}{\xi}\right)\chi'\left(\frac{y}{\xi}\right)\rho^2 dxdy \\ &\quad - \int_0^{x_0}\int_0^\infty 2yg_y^2\chi^2\left(\frac{y}{\xi}\right)\rho\rho_y dxdy \\ &\lesssim \left\|y\chi\left(\frac{y}{\xi}\right)g_{yy}\rho\right\|_{L_{x,y}^2}^2 + \frac{1}{\xi^2}\|\bar{u}g_y\rho\|_{L_{x,y}^2}^2 \\ &\lesssim \xi\|\sqrt{\bar{u}}g_{yy}\rho\|_{L_{x,y}^2}^2 + \frac{L}{\xi^2}\|g\|_{\Xi_0}^2. \end{aligned}$$

So we have

$$\|g_y\rho\|^2_{L^2_{x,y}} \lesssim \xi\|\sqrt{\bar{u}}g_{yy}\rho\|^2_{L^2_{x,y}} + \frac{L}{\xi^2}\|g\|^2_{\Xi_0}.$$

Select $\xi = L^{\frac{1}{3}}$, and then

$$\|g_y\rho\|^2_{L^2_{x,y}} \lesssim L^{\frac{1}{3}}\|g\|^2_{\Xi_0}. \tag{A.17}$$

So the first term of the L^2 inner product of the equation in (A.13) and $g_y\rho^2$ is

$$\int_0^{x_0}\int_0^{\infty}[\bar{u}^2g_y]_x g_y\rho^2 dxdy = \frac{1}{2}\|\bar{u}g_y\rho\|^2_{L^2_y(x=x_0)} - \frac{1}{2}\|\bar{u}g_y\rho\|^2_{L^2_y(x=0)} + O(L^{\frac{1}{3}}\|g\|^2_{\Xi_0}). \tag{A.18}$$

The second term of the L^2 inner product of the equation in (A.13) and $g_y\rho^2$ is

$$\begin{aligned}-\int_0^{x_0}\int_0^{\infty}\partial_y^3[\bar{u}g]g_y\rho^2 dxdy = &\int_0^{x_0}\int_0^{\infty}\partial_y^2[\bar{u}g]g_{yy}\rho^2 dxdy + \int_0^{x_0}\int_0^{\infty}2\partial_y^2[\bar{u}g]g_y\rho_y\rho dxdy\\ &+ \langle\partial_y^2[\bar{u}g], g_y\rho^2\rangle_{L^2_x(y=0)},\end{aligned}$$

where

$$\begin{aligned}\int_0^{x_0}\int_0^{\infty}\partial_y^2[\bar{u}g]g_{yy}\rho^2 dxdy &= \int_0^{x_0}\int_0^{\infty}(\bar{u}g_{yy} + 2\bar{u}_yg_y + \bar{u}_{yy}g)g_{yy}\rho^2 dxdy\\
&= \|\sqrt{\bar{u}}g_{yy}\rho\|^2_{L^2_{x,y}} - \langle\bar{u}_yg_y, g_y\rangle_{L^2_x(y=0)} + \int_0^{x_0}\int_0^{\infty}(\bar{u}_y\rho^2)_y g_y^2 dxdy\\
&\quad - \int_0^{x_0}\int_0^{\infty}\bar{u}_{yy}g_y^2\rho^2 dxdy - \int_0^{x_0}\int_0^{\infty}(\bar{u}_{yy}\rho^2)_y g g_y dxdy\\
&= \|\sqrt{\bar{u}}g_{yy}\rho\|^2_{L^2_{x,y}} - \langle\bar{u}_yg_y, g_y\rangle_{L^2_x(y=0)}\\
&\quad + O\left(\|g_y\rho\|^2_{L^2_{x,y}} + \|y(\bar{u}_{yy}\rho^2)_y\|_{L^\infty}\left\|\frac{g}{y}\right\|_{L^2_{x,y}}\|g_y\|_{L^2_{x,y}}\right)\\
&= \|\sqrt{\bar{u}}g_{yy}\rho\|^2_{L^2_{x,y}} - \langle\bar{u}_yg_y, g_y\rangle_{L^2_x(y=0)} + O(\|g_y\rho\|^2_{L^2_{x,y}}),\\
\int_0^{x_0}\int_0^{\infty}2\partial_y^2[\bar{u}g]g_y\rho_y\rho dxdy &= \int_0^{x_0}\int_0^{\infty}\bar{u}(g_y^2)_y\rho_y\rho dxdy + \int_0^{x_0}\int_0^{\infty}4\bar{u}_yg_y^2\rho_y\rho dxdy\\
&\quad + \int_0^{x_0}\int_0^{\infty}2\bar{u}_{yy}gg_y\rho_y\rho dxdy\\
&= O\left(\|g_y\rho\|^2_{L^2_{x,y}} + \|y\bar{u}_{yy}\rho_y\rho\|_{L^\infty}\left\|\frac{g}{y}\right\|_{L^2_{x,y}}\|g_y\|_{L^2_{x,y}}\right)\\
&= O(\|g_y\rho\|^2_{L^2_{x,y}}),\end{aligned}$$

and

$$\langle\partial_y^2[\bar{u}g], g_y\rho^2\rangle_{L^2_x(y=0)} = 2\langle\bar{u}_yg_y, g_y\rho^2\rangle_{L^2_x(y=0)}.$$

So we obtain

$$-\int_0^{x_0}\int_0^{\infty}\partial_y^3[\bar{u}g]g_y\rho^2 dxdy = \|\sqrt{\bar{u}}g_{yy}\rho\|^2_{L^2_{x,y}} + \langle\bar{u}_yg_y, g_y\rho^2\rangle_{L^2_x(y=0)} + O(L^{\frac{1}{3}}\|g\|^2_{\Xi_0}). \tag{A.19}$$

The third term of the L^2 inner product of the equation in (A.13) and $g_y\rho^2$ is

$$\begin{aligned}\int_0^{x_0}\int_0^{\infty}\bar{v}(\bar{u}g)_{yy}g_y\rho^2 dxdy &= \int_0^{x_0}\int_0^{\infty}\bar{v}(\bar{u}g_{yy} + 2\bar{u}_yg_y + \bar{u}_{yy}g)g_y\rho^2 dxdy\\
&= -\frac{1}{2}\int_0^{x_0}\int_0^{\infty}(\bar{v}\bar{u}\rho^2)_y g_y^2 dxdy + O\Bigg(\|\bar{v}\bar{u}_y\|_{L^\infty}\|g_y\rho\|^2_{L^2_{x,y}}\end{aligned}$$

$$
\begin{aligned}
&+ \|y\bar{v}\bar{u}_{yy}\rho^2\|_{L^\infty}\left\|\frac{g}{y}\right\|_{L^2_{x,y}}\|g_y\|_{L^2_{x,y}}\bigg)\\
&= O\bigg(\left\|\frac{(\bar{v}\bar{u}\rho^2)_y}{\rho^2}\right\|_{L^\infty}\|g_y\rho\|^2_{L^2_{x,y}} + \|g_y\rho\|^2_{L^2_{x,y}}\bigg)\\
&= O(L^{\frac{1}{3}}\|g\|^2_{\Xi_0}). \qquad \text{(A.20)}
\end{aligned}
$$

The last term of the L^2 inner product of the equation in (A.13) and $g_y\rho^2$ is

$$
\int_0^{x_0}\int_0^\infty \bar{v}_{yy}\bar{u}gg_y\rho^2dxdy = O\bigg(\|y\bar{v}_{yy}\bar{u}\rho^2\|_{L^\infty}\left\|\frac{g}{y}\right\|_{L^2_{x,y}}\|g_y\|_{L^2_{x,y}}\bigg) = O(\|g_y\rho\|^2_{L^2_{x,y}}). \qquad \text{(A.21)}
$$

Collecting (A.18)–(A.21), we have

$$
\begin{aligned}
&\frac{1}{2}\|\bar{u}g_y\rho\|^2_{L^2_y(x=x_0)} + \|\sqrt{\bar{u}}g_{yy}\rho\|^2_{L^2_{x,y}} + \langle\bar{u}_yg_y, g_y\rho^2\rangle_{L^2_x(y=0)}\\
&\quad = O(L^{\frac{1}{3}}\|g\|^2_{\Xi_0}) + \frac{1}{2}\|\bar{u}g_y\rho\|^2_{L^2_y(x=0)} + \int_0^{x_0}\int_0^\infty hg_y\rho^2dxdy. \qquad \text{(A.22)}
\end{aligned}
$$

Taking the supremum of $0 \leqslant x_0 \leqslant L$ and noticing that L is small enough, we have

$$
\sup_{0\leqslant x_0\leqslant L}\|\bar{u}g_y\rho\|^2_{L^2_y(x=x_0)} + \|\sqrt{\bar{u}}g_{yy}\rho\|^2_{L^2_{x,y}} + \langle\bar{u}_yg_y, g_y\rho^2\rangle_{L^2_x(y=0)} \lesssim \|\bar{u}g_y\rho\|^2_{L^2_y(x=0)} + \|h\rho\|^2_{L^2_{x,y}}. \qquad \text{(A.23)}
$$

The inequality (A.16) is similar to (A.15). The differential equation (A.13) with respect to x is

$$
\begin{aligned}
&\partial_x[\bar{u}^2g_{xy}] - \partial_y^3[\bar{u}g_x] + \bar{v}\partial_y^2(\bar{u}g_x) - \bar{u}\bar{v}_{yy}g_x + \partial_x[2\bar{u}\bar{u}_xg_y] - \partial_y^3[\bar{u}_xg]\\
&\quad + \bar{v}_x\partial_y^2(\bar{u}g) + \bar{v}\partial_y^2(\bar{u}_xg) - \bar{u}_x\bar{v}_{yy}g - \bar{u}\bar{v}_{xyy}g = h_x. \qquad \text{(A.24)}
\end{aligned}
$$

Taking $g_{xy}\frac{\rho^2}{\langle y\rangle^2}$ as the test function, like (A.22), we obtain

$$
\begin{aligned}
&\int_0^{x_0}\int_0^\infty [\partial_x[\bar{u}^2g_{xy}] - \partial_y^3[\bar{u}g_x] + \bar{v}\partial_y^2(\bar{u}g_x) - \bar{u}\bar{v}_{yy}g_x]g_{xy}\frac{\rho^2}{\langle y\rangle^2}dxdy\\
&\quad = \frac{1}{2}\bigg(\left\|\bar{u}g_{xy}\frac{\rho}{\langle y\rangle}\right\|^2_{L^2_y(x=x_0)} - \left\|\bar{u}g_{xy}\frac{\rho}{\langle y\rangle}\right\|^2_{L^2_y(x=0)}\bigg) + \left\|\sqrt{\bar{u}}g_{xyy}\frac{\rho}{\langle y\rangle}\right\|^2_{L^2_{x,y}}\\
&\qquad + \left\langle\bar{u}_yg_{xy}, g_{xy}\frac{\rho}{\langle y\rangle}\right\rangle_{L^2_x(y=0)} + O(L^{\frac{1}{3}}\|g\|^2_{\Xi_1}) \qquad \text{(A.25)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{x_0}\int_0^\infty [\partial_x(2\bar{u}\bar{u}_xg_y) + \bar{v}_x\partial_y^2(\bar{u}g) + \bar{v}\partial_y^2(\bar{u}_xg) - \bar{u}_x\bar{v}_{yy}g - \bar{u}\bar{v}_{xyy}g]g_{xy}\frac{\rho^2}{\langle y\rangle^2}dxdy\\
&\quad = O\bigg(\|\sqrt{\bar{u}}g_{yy}\rho\|^2_{L^2_{x,y}} + \|g_y\rho\|^2_{L^2_{x,y}} + \left\|g_{xy}\frac{\rho}{\langle y\rangle}\right\|^2_{L^2_{x,y}}\bigg)\\
&\quad = O(\|g\|^2_{\Xi_0} + L^{\frac{1}{3}}\|g\|^2_{\Xi_1}). \qquad \text{(A.26)}
\end{aligned}
$$

The difficult term is

$$
\begin{aligned}
\int_0^{x_0}\int_0^\infty -\partial_y^3[\bar{u}_xg]g_{xy}\frac{\rho^2}{\langle y\rangle^2}dxdy &= -\int_0^{x_0}\int_0^\infty (\bar{u}_xg_{yyy} + 3\bar{u}_{xy}g_{yy})g_{xy}\frac{\rho^2}{\langle y\rangle^2}dxdy\\
&\quad + O\bigg(\|g_y\rho\|^2_{L^2_{x,y}} + \left\|g_{xy}\frac{\rho}{\langle y\rangle}\right\|^2_{L^2_{x,y}}\bigg)\\
&= O\bigg(\bigg(\left\|\bar{u}g_{yyy}\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}} + \left\|g_{yy}\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}}\bigg)\left\|g_{xy}\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}}
\end{aligned}
$$

$$+ \|g_y\rho\|_{L^2_{x,y}}^2 + \left\|g_{xy}\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}}^2\bigg). \tag{A.27}$$

From the equation (A.12), we have

$$\left\|\phi_{yyy}\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}}^2 = O\bigg(\|g\|_{\Xi_0}^2 + L^{\frac{1}{3}}\|g\|_{\Xi_1}^2 + \left\|h\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}}^2\bigg).$$

Notice the fact that

$$\left\|\chi\partial_y^2\left(\frac{\phi}{y}\right)\right\|_{L^2_{x,y}} = O(\|\phi_{yyy}\|_{L^2_{x,y}} + \|\phi_{yy}\|_{L^2_{x,y}}).$$

Similarly, we can get $\|\chi g_{yy}\|_{L^2_{x,y}} = O(\|\phi_{yyy}\|_{L^2_{x,y}} + \|\phi_{yy}\|_{L^2_{x,y}})$, so we have

$$\begin{aligned}\left\|g_{yy}\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}}^2 &= \|g_{yy}\chi\|_{L^2_{x,y}}^2 + \left\|g_{yy}(1-\chi)\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}}^2 \\ &= O\bigg(\|\phi_{yyy}\|_{L^2_{x,y}}^2 + \|\phi_{yy}\|_{L^2_{x,y}}^2 + \left\|\sqrt{\bar{u}}g_{yy}\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}}^2\bigg) \\ &= O\bigg(\|g\|_{\Xi_0}^2 + L^{\frac{1}{3}}\|g\|_{\Xi_1}^2 + \left\|h\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}}^2\bigg)\end{aligned}$$

and

$$\begin{aligned}\left\|\bar{u}g_{yyy}\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}}^2 &= \left\|(\phi_{yyy} - 3\bar{u}_y g_{yy} - 3\bar{u}_{yy}g_y - \bar{u}_{yyy}g)\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}}^2 \\ &= O\bigg(\|g\|_{\Xi_0}^2 + L^{\frac{1}{3}}\|g\|_{\Xi_1}^2 + \left\|h\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}}^2\bigg).\end{aligned}$$

We conclude (A.27) as

$$\int_0^{x_0}\int_0^\infty -\partial_y^3[\bar{u}_x g]g_{xy}\frac{\rho^2}{\langle y\rangle^2}dxdy = O\bigg(\|g\|_{\Xi_0}^2 + L^{\frac{1}{3}}\|g\|_{\Xi_1}^2 + \left\|h\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}}^2\bigg). \tag{A.28}$$

Collecting (A.25), (A.26) and (A.28), we have

$$\begin{aligned}&\frac{1}{2}\left\|\bar{u}g_{xy}\frac{\rho}{\langle y\rangle}\right\|_{L^2_y(x=x_0)}^2 + \left\|\sqrt{\bar{u}}g_{xyy}\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}}^2 + \left\langle \bar{u}_y g_{xy}, g_{xy}\frac{\rho}{\langle y\rangle}\right\rangle_{L^2_x(y=0)} \\ &\quad = \frac{1}{2}\left\|\bar{u}g_{xy}\frac{\rho}{\langle y\rangle}\right\|_{L^2_y(x=0)}^2 + O\bigg(\|g\|_{\Xi_0}^2 + L^{\frac{1}{3}}\|g\|_{\Xi_1}^2 + \left\|h_x\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}}^2 + \left\|h\frac{\rho}{\langle y\rangle}\right\|_{L^2_{x,y}}^2\bigg).\end{aligned} \tag{A.29}$$

So we finish the proof of (A.16). □

Lemma A.1 shows if g satisfies the linear parabolic-type equation (A.13), then $\|g\|_{\Xi_0}$ and $\|g\|_{\Xi_1}$ can be dominated by its initial data and h, and we can use the standard method to prove the local existence of the solution. Following this way, we can also get the estimates of the high-order derivatives to show the smoothness of the solution. Because the equation (A.13) is linear, the local well-posedness means the global well-posedness, and the equation (A.13) admits a smooth solution g even if L is large.