SCIENCE CHINA
Mathematics

• ARTICLES •

August 2023 Vol. 66 No. 8: 1813–1830
https://doi.org/10.1007/s11425-022-2005-x

Hereditary uniform property Γ

Huaxin Lin[1,2]

[1]*Department of Mathematics, East China Normal University, Shanghai 200062, China;*
[2]*Department of Mathematics, University of Oregon, Eugene, OR 97405, USA*[†]

Email: hlin@uoregon.edu

Received February 24, 2022; accepted July 12, 2022; published online February 22, 2023

Abstract We study the uniform property Γ for separable simple C^*-algebras which have quasitraces and may not be exact. We show that a stably finite separable simple C^*-algebra A with the strict comparison and uniform property Γ has tracial approximate oscillation zero and stable rank one. Moreover in this case, its hereditary C^*-subalgebras also have a version of uniform property Γ. If a separable non-elementary simple amenable C^*-algebra A with strict comparison has this hereditary uniform property Γ, then A is $\mathcal{Z}$-stable.

Keywords simple C^*-algebra, uniform property Γ, tracial oscillation zero

MSC(2020) 46L35, 46L05

Citation: Lin H X. Hereditary uniform property Γ. Sci China Math, 2023, 66: 1813–1830, https://doi.org/10.1007/s11425-022-2005-x

1 Introduction

The uniform property Γ was recently introduced in [6] in the study of regularity properties for simple nuclear C^*-algebras, specifically, properties of the finite nuclear dimension and $\mathcal{Z}$-stability (see [19] and also [17,32]). More recently, it is shown in [5] that for a unital separable nuclear simple C^*-algebra A, A has the strict comparison and uniform property Γ if and only if A is $\mathcal{Z}$-stable, and if and only if A has the finite nuclear dimension, which is a significant recent advance towards the resolution of the Toms-Winter conjecture.

The uniform property Γ is originally only defined for unital C^*-algebras, or those C^*-algebras whose tracial state space is compact. In [4], a stabilized uniform property Γ was introduced and it is shown that if A is a (non-unital) separable simple nuclear C^*-algebra with strict comparison which has the stable rank one and stabilized uniform property Γ, then A is $\mathcal{Z}$-stable.

In this paper, we study the uniform property Γ for separable simple C^*-algebras using quasitraces instead of traces. Simple C^*-algebras with the strict comparison and uniform property Γ have a very nice matricial structure (see Theorem 3.3). We also find that if A has the strict comparison and uniform property Γ, then A has tracial approximate oscillation zero, and the canonical map $\Gamma: \mathrm{Cu}(A) \to \mathrm{LAff}_+(\widetilde{QT}(A))$ is surjective and has stable rank one, without assuming that A is amenable. In particular, $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes \mathcal{Z})$. Moreover, in this case, a version of the uniform property Γ holds for hereditary C^*-subalgebras. This property is called the hereditary uniform property Γ (see Definition 4.1),

[†]Current address

which is defined for C^*-algebras whose sets of normalized 2-quasitraces may not be compact, or even empty (but for C^*-algebras having densely defined non-zero traces). Therefore, the uniform property Γ is a strong condition even in the absence of amenability. However, there are separable simple C^*-algebras which have the strict comparison and hereditary uniform property Γ but are not $\mathcal{Z}$-stable (see Remark 4.7).

Regarding the Toms-Winter conjecture, we also obtain a similar conclusion as in [5] (for non-unital simple C^*-algebras). To be more specific, let A be a (non-unital) stably finite separable non-elementary simple nuclear C^*-algebra with strict comparison. Following [5], we show that A has the hereditary uniform property Γ if and only if A is $\mathcal{Z}$-stable. This result is similar to the statement in [4] for the non-unital case but we do not assume *a priori*, that A has stable rank one, or $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes \mathcal{K})$ (see Remark 4.5 and [29]). This is possible because we show that if A has the strict comparison and hereditary uniform property Γ, then A has tracial approximate oscillation zero. We also observe that if A is tracially approximately divisible, then A has the hereditary uniform property Γ. If A is a separable simple non-elementary amenable C^*-algebra with strict comparison, the converse also holds as, under the assumption that A is amenable, tracial approximate divisibility is equivalent to $\mathcal{Z}$-stability (which is essentially a restatement of Matui and Sato [24]) (see also [7]).

2 Preliminaries

Definition 2.1. Let A be a C^*-algebra and $F \subset A$ be a subset of A. Denote by $\mathrm{Her}(F)$ the hereditary C^*-subalgebra of A generated by F. Denote by $A^{\mathbf{1}}$ the unit ball of A, and by A_+ the set of all positive elements in A. Put $A_+^{\mathbf{1}} := A_+ \cap A^{\mathbf{1}}$. Denote by $\widetilde{A}$ the minimal unitization of A. Let $\mathrm{Ped}(A)$ denote the Pedersen ideal of A, $\mathrm{Ped}(A)_+ := \mathrm{Ped}(A) \cap A_+$ and $\mathrm{Ped}(A)_+^{\mathbf{1}} := \mathrm{Ped}(A) \cap A_+^{\mathbf{1}}$. Denote by $T(A)$ the tracial state space of A.

Definition 2.2. Let A and B be C^*-algebras and $\varphi : A \to B$ be a linear map. The map φ is said to be positive if $\varphi(A_+) \subset B_+$. The map φ is said to be completely positive contractive, abbreviated to c.p.c., if $\|\varphi\| \leqslant 1$ and $\varphi \otimes \mathrm{id} : A \otimes M_n \to B \otimes M_n$ is positive for all $n \in \mathbb{N}$. A c.p.c. map $\varphi : A \to B$ is called order zero, if for any $x, y \in A_+$, $xy = 0$ implies $\varphi(x)\varphi(y) = 0$ (see [34, Definition 2.3]). If $ab = ba = 0$, we also write $a \perp b$.

In what follows, $\{e_{i,j}\}_{i,j=1}^n$ (or just $\{e_{i,j}\}$, if there is no confusion) stands for a system of matrix units for M_n and $\iota \in C_0((0,1])$ denotes the identity function on $(0,1]$, i.e., $\iota(t) = t$ for all $t \in (0,1]$.

Notation 2.3. Let $\epsilon > 0$. Define a continuous function $f_\epsilon : [0,+\infty) \to [0,1]$ by

$$f_\epsilon(t) \begin{cases} = 0, & t \in [0, \epsilon/2], \\ = 1, & t \in [\epsilon, \infty), \\ \text{is linear}, & t \in [\epsilon/2, \epsilon]. \end{cases}$$

Definition 2.4. Let A be a C^*-algebra and $a, b \in (A \otimes \mathcal{K})_+$. We write $a \lesssim b$ if there is an $x_n \in A \otimes \mathcal{K}$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \|a - x_n^* b x_n\| = 0$. We write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$ both hold. The Cuntz relation $\sim$ is an equivalence relation. Set $\mathrm{Cu}(A) = (A \otimes \mathcal{K})_+/\sim$. Let $\langle a \rangle$ denote the equivalence class of a. We write $[a] \leqslant [b]$ if $a \lesssim b$.

Definition 2.5. Let A be a σ-unital C^*-algebra. A densely defined 2-quasitrace is a 2-quasitrace defined on $\mathrm{Ped}(A)$ (see [1, Definition II.1.1]). Denote by $\widetilde{QT}(A)$ the set of densely defined quasitraces on $A \otimes \mathcal{K}$. In what follows, we identify A with $A \otimes e_{1,1}$, whenever it is convenient. Let $\tau \in \widetilde{QT}(A)$. Then $\tau(a) \neq \infty$ for any $a \in \mathrm{Ped}(A)_+ \setminus \{0\}$.

We endow $\widetilde{QT}(A)$ with the topology in which a net $\{\tau_i\}$ converges to τ if $\{\tau_i(a)\}$ converges to $\tau(a)$ for all $a \in \mathrm{Ped}(A)$ (see also [11, p. 985, (4.1)]).

Denote by $QT(A)$ the set of those $\tau \in \widetilde{QT}(A)$ such that $\|\tau_A\| = 1$.

Note that for each $a \in (A \otimes \mathcal{K})_+$ and $\varepsilon > 0$, $f_\varepsilon(a) \in \mathrm{Ped}(A \otimes \mathcal{K})_+$. Define

$$\widehat{[a]}(\tau) := d_\tau(a) = \lim_{\varepsilon\to 0} \tau(f_\varepsilon(a)) \quad \text{for all } \tau \in \widetilde{QT}(A). \tag{2.1}$$

Definition 2.6. Let A be a simple C^*-algebra. Then A is said to have (Blackadar's) strict comparison, if given any $a, b \in (A \otimes \mathcal{K})_+$, one has $a \lesssim b$, whenever

$$d_\tau(a) < d_\tau(b) \quad \text{for all } \tau \in \widetilde{QT}(A) \setminus \{0\}. \tag{2.2}$$

Definition 2.7. Let A be a C^*-algebra with $\widetilde{QT}(A) \setminus \{0\} \neq \emptyset$. Let $S \subset \widetilde{QT}(A)$ be a convex subset. Set (if $0 \notin S$, we ignore the condition $f(0) = 0$)

$$\mathrm{Aff}_+(S) = \{f : C(S, \mathbb{R})_+ : f \text{ affine}, \, f(s) > 0 \text{ for } s \neq 0, \, f(0) = 0\} \cup \{0\}, \tag{2.3}$$

$$\mathrm{LAff}_+(S) = \{f : S \to [0, \infty] : \exists \{f_n\}, \, f_n \nearrow f, \, f_n \in \mathrm{Aff}_+(S)\}. \tag{2.4}$$

For a simple C^*-algebra A and each $a \in (A \otimes \mathcal{K})_+$, the function $\hat{a}(\tau) = \tau(a)$ $(\tau \in S)$ is in general in $\mathrm{LAff}_+(S)$. If $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$, then $\hat{a} \in \mathrm{Aff}_+(S)$. For $\widehat{[a]}(\tau) = d_\tau(a)$ defined above, we have $\widehat{[a]} \in \mathrm{LAff}_+(\widetilde{QT}(A))$.

We write $\Gamma : \mathrm{Cu}(A) \to \mathrm{LAff}_+(\widetilde{QT}(A))$ for the canonical map defined by $\Gamma([a])(\tau) = \widehat{[a]} = d_\tau(a)$ for all $\tau \in \widetilde{QT}(A)$.

In the case where A is algebraically simple (i.e., A is a simple C^*-algebra and $A = \mathrm{Ped}(A)$), Γ also induces a canonical map $\Gamma_1 : \mathrm{Cu}(A) \to \mathrm{LAff}_+(\overline{QT(A)}^w)$, where $\overline{QT(A)}^w$ is the weak $*$-closure of $QT(A)$. Since in this case, $\mathbb{R}_+ \cdot \overline{QT(A)}^w = \widetilde{QT}(A)$, the map Γ is surjective if and only if Γ_1 is surjective. We point out that in this case, $0 \notin \overline{QT(A)}^w$ and $\overline{QT(A)}^w$ is compact (see [14, Proposition 2.9]).

The following is known to experts.

Proposition 2.8 (See [1, II.4.4])**.** *Let A be a separable C^*-algebra. If $QT(A)$ is nonempty and compact, then $QT(A)$ is a Choquet simplex.*

Proof. If A is unital, by [1, II.4.4], $QT(A)$ is a Choquet simplex. If A is not unital, by [1, II.2.5], every 2-quasitrace extends to a 2-quasitrace on A with $\tau(1_{\widetilde{A}}) = \|\tau\|$. We then view $QT(A)$ as a closed convex subset of Choquet simplex $QT(\widetilde{A})$. On the other hand, any $\tau \in QT(\widetilde{A})$ has the form $\tau = \alpha\tau_0 + (1-\alpha)\tau_A$, where $0 \leqslant \alpha \leqslant 1$, $\tau_A \in QT(A)$ and τ_0 is the unique tracial state which vanishes on A.

By the Choquet theorem, α and τ_A are uniquely determined by τ. In particular, $QT(A)$ is a face of $QT(\widetilde{A})$. Now suppose that $\tau \in QT(\widetilde{A})$. Then there exists a unique (probability) boundary measure μ on $\partial_e(QT(\widetilde{A}))$ such that

$$f(\tau) = \int_{\partial_e(QT(\widetilde{A}))} f(s) d\mu \quad \text{for all } f \in \mathrm{Aff}(QT(\widetilde{A})). \tag{2.5}$$

If $\mu(\{\tau_0\}) = \alpha > 0$, then $\tau = \alpha\tau_0 + (1-\alpha)\tau_A$ for some $\tau_A \in QT(A)$. If $\tau \in QT(A)$, then $\alpha = 0$. In other words, μ is concentrated on $\partial_e(QT(A))$. We have just shown that every $\tau \in QT(A)$ is the barycenter of a unique normalized extremal boundary measure. So $QT(A)$ is a Choquet simplex. □

Definition 2.9. Let $l^\infty(A)$ be the C^*-algebra of bounded sequences of A. Recall that

$$c_0(A) := \left\{\{a_n\} \in l^\infty(A) : \lim_{n\to\infty} \|a_n\| = 0\right\}$$

is a (closed two-sided) ideal of $l^\infty(A)$. Let $A_\infty := l^\infty(A)/c_0(A)$ and $\pi^\infty : l^\infty(A) \to A_\infty$ be the quotient map. We view A as a subalgebra of $l^\infty(A)$ via the canonical map $\iota : a \mapsto \{a, a, \ldots\}$ for all $a \in A$. In what follows, we may identify a with the constant sequence $\{a, a, \ldots\}$ in $l^\infty(A)$ whenever it is convenient without further warning.

Put $A' = \{x = \{x_n\} \in l^\infty(A) : \lim_{n\to\infty} \|x_n a - a x_n\| = 0\}$.

Definition 2.10. Let A be a C^*-algebra $QT(A) \neq \{0\}$. Let $\tau \in \widetilde{QT}(A) \setminus \{0\}$. Define for each $x \in A$,

$$\|x\|_{_{2,\tau}} = \tau(x^*x)^{1/2}. \tag{2.6}$$

Let $S \subset \widetilde{QT}(A) \setminus \{0\}$ be a compact subset. Define

$$\|x\|_{_{2,S}} = \sup\{\tau(x^*x)^{1/2} : \tau \in S\}. \tag{2.7}$$

Put $I_{S,\mathbb{N}} = \{\{x_n\} \in l^\infty(A) : \lim_{n\to\infty} \|x\|_{_{2,S}} = 0\}$.

We quote the following proposition which follows from [1, II.2.2 and Theorem I.17].

Proposition 2.11 (See [18, Proposition 3.2]). *Let A be a C^*-algebra, $\tau \in QT(A)$ and $I = \{x \in A : \tau(x^*x) = 0\}$. Then I is a (closed two-sided) ideal and there is a unique 2-quasitrace $\bar{\tau}$ on A/I such that*

$$\tau(x) = \bar{\tau}(\rho(x)) \quad \textit{for all } x \in A, \tag{2.8}$$

where $\rho : A \to A/I$ is the quotient map.

Definition 2.12. Let $\varpi \in \beta(\mathbb{N}) \setminus \mathbb{N}$ be a free ultrafilter. Set

$$c_{0,\varpi} = \Big\{\{x_n\} \in l^\infty(A) : \lim_{n\to\omega} \|x_n\| = 0\Big\}. \tag{2.9}$$

Denote by $\pi_\infty : l^\infty(A) \to l^\infty(A)/c_{0,\varpi}$ the quotient map. Let $S \subset \widetilde{QT}(A)$ be a compact subset. Define

$$I_{S,\varpi} = \Big\{\{x_n\} \in l^\infty(A) : \lim_{n\to\varpi} \|x_n\|_{2,S} = 0\Big\}. \tag{2.10}$$

It is a (closed two-sided) ideal. In the case where $A = \mathrm{Ped}(A)$, we usually consider $I_{\overline{QT(A)}^w,\varpi}$. If A has the continuous scale, we consider $I_{QT(A),\varpi}$.

Denote by $\Pi_\varpi : l^\infty(A) \to l^\infty(A)/I_{\overline{QT(A)}^w,\varpi}$ the quotient map. We also write $\Pi : l^\infty(A) \to l^\infty(A)/I_{QT(A),\mathbb{N}}$ for the quotient map.

For convenience, abusing the notation, we may also write A' for $\Pi(A')$ as well as $\Pi_\varpi(A')$.

If $\tau_n \in QT(A)$, for $x = \{x_n\} \in l^\infty(A)$, define

$$\tau_\varpi(x) = \lim_{n\to\varpi} \tau_n(x_n). \tag{2.11}$$

It is a 2-quasitrace on $l^\infty(A)$.

Fix $\{\tau_n\} \subset QT(A)$. Let $J = \{\{x_n\} \in l^\infty(A) : \tau_\varpi(\{x_n^*x_n\}) = 0\}$. Then J is a (closed two-sided) ideal of $l^\infty(A)$ and $\tau_\varpi|_J = 0$. If $x = \{x_n\} \in (I_{\overline{QT(A)}^w,\varpi})_{\text{s.a.}}$, then

$$\lim_{n\to\varpi} |\tau_n(x_n)|^2 \leqslant \lim_{n\to\varpi} \tau_n(x_n^*x_n) \leqslant \lim_{n\to\varpi} \|x_n^*x_n\|^2_{2,\overline{QT(A)}^w} = 0. \tag{2.12}$$

In other words, $\tau_\varpi(x) = 0$ and $x \in I_{\overline{QT(A)}^w,\varpi}$.

Since τ_ϖ is a 2-quasitrace on $l^\infty(A)$, by [18, Proposition 4.2] (see also Proposition 2.11), $\tau_\varpi = \tau_\varpi \circ \pi_J$, where $\pi_J : l^\infty(A) \to l^\infty(A)/J$ is the quotient map. In particular, $\tau_\varpi(x + j) = \tau_\varpi(x)$ for all $x \in l^\infty(A)$ and $j \in J$. Since we have shown $I_{\overline{QT(A)}^w,\varpi} \subset J$, we may also view τ_ϖ as a normalized 2-quasitrace on $l^\infty(A)/I_{\overline{T(A)}^w,\varpi}$. Similarly, we may view τ_ϖ as a normalized 2-quasitrace of $l^\infty(A)/c_{0,\varpi}$.

If $\tau_n = \tau$ for all $n \in \mathbb{N}$, we may write τ instead of τ_ϖ.

Denote by $QT_\varpi(A)$ the set $\{\tau_\varpi : \{\tau_n\} \subset QT(A)\}$.

The following is a variation of [1, II.2.5]. Note that δ below depends on ε but not τ.

Lemma 2.13 (See [1, II.2.5]). *Let A be a separable C^*-algebra with $QT(A) \neq \emptyset$. Then for any $\varepsilon > 0$, there exists a $\delta > 0$ satisfying the following: for any normal elements $a, b \in A^{\mathbf{1}}$ such that $\|ab - ba\|_{2,\overline{QT(A)}^w} < \delta$, then for any $\tau \in QT(A)$,*

$$|\tau(a + b) - \tau(a) + \tau(b)| < \varepsilon. \tag{2.13}$$

Proof. Suppose not, and then for some $\varepsilon_0 > 0$, there exist a sequence of pairs of normal elements $a_n, b_n \in A^{\mathbf{1}}$ and a sequence $\{\tau_n\} \subset QT(A)$ such that $\|a_nb_n - b_na_n\|_{2,\overline{QT(A)}^w} < 1/n$ but

$$|\tau_n(a_nb_n) - \tau_n(a_n) + \tau_n(b_n)| \geqslant \varepsilon_0, \quad n \in \mathbb{N}. \tag{2.14}$$

Put $a = \Pi_\varpi(\{a_n\})$ and $b = \Pi_\varpi(\{b_n\})$. Then a and b are normal and $ab = ba$. Define $\tau_\varpi(\{x_n\}) = \lim_{n\to\varpi} \tau_n(x_n)$ for $\{x_n\} \in l^\infty(A)$. View $\tau_\varpi \in QT(l^\infty(A)/I_{\overline{T(A)}^w,\varpi})$. Then $\tau_\varpi(a + b) = \tau_\varpi(a) + \tau_\varpi(b)$. This contradicts (2.14). □

Proposition 2.14 (See [5, Proposition 3.1], [28, Lemma 4.2(ii)] and [12, Proposition 4.3.6]). *Let A be a separable C^*-algebra with $QT(A) \neq \emptyset$ and $K \subset \partial_e(QT(A))$ be a compact subset. Then for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$ and the finite subset $\mathcal{G} \subset A$ satisfying the following: supposing that $b \in A^{\mathbf{1}}$ such that*

$$\|cb - bc\|_{2,K} < \delta \quad \text{for all } \tau \in K \text{ and } c \in \mathcal{G}, \tag{2.15}$$

then for all $a \in \mathcal{F}$,

$$\sup\{|\tau(ab) - \tau(a)\tau(b)| : \tau \in K\} < \varepsilon. \tag{2.16}$$

Proof. One notes that the proof of [5, Proposition 3.1] works for $QT(A)$. Then this proposition follows from a similar proof. □

Definition 2.15 (See [14, Definitions 4.1, 4.7 and 5.1]). Let A be a C^*-algebra with $\widetilde{QT}(A) \setminus \{0\} \neq \emptyset$. Let $S \subset \widetilde{QT}(A) \setminus \{0\}$ be a compact subset. Define for each $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$,

$$\omega(a)\,|_S = \inf\{\sup\{d_\tau(a) - \tau(c) : \tau \in S\} : c \in \overline{a(A \otimes \mathcal{K})a},\ 0 \leqslant c \leqslant 1\} \tag{2.17}$$

(see [9, A1]). The number $\omega(a)\,|_S$ is called the (tracial) oscillation of a on S.

We are only interested in the case where $\mathbb{R}_+ \cdot S = \widetilde{QT}(A)$. Let $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$. We write $\Omega^T(a) = 0$ if there exists a sequence $c_n \in \mathrm{Her}(a)^{\mathbf{1}}_+$ with $\lim_{n\to\infty} \omega(c_n)\,|_S = 0$ such that $\lim_{n\to\infty} \|a - c_n\|_{2,S} = 0$. Note that $\Omega^T(a) = 0$ does not depend on the choice of S (as long as $\mathbb{R}_+ \cdot S = \widetilde{QT}(A)$ [14, Definition 4.7]).

A separable simple C^*-algebra A is said to have T-tracial approximate oscillation zero, if for any $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$, $\Omega^T(a) = 0$. We say that A has tracial approximate oscillation zero if A has T-tracial approximate oscillation zero and strict comparison.

If A is a separable simple C^*-algebra and $b \in \mathrm{Ped}(A)_+$, then by Brown's stable isomorphism theorem, $\mathrm{Her}(b) \otimes \mathcal{K} \cong A \otimes \mathcal{K}$. So we may view $a \in \mathrm{Ped}(\mathrm{Her}(b) \otimes \mathcal{K})_+$. Note that $\mathrm{Her}(b)$ is algebraically simple. We often assume that A is algebraically simple and choose S to be $\overline{QT(A)}^w$. In that case, we omit S.

3 Uniform property Γ

Let us recall the definition of the uniform property Γ. We fix a free ultrafilter $\varpi \in \beta(\mathbb{N}) \setminus \mathbb{N}$.

Definition 3.1 (See [6, Definition 2.1] and [16, Definition 2.1]). Let A be a separable C^*-algebra with nonempty compact $QT(A)$. We say that A has the uniform property Γ, if for any $n \in \mathbb{N}$, there exist pairwise orthogonal projections $p_1, p_2, \ldots, p_n \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$ (see Definition 2.9) such that for $1 \leqslant i \leqslant n$,

$$\tau(p_i a) = \frac{1}{n}\tau(a) \quad \text{for all } a \in A \text{ and } \tau \in QT_\varpi(A). \tag{3.1}$$

It should be noted that we do not assume all the 2-quasitraces are traces. Let $p = \sum_{i=1}^n p_i$. Then p is a projection and $\tau(pa) = \tau(a)$ for all $\tau \in QT_\varpi(A)$ and $a \in A$. Suppose that $c_k \in (l^\infty(A) \cap A')^{\mathbf{1}}_+$ such that $\Pi_\varpi(\{c_k\}) = p$. Then for all $a \in A_+$,

$$\|ac_k - a\|^2_{2,QT(A)} \leqslant \sup\{\tau(a - a^{1/2}c_k a^{1/2}) : \tau \in QT(A)\} \to 0 \quad \text{as } k \to \varpi. \tag{3.2}$$

It follows that $\Pi_\varpi(\iota(a))p = \Pi_\varpi(\iota(a))$ for all $a \in A$. Let $e \in A^{\mathbf{1}}_+$ be a strictly positive element of A. Then $d_\tau(e_A) = 1$ for all $\tau \in QT(A)$. By the Dini theorem, $\tau(e^{1/k}) \nearrow d_\tau(e)$ uniformly on $QT(A)$. By [1, II.2.5], we extend each $\tau \in QT(A)$ to a 2-quasitrace in $QT(\widetilde{A})$ which we still write τ (so $\tau(1_{\widetilde{A}}) = 1$ [1, II.2.5]), if A is not unital. Therefore, for any $\{a_k\} \in l^\infty(A)^{\mathbf{1}}$,

$$\lim_{k\to\infty} \|a_k(1_{\widetilde{A}} - e^{1/k})\|_{2,QT(A)} \leqslant \lim_{k\to\infty} \|1_{\widetilde{A}} - e^{1/k}\|_{2,QT(A)} = 0 \tag{3.3}$$

(see [18, Lemma 3.5] and [14, Definition 2.16]). It follows that $l^\infty(A)/I_{_{QT(A),\varpi}}$ has a unit $E := \Pi_\varpi(\{e^{1/k}\})$. Suppose that $E - p \neq 0$. Then there would be a non-zero element $b = \{b_n\} \in l^\infty(A)^{\mathbf{1}}_+$ such that $p\Pi_\varpi(b) = 0$. Then for all $k \in \mathbb{N}$,

$$\Pi_\varpi(\iota(e^{1/k}))\Pi_\varpi(b) = \Pi_\varpi(\iota(e^{1/k}))p\Pi_\varpi(b) = 0, \tag{3.4}$$

or

$$\Pi_\varpi(E - \iota(e^{1/k}))\Pi_\varpi(b) = \Pi_\varpi(b). \tag{3.5}$$

However, since $\tau(e^{1/k}) \nearrow 1$ uniformly on $QT(A)$, for any $\varepsilon > 0$, there exists a $k \in \mathbb{N}$ such that $\|E - \iota(e^{1/k})\|_{_{QT(A),\varpi}} < \varepsilon$. Hence,

$$\|\Pi_\varpi(b)\| < \varepsilon. \tag{3.6}$$

It follows that

$$p = E = 1_{l^\infty(A)/I_{_{QT(A),\varpi}}}.$$

Note that we follow the same spirit in [6], so the uniform property Γ, as in [6, Definition 2.1] (see also [5]), is only defined for separable C^*-algebras with compact $QT(A)$. It is worth mentioning that if A is a σ-unital simple C^*-algebra with nonempty compact $QT(A)$ and strict comparison, then (by the Dini theorem) A has the continuous scale. It follows that A is algebraically simple (see [20, Theorem 3.3]).

Proposition 3.2 (See [5, Corollary 3.2]). *Let A be a separable simple C^*-algebra with nonempty compact $QT(A)$. If A has the uniform property Γ, then for any $n \in \mathbb{N}$, there are mutually orthogonal projections $p_1, p_2, \ldots, p_n \in (l^\infty(A) \cap A')/I_{_{QT(A),\varpi}}$ such that for $1 \leqslant i \leqslant n$,*

$$\tau(p_i) = \frac{1}{n} \quad \text{for all } \tau \in QT_\varpi(A). \tag{3.7}$$

Conversely, suppose that $\partial_e(T(A))$ is σ-compact and that there are mutually orthogonal projections

$$p_1, p_2, \ldots, p_n \in (l^\infty(A) \cap A')/I_{_{QT(A),\varpi}}$$

such that for $1 \leqslant i \leqslant n$, (3.7) *holds. Then for any $a \in A$ and $1 \leqslant i \leqslant n$,*

$$\tau(p_i a) = \frac{1}{n}\tau(a) \quad \text{for all } \tau \in QT(A). \tag{3.8}$$

Note that in (3.8), $\tau \in QT(A)$ not in $QT_\varpi(A)$.

Proof of Proposition 3.2. Suppose that A has the uniform property Γ. Then for any $n \in \mathbb{N}$, there exist mutually orthogonal projections $p_1, p_2, \ldots, p_n \in (l^\infty(A) \cap A')/I_{_{QT(A),\varpi}}$ such that for $1 \leqslant i \leqslant n$,

$$\tau(p_i a) = \frac{1}{n}\tau(a) \quad \text{for all } \tau \in QT_\varpi(A). \tag{3.9}$$

Let $\{p_i^{(m)}\} \in (l^\infty(A) \cap A')^{\mathbf{1}}_+$ be such that $\Pi_\varpi(\{p_i^{(m)}\}) = p_i$, $1 \leqslant i \leqslant n$. Choose a strictly positive element $e \in A^{\mathbf{1}}_+$. Let $\varepsilon \in (0, 1/2)$. Since $QT(A)$ is compact, by the Dini theorem, there exists a $k \in \mathbb{N}$ such that

$$\sup\{1 - \tau(e^{1/k}) : \tau \in QT(A)\} < \varepsilon. \tag{3.10}$$

It follows that for all $1 \leqslant i \leqslant n$,

$$\tau(p_i) \geqslant \tau(e^{1/k}p_i) = \frac{1}{n}\tau(e^{1/k}) > \frac{1}{n} - \frac{\varepsilon}{n} \quad \text{for all } \tau \in QT_\varpi(A). \tag{3.11}$$

Letting $\varepsilon \to 0$, we obtain that for all $1 \leqslant i \leqslant n$,

$$\tau(p_i) \geqslant \frac{1}{n} \quad \text{for all } \tau \in QT_\varpi(A). \tag{3.12}$$

Since $\sum_{i=1}^n p_i = 1$, it follows that $\tau(p_i) = \frac{1}{n}$ for all $\tau \in QT_\varpi(A)$.

For the second part of this proposition, suppose that there are mutually orthogonal projections $p_1, p_2, \ldots, p_n \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$ such that for $1 \leqslant i \leqslant n$, (3.7) holds. Let $a \in A$. We show that for any $\tau \in QT(A)$, (3.8) holds. It suffices to show this for the case $a \in A_+^{\mathbf{1}}$.

Suppose not, and then there are $a \in A_+^{\mathbf{1}}$ and $\tau \in QT(A)$ such that

$$\left|\frac{1}{n}\tau(a) - \tau(p_i a)\right| > \sigma \tag{3.13}$$

for some $1 > \sigma > 0$.

Choose $\varepsilon \in (0, \sigma/16)$. By the Choquet theorem, there exists a probability Borel measure μ_τ on $QT(A)$ concentrated on $\partial_e(QT(A))$ such that for any $f \in \mathrm{Aff}(QT(A))$,

$$f(\tau) = \int_{\partial_e(QT(A))} f d\mu_\tau. \tag{3.14}$$

Since $\partial_e(QT(A))$ is σ-compact, there exists a compact subset $K \subset \partial_e(QT(A))$ such that

$$\mu(\partial_e(QT(A)) \setminus K) < \varepsilon. \tag{3.15}$$

It follows from Proposition 2.14 (see also [5, Proposition 3.1]) that there are $\delta > 0$ and the finite subset $\mathcal{G} \subset A$ such that if $b \in A_+^{\mathbf{1}}$ such that $\|[x, b]\| < \delta$ for all $x \in \mathcal{G}$, then

$$\sup\{|t(ab) - t(a)t(b)| : \tau \in K\} < \varepsilon. \tag{3.16}$$

Let $\{p_i^{(m)}\} \in (l^\infty(A) \cap A')_+^{\mathbf{1}}$ be such that $\Pi_\varpi(\{p_i^{(m)}\}) = p_i$ $(1 \leqslant i \leqslant n)$. For any $\mathcal{P} \in \varpi$, there is an $m \in \mathcal{P}$ such that

$$\left|\frac{1}{n}\tau(a) - \tau(p_i^{(m)} a)\right| > \sigma/2, \tag{3.17}$$

$$\|[x, p_i^{(m)}]\| < \delta \quad \text{for all } x \in \mathcal{G}, \tag{3.18}$$

$$\sup\left\{\left|t(p_i^{(m)}) - \frac{1}{n}\right| : t \in T(A)\right\} < \varepsilon. \tag{3.19}$$

Then by the choice of δ, we estimate that

$$\begin{aligned}
\left|\frac{1}{n}\tau(a) - \tau(ap_i^{(m)})\right| &= \left|\int_{\partial_e(QT(A))} \left(\frac{1}{n}\widehat{a} - \widehat{ap_i^{(m)}}\right) d\mu_\tau\right| \\
&\leqslant \int_{\partial_e(QT(A))} \left|\frac{1}{n}\widehat{a} - \widehat{ap_i^{(m)}}\right| d\mu_\tau \\
&< \int_K \left|\frac{1}{n}\widehat{a} - \widehat{ap_i^{(m)}}\right| d\mu_\tau + 2\varepsilon \text{ (by (3.15))} \\
&< \int_K \left|\frac{1}{n}\widehat{a} - \frac{1}{n}\widehat{a}\right| d\mu_\tau + 4\varepsilon = 4\varepsilon < \sigma/2 \text{ (by (3.16) and (3.19))}.
\end{aligned}$$

This contradicts (3.17) and the proof is completed. □

If A has strict comparison, then the uniform property Γ provides a unital homomorphism $\varphi : M_n \to l^\infty(A)/I_{QT(A),\varpi}$ as follows.

Theorem 3.3. *Let A be a non-elementary separable simple C^*-algebra with strict comparison and nonempty compact $QT(A)$. If A has the uniform property Γ, then for any $n \in \mathbb{N}$, there is a unital homomorphism $\varphi : M_n \to l^\infty(A)/I_{QT(A),\varpi}$ such that $\varphi(e_{i,i}) \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$ and for all $1 \leqslant i \leqslant n$,*

$$\tau(a\varphi(e_{i,i})) = \frac{1}{n}\tau(a) \quad \textit{for all } a \in A \textit{ and } \tau \in QT_\varpi(A). \tag{3.20}$$

Proof. By [1, II.2.5], we extend each $\tau \in QT(A)$ to a 2-quasitrace in $QT(\widetilde{A})$ with $\tau(1_{\widetilde{A}}) = 1$ (if A is not unital).

Fix an integer $n \in \mathbb{N}$ with $n \geqslant 2$. Let $l \in \mathbb{N}$. Choose an integer $m(l) \in \mathbb{N}$ such that

$$\left|\frac{n}{m(l)}\right| < \frac{1}{2(n+l)^2}, \quad l = 1, 2, \dots \tag{3.21}$$

Let $K = nm(l) + n(n+1)/2$.

Since A has the uniform property Γ, there exist projections $p_{1,l}, p_{2,l}, \dots, p_{K,l} \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$ such that for $1 \leqslant i \leqslant n$,

$$\sum_{i=1}^K p_{i,l} = 1_{(l^\infty(A)\cap A')/I_{QT(A),\varpi}}, \tag{3.22}$$

$$\tau(p_{i,l}a) = \frac{1}{K}\tau(a) \quad \text{and} \quad \tau(p_{i,l}) = \frac{1}{K} \quad \text{for all } a \in A \text{ and } \tau \in QT_\varpi(A). \tag{3.23}$$

We write $P_{i,l} = \{p_{i,l}^{(k)}\}$, where $\{p_{i,l}^{(k)}\} \in (l^\infty(A) \cap A')_+^{\mathbf{1}}$ such that $\Pi_\varpi(P_i) = p_{i,l}$, $1 \leqslant i \leqslant K$. Moreover, $p_{i,l}^{(k)} \perp p_{j,l}^{(k)}$, if $i \neq j$ and $1 \leqslant i, j \leqslant K$. By replacing $p_{i,l}^{(k)}$ by $f_{1/4}(p_{i,l}^{(k)})$ if necessary, we may assume that $\{p_{i,l}^{(k)}\}$ is a permanent projection lifting of $p_{i,l}$ ($1 \leqslant i \leqslant n$) (see [14, Proposition 6.2] and [23, Proposition 2.21]). Therefore, by [14, (1) and (2) of Proposition 6.2] (see also [23, Proposition 2.21]), we may assume that

$$\lim_{k\to\varpi} \sup\{\tau(p_{i,l}^{(k)}) - \tau(f_{1/4}(p_{i,l}^{(k)})p_{i,l}^{(k)}) : \tau \in QT(A)\} = 0, \tag{3.24}$$

$$\lim_{k\to\varpi} \sup\{d_\tau(p_{i,l}^{(k)}) - \tau((p_{i,l}^{(k)})^2) : \tau \in QT(A)\} = 0. \tag{3.25}$$

Since $\tau((p_{i,l}^{(k)})^2) \leqslant \tau((p_{i,l}^{(k)}))$ for all $\tau \in QT(A)$, we obtain

$$\lim_{k\to\varpi} \sup\{d_\tau(p_{i,l}^{(k)}) - \tau((p_{i,l}^{(k)})) : \tau \in QT(A)\} = 0. \tag{3.26}$$

Since $p_{i,l}$ is a projection, $f_{1/4}(p_{i,l}) = p_{i,l}$ ($1 \leqslant i \leqslant n$). Consequently,

$$\lim_{k\to\varpi} \|p_{i,l}^{(k)} - f_{1/4}((p_{i,l}^{(k)}))\|_{2,QT(A)} = 0. \tag{3.27}$$

Note that (recalling that $p_{i,l}^{(k)}$ commutes with $f_{1/4}((p_{i,l}^{(k)}))$)

$$|\tau(p_{i,l}^{(k)}) - \tau(f_{1/4}((p_{i,l}^{(k)})))| \leqslant \tau(1_{\widetilde{A}})^{1/2}\tau((p_{i,l}^{(k)} - f_{1/4}((p_{i,l}^{(k)})))^2)^{1/2} \quad \text{for all } \tau \in QT(A). \tag{3.28}$$

By (3.27), we have

$$\lim_{k\to\varpi} \sup\{|\tau(p_{i,l}^{(k)}) - \tau(f_{1/4}((p_{i,l}^{(k)})))| : \tau \in QT(A)\} = 0. \tag{3.29}$$

Let $q_{1,l}$ be $m(l)+1$ copies of $p_{i,l}$'s, $q_{2,l}$ be $m(l)+2$ copies of $p_{i,l}$'s, $\dots$, and $q_{n,l}$ be $m(l)+n$ copies of p_i's. Then

$$\sum_{i=1}^n q_{i,l} = \sum_{i=1}^K p_{i,l} \quad \text{and} \quad \tau\bigg(\sum_{i=1}^n q_{i,l}\bigg) = \frac{nm(l) + n(n+1)/2}{K} = 1 \quad \text{for all } \tau \in QT_\varpi(A). \tag{3.30}$$

Write $q_{i,l} = \Pi(\{c_{i,l}^{(k)}\})$, where $c_{i,l}^{(k)}$ is the sum of $m(l)+i$ copies of $p_{i,l}^{(k)}$. Then (by (3.23))

$$\lim_{k\to\varpi} \sup\left\{\left|\tau(ac_{i,l}^{(k)}) - \frac{m(l)+i}{K}\tau(a)\right| : \tau \in QT(A)\right\} = 0, \tag{3.31}$$

$$\lim_{k\to\varpi} \sup\left\{\left|\tau(c_{i,l}^{(k)}) - \frac{m(l)+i}{K}\right| : \tau \in QT(A)\right\} = 0 \tag{3.32}$$

for all $a \in A$. Note that for each fixed n and $1 \leqslant i \leqslant n$,

$$\lim_{l\to\infty} \frac{m(l)+i}{K} = \frac{1}{n}. \tag{3.33}$$

Let $\{\mathcal{F}_k\}$ be an increasing sequence of finite subsets of A such that $\bigcup_{k=1}^{\infty} \mathcal{F}_k$ is dense in A. Then for each $l \in \mathbb{N}$, by (3.32), (3.26) and (3.29) as well as (3.31) (recalling also $p_{i,l} \in A'$), we find an integer $k(l) \in \mathbb{N}$ such that $k(l) < k(l+1)$,

$$d_\tau(c_{1,l}^{(k(l))}) < d_\tau(c_{2,l}^{(k(l))}) < \cdots < d_\tau(c_{n,l}^{(k(l))}) \quad \text{for all } \tau \in QT(A), \tag{3.34}$$

$$\tau(f_{1/4}(c_{i,l}^{(k(l))})) > \frac{1}{n} - 1/(2(n+l))^2 \quad \text{for all } \tau \in QT(A), \tag{3.35}$$

$$\sup\left\{\left|\tau(ac_{i,l}^{(k(l))}) - \frac{m(l)+i}{K}\tau(a)\right| : \tau \in QT(A)\right\} < 1/l, \tag{3.36}$$

$$\sup\{|\tau(p_{i,l}^{(k)}) - \tau(f_{1/4}((p_{i,l}^{(k)})))| : \tau \in QT(A)\} < 1/l, \tag{3.37}$$

$$\|[c_{i,l}^{k(l)}, b]\| < 1/l \quad \text{for all } b \in \mathcal{F}_k \text{ and } 1 \leqslant i \leqslant n. \tag{3.38}$$

Since A has strict comparison, by (3.34), we obtain $x_{i,l} \in A$ such that

$$x_{i,l}^* x_{i,l} = f_{1/4}(c_{1,l}^{(k(l))}) \quad \text{and} \quad x_{i,l}x_{i,l}^* \in \mathrm{Her}(c_{i,l}^{(k(l))}), \quad i = 2, 3, \ldots, n. \tag{3.39}$$

Recall that $c_{i,l}^{(k)} \perp c_{j,l}^{(k(l))}$, if $i \neq j$ and $1 \leqslant i, j \leqslant n$. Write $x_{i,l} = u_{i,l} f_{1/4}(c_{1,l}^{(k(l))})^{1/2}$, $1 \leqslant i \leqslant n$.

This provides a homomorphism $\varphi^{(l)} : C_0((0,1]) \otimes M_n \to A$ such that

$$\varphi^{(l)}(\jmath \otimes e_{1,1}) = (x_{2,l}^* x_{2,l})^{1/2} = (f_{1/4}(c_{1,l}^{(k(l))}))^{1/2}, \tag{3.40}$$

$$\varphi^{(l)}(\jmath \otimes e_{1,j}) = x_{j,l}, \quad \varphi^{(l)}(\iota \otimes e_{j,1}) = x_{j,l}^*, \quad 2 \leqslant j \leqslant n, \tag{3.41}$$

$$\varphi^{(l)}(\jmath \otimes e_{i,j}) = u_{i,l} f_{1/4}(c_{1,l}^{(k(l))}) u_{j,l}^*, \quad 2 \leqslant i, j \leqslant n, \tag{3.42}$$

$$\varphi^{(l)}(\jmath \otimes e_{i,i}) = (x_{i,l}x_{i,l}^*)^{1/2}, \quad i > 1, \tag{3.43}$$

$$\varphi^{(l)}(\jmath \otimes 1_n) = f_{1/4}(c_{1,l}^{k(l)})^{1/2} + \sum_{i=2}^{n} (x_{i,l}x_{i,l}^*)^{1/2}, \tag{3.44}$$

where $\jmath$ is the identity function on $[0,1]$. Define $\psi^{(l)} : M_n \to A$ by $\psi^{(l)}(e_{i,j}) = \varphi^{(l)}(\jmath \otimes e_{i,j})$ $(1 \leqslant i, j \leqslant n)$. Then $\psi^{(l)}$ is an order zero c.p.c. map. We also have (as $l \to \infty$)

$$\|\psi^{(l)}(e_{i,i}) - c_{i,l}^{(k(l))}\|_{2,QT(A)} \to 0, \tag{3.45}$$

$$\left\|\psi^{(l)}(1_n) - \sum_{i=1}^{n} c_{i,l}^{(k(l))}\right\|_{2,QT(A)} \to 0. \tag{3.46}$$

Define $\Psi = \{\psi^{(l)}\} : M_n \to l^\infty(A)$ and $\varphi = \Pi_\varpi \circ \Psi : M_n \to l^\infty(A)/I_{QT(A),\varpi}$. Then φ is an order zero c.p.c. map. By (3.46), it is unital. Hence, φ is a unital homomorphism. Combining (3.36) with (3.33), we obtain

$$\tau(a\varphi(1_n)) = \frac{1}{n}\tau(a) \quad \text{for all } a \in A \text{ and } \tau \in QT_\varpi(A). \tag{3.47}$$

Note that by (3.38), $\{c_{i,l}^{(k(l))}\} \in A'$. Thus, by (3.45), we have $\varphi(e_{i,i}) \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$. □

Proposition 3.4. *Let A be a separable C^*-algebra with nonempty compact $QT(A)$. Suppose that A has the uniform property Γ. Then for any $k \in \mathbb{N}$, $M_k(A)$ also has the uniform property Γ.*

Proof. Fix $k \in \mathbb{N}$. Let $n \in \mathbb{N}$. Since A has the uniform property Γ, there are mutually orthogonal projections $p_1, p_2, \ldots, p_n \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$ such that $\sum_{i=1}^{n} p_i = 1$ and

$$\tau(ap_i) = \frac{1}{n}\tau(a) \quad \text{for all } a \in A \text{ and } \tau \in QT_\varpi(A). \tag{3.48}$$

Put $q_i = p_i \otimes 1_{M_k}$, $i = 1, 2, \ldots, n$. Then q_i's are projections and $\sum_{i=1}^n q_i = 1_{M_k(C)}$, where $C = l^\infty(A)/I_{QT(A),\varpi}$, and for any $b = (a_{i,j})_{k\times k} \in M_k(A)$, $q_i b = bq_i$ and $\tau(bq_i) = \frac{1}{n}\tau(b)$ for all $\tau \in QT_\varpi(M_k(A))$. □

Theorem 3.5. *Let A be a non-elementary separable simple C^*-algebra with strict comparison and nonempty compact $QT(A)$. Suppose that A has the uniform property Γ. Then Γ is surjective (see Definition 2.7).*

Proof. Fix $a \in A_+^{\mathbf{1}} \setminus \{0\}$ and $n \in \mathbb{N}$. There is an $r \in (0, 1/2)$ such that $f_r(a) > 0$. Set

$$\sigma_0 = \inf\{\tau(f_r(a)) : \tau \in QT(A)\} > 0. \tag{3.49}$$

Choose $m \in \mathbb{N}$ such that $1/m < \sigma_0/8(n+1)$. Since A has the uniform property Γ, there is a projection $p \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$ such that

$$\tau(bp) = \frac{1}{nm}\tau(b) \quad \text{for all } \tau \in QT_\varpi(A) \text{ and } b \in A. \tag{3.50}$$

Fix $\varepsilon \in (0, r/2)$. Then for $\eta \in \{\varepsilon, \varepsilon/2, \varepsilon/4, \varepsilon/8\}$,

$$\tau(f_\eta(a)p) = \frac{1}{nm}\tau(f_\eta(a)) \quad \text{for all } \tau \in QT_\varpi(A). \tag{3.51}$$

Choose $\delta \in (0, 1/(8(n+1)m)^2)$. Recall that $p \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$. Therefore (by lifting p to a sequence in $l^\infty(A) \cap A'$), we obtain an element $e \in A_+^{\mathbf{1}}$ such that for any $\eta \in \{\varepsilon, \varepsilon/2, \varepsilon/4, \varepsilon/8\}$ and all $\tau \in QT(A)$,

$$\frac{1}{nm}\tau(f_\eta(a)) + \frac{1}{2(n+1)m^3} > \tau(ef_\eta(a)e) > \frac{1}{nm}\tau(f_\eta(a)) - \frac{1}{2(n+1)m^2}. \tag{3.52}$$

Put $c := ef_{\varepsilon/4}(a)e$. Then by (3.52),

$$d_\tau(c) \geqslant \tau(ef_{\varepsilon/4}(a)e) > \frac{1}{nm}\tau(f_{\varepsilon/4}(a)) - 1/2(n+1)m^2 \quad \text{for all } \tau \in QT(A). \tag{3.53}$$

Choose $b \in (A \otimes \mathcal{K})_+^{\mathbf{1}}$ such that $[b] = (m-1)[c]$. Then for all $\tau \in QT(A)$,

$$\begin{aligned}(n+1)\widehat{[b]} = (n+1)(m-1)\widehat{[c]} &> \frac{(n+1)(m-1)}{nm}(\tau(f_{\varepsilon/4}(a))) - 1/2m \\ &> \tau(f_{\varepsilon/4}(a)) + \frac{1}{n}\tau(f_{\varepsilon/4}(a)) - \frac{1}{m} - \frac{1}{nm} - \frac{1}{2m} \\ &\geqslant \tau(f_{\varepsilon/4}(a)) + \frac{\sigma_0}{n} - \frac{1}{m} - \frac{1}{nm} - \frac{1}{2m} \\ &> \tau(f_{\varepsilon/4}(a)) \geqslant d_\tau(f_\varepsilon(a)).\end{aligned} \tag{3.54}$$

Since A has strict comparison,

$$(n+1)[b] \geqslant [f_\varepsilon(a)]. \tag{3.55}$$

By (3.52), for all $\tau \in QT(A)$, we also have

$$\begin{aligned}n\widehat{[b]} = n(m-1)\widehat{[c]} &\leqslant \frac{m-1}{m}\tau(f_{\varepsilon/4}(a)) + \frac{1}{2m^2} \\ &\leqslant \tau(f_{\varepsilon/4}(a)) - \left(\frac{\sigma_0}{m} - \frac{1}{2m^2}\right) \leqslant \tau(f_{\varepsilon/4}(a)) \leqslant d_\tau(a).\end{aligned} \tag{3.56}$$

It follows that

$$n[b] \leqslant [a]. \tag{3.57}$$

By Proposition 3.4, (3.55) and (3.57) also hold for any $a \in M_n(A)_+$. It follows that (3.55) and (3.57) hold for any $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$. We use an argument of Robert [26] to finish the proof.

Let $x' \ll x \in \mathrm{Cu}(A)$. Choose $a \in (A \otimes \mathcal{K})_+^{\mathbf{1}}$ such that $x = [a]$. Then for some $\varepsilon \in (0, 1/2)$, $x' \leqslant [f_\varepsilon(a)]$. Now $f_{\varepsilon/2}(a) \in \mathrm{Ped}(A \otimes \mathcal{K})_+$. By what has been proved, there is a $b \in \mathrm{Ped}(A \otimes K)_+$ such that

$$x' \leqslant [f_\varepsilon(a)] \leqslant [f_\varepsilon(f_{\varepsilon/2}(a))] \leqslant (n+1)[b] \quad \text{and} \quad n[b] \leqslant [f_{\varepsilon/2}(a)] \leqslant [a]. \tag{3.58}$$

It follows that A satisfies the property (D) in [13, Definition 5.5]. Then by an argument of Robert (see the proof of [26, Proposition 6.2.1]), Γ is surjective (see [13, Lemma 5.6]). □

Lemma 3.6. *Let A be a separable algebraically simple C^*-algebra with $QT(A) \neq \emptyset$ which has strict comparison and for which the canonical map Γ is surjective. Suppose that there are n mutually orthogonal elements $a_1, a_2, \ldots, a_n, a_{n+1} \in A_+^{\mathbf{1}}$ such that for some*

$$0 < \eta_1 < \bar{\eta}_1 < \eta_2 < \bar{\eta}_2 < \cdots < \eta_n < \bar{\eta}_n < \eta_{n+1} < \delta/2 \tag{3.59}$$

and $\delta \in (0, 1/2)$,

$$d_\tau(f_{\eta_2}(a_2)) < d_\tau(a_1), \tag{3.60}$$

$$d_\tau(f_{\eta_{i+1}}(a_{i+1})) < d_\tau(f_{\bar{\eta}_i}(a_i)) \quad \text{for all } \tau \in \overline{QT(A)}^w, \quad 2 \leqslant i \leqslant n. \tag{3.61}$$

Then for any $\sigma \in (0, 1/2)$, there is a $d \in \mathrm{Her}(\sum_{i=1}^{n+1} a_i)_+^{\mathbf{1}}$ such that

$$\sum_{i=2}^{n+1} f_\delta(a_i) \leqslant d \quad \text{and} \quad \omega(d) < \sigma. \tag{3.62}$$

Proof. We prove this by induction on n (for any $\sigma \in (0, 1/2)$). For $n = 1$, since A has strict comparison, there is an $x \in \mathrm{Her}(a)$, where $a = \sum_{i=1}^{n+1} a_i$ such that

$$x^*x = f_{\delta_1}(a_2) \quad \text{and} \quad xx^* \in \mathrm{Her}(a_1), \tag{3.63}$$

where $\eta_2 < \delta_1 < \bar{\eta}_2 < \delta/2$. Put $C_1 := \mathrm{Her}(x^*x + xx^*)$. Define $\psi : C_0((0,1]) \otimes M_2 \to C_1$ by $\psi(\iota \otimes e_{1,1}) = (xx^*)^{1/2}$, $\psi(\iota \otimes e_{2,2}) = (x^*x)^{1/2}$, $\psi(\iota \otimes e_{1,2}) = x$ and $\psi(\iota \otimes e_{2,1}) = x^*$. Thus (see, for example, [14, Proposition 8.3]), we may write $C_1 = M_2(\mathrm{Her}(x^*x))$. Then for any $0 < \varepsilon'' < \varepsilon' < \eta_1/2$, by [14, Lemma 8.9], there exist $c_1 \in \mathrm{Her}(f_{\varepsilon''}(x^*x))_+^{\mathbf{1}}$ and a unitary $U_1 \in \widetilde{C_1}$ such that with $b_1 = U_1^* \mathrm{diag}(0, c) U_1$,

(1) $f_{\varepsilon'}(x^*x) \leqslant b_1$;

(2) $d_\tau(f_{\varepsilon'}(x^*x)) \leqslant d_\tau(b_1) \leqslant d_\tau(f_{\varepsilon''}(x^*x))$ for all $\tau \in \overline{QT(A)}^w$;

(3) for some $\delta_1' \in (0, 1/2)$,

$$d_\tau(b_1) - \tau(f_{\delta_1'}(b_1)) < \sigma/2(n+1) \quad \text{for all } \tau \in \overline{QT(A)}^w; \tag{3.64}$$

(4) $U_1^*(g_{\varepsilon''/2}(x^*x) + xx^*)U \in B_1$, where $B_1 := (\mathrm{Her}(b_1)^\perp) \cap C_1$. Note that $b_1 \in C_1 \subset \mathrm{Her}(a_1 + a_2)$, and by (1) above, $f_\delta(a_2) \leqslant b_1$.

Let a_2'' be a strictly positive element of B_1. Then $a_2'' \in \mathrm{Her}(a)_+^{\mathbf{1}}$ and

$$d_\tau(a_2'') > d_\tau(g_{\varepsilon''/2}(x^*x) + xx^*) > d_\tau(f_{\bar{\eta}_2}(a_2)) \quad \text{for all } \tau \in \overline{QT(A)}^w. \tag{3.65}$$

Therefore, this lemma holds for $n = 1$.

We assume that this lemma holds for $n-1$ (for any $\sigma \in (0, 1/2)$). We keep the notation just introduced. Then $a_2'' \perp a_i$, $i = 3, 4, \ldots, n+1$. Moreover, by (3.65),

$$d_\tau(f_{\bar{\eta}_3}(a_3)) < d_\tau(a_2'') \quad \text{for all } \tau \in \overline{QT(A)}^w. \tag{3.66}$$

Put $a' := a_2'' + a_3 + a_4 + \cdots + a_{n+1}$. Then by the inductive assumption (choosing $\sigma/2(n+1)$ instead of σ), we obtain $b_2 \in \mathrm{Her}(a')_+^{\mathbf{1}}$ such that

$$f_\delta\left(\sum_{i=3}^{n+1} a_i v\right) \leqslant b_2 \quad \text{and} \quad \omega(b_2) < \sigma/2(n+1) \quad \text{for all } \tau \in \overline{QT(A)}^w. \tag{3.67}$$

Note that $b_1 \perp b_2$, and by [14, Proposition 4.4], $\omega(b_1+b_2) < \sigma$. Moreover,

$$\sum_{i=2}^{n+1} f_\delta(a_i) \leqslant b_1 + b_2. \tag{3.68}$$

This completes the induction and this lemma follows. □

Theorem 3.7. *Let A be a separable simple C^*-algebra with strict comparison and nonempty compact $QT(A)$. Suppose that A also has the uniform property Γ. Then*

(i) *the map Γ is surjective*;

(ii) *A has tracial approximate oscillation zero*;

(iii) *A has stable rank one*;

(iv) *A has the property* (*TM*) (*see* [14, *Theorem* 1.1]).

Proof. We have shown that (i) holds (see Theorem 3.5). It follows from [14, Theorem 1.1] that (ii)–(iv) are equivalent. We show that (ii) holds.

We need to show that for any $a \in \mathrm{Ped}(A\otimes\mathcal{K})_+^{\mathbf{1}}$, $\Omega^T(a) = 0$.

Let $\varepsilon > 0$. There is an $m \in \mathbb{N}$ such that $\|a - a^{1/2}E_m a^{1/2}\| < \varepsilon/2$, where $E_m = \sum_{i=1}^m e_{i,i}$ and $\{e_{i,j}\}$ is a system of matrix units for $\mathcal{K}$. Note that $a^{1/2}E_m a^{1/2} \in \mathrm{Her}(a)$. Therefore, to show that $\Omega^T(a) = 0$, it suffices to show that $\Omega^T(a^{1/2}E_m a^{1/2}) = 0$. Put $z = E_m a^{1/2}$. Then $z^*z = a^{1/2}E_m a^{1/2}$ and $zz^* = E_m a E_m$.

Therefore, it suffices to show that $\Omega^T(E_m a E_m) = 0$. Consequently, it suffices to show that $\Omega^T(a) = 0$ for any $a \in M_m(A)_+^{\mathbf{1}}$. Since by Proposition 3.4, $M_m(A)$ also has the uniform property Γ, without loss of generality, we may assume that $a \in A_+^{\mathbf{1}}$.

Therefore, it suffices to show that for any $a \in A_+^{\mathbf{1}}$, $\Omega^T(a) = 0$. If $0 \in \overline{\mathbb{R}_+ \setminus \mathrm{sp}(a)}$, then $\Omega^T(a) = 0$. Hence, we may assume that there is an $\varepsilon_0 \in (0, 1/2)$ such that $[0, \varepsilon_0] \subset \mathrm{sp}(a)$.

Let $\varepsilon, \sigma \in (0, \varepsilon_0/2)$. By [14, Proposition 5.7], it suffices to show that there is a $d \in \mathrm{Her}(a)_+^{\mathbf{1}}$ such that

$$\|a - ad\|_{2,QT(A)} < \varepsilon \quad \text{and} \quad \omega(d) < \sigma. \tag{3.69}$$

Fix any $\eta \in (0, (\varepsilon/8)^3)$. Choose $n \in \mathbb{N}$ such that $1/n < (\eta/8)^3$.

By Theorem 3.3, there is a unital homomorphism $\varphi: M_{n+1} \to l^\infty(A)/I_{QT(A),\varpi}$ such that $\varphi(e_{i,i}) \in (l^\infty(A)\cap A')/I_{QT(A),\varpi}$, $1 \leqslant i \leqslant n+1$. There exists an order zero c.p.c. map $\Phi = \{\varphi_k\} : M_{n+1} \to l^\infty(A)$ such that $\Pi_\varpi \circ \Phi = \varphi$, and for all $1 \leqslant i \leqslant n+1$,

$$\tau(b\varphi(e_{i,i})) = \frac{1}{n+1}\tau(b) \quad \text{for all } b \in A \text{ and } \tau \in QT_\varpi(A). \tag{3.70}$$

Choose

$$0 < r_1 < r_2/2 < r_2 < \cdots < r_{3n+2} < r_{3(n+1)}/2 < r_{3(n+1)} < \eta/2. \tag{3.71}$$

It follows that (recalling that $\varphi(e_{i,i}) \in (l^\infty(A)\cap A')/I_{QT(A),\varpi}$) for all $1 \leqslant j \leqslant 3(n+1)$ and $1 \leqslant i \leqslant n+1$,

$$\lim_{k\to\varpi}\left(\sup_{\tau\in QT(A)}\left|\tau(f_{r_j}(a)\varphi_k(e_{i,i})) - \frac{1}{n+1}\tau(f_{r_j}(a))\right|\right) = 0, \tag{3.72}$$

$$\lim_{k\to\varpi}\|f_{r_j}(a^{1/2}\varphi_k(e_{i,i})a^{1/2}) - f_{r_j}(a)\varphi_k(e_{i,i})\|_{2,QT(A)} = 0, \tag{3.73}$$

$$\lim_{k\to\varpi}\|f_{r_j}(a^{1/2}\varphi_k(e_{i,i})a^{1/2}) - f_{r_j}(\varphi_k(e_{i,i})a\varphi_k(e_{i,i}))\|_{2,QT(A)} = 0. \tag{3.74}$$

Since $\Pi_\varpi(\iota(a^{1/2}))\varphi(e_{i,i})\Pi_\varpi(\iota(a^{1/2})) = \varphi(e_{i,i})\Pi_\varpi(\iota(a))\varphi(e_{i,i})$ for $1 \leqslant i \leqslant n+1$, there are, for each $k \in \mathbb{N}$, mutually orthogonal elements $a_{i,k} \in \mathrm{Her}(a)_+^{\mathbf{1}}$ $(1 \leqslant i \leqslant n+1)$ such that

$$\Pi_\varpi(\{a_{i,k}\}) = \Pi_\varpi(\iota(a^{1/2}))\varphi(e_{i,i})\Pi_\varpi(\iota(a^{1/2})), \tag{3.75}$$

$$\Pi_\varpi(f_{r_j}(\{a_{i,k}\})) = \Pi_\varpi(f_{r_j}(\iota(a^{1/2})))\varphi(e_{i,i})\Pi_\varpi(\iota(a^{1/2})). \tag{3.76}$$

Therefore, for $1 \leqslant j \leqslant 3(n+1)$,

$$\lim_{k\to\varpi}\left(\sup_{\tau\in QT(A)}\left|\tau(f_{r_j}(a_{i,k}))-\frac{1}{n+1}\tau(f_{r_j}(a))\right|\right)=0. \tag{3.77}$$

Since A is simple, $QT(A)$ is compact and $[0,\varepsilon_0]\subset \mathrm{sp}(a)$, for any $g\in C_0((0,1])_+^{\mathbf{1}}$ with $g|_{[0,\varepsilon_0]}\neq 0$, we have

$$\inf\{\tau(g(a)):\tau\in QT(A)\}>0. \tag{3.78}$$

Then by (3.77), there exists a $\mathcal{P}\in\varpi$ such that for any $k\in\mathcal{P}$,

$$\tau(f_{r_{3j+1}}(a_{i+1,k}))<\tau(f_{r_{3j}}(a_{i,k}))<\frac{1}{n}\quad\text{for all } \tau\in QT(A), \tag{3.79}$$

$1\leqslant i\leqslant n$. It follows that

$$d_\tau(f_{r_{3j+2}}(a_{i+1,k}))<d_\tau(f_{r_{3j}}(a_{i,k}))\quad\text{for all } \tau\in QT(A). \tag{3.80}$$

Keep in mind that (3.71) holds. We also have $a_{i,k}\perp a_{i+1,k}$ ($1\leqslant i\leqslant n$). Put $a':=\sum_{i=1}^{n+1}a_{i,k}$ and $c=\sum_{i=2}^{n+1}a_{i,k}$. Then by Lemma 3.6, we obtain $d\in \mathrm{Her}(a')_+^{\mathbf{1}}$ such that

$$f_\eta(c)\leqslant d\quad\text{and}\quad \omega(d)<\sigma. \tag{3.81}$$

Note that $a_{i,k}\in\mathrm{Her}(a)$. Therefore $c\in\mathrm{Her}(a)$. We also have $d\in\mathrm{Her}(a)$. By (3.75) and the fact that φ is unital, we may assume that

$$\|a-a'\|_{2,QT(A)}<\left(\frac{\varepsilon}{8}\right)^3. \tag{3.82}$$

Then (see [18, Lemma 3.5] and [14, Definition 2.16])

$$\|a-c\|_{2,QT(A)}^{2/3}\leqslant\|a-a'\|_{2,QT(A)}^{2/3}+\|a'-c\|_{2,QT(A)}^{2/3}<\left(\frac{\varepsilon}{8}\right)^2+\left(\frac{1}{n+1}\right)^{2/3}. \tag{3.83}$$

It follows that

$$\|a-ad\|_{2,QT(A)}^{2/3}\leqslant\|a-c\|_{2,QT(A)}^{2/3}+\|d\|\|a-c\|_{2,QT(A)}^{2/3}+\|c-cd\|_{2,QT(A)}^{2/3}<(\varepsilon)^2. \tag{3.84}$$

Thus (3.69) holds and the theorem follows. □

We now consider simple C^*-algebras A for which $QT(A)$ may not be compact.

4 Hereditary uniform property Γ

Definition 4.1 (See [5, Definition 2.1]). Let A be a separable simple C^*-algebra with $\widetilde{QT}(A)\setminus\{0\}\neq\emptyset$. C^*-algebra A is said to have the hereditary uniform property Γ, if for any $e\in\mathrm{Ped}(A\otimes\mathcal{K})_+\setminus\{0\}$ and any $n\in\mathbb{N}$, there exist pairwise orthogonal projections $p_1,p_2,\dots,p_n\in(l^\infty(A_e)\cap(A_e)')/I_{\overline{QT(A_e)}^w,\varpi}$, where $A_e=\overline{e(A\otimes\mathcal{K})e}$ such that for $1\leqslant i\leqslant n$,

$$\tau(p_ia)=\frac{1}{n}\tau(a)\quad\text{for all } a\in A_e \text{ and } \tau\in QT_\varpi^w(A_e), \tag{4.1}$$

where $QT_\varpi^w(A_e)=\{\tau_\varpi:\{\tau_n\}\subset\overline{QT(A_e)}^w\}$.

Proposition 4.2 (See [31, Proposition 2.2]). *Let A be a separable simple C^*-algebra with $\widetilde{QT}(A) \setminus \{0\} \neq \emptyset$. Then the following are equivalent*:

(i) *A has the hereditary uniform property Γ*;

(ii) *for any $e \in \mathrm{Per}(A \otimes K)_+ \setminus \{0\}$, any finite subset $\mathcal{F} \subset A_e = \overline{e(A \otimes \mathcal{K})e}$, any $\varepsilon > 0$ and any $n \in \mathbb{N}$, there exist pairwise orthogonal elements $e_1, e_2, \ldots, e_n \in (A_e)^{\mathbf{1}}_+$ such that for $1 \leqslant i \leqslant n$ and $a \in A_e$, we have*

$$\|[x, e_i]\|_{2,\overline{QT(A_e)}^w} < \varepsilon, \quad \sup_{\overline{QT(A_e)}^w} \left|\tau(ae_i) - \frac{1}{n}\tau(a)\right| < \varepsilon, \quad \|e_i - e_i^2\|_{2,\overline{QT(A_e)}^w} < \varepsilon; \tag{4.2}$$

(iii) *for any $e \in \mathrm{Per}(A \otimes K)_+$, any finite subset $\mathcal{F} \subset A_e = \overline{e(A \otimes \mathcal{K})e}$, any $\varepsilon > 0$ and any $n \in \mathbb{N}$, there exist pairwise orthogonal elements $e_1, e_2, \ldots, e_n \in (A_e)+^{\mathbf{1}}$ such that for $1 \leqslant i \leqslant n$ and $a \in A_e$, we have*

$$\|[x, e_i]\| < \varepsilon, \quad \sup_{\overline{QT(A_e)}^w} \left|\tau(ae_i) - \frac{1}{n}\tau(a)\right| < \varepsilon, \quad \|e_i - e_i^2\|_{2,\overline{QT(A_e)}^w} < \varepsilon. \tag{4.3}$$

Proof. The proof is just a repetition of that of [31, Proposition 2.1]. □

Theorem 4.3. *Let A be a separable non-elementary simple C^*-algebra with strict comparison and nonempty compact $QT(A)$. Suppose that A has the uniform property Γ. Then A has the hereditary uniform property Γ.*

Proof. Let $e_A \in A_+$ be a strictly positive element of A and let $e \in \mathrm{Ped}(A \otimes \mathcal{K})^{\mathbf{1}}_+ \setminus \{0\}$. We view A as a hereditary C^*-subalgebra of $A \otimes \mathcal{K}$. Put $A_1 = \overline{e(A \otimes \mathcal{K})e}$. There is an $\varepsilon \in (0, 1/2)$ such that $f_\varepsilon(e_A) \neq 0$. Note that $f_\varepsilon(e_A) \in \mathrm{Ped}(A \otimes \mathcal{K})$. Since $e \in \mathrm{Ped}(A \otimes \mathcal{K})$, there is a $K \in \mathbb{N}$ such that $[e] \leqslant K[f_\varepsilon(e_A)] \leqslant K[e_A]$. By Theorem 3.7, A has stable rank one. So does $A \otimes \mathcal{K}$. It follows from [26, Proposition 2.1.2] that there is an $x \in A \otimes \mathcal{K}$ such that

$$x^*x = e \quad \text{and} \quad xx^* \in M_K(A). \tag{4.4}$$

Thus there is an isomorphism ψ from A_1 to a hereditary C^*-subalgebra of $M_K(A)$ with $\psi(e) \sim e$ (see [8, 1.4]). Therefore, without loss of generality, we may assume that $e \in M_K(A)^{\mathbf{1}}_+$. Since $M_K(A)$ also has the uniform property Γ (see Proposition 3.4), to simplify the notation, we may further assume that $e \in A^{\mathbf{1}}_+$.

Fix $n \in \mathbb{N}$. Let $p_1, p_2, \ldots, p_n \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$ be mutually orthogonal projections such that for all $a \in A$,

$$\tau(p_i a) = \frac{1}{n}\tau(a) \quad \text{for all } \tau \in QT_\varpi(A), \quad 1 \leqslant i \leqslant n. \tag{4.5}$$

Let $p_i^{(k)} \in A^{\mathbf{1}}_+$ be such that $p_i^{(k)} \perp p_j^{(k)}$ if $i \neq j$, $\{p_i^{(k)}\}_{k \in \mathbb{N}} \subset A'$ and $\Pi_\varpi(\{p_i^{(k)}\}) = p_i$, $1 \leqslant i, j \leqslant n$.

Since by Theorem 3.7, A has tracial approximate oscillation zero, there is a sequence $\{a_k\}$ in A_1 with $0 \leqslant a_k \leqslant 1$ such that for any $b \in A_1$,

$$\lim_{k\to\infty} \|b - ba_k\|_{2,QT(A)} = 0 \quad \text{and} \quad \lim_{k\to\infty} \omega(a_k) = 0. \tag{4.6}$$

It follows from [14, Proposition 6.2] that there exists $\{j(k)\} \subset \mathbb{N}$ such that $\Pi(\{a_k^{1/j(k)}\}) = q$ is a projection (recalling that $\Pi : l^\infty(A) \to l^\infty(A)/I_{\overline{QT(A)}^w,\mathbb{N}}$ is the quotient map). Put $c_k = a_k^{1/j(k)}$, $k \in \mathbb{N}$. Note that for any $b \in A^{\mathbf{1}}_+$,

$$\Pi(\iota(b)) = \Pi(\iota(b^{1/2})\{a_k\}\iota(b^{1/2})) \leqslant \Pi(\iota(b^{1/2})\{c_k\}\iota(b^{1/2})) \leqslant \Pi(\iota(b)). \tag{4.7}$$

It follows that for any $b \in A_1$,

$$\lim_{k\to\infty} \|b - bc_k\|_{2,QT(A)} = 0 = \lim_{k\to\infty} \|b - b^{1/2}c_k b^{1/2}\|_{2,QT(A)}. \tag{4.8}$$

In particular, $\{c_k\} \in (A_1)'$. Let $\{\mathcal{F}_k\}$ be an increasing sequence of finite subsets of A_1 such that its union is dense in A_1. Without loss of generality, we may assume that for all $k \in \mathbb{N}$,

$$\|bc_k - b\|_{2,QT(A)} < 1/k \quad \text{and} \quad \|c_k b - b\|_{2,QT(A)} < 1/k \quad \text{for all } b \in \mathcal{F}_k. \tag{4.9}$$

Put $\mathcal{G}_k = \mathcal{F}_k \cup \{c_1, c_2, \dots, c_k\}$. For each $k \in \mathbb{N}$, there exists a $\mathcal{P}_k \in \varpi$ such that for all $m \in \mathcal{P}_k$,

$$\|p_i^{(m)} - (p_i^{(m)})^2\|_{2,QT(A)} < 1/k, \tag{4.10}$$

$$\sup\left\{\left|\tau(p_i^{(m)}b) - \frac{1}{n}\tau(b)\right| : \tau \in QT(A)\right\} < 1/k \quad \text{and} \quad \|[p_i^{(m)}, b]\| < 1/k \tag{4.11}$$

for all $b \in \mathcal{G}_k$ and $1 \leqslant i \leqslant n$. We may assume that $\mathcal{P}_k \subset \mathcal{P}_{k+1}$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, choose $m(k) \in \mathcal{P}_k$ such that $m(k) < m(k+1)$ for all $k \in \mathbb{N}$. Define $d_i^{(k)} = p_i^{(m(k))}$, $k \in \mathbb{N}$ and $1 \leqslant i \leqslant n$. Then $d_i = \Pi(\{d_i^{(k)}\})$ is a projection, and $d_i d_j = 0$ if $i \neq j$ $(1 \leqslant i, j \leqslant n)$. Moreover,

$$\|d_i^{(k)} - (d_i^{(k)})^2\|_{2,QT(A)} < 1/k, \tag{4.12}$$

$$\sup\left\{\left|\tau(d_i^{(k)}b) - \frac{1}{n}\tau(b)\right| : \tau \in QT(A)\right\} < 1/k, \tag{4.13}$$

$$\|[d_i^{(k)}, c_k]\| < 1/k, \tag{4.14}$$

$$\|[d_i^{(k)}, b]\| < 1/k, \quad b \in \mathcal{F}_k, \quad 1 \leqslant i \leqslant n. \tag{4.15}$$

It follows (by (4.14)) that

$$d_i q = q d_i, \quad 1 \leqslant i \leqslant n. \tag{4.16}$$

Put $q_i = d_i q$, $i \in \mathbb{N}$. Then (also by (4.12)), $\{q_i : 1 \leqslant i \leqslant n\}$ are mutually orthogonal projections in $l^\infty(A)/I_{QT(A),\mathbb{N}}$. For any $b \in A_1$, by (4.9), $q\Pi(\iota(b)) = \Pi(\iota(b))q = \Pi(\iota(b))$ in $l^\infty(A)/I_{QT(A),\mathbb{N}}$. Then for any $\tau \in QT_\varpi(A)$,

$$|\tau(d_i q b) - \tau(d_i b)| = 0. \tag{4.17}$$

It follows that for $1 \leqslant i \leqslant n$,

$$\lim_{k\to\varpi} \sup\{|\tau((d_i^{(k)}c_k)b) - \tau(d_i^{(k)}b)| : \tau \in QT(A)\} = 0. \tag{4.18}$$

Then by (4.13),

$$\lim_{k\to\varpi} \sup\left\{\left|\tau((d_i^{(k)}c_k)b) - \frac{1}{n}\tau(b)\right| : \tau \in QT(A)\right\} = 0. \tag{4.19}$$

This also implies that for $1 \leqslant i \leqslant n$,

$$\tau(q_i b) = \frac{1}{n}\tau(b) \quad \text{for all } \tau \in QT_\varpi(A_1) \text{ and } b \in A_1. \tag{4.20}$$

Put

$$J = \Big\{\{b_k\} \in l^\infty(A_1) : \lim_{k\to\infty} \|b_k\|_{2,\overline{QT(A_1)}^w} = 0\Big\}.$$

Note that $\widetilde{QT}(A_1) = \mathbb{R}_+ \cdot \overline{QT(A_1)}^w$. Since $QT(A)$ is a basis for $\widetilde{QT}(A)$, we then have (see also [14, Proposition 2.18])

$$l^\infty(A_1) \cap I_{QT(A),\mathbb{N}} = J. \tag{4.21}$$

By (4.8), (4.15) and (4.16),

$$q_i \Pi(\iota(b)) = \Pi(\iota(b)) q_i, \quad 1 \leqslant i \leqslant n. \tag{4.22}$$

It remains to show that $q_i \in (l^\infty(A_1) \cap (A_1)')/J$.

By central surjectivity of Sato [28] (since we do not assume that A is even exact, we apply [13, Proposition 3.10] (see also [13, Proposition 3.8] and [14, Proposition 2.18])), we may assume that $q_i \in (l^\infty(A) \cap A')/_{I_{QT(A),\mathbb{N}}}$. The new lifting may be written as $\Pi(\{e_i^{(k)}\}) = q_i$, where $e_i^{(k)} \perp e_j^{(k)}$ for $i \neq j$ $(1 \leqslant i \leqslant n)$ and $\{e_i^{(k)}\} \in (A')_+^{\mathbf{1}}$ and $e_i^{(k)} = d_i^{(k)} c_k + h_k$ for some $\{h_k\} \in I_{QT(A),\mathbb{N}}$. Put $f_i^{(k)} = c_k e_i^{(k)} c_k$, $1 \leqslant i \leqslant n$, $k \in \mathbb{N}$. Then $f_i^{(k)} \in (A_1)'$, since $\{c_k\} \in (A_1)'$. We still have $\Pi(\{f_i^{(k)}\}) = q_i$, $1 \leqslant i \leqslant n$. In other words, $q_i \in (l^\infty(A_1) \cap (A_1)')/J$, $1 \leqslant i \leqslant n$. This completes the proof. □

Proposition 4.4. *Let A be a separable simple C^*-algebra with nonempty compact $QT(A)$. Suppose that A has the hereditary uniform property Γ. Then A has the uniform property Γ.*

Proof. Choose any strictly positive element $e \in \mathrm{Ped}(A)_+ \setminus \{0\}$. Then $A_e = A$. Then (3.1) is the same as (4.1). □

Remark 4.5. Theorem 4.3 states that if a separable simple C^*-algebra A with strict comparison has the uniform property Γ, then (4.1) holds for each $e \in \mathrm{Ped}(A \otimes \mathcal{K})_+^{\mathbf{1}}$. This fact may be regarded as the statement that in this case, the uniform property Γ carries to hereditary C^*-subalgebras as well as $A \otimes \mathcal{K}$, if we restrict ourselves to hereditary C^*-subalgebras of $A \otimes \mathcal{K}$ which are algebraically simple, or rather, to those hereditary C^*-subalgebras of $A \otimes \mathcal{K}$ whose quasitraces are bounded. Recall that the uniform property Γ is originally only defined on C^*-algebras with compact $T(A)$ (see [6, Definition 2.1]). It seems to us that Definition 4.1 is an appropriate generalization of the uniform property Γ to separable simple C^*-algebras which do not have the continuous scale. A more general version of uniform property Γ (where p_i's are not required to be projections) which is called the stabilized uniform property Γ was introduced in [4]. However, we prefer to keep the condition that each p_i is a projection intact. The proof of Theorem 4.3 uses the notion of tracial approximate oscillation zero. Theorem 4.9 below shows that if A has the strict comparison and hereditary uniform property Γ, then this is also automatic. In particular, A has stable rank one.

Let A be a separable simple C^*-algebra with $T(A) = QT(A) \neq \emptyset$ which has strict comparison. Suppose that A has the stabilized uniform property Γ in the sense of [4, Definition 2.5]. Suppose that $K_0(A)_+ \neq \{0\}$. Then there is a projection $e \in A \otimes \mathcal{K} \setminus \{0\}$. Put $A_1 = e(A \otimes \mathcal{K})e$. Then A_1 is unital. Since A_1 also has the stabilized uniform property Γ, A_1 has the uniform property Γ (see [4, Proposition 2.6]). By Theorem 4.3, A has the hereditary uniform property Γ. More generally, if there is an $e \in \mathrm{Ped}(A \otimes \mathcal{K})_+ \setminus \{0\}$ such that $d_\tau(e)$ is continuous, set $A_1 = e(A \otimes \mathcal{K})e$. Then $T(A_1)$ is compact. Thus the same argument also implies that A_1 has the hereditary uniform property Γ. This is the case if $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes \mathcal{Z})$. So under the assumption that $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes \mathcal{Z})$, the stabilized uniform property Γ is the same as the hereditary uniform property Γ.

Theorem 4.6. *Let A be a finite separable non-elementary simple C^*-algebra which are tracially approximately divisible* (*see, for example,* [15, *Definition* 5.2]). *Then A has the hereditary uniform property Γ.*

Proof. It follows from [13, Corollary 6.5] and the proof of [13, Theorem 5.2] that $W(A)$ is almost unperforated and by [27, Corollary 5.1] (see also [15, Proposition 4.9]) A has a non-zero 2-quasitrace. By [13, Theorem 5.7], the map Γ is surjective. Choose $e \in \mathrm{Ped}(A \otimes \mathcal{K})_+^{\mathbf{1}} \setminus \{0\}$ such that $d_\tau(e)$ is continuous on $\overline{QT(A)}^w$ and $d_\tau(e) < r$ for all $\tau \in \overline{QT(A)}^w$ and $r \in (0, 1/2)$. By [13, Theorem 6.7], A has stable rank one. So we may assume that $e \in \mathrm{Ped}(A)_+$.

Put $A_1 = \mathrm{Her}(e)$. Then A_1 has the continuous scale (see, for example, [10, Theorem 5.3]). By [15, Theorem 5.5], A_1 is tracially approximately divisible. Now $QT(A_1)$ is compact and A_1 has strict comparison (see [13, Theorem 5.2]).

Now fix $n \in \mathbb{N}$. By [13, Theorem 4.11], there is a unital homomorphism $\psi : M_n \to (l^\infty(A_1) \cap (A_1)')/I_{QT(A_1),\varpi}$ (noting that $I_{QT(A_1),\mathbb{N}} \subset I_{QT(A_1),\varpi}$). Let $p_i = \psi(e_{i,i})$, $1 \leqslant i \leqslant n$. Then $p_i \in (l^\infty(A_1) \cap (A_1)')/I_{QT(A_1),\varpi}$, $1 \leqslant i \leqslant n$, and for any $a \in A$,

$$\tau(p_i a) = \tau(\varphi(e_{i,i})a) = \frac{1}{n}\tau(a) \quad \text{for all } \tau \in QT_\varpi(A_1), \quad 1 \leqslant i \leqslant n. \tag{4.23}$$

In other words, A_1 has the uniform property Γ. By Theorem 4.3, A_1 has the hereditary uniform property Γ. By Brown's stable isomorphism theorem [2], A has the hereditary uniform property Γ. $\square$

Remark 4.7. It is known that separable simple C^*-algebras with tracial rank zero are tracially approximately divisible (see [21, Lemma 6.10]). In fact, any separable simple C^*-algebra A with tracial rank at most one is tracially approximately divisible (see the proof of [22, Theorem 5.4]). Therefore, by Theorem 4.6, these C^*-algebras have the hereditary uniform property Γ (and strict comparison) but they may not be $\mathcal{Z}$-stable (see [25] and [13, Example 6.10]).

Theorem 4.8. *Let A be a separable simple C^*-algebra with strict comparison and $\widetilde{QT}(A) \setminus \{0\} \neq \emptyset$. Suppose that A has the hereditary uniform property Γ. Then the map $\Gamma : \mathrm{Cu}(A) \to \mathrm{LAff}_+(\widetilde{QT}(A))$ is surjective.*

Proof. The proof is almost the same as that of Theorem 3.5. But $QT(A)$ will be replaced by $\overline{QT(A)}^w$. The formula (3.49) holds with $QT(A)$ being replaced by $\overline{QT(A)}^w$. The formula (3.50) holds with $QT_{\varpi}(A)$ being replaced by $QT_{\varpi}^w(A)$. Inequalities (3.52) also hold with $QT(A)$ being replaced by $\overline{QT(A)}^w$. Moreover, we also have (3.53) holds with $QT(A)$ being replaced by $\overline{QT(A)}^w$. We then have

$$n[b] \leqslant [a] \quad \text{and} \quad [f_{\varepsilon}(a)] \leqslant (n+1)[b] \tag{4.24}$$

as in the proof of Theorem 3.5. Note that this holds for any $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+^{\mathbf{1}}$ since we assume that A has the hereditary uniform property Γ and we may begin with an element $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+^{\mathbf{1}}$. Then the same argument of Robert [26] as in the proof of Theorem 3.5 implies that the map Γ is surjective. $\square$

Theorem 4.9. *Let A be a separable simple C^*-algebra with strict comparison and $\widetilde{QT}(A) \setminus \{0\} \neq \emptyset$. Suppose that A has the hereditary uniform property Γ. Then A has tracial approximate oscillation zero and stable rank one.*

Proof. It follows from Theorem 4.8 that the map Γ is surjective. Choose $e \in \mathrm{Ped}(A)_+^{\mathbf{1}} \setminus \{0\}$ such that $d_\tau(e)$ is continuous on $\widetilde{QT}(A)$. Then $\mathrm{Her}(e)$ has the continuous scale (see, for example, [10, Theorem 5.3]). Since A has the hereditary uniform property Γ, $\mathrm{Her}(e)$ has the uniform property Γ. It follows from Theorem 3.7 that $\mathrm{Her}(e)$ has tracial approximate oscillation zero and stable rank one. By Brown's stable isomorphism theorem, A has tracial approximate oscillation zero and stable rank one. $\square$

Towards the Toms-Winter conjecture, as in [4, 5], we have the following theorem.

Theorem 4.10. *Let A be a stably finite separable non-elementary amenable simple C^*-algebra. Then the following are equivalent:*

(1) *A has the strict comparison and hereditary uniform property Γ;*

(2) $A \cong A \otimes \mathcal{Z}$;

(3) *A has the finite nuclear dimension.*

Proof. The equivalence of (2) and (3) has been proved (see [3, 6, 24, 30, 33]).

To see (2) $\Rightarrow$ (1), let A be $\mathcal{Z}$-stable. It is proved in [27] that A has strict comparison. By [15, Theorem 5.9], A is tracially approximately divisible (see also [13, Theorem 5.2]). Then by Theorem 4.6, A has the hereditary uniform property Γ.

For (1) $\Rightarrow$ (2), we note that by Theorem 3.7, the map Γ is surjective. Choose $e \in \mathrm{Ped}(A)_+ \setminus \{0\}$ such that $A_1 = \mathrm{Her}(e)$ has the continuous scale. Thus, by Proposition 4.4, A_1 has the uniform property Γ. It follows from [5, Theorem 4.6] that A_1 is uniformly McDuff. By [10, Theorem 5.3], $T(A_1)$ is compact and A_1 has strict comparison. Then by a version of Matui-Sato's result (see, for example, [7, Proposition 4.4]), A_1 is $\mathcal{Z}$-stable and hence A is $\mathcal{Z}$-stable. $\square$

Acknowledgements This work was supported by National Science Foundation of USA (Grant No. DMS-1954600) and the Research Center for Operator Algebras in East China Normal University which is supported by Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, Science and Technology Commission of Shanghai Municipality (Grant No. 22DZ2229014).

References

1 Blackadar B, Handelman D. Dimension functions and traces on C^*-algebra. J Funct Anal, 1982, 45: 297–340
2 Brown L G. Stable isomorphism of hereditary subalgebras of C^*-algebras. Pacific J Math, 1977, 71: 335–348
3 Castillejos J, Evington S. Nuclear dimension of simple stably projectionless C^*-algebras. Anal PDE, 2020, 13: 2205–2240
4 Castillejos J, Evington S. Stabilising uniform property Γ. Proc Amer Math Soc, 2022, 149: 4725–4737
5 Castillejos J, Evington S, Tikuisis A, White S. Uniform property Γ. Int Math Res Not IMRN, 2022, 2022: 9864–9908
6 Castillejos J, Evington S, Tikuisis A, White S, Winter W. Nuclear dimension of simple C^*-algebras. Invent Math, 2021, 224: 245–290
7 Castillejos J, Li K, Szabo G. On tracial $\mathcal{Z}$-stability of simple non-unital C^*-algebras. Canad J Math, 2023, in press
8 Cuntz J. The structure of multiplication and addition in simple C^*-algebras. Math Scand, 1977, 40: 215–233
9 Elliott G A, Gong G H, Lin H X, Niu Z. The classification of simple separable KK-contractible C^*-algebras with finite nuclear dimension. J Geom Phys, 2020, 158: 103861
10 Elliott G A, Gong G H, Lin H X, Niu Z. Simple stably projectionless C^*-algebras of generalized tracial rank one. J Noncommut Geom, 2020, 14: 251–347
11 Elliott G A, Robert L, Santiago L. The cone of lower semicontinuous traces on a C^*-algebra. Amer J Math, 2011, 133: 969–1005
12 Evington S. W^*-bundles. PhD Thesis. Glasgow: University of Glasgow, 2018
13 Fu X L, Li K, Lin H X. Tracial approximate divisibility and stable rank one. J Lond Math Soc (2), 2023, in press
14 Fu X L, Lin H X. Tracial approximate oscillation zero and stable rank one. arXiv:2112.14007, 2021
15 Fu X L, Lin H X. Non-amenable simple C^*-algebras with tracial approximation. Forum Math Sigma, 2022, 10: e14
16 Gong G H, Jiang X H, Su H B. Obstructions to $\mathcal{Z}$-stability for unital simple C^*-algebras. Canad Math Bull, 2000, 43: 418–426
17 Gong G H, Lin H X, Niu Z. A classification of finite simple amenable $\mathcal{Z}$-stable C^*-algebras, II: C^*-algebras with rational generalized tracial rank one. C R Math Acad Sci Soc R Can, 2020, 42: 451–539
18 Haagerup U. Quasitraces on exact C^*-algebras are traces. C R Math Acad Sci Soc R Can, 2014, 36: 67–92
19 Jiang X H, Su H B. On a simple unital projectionless C^*-algebra. Amer J Math, 1999, 121: 359–413
20 Lin H X. Simple C^*-algebras with continuous scales and simple corona algebras. Proc Amer Math Soc, 1991, 112: 871–880
21 Lin H X. Tracially AF C^*-algebras. Trans Amer Math Soc, 2001, 353: 693–722
22 Lin H X. Simple nuclear C^*-algebras of tracial topological rank one. J Funct Anal, 2007, 251: 601–679
23 Lin H X. Tracial oscillation zero and $\mathcal{Z}$-stability. arXiv:2112.12036, 2021
24 Matui H, Sato Y. Strict comparison and $\mathcal{Z}$-absorption of nuclear C^*-algebras. Acta Math, 2012, 209: 179–196
25 Niu Z, Wang Q Y. A tracially AF algebra which is not $\mathcal{Z}$-absorbing. Münster J Math, 2021, 14: 41–57
26 Robert L. Classification of inductive limits of 1-dimensional NCCW complexes. Adv Math, 2012, 231: 2802–2836
27 Rørdam M. The stable rank and real rank of $\mathcal{Z}$-absorbing C^*-algebras. Internat J Math, 2004, 15: 1065–1084
28 Sato Y. Trace spaces of simple nuclear C^*-algebras with finite-dimensional extreme boundary. arXiv:1209.3000, 2012
29 Thiel H. Ranks of operators in simple C^*-algebras with stable rank one. Comm Math Phys, 2020, 377: 37–76
30 Tikuisis A. Nuclear dimension, $\mathcal{Z}$-stability, and algebraic simplicity for stably projectionless C^*-algebras. Math Ann, 2014, 358: 729–778
31 Toms A, White S, Winter W. $\mathcal{Z}$-stability and finite-dimensional tracial boundaries. Int Math Res Not IMRN, 2015, 2015: 2702–2727
32 Toms A, Winter W. Strongly self-absorbing C^*-algebras. Trans Amer Math Soc, 2007, 359: 3999–4029
33 Winter W. Nuclear dimension and $\mathcal{Z}$-stability of pure C^*-algebras. Invent Math, 2012, 187: 259–342
34 Winter W, Zacharias J. The nuclear dimension of C^*-algebras. Adv Math, 2010, 224: 461–498