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Nonconforming finite element Stokes complexes in three dimensions

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Abstract Two nonconforming finite element Stokes complexes starting from the conforming Lagrange element and ending with the nonconforming P_1-P_0 element for the Stokes equation in three dimensions are studied. Commutative diagrams are also shown by combining nonconforming finite element Stokes complexes and interpolation operators. The lower order H(gradcurl)-nonconforming finite element only has 14 degrees of freedom, whose basis functions are explicitly given in terms of the barycentric coordinates. The H(gradcurl)-nonconforming elements are applied to solve the quad-curl problem, and the optimal convergence is derived. By the nonconforming finite element Stokes complexes, the mixed finite element methods of the quad-curl problem are decoupled into two mixed methods of the Maxwell equation and the nonconforming P_1-P_0 element method for the Stokes equation, based on which a fast solver is discussed. Numerical results are provided to verify the theoretical convergence rates.

Keywords nonconforming finite element Stokes complex, quad-curl problem, error analysis, decoupling, fast solver

MSC(2020) 65N30, 65N12, 65N22

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1 Introduction

In this paper, we consider the nonconforming finite element discretization of the following Stokes complex in three dimensions:

$$\mathbb{R} \xrightarrow{\subset} H^1(\Omega) \xrightarrow{\nabla} \boldsymbol{H}(\mathrm{gradcurl},\Omega) \xrightarrow{\mathrm{curl}} \boldsymbol{H}^1(\Omega;\mathbb{R}^3) \xrightarrow{\mathrm{div}} L^2(\Omega) \to 0, \tag{1.1}$$

where $\boldsymbol{H}(\mathrm{gradcurl},\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\Omega) : \mathrm{curl}\,\boldsymbol{v} \in \boldsymbol{H}^1(\Omega;\mathbb{R}^3)\}$. Conforming finite element Stokes complexes on triangles and rectangles in two dimensions are devised in [12, 31, 44]. Conforming finite element Stokes complexes on split meshes in three dimensions are advanced in [15, 32]. We refer to [5] for a conforming virtual element discretization of the Stokes complex (1.1). Recently, conforming finite element Stokes complexes on tetrahedrons in three dimensions using pure polynomials as shape functions are devised in [13], in which the dimension of each finite element space is high, and super-smooth degrees of freedom appear. In [45], $\boldsymbol{H}$(gradcurl)-conforming finite elements of the first kind in three dimensions are constructed with the degree of polynomials $k \geqslant 6$. The number of the degrees of freedom for the lowest order element in [45] is 315, which is reduced to 18 by enriching the shape function space with macro-element bubble functions in [32]. Nonconforming elements to discretize $\boldsymbol{H}(\mathrm{gradcurl}, \Omega)$ are another choices to reduce the high dimensions of the conforming element spaces. The $\boldsymbol{H}$(gradcurl)-nonconforming Zheng-Hu-Xu element in [48] has only 20 degrees of freedom, which is the first $\boldsymbol{H}$(gradcurl)-nonconforming finite element.

We construct an $\boldsymbol{H}$(gradcurl)-nonconforming finite element possessing fewer degrees of freedom than the Zheng-Hu-Xu element, but preserving the same approximation error in the energy norm. The finite element discretization of $\boldsymbol{H}^1(\Omega; \mathbb{R}^3) \times L^2(\Omega)$ in the Stokes complex (1.1) should be a stable divergence-free pair for the Stokes equation, which suggests us to use the nonconforming linear element and the piecewise constant to discretize $\boldsymbol{H}^1(\Omega; \mathbb{R}^3)$ and $L^2(\Omega)$, respectively. On the other hand, the direct sum decomposition $\mathbb{P}_k(K; \mathbb{R}^3) = \nabla \mathbb{P}_{k+1}(K) \oplus ((\boldsymbol{x} - \boldsymbol{x}_K) \times \mathbb{P}_{k-1}(K; \mathbb{R}^3))$ (see [2,3]) implies that the curl operator curl : $(\boldsymbol{x} - \boldsymbol{x}_K) \times \mathbb{P}_1(K; \mathbb{R}^3) \to \mathbb{P}_1(K; \mathbb{R}^3)$ is injective. This motivates us to take the space of shape functions $\boldsymbol{W}_k(K) = \nabla \mathbb{P}_{k+1}(K) \oplus ((\boldsymbol{x} - \boldsymbol{x}_K) \times \mathbb{P}_1(K; \mathbb{R}^3))$ with $k = 0, 1$. Note that $\boldsymbol{W}_1(K)$ is exactly the space of shape functions of the Zheng-Hu-Xu element, and hence we give a new understanding of the Zheng-Hu-Xu element by the space decomposition. The dimension of $\boldsymbol{W}_0(K)$ is 14, which is six fewer than the dimension of $\boldsymbol{W}_1(K)$. The degrees of freedom $\mathcal{N}_0(K)$ for $\boldsymbol{W}_0(K)$ are given by

$$\begin{aligned} &\int_e \boldsymbol{v} \cdot \boldsymbol{t}_e ds \quad \text{on each } e \in \mathcal{E}(K), \\ &\int_F (\mathrm{curl}\, \boldsymbol{v}) \times \boldsymbol{n} ds \quad \text{on each } F \in \mathcal{F}(K). \end{aligned}$$

By comparing the degrees of freedom, we see that the lower order nonconforming element $(K, \boldsymbol{W}_0(K), \mathcal{N}_0(K))$ for $\boldsymbol{H}(\mathrm{gradcurl}, \Omega)$ is very similar to the Morley-Wang-Xu element (see [43]) for $H^2(\Omega)$. The explicit expressions of the basis functions of $\boldsymbol{W}_0(K)$ are shown in terms of the barycentric coordinates.

Then we combine the conforming $(k+1)$-th order Lagrange element space V_{h0}^g, the $\boldsymbol{H}$(gradcurl)-nonconforming finite element space $\boldsymbol{W}_{h0}$ including the Zheng-Hu-Xu element and the lower order one constructed in this paper, the nonconforming linear element space $\boldsymbol{V}_{h0}^s$, and the piecewise constant space $\mathcal{Q}_{h0}$ to build up the nonconforming finite element Stokes complexes

$$0 \xrightarrow{\subset} V_{h0}^g \xrightarrow{\nabla} \boldsymbol{W}_{h0} \xrightarrow{\mathrm{curl}_h} \boldsymbol{V}_{h0}^s \xrightarrow{\mathrm{div}_h} \mathcal{Q}_{h0} \to 0. \tag{1.2}$$

The divergence-free subspace of the nonconforming linear element space $\boldsymbol{V}_{h0}^s$ is explicitly characterized due to this nonconforming finite element Stokes complex, which essentially extends the result of Falk and Morley [21] to three dimensions. Recently, this nonconforming finite element Stokes complex is applied to prove the quasi-orthogonality of the adaptive finite element method for the quad-curl problem in [9]. Furthermore, we develop the commutative diagram for Stokes complex (see (1.1)), i.e.,

$$\begin{array}{ccccccccccc} 0 & \xrightarrow{\subset} & H_0^1(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_0(\mathrm{gradcurl}, \Omega) & \xrightarrow{\mathrm{curl}} & \boldsymbol{H}_0^1(\Omega; \mathbb{R}^3) & \xrightarrow{\mathrm{div}} & L_0^2(\Omega) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle I_h^{SZ}} & & \downarrow{\scriptstyle \boldsymbol{\Pi}_h^{gc}} & & \downarrow{\scriptstyle \boldsymbol{I}_h^{s}} & & \downarrow{\scriptstyle I_h^{L^2}} & & \\ 0 & \xrightarrow{\subset} & V_{h0}^g & \xrightarrow{\nabla} & \boldsymbol{W}_{h0} & \xrightarrow{\mathrm{curl}_h} & \boldsymbol{V}_{h0}^s & \xrightarrow{\mathrm{div}_h} & \mathcal{Q}_{h0} & \longrightarrow & 0, \end{array}$$

where I_h^{SZ} is the Scott-Zhang interpolation operator (see [41]), $\boldsymbol{\Pi}_h^{gc}$ is a quasi-interpolation operator, and both $\boldsymbol{I}_h^s$ and $I_h^{L^2}$ are the standard interpolation operators based on the degrees of freedom.

The $\boldsymbol{H}$(gradcurl)-nonconforming element together with the Lagrange element is then applied to solve the quad-curl problem. The discrete Poincaré inequality is established for the $\boldsymbol{H}$(gradcurl)-nonconforming element space $\boldsymbol{W}_{h0}$, as a result the coercivity on the weak divergence-free space follows. Then we acquire the discrete stability of the bilinear form from the evident discrete inf-sup condition, and derive the optimal convergence of the nonconforming mixed finite methods. Since the interpolation operator $\boldsymbol{I}_h^{gc}$ is not well defined on $\boldsymbol{H}_0(\mathrm{gradcurl}, \Omega)$, in the error analysis we exploit a quasi-interpolation operator $\boldsymbol{\Pi}_h^{gc}$

defined on $\boldsymbol{H}_0(\mathrm{gradcurl},\Omega)$, which is constructed by combining a regular decomposition for the space $\boldsymbol{H}_0(\mathrm{gradcurl},\Omega)$, the interpolation operator $\boldsymbol{I}_h^{gc}$ and the Scott-Zhang interpolation operator (see [41]).

By the nonconforming finite element Stokes complex (1.2), we equivalently decouple the mixed finite element method of the quad-curl problem into two mixed methods of the Maxwell equation and the nonconforming P_1-P_0 element method for the Stokes equation, as the decoupling of the quad-curl problem in the continuous level (see [11, 47]). A fast solver based on this equivalent decoupling is discussed for the mixed finite element method of the quad-curl problem.

In addition to the Stokes complex (1.1), another kind of Stokes complex (see [34]) is

$$\mathbb{R} \xrightarrow{\subset} H^2(\Omega) \xrightarrow{\mathrm{curl}} \boldsymbol{H}^1(\Omega;\mathbb{R}^2) \xrightarrow{\mathrm{div}} L^2(\Omega) \to 0 \tag{1.3}$$

in two dimensions, and

$$\mathbb{R} \xrightarrow{\subset} H^2(\Omega) \xrightarrow{\nabla} \boldsymbol{H}^1(\mathrm{curl},\Omega) \xrightarrow{\mathrm{curl}} \boldsymbol{H}^1(\Omega;\mathbb{R}^3) \xrightarrow{\mathrm{div}} L^2(\Omega) \to 0 \tag{1.4}$$

in three dimensions, where $\boldsymbol{H}^1(\mathrm{curl},\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega;\mathbb{R}^3) : \mathrm{curl}\,\boldsymbol{v} \in \boldsymbol{H}^1(\Omega;\mathbb{R}^3)\}$. We refer to [4, 15, 21, 22, 26–28, 35, 36, 46] for some finite element discretizations of the Stokes complex (1.3) in two dimensions, and [13, 23, 27, 39, 42] for some finite element discretizations of the Stokes complex (1.4) in three dimensions. The finite elements corresponding to the Stokes complexes (1.3)–(1.4) are not suitable to discretize the quad-curl problem, since $\nabla H^1(\Omega) \subset \boldsymbol{H}(\mathrm{gradcurl},\Omega)$ is not a subspace of $\boldsymbol{H}^1(\mathrm{curl},\Omega)$.

The rest of this paper is organized as follows. In Section 2, we devise a lower order $\boldsymbol{H}$(gradcurl)-nonconforming finite element. Nonconforming finite element Stokes complexes are developed in Section 3. In Section 4, we propose the nonconforming mixed finite element method for the quad-curl problem. The decoupling of the mixed finite element method and a fast solver are discussed in Section 5. Numerical results are presented in Section 6. In Appendix A, we give the regularity of the quad-curl problem on convex domains.

2 $\boldsymbol{H}$(gradcurl)-nonconforming finite elements

In this section, we present $\boldsymbol{H}$(gradcurl)-nonconforming finite elements.

2.1 Notation

Given a bounded domain $G \subset \mathbb{R}^3$ and a nonnegative integer m, let $H^m(G)$ be the usual Sobolev space of functions on G, and $\boldsymbol{H}^m(G;\mathbb{R}^3)$ be the vector version of $H^m(G)$. The corresponding norm and the semi-norm are denoted, respectively, by $\|\cdot\|_{m,G}$ and $|\cdot|_{m,G}$. Let $(\cdot,\cdot)_G$ be the standard inner product on $L^2(G)$ or $\boldsymbol{L}^2(G;\mathbb{R}^3)$. If G is Ω, we abbreviate $\|\cdot\|_{m,G}$, $|\cdot|_{m,G}$ and $(\cdot,\cdot)_G$ by $\|\cdot\|_m$, $|\cdot|_m$ and $(\cdot,\cdot)$, respectively. Denote by $H_0^m(G)(\boldsymbol{H}_0^m(G;\mathbb{R}^3))$ the closure of $C_0^\infty(G)(\boldsymbol{C}_0^\infty(G;\mathbb{R}^3))$ with respect to the norm $\|\cdot\|_{m,G}$. Let $\mathbb{P}_m(G)$ stand for the set of all the polynomials in G with the total degree no more than m, and $\mathbb{P}_m(G;\mathbb{R}^3)$ be the vector version of $\mathbb{P}_m(G)$. Let $Q_G^m : L^2(G) \to \mathbb{P}_m(G)$ be the L^2-orthogonal projector, and its vector version be denoted by $\boldsymbol{Q}_G^m$. Set $Q_G := Q_G^0$. The gradient operator, the curl operator and the divergence operator are denoted by ∇, curl and div, respectively. Define Sobolev spaces $\boldsymbol{H}(\mathrm{curl},G)$, $\boldsymbol{H}_0(\mathrm{curl},G)$, $\boldsymbol{H}(\mathrm{div},G)$, $\boldsymbol{H}_0(\mathrm{div},G)$ and $L_0^2(G)$ in the standard way.

Assume that $\Omega \subset \mathbb{R}^3$ is a contractible polyhedron. Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of tetrahedral meshes of Ω. For each element $K \in \mathcal{T}_h$, denote by $\boldsymbol{n}_K$ the unit outward normal vector to ∂K, which will be abbreviated as $\boldsymbol{n}$ for simplicity. Let $\mathcal{F}_h$, $\mathcal{F}_h^i$, $\mathcal{E}_h$ and $\mathcal{V}_h$ be the union of all the faces, interior faces, edges and vertices of the partition $\mathcal{T}_h$, respectively. We fix a unit normal vector $\boldsymbol{n}_F$ for each face $F \in \mathcal{F}_h$, and a unit tangent vector $\boldsymbol{t}_e$ for each edge $e \in \mathcal{E}_h$. For any $K \in \mathcal{T}_h$, denote by $\mathcal{F}(K)$, $\mathcal{E}(K)$ and $\mathcal{V}(K)$ the set of all the faces, edges and vertices of K, respectively. For any $F \in \mathcal{F}_h$, let $\mathcal{E}(F)$ be the set of all the edges of F. For each $e \in \mathcal{E}(F)$, denote by $\boldsymbol{n}_{F,e}$ the unit vector being parallel to F and outward normal to ∂F. Set $\boldsymbol{t}_{F,e} := \boldsymbol{n}_F \times \boldsymbol{n}_{F,e}$, where $\times$ is the exterior product. For elementwise smooth function $\boldsymbol{v}$, define

$$\|\boldsymbol{v}\|_{1,h}^2 := \sum_{K\in\mathcal{T}_h} \|\boldsymbol{v}\|_{1,K}^2, \quad |\boldsymbol{v}|_{1,h}^2 := \sum_{K\in\mathcal{T}_h} |\boldsymbol{v}|_{1,K}^2.$$

Let ∇_h, curl_h and div_h be, respectively, the elementwise version of ∇, curl and div with respect to $\mathcal{T}_h$. Throughout this paper, we use "$\lesssim \cdots$" to mean that "$\leqslant C \cdots$", where C is a generic positive constant independent of h, which may take different values at different appearances. In addition, $A \eqsim B$ means that $A \lesssim B$ and $B \lesssim A$.

2.2 Nonconforming finite elements

We focus on constructing nonconforming finite elements for the space $\boldsymbol{H}(\operatorname{gradcurl}, \Omega)$ in this subsection. To this end, recall the direct sum of the polynomial space (see [2, 3])

$$\mathbb{P}_k(K; \mathbb{R}^3) = \nabla \mathbb{P}_{k+1}(K) \oplus ((\boldsymbol{x} - \boldsymbol{x}_K) \times \mathbb{P}_{k-1}(K; \mathbb{R}^3)), \quad \forall\, K \in \mathcal{T}_h, \tag{2.1}$$

where $\boldsymbol{x}_K$ is the barycenter of K. The decomposition (2.1) implies that $\operatorname{curl} : (\boldsymbol{x} - \boldsymbol{x}_K) \times \mathbb{P}_{k-1}(K; \mathbb{R}^3) \to \mathbb{P}_{k-1}(K; \mathbb{R}^3)$ is injective. We intend to use the nonconforming linear element to discretize $\boldsymbol{H}^1(\Omega; \mathbb{R}^3)$, and then the decomposition (2.1) and the complex (1.1) motivate us that the space of shape functions to discrete $\boldsymbol{H}(\operatorname{gradcurl}, \Omega)$ should include $(\boldsymbol{x} - \boldsymbol{x}_K) \times \mathbb{P}_1(K; \mathbb{R}^3)$. The direct sum in (2.1) also suggests to enrich $(\boldsymbol{x} - \boldsymbol{x}_K) \times \mathbb{P}_1(K; \mathbb{R}^3)$ with $\nabla \mathbb{P}_l(K)$ for some positive integer l to get the space of shape functions. Hence for each $K \in \mathcal{T}_h$, define the space of shape functions as

$$\boldsymbol{W}_k(K) := \nabla \mathbb{P}_{k+1}(K) \oplus ((\boldsymbol{x} - \boldsymbol{x}_K) \times \mathbb{P}_1(K; \mathbb{R}^3)) \quad \text{for } k = 0, 1.$$

By the decomposition (2.1), we have $\mathbb{P}_k(K; \mathbb{R}^3) \subset \boldsymbol{W}_k(K) \subset \mathbb{P}_2(K; \mathbb{R}^3)$ and

$$\dim \boldsymbol{W}_k(K) = \begin{cases} 14, & k = 0, \\ 20, & k = 1. \end{cases}$$

Then we choose the following local degrees of freedom $\mathcal{N}_k(K)$:

$$\int_e \boldsymbol{v} \cdot \boldsymbol{t}_e q \, ds, \quad \forall\, q \in \mathbb{P}_k(e) \text{ on each } e \in \mathcal{E}(K), \tag{2.2}$$

$$\int_F (\operatorname{curl} \boldsymbol{v}) \times \boldsymbol{n} \, ds \quad \text{on each } F \in \mathcal{F}(K). \tag{2.3}$$

The degrees of freedom (2.2)–(2.3) are inspired by the degrees of freedom of the nonconforming linear element and the Nédélec element (see [37, 38]). Note that the triple $(K, \boldsymbol{W}_1(K), \mathcal{N}_1(K))$ is exactly the nonconforming finite element in [48]. In this paper, we embed this nonconforming finite element into the discrete Stokes complex. We also construct the lowest order triple $(K, \boldsymbol{W}_0(K), \mathcal{N}_0(K))$.

Lemma 2.1. *The degrees of freedom* (2.2)–(2.3) *are unisolvent for the shape function space* $\boldsymbol{W}_k(K)$.

Proof. Notice that the number of the degrees of freedom (2.2)–(2.3) is same as the dimension of $\boldsymbol{W}_k(K)$. It is sufficient to show that $\boldsymbol{v} = \boldsymbol{0}$ for any $\boldsymbol{v} \in \boldsymbol{W}_k(K)$ with vanishing degrees of freedom (2.2)–(2.3).

For each $F \in \mathcal{F}(K)$, apply the integration by parts on the face F to obtain

$$\begin{aligned} \int_F (\operatorname{curl} \boldsymbol{v}) \cdot \boldsymbol{n}_F \, ds &= \int_F \operatorname{div}(\boldsymbol{v} \times \boldsymbol{n}_F) \, ds = \sum_{e \in \mathcal{E}(F)} \int_e (\boldsymbol{v} \times \boldsymbol{n}_F)|_F \cdot \boldsymbol{n}_{F,e} \, ds \\ &= \sum_{e \in \mathcal{E}(F)} \int_e \boldsymbol{v} \cdot (\boldsymbol{n}_F \times \boldsymbol{n}_{F,e}) \, ds = \sum_{e \in \mathcal{E}(F)} \int_e \boldsymbol{v} \cdot \boldsymbol{t}_{F,e} \, ds. \end{aligned} \tag{2.4}$$

We get from the vanishing degrees of freedom (2.2) that

$$\int_F (\operatorname{curl} \boldsymbol{v}) \cdot \boldsymbol{n}_F \, ds = 0,$$

which together with the vanishing degrees of freedom (2.3) implies

$$\int_F \operatorname{curl} \boldsymbol{v} \, ds = \boldsymbol{0}.$$

Since $\operatorname{curl}\boldsymbol{v} \subseteq \mathbb{P}_1(K;\mathbb{R}^3)$, we acquire from the unisolvence of the nonconforming linear element that $\operatorname{curl}\boldsymbol{v} = \boldsymbol{0}$. Employing the fact that $\operatorname{curl} : (\boldsymbol{x}-\boldsymbol{x}_K)\times\mathbb{P}_1(K;\mathbb{R}^3) \to \mathbb{P}_1(K;\mathbb{R}^3)$ is injective, we see that there exists a $q \in \mathbb{P}_{k+1}(K)$ such that $\boldsymbol{v} = \nabla q$. By the vanishing degrees of freedom (2.2), it holds that $\partial_{t_e} q = 0$, which implies that we can choose $q \in \mathbb{P}_{k+1}(K)$ such that $q\,|_e = 0$ for each $e \in \mathcal{E}(K)$. Noting that $k = 0, 1$, we acquire $q = 0$ and $\boldsymbol{v} = \boldsymbol{0}$. □

By comparing the degrees of freedom, we see that the lower order nonconforming element $(K, \boldsymbol{W}_0(K), \mathcal{N}_0(K))$ for $\boldsymbol{H}(\operatorname{gradcurl},\Omega)$ is very similar to the Morley-Wang-Xu element (see [43]) for $H^2(\Omega)$.

Next, we give a norm equivalence of the space $\boldsymbol{W}_k(K)$. To this end, recall the Poincaré operator $\mathcal{K}_K : \mathbb{P}_1(K;\mathbb{R}^3) \to (\boldsymbol{x}-\boldsymbol{x}_K)\times\mathbb{P}_1(K;\mathbb{R}^3)$ in [25, 29], i.e.,

$$\mathcal{K}_K\boldsymbol{q} := -(\boldsymbol{x}-\boldsymbol{x}_K)\times\int_0^1 t\boldsymbol{q}(t(\boldsymbol{x}-\boldsymbol{x}_K)+\boldsymbol{x}_K)dt.$$

Then we have the identity (see [25, Theorem 2.1])

$$\operatorname{curl}\mathcal{K}_K(\operatorname{curl}\boldsymbol{v}) = \operatorname{curl}\boldsymbol{v}, \quad \forall\,\boldsymbol{v} \in \boldsymbol{W}_k(K). \tag{2.5}$$

By the inverse inequality, we have

$$\|\mathcal{K}_K\boldsymbol{q}\|_{0,K} \lesssim h_K^{5/2}\|\boldsymbol{q}\|_{L^\infty(K)} \lesssim h_K\|\boldsymbol{q}\|_{0,K}, \quad \forall\,\boldsymbol{q} \in \mathbb{P}_1(K;\mathbb{R}^3). \tag{2.6}$$

Lemma 2.2. *For $\boldsymbol{v} \in \boldsymbol{W}_k(K)$, there exists a $q \in \mathbb{P}_{k+1}(K)$ such that*

$$\boldsymbol{v} = \nabla q + \mathcal{K}_K(\operatorname{curl}\boldsymbol{v}), \tag{2.7}$$

$$\|q\|_{0,K}^2 \lesssim h_K^4\|\operatorname{curl}\boldsymbol{v}\|_{0,K}^2 + h_K^4\sum_{e\in\mathcal{E}(K)}\|Q_e^k(\boldsymbol{v}\cdot\boldsymbol{t}_e)\|_{0,e}^2. \tag{2.8}$$

Proof. Take a vertex $\delta \in \mathcal{V}(K)$. Due to (2.5), $\boldsymbol{v} - \mathcal{K}_K(\operatorname{curl}\boldsymbol{v}) \in \boldsymbol{W}_k(K)\cap\ker(\operatorname{curl})$, which means $\boldsymbol{v} - \mathcal{K}_K(\operatorname{curl}\boldsymbol{v}) \in \nabla\mathbb{P}_{k+1}(K)$. Choose $q \in \mathbb{P}_{k+1}(K)$ such that $\boldsymbol{v} - \mathcal{K}_K(\operatorname{curl}\boldsymbol{v}) = \nabla q$ and $q(\delta) = 0$. By the fact that $q \in \mathbb{P}_2(K)$, the norm equivalence of the Lagrange element and the inverse inequality, we have

$$\begin{aligned}\|q\|_{0,K}^2 &\lesssim h_K^2\sum_{e\in\mathcal{E}(K)}\|q\|_{0,e}^2 \lesssim h_K^3\sum_{e\in\mathcal{E}(K)}\|q\|_{L^\infty(e)}^2 = h_K^3\sum_{e\in\mathcal{E}(K)}\|q(\boldsymbol{x})-q(\delta)\|_{L^\infty(e)}^2\\ &\lesssim h_K^3\sum_{e\in\mathcal{E}(K)}h_e^2\|\partial_t q\|_{L^\infty(e)}^2 \lesssim h_K^4\sum_{e\in\mathcal{E}(K)}\|\partial_t q\|_{0,e}^2.\end{aligned}$$

Since $\partial_t q = Q_e^k(\partial_t q) = Q_e^k(\boldsymbol{v}\cdot\boldsymbol{t}_e) + Q_e^k(\mathcal{K}_K(\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{t}_e)$ on edge e, we get from the inverse inequality that

$$\begin{aligned}\|q\|_{0,K}^2 &\lesssim h_K^4\sum_{e\in\mathcal{E}(K)}(\|Q_e^k(\boldsymbol{v}\cdot\boldsymbol{t}_e)\|_{0,e}^2 + \|\mathcal{K}_K(\operatorname{curl}\boldsymbol{v})\|_{0,e}^2)\\ &\lesssim h_K^2\|\mathcal{K}_K(\operatorname{curl}\boldsymbol{v})\|_{0,K}^2 + h_K^4\sum_{e\in\mathcal{E}(K)}\|Q_e^k(\boldsymbol{v}\cdot\boldsymbol{t}_e)\|_{0,e}^2.\end{aligned}$$

Finally, we conclude (2.8) from (2.6). □

Lemma 2.3. *For $\boldsymbol{v} \in \boldsymbol{W}_k(K)$, we have the norm equivalence*

$$\|\boldsymbol{v}\|_{0,K}^2 \eqsim h_K^2\sum_{e\in\mathcal{E}(K)}\|Q_e^k(\boldsymbol{v}\cdot\boldsymbol{t}_e)\|_{0,e}^2 + h_K^3\sum_{F\in\mathcal{F}(K)}\|\boldsymbol{Q}_F^0((\operatorname{curl}\boldsymbol{v})\times\boldsymbol{n})\|_{0,F}^2. \tag{2.9}$$

Proof. Since $\operatorname{curl}\boldsymbol{v} \in \mathbb{P}_1(K;\mathbb{R}^3)$, by the norm equivalence of the nonconforming P_1 element,

$$\begin{aligned}\|\operatorname{curl}\boldsymbol{v}\|_{0,K}^2 &\lesssim h_K\sum_{F\in\mathcal{F}(K)}\|\boldsymbol{Q}_F^0(\operatorname{curl}\boldsymbol{v})\|_{0,F}^2\\ &\lesssim h_K\sum_{F\in\mathcal{F}(K)}(\|\boldsymbol{Q}_F^0((\operatorname{curl}\boldsymbol{v})\times\boldsymbol{n})\|_{0,F}^2 + \|Q_F^0((\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{n})\|_{0,F}^2).\end{aligned}$$

From (2.4), we get

$$h_K\|Q_F^0((\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{n})\|_{0,F}^2 \lesssim h_K^3|Q_F^0((\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{n})|^2 \lesssim h_K\sum_{e\in\mathcal{E}(F)}|Q_e^0(\boldsymbol{v}\cdot\boldsymbol{t}_e)|^2$$
$$\lesssim \sum_{e\in\mathcal{E}(F)}\|Q_e^0(\boldsymbol{v}\cdot\boldsymbol{t}_e)\|_{0,e}^2 \leqslant \sum_{e\in\mathcal{E}(F)}\|Q_e^k(\boldsymbol{v}\cdot\boldsymbol{t}_e)\|_{0,e}^2.$$

Combining the last two inequalities, we have

$$\|\operatorname{curl}\boldsymbol{v}\|_{0,K}^2 \lesssim \sum_{e\in\mathcal{E}(K)}\|Q_e^k(\boldsymbol{v}\cdot\boldsymbol{t}_e)\|_{0,e}^2 + h_K\sum_{F\in\mathcal{F}(K)}\|\boldsymbol{Q}_F^0((\operatorname{curl}\boldsymbol{v})\times\boldsymbol{n})\|_{0,F}^2. \tag{2.10}$$

Applying Lemma 2.2 to $\boldsymbol{v}$, we derive from (2.7), the inverse inequality, (2.6) and (2.8) that

$$\|\boldsymbol{v}\|_{0,K}^2 \leqslant 2\|\nabla q\|_{0,K}^2 + 2\|\mathcal{K}_K(\operatorname{curl}\boldsymbol{v})\|_{0,K}^2 \lesssim h_K^{-2}\|q\|_{0,K}^2 + h_K^2\|\operatorname{curl}\boldsymbol{v}\|_{0,K}^2$$
$$\lesssim h_K^2\|\operatorname{curl}\boldsymbol{v}\|_{0,K}^2 + h_K^2\sum_{e\in\mathcal{E}(K)}\|Q_e^k(\boldsymbol{v}\cdot\boldsymbol{t}_e)\|_{0,e}^2.$$

Then we acquire from (2.10) that

$$\|\boldsymbol{v}\|_{0,K}^2 \lesssim h_K^2\sum_{e\in\mathcal{E}(K)}\|Q_e^k(\boldsymbol{v}\cdot\boldsymbol{t}_e)\|_{0,e}^2 + h_K^3\sum_{F\in\mathcal{F}(K)}\|\boldsymbol{Q}_F^0((\operatorname{curl}\boldsymbol{v})\times\boldsymbol{n})\|_{0,F}^2.$$

The other side of (2.9) follows from the inverse inequality. □

2.3 Basis functions

We figure out the basis functions of $\boldsymbol{W}_0(K)$ in this subsection. We refer to [48] for the basis functions of $\boldsymbol{W}_1(K)$. Let λ_1, λ_2, λ_3 and λ_4 be the barycentric coordinates of the point $\boldsymbol{x}$ with respect to the vertices $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ and $\boldsymbol{x}_4$ of the tetrahedron K, respectively. Let F_l be the face of K opposite to $\boldsymbol{x}_l$. The vertices of F_l are denoted by $\boldsymbol{x}_{l_1}$, $\boldsymbol{x}_{l_2}$ and $\boldsymbol{x}_{l_3}$ with $l_1 < l_2 < l_3$. Set $\boldsymbol{t}_{ij} := \boldsymbol{x}_j - \boldsymbol{x}_i$, which is a tangential vector to the edge e_{ij} with the vertices $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$, and similarly define other tangential vectors with different subscripts. For ease of presentation, let

$$M_{e_{ij}}(\boldsymbol{v}) := \frac{1}{|e_{ij}|}\int_{e_{ij}}\boldsymbol{v}\cdot\boldsymbol{t}_{ij}ds, \quad \boldsymbol{M}_{F_l}(\boldsymbol{v}) := \int_{F_l}(\operatorname{curl}\boldsymbol{v})\times\boldsymbol{n}_l ds,$$
$$M_{F_l,1}(\boldsymbol{v}) := \frac{1}{2|F_l|(\nabla\lambda_{l_1}\times\nabla\lambda_{l_2})\cdot\boldsymbol{n}_l}\int_{F_l}(\operatorname{curl}\boldsymbol{v})\cdot((\boldsymbol{n}_l\times\nabla\lambda_{l_2})\times\boldsymbol{n}_l)ds,$$
$$M_{F_l,2}(\boldsymbol{v}) := \frac{1}{2|F_l|(\nabla\lambda_{l_2}\times\nabla\lambda_{l_1})\cdot\boldsymbol{n}_l}\int_{F_l}(\operatorname{curl}\boldsymbol{v})\cdot((\boldsymbol{n}_l\times\nabla\lambda_{l_1})\times\boldsymbol{n}_l)ds.$$

The degrees of freedom $M_{F_l,1}(\boldsymbol{v})$ and $M_{F_l,2}(\boldsymbol{v})$ are equivalent to $\boldsymbol{M}_{F_l}(\boldsymbol{v})$, i.e., (2.3).

2.3.1 *Basis functions corresponding to the face degrees of freedom*

Define

$$\boldsymbol{\varphi}_{F_l,i} := \frac{1}{4}(8\lambda_l - 3)(\boldsymbol{x}-\boldsymbol{x}_K)\times(\boldsymbol{n}_l\times\nabla\lambda_{l_i}) + \frac{1}{4}(\boldsymbol{x}_l-\boldsymbol{x}_K)\cdot\boldsymbol{n}_l\nabla\lambda_{l_i} + \frac{1}{16}\boldsymbol{n}_l$$
$$= \frac{1}{16}(8\lambda_l-3)[(4\lambda_{l_i}-1)\boldsymbol{n}_l - 4(\boldsymbol{x}-\boldsymbol{x}_K)\cdot\boldsymbol{n}_l\nabla\lambda_{l_i}]$$
$$+ \frac{1}{4}(\boldsymbol{x}_l-\boldsymbol{x}_K)\cdot\boldsymbol{n}_l\nabla\lambda_{l_i} + \frac{1}{16}\boldsymbol{n}_l$$

for $i = 1, 2$. We show that $\boldsymbol{\varphi}_{F_l,1}$ and $\boldsymbol{\varphi}_{F_l,2}$ are the basis functions being dual to $M_{F_l,1}(\boldsymbol{v})$ and $M_{F_l,2}(\boldsymbol{v})$, respectively.

Lemma 2.4. *Functions $\boldsymbol{\varphi}_{F_l,1}$ and $\boldsymbol{\varphi}_{F_l,2}$ are the basis functions of $\boldsymbol{W}_0(K)$ being dual to $M_{F_l,1}(\boldsymbol{v})$ and $M_{F_l,2}(\boldsymbol{v})$, respectively, i.e.,*

$$M_e(\boldsymbol{\varphi}_{F_l,1}) = M_e(\boldsymbol{\varphi}_{F_l,2}) = 0, \quad \forall\, e \in \mathcal{E}(K), \tag{2.11}$$

$$\boldsymbol{M}_F(\boldsymbol{\varphi}_{F_l,1}) = \boldsymbol{M}_F(\boldsymbol{\varphi}_{F_l,2}) = \boldsymbol{0}, \quad \forall\, F \in \mathcal{F}(K)\backslash\{F_l\}, \tag{2.12}$$

$$M_{F_l,2}(\boldsymbol{\varphi}_{F_l,1}) = M_{F_l,1}(\boldsymbol{\varphi}_{F_l,2}) = 0, \quad M_{F_l,1}(\boldsymbol{\varphi}_{F_l,1}) = M_{F_l,2}(\boldsymbol{\varphi}_{F_l,2}) = 1. \tag{2.13}$$

Proof. Apparently, $\boldsymbol{\varphi}_{F_l,1} \cdot \boldsymbol{t}_{l_2l_3} = 0$. By $\boldsymbol{n}_l \cdot \boldsymbol{t}_{l_1l_2} = 0$, $\nabla\lambda_{l_1} \cdot \boldsymbol{t}_{l_1l_2} = -1$ and $\lambda_l\,|_{e_{l_1l_2}} = 0$, we get

$$\begin{aligned} M_{e_{l_1l_2}}(\boldsymbol{\varphi}_{F_l,1}) &= \frac{1}{4|e_{l_1l_2}|}\int_{e_{l_1l_2}} ((8\lambda_l - 3)(\boldsymbol{x} - \boldsymbol{x}_K) \cdot \boldsymbol{n}_l - (\boldsymbol{x}_l - \boldsymbol{x}_K) \cdot \boldsymbol{n}_l) ds \\ &= -\frac{3}{4}(\boldsymbol{x}_{l_1} - \boldsymbol{x}_K) \cdot \boldsymbol{n}_l - \frac{1}{4}(\boldsymbol{x}_l - \boldsymbol{x}_K) \cdot \boldsymbol{n}_l. \end{aligned}$$

Noting that

$$\boldsymbol{x}_l - \boldsymbol{x}_K + 3(\boldsymbol{x}_{l_1} - \boldsymbol{x}_K) = \boldsymbol{x}_l + 3\boldsymbol{x}_{l_1} - 4\boldsymbol{x}_K = 2\boldsymbol{x}_{l_1} - \boldsymbol{x}_{l_2} - \boldsymbol{x}_{l_3}$$

is parallel to the face F_l, we have

$$(\boldsymbol{x}_l - \boldsymbol{x}_K + 3(\boldsymbol{x}_{l_1} - \boldsymbol{x}_K)) \cdot \boldsymbol{n}_l = 0.$$

Hence, $M_{e_{l_1l_2}}(\boldsymbol{\varphi}_{F_l,1}) = 0$. Since $\boldsymbol{n}_l = \frac{\boldsymbol{n}_l\cdot\nabla\lambda_l}{|\nabla\lambda_l|^2}\nabla\lambda_l$, $4(\boldsymbol{x} - \boldsymbol{x}_K) \cdot \nabla\lambda_l = 4\lambda_l - 1$ and $(\lambda_{l_1} + \lambda_l)\,|_{e_{l_1l}} = 1$, it follows that

$$\begin{aligned} M_{e_{l_1l}}(\boldsymbol{\varphi}_{F_l,1}) &= \frac{\boldsymbol{n}_l \cdot \nabla\lambda_l}{8|e_{l_1l}||\nabla\lambda_l|^2}\int_{e_{l_1l}} (8\lambda_l - 3) ds + \frac{\boldsymbol{n}_l \cdot \nabla\lambda_l}{16|\nabla\lambda_l|^2}(1 - 4(\boldsymbol{x}_l - \boldsymbol{x}_K) \cdot \nabla\lambda_l) \\ &= \frac{\boldsymbol{n}_l \cdot \nabla\lambda_l}{16|\nabla\lambda_l|^2}(3 - 4(\boldsymbol{x}_l - \boldsymbol{x}_K) \cdot \nabla\lambda_l) = 0. \end{aligned}$$

Similarly, we can show that $M_e(\boldsymbol{\varphi}_{F_l,1}) = 0$ for other edges and $M_e(\boldsymbol{\varphi}_{F_l,2}) = 0$. Hence, (2.11) holds.

On the other hand, by the identity

$$\operatorname{curl}((\boldsymbol{x} - \boldsymbol{x}_K) \times \boldsymbol{q}) = (\boldsymbol{x} - \boldsymbol{x}_K) \operatorname{div} \boldsymbol{q} - ((\boldsymbol{x} - \boldsymbol{x}_K) \cdot \nabla)\boldsymbol{q} - 2\boldsymbol{q},$$

we find that for $i = 1, 2$,

$$\operatorname{curl} \boldsymbol{\varphi}_{F_l,i} = \frac{1}{4}\operatorname{curl}((8\lambda_l - 3)(\boldsymbol{x} - \boldsymbol{x}_K) \times (\boldsymbol{n}_l \times \nabla\lambda_{l_i})) = 2(1 - 3\lambda_l)\boldsymbol{n}_l \times \nabla\lambda_{l_i}.$$

We conclude (2.12)–(2.13) by the fact that $1 - 3\lambda_l$ is the basis function of the nonconforming P_1 element. □

2.3.2 *Basis functions corresponding to the edge degrees of freedom*

Next, we construct the basis function corresponding to the degree of freedom $M_{e_{ij}}(\boldsymbol{v})$. Recall the basis function of the lowest order Nédélec element of the first kind $\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i$. Thanks to (2.11)–(2.12), the function $\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i$ can be modified by $\boldsymbol{\varphi}_{F_l,1}$ and $\boldsymbol{\varphi}_{F_l,2}$ to derive the basis function of $\boldsymbol{W}_0(K)$ corresponding to the degree of freedom $M_{e_{ij}}(\boldsymbol{v})$.

Lemma 2.5. *Let*

$$\boldsymbol{\varphi}_{e_{ij}} := \lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i + \sum_{l=1}^{4}(c_{l,1}^{ij}\boldsymbol{\varphi}_{F_l,1} + c_{l,2}^{ij}\boldsymbol{\varphi}_{F_l,2})$$

with constants

$$c_{l,1}^{ij} := \frac{1}{(\nabla\lambda_{l_1} \times \nabla\lambda_{l_2}) \cdot \boldsymbol{n}_l}(\nabla\lambda_j \times \nabla\lambda_i) \cdot ((\boldsymbol{n}_l \times \nabla\lambda_{l_2}) \times \boldsymbol{n}_l)$$

and

$$c_{l,2}^{ij} := \frac{1}{(\nabla\lambda_{l_2} \times \nabla\lambda_{l_1}) \cdot \boldsymbol{n}_l}(\nabla\lambda_j \times \nabla\lambda_i) \cdot ((\boldsymbol{n}_l \times \nabla\lambda_{l_1}) \times \boldsymbol{n}_l).$$

Then

$$M_{e_{ij}}(\boldsymbol{\varphi}_{e_{ij}}) = 1, \quad M_e(\boldsymbol{\varphi}_{e_{ij}}) = 0, \quad \boldsymbol{M}_F(\boldsymbol{\varphi}_{e_{ij}}) = \boldsymbol{0}$$

for each $e \in \mathcal{E}(K) \backslash \{e_{ij}\}$ *and* $F \in \mathcal{F}(K)$.

Proof. The identities $M_{e_{ij}}(\boldsymbol{\varphi}_{e_{ij}}) = 1$ and $M_e(\boldsymbol{\varphi}_{e_{ij}}) = 0$ follow from (2.11) and the fact that

$$M_{e_{ij}}(\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i) = 1, \quad M_e(\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i) = 0, \quad \forall\, e \in \mathcal{E}(K) \backslash \{e_{ij}\}.$$

On the other hand, we get from (2.12)–(2.13) and $\mathrm{curl}(\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i) = 2 \nabla \lambda_i \times \nabla \lambda_j$ that

$$M_{F_l, r}(\boldsymbol{\varphi}_{e_{ij}}) = M_{F_l, r}(\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i) + c_{l,r}^{ij} = 0$$

for $r = 1, 2$. □

In summary, we arrive at the basis functions being dual to the degrees of freedom $M_{F_l,1}(\boldsymbol{v})$, $M_{F_l,2}(\boldsymbol{v})$ and $M_{e_{ij}}(\boldsymbol{v})$.

(1) Two basis functions on each face F_l ($1 \leqslant l \leqslant 4$),

$$\boldsymbol{\varphi}_{F_l, i} = \frac{1}{4}(8\lambda_l - 3)(\boldsymbol{x} - \boldsymbol{x}_K) \times (\boldsymbol{n}_l \times \nabla \lambda_{l_i}) + \frac{1}{4}(\boldsymbol{x}_l - \boldsymbol{x}_K) \cdot \boldsymbol{n}_l \nabla \lambda_{l_i} + \frac{1}{16} \boldsymbol{n}_l$$

for $i = 1, 2$, where $\boldsymbol{x}_K$ is the barycenter of K.

(2) One basis function on each edge e_{ij} ($1 \leqslant i < j \leqslant 4$),

$$\boldsymbol{\varphi}_{e_{ij}} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i + \sum_{l=1}^{4} (c_{l,1}^{ij} \boldsymbol{\varphi}_{F_l,1} + c_{l,2}^{ij} \boldsymbol{\varphi}_{F_l,2})$$

with constants

$$c_{l,1}^{ij} := \frac{1}{(\nabla \lambda_{l_1} \times \nabla \lambda_{l_2}) \cdot \boldsymbol{n}_l} (\nabla \lambda_j \times \nabla \lambda_i) \cdot ((\boldsymbol{n}_l \times \nabla \lambda_{l_2}) \times \boldsymbol{n}_l)$$

and

$$c_{l,2}^{ij} := \frac{1}{(\nabla \lambda_{l_2} \times \nabla \lambda_{l_1}) \cdot \boldsymbol{n}_l} (\nabla \lambda_j \times \nabla \lambda_i) \cdot ((\boldsymbol{n}_l \times \nabla \lambda_{l_1}) \times \boldsymbol{n}_l).$$

3 Nonconforming finite element Stokes complexes

We consider the nonconforming finite element discretization of the Stokes complex (1.1) in this section. The homogeneous version of the Stokes complex (1.1) is

$$0 \xrightarrow{\subset} H_0^1(\Omega) \xrightarrow{\nabla} \boldsymbol{H}_0(\mathrm{gradcurl}, \Omega) \xrightarrow{\mathrm{curl}} \boldsymbol{H}_0^1(\Omega; \mathbb{R}^3) \xrightarrow{\mathrm{div}} L_0^2(\Omega) \to 0,$$

where

$$\boldsymbol{H}_0(\mathrm{gradcurl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega) : \mathrm{curl}\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega; \mathbb{R}^3)\}.$$

Since $\boldsymbol{H}_0(\mathrm{curl}, \Omega) \cap \boldsymbol{H}_0(\mathrm{div}, \Omega) = \boldsymbol{H}_0^1(\Omega; \mathbb{R}^3)$, it holds that

$$\boldsymbol{H}_0(\mathrm{gradcurl}, \Omega) = \boldsymbol{H}_0(\mathrm{curl}^2, \Omega),$$

where

$$\boldsymbol{H}_0(\mathrm{curl}^2, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega) : \mathrm{curl}\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega)\}.$$

We can use the Lagrange element, the nonconforming linear element and the piecewise constant to discretize $H^1(\Omega)$, $\boldsymbol{H}^1(\Omega; \mathbb{R}^3)$ and $L^2(\Omega)$ in the Stokes complex (1.1), respectively. Take the Lagrange element space

$$V_h^g := \{v_h \in H^1(\Omega) : v_h|_K \in \mathbb{P}_{k+1}(K) \text{ for each } K \in \mathcal{T}_h\}$$

with $k = 0, 1$, the nonconforming linear element space

$$\boldsymbol{V}_h^s := \left\{\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega; \mathbb{R}^3) : \boldsymbol{v}_h|_K \in \mathbb{P}_1(K; \mathbb{R}^3) \text{ for each } K \in \mathcal{T}_h \text{ and } \int_F [\![\boldsymbol{v}_h]\!] ds = \boldsymbol{0} \text{ for each } F \in \mathcal{F}_h^i\right\}$$

and the piecewise constant space

$$\mathcal{Q}_h := \{q_h \in L^2(\Omega) : q_h|_K \in \mathbb{P}_0(K) \text{ for each } K \in \mathcal{T}_h\}.$$

Here, $[\![\boldsymbol{v}_h]\!]$ is the jump of $\boldsymbol{v}_h$ across F. Define the global $\boldsymbol{H}$(gradcurl)-nonconforming element space

$$\begin{aligned}\boldsymbol{W}_h := \{&\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega;\mathbb{R}^3) : \boldsymbol{v}_h|_K \in \boldsymbol{W}_k(K) \text{ for each } K \in \mathcal{T}_h, \text{ and all the}\\ &\text{degrees of freedom (2.2)–(2.3) are single-valued}\}.\end{aligned}$$

According to the proof of Lemma 2.1, it holds that

$$\int_F [\![\operatorname{curl} \boldsymbol{v}_h]\!] ds = \boldsymbol{0}, \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{W}_h, \quad F \in \mathcal{F}_h^i. \tag{3.1}$$

To prove the exactness of the nonconforming discrete Stokes complexes, we need the help of the Nédélec element spaces (see [37, 38])

$$\boldsymbol{V}_h^c := \{\boldsymbol{v}_h \in \boldsymbol{H}(\operatorname{curl},\Omega) : \boldsymbol{v}_h|_K \in \boldsymbol{V}_k^c(K) \text{ for each } K \in \mathcal{T}_h\},$$

where $\boldsymbol{V}_k^c(K) := \mathbb{P}_k(K;\mathbb{R}^3) + (\boldsymbol{x} - \boldsymbol{x}_K) \times \mathbb{P}_0(K;\mathbb{R}^3)$ with $k = 0, 1$. Apparently, $\boldsymbol{V}_k^c(K) \subset \boldsymbol{W}_k(K)$. The degrees of freedom for $\boldsymbol{V}_k^c(K)$ are

$$\int_e \boldsymbol{v} \cdot \boldsymbol{t}_e q ds, \quad \forall\, q \in \mathbb{P}_k(e) \text{ on each } e \in \mathcal{E}(K). \tag{3.2}$$

It is observed that the degrees of freedom (3.2) are exactly the same as (2.2). By the finite element de Rham complexes (see [2, 3]), we have

$$\boldsymbol{V}_h^c \cap \ker(\operatorname{curl}) = \nabla V_h^g. \tag{3.3}$$

The notation $\ker(\mathcal{A})$ means the kernel space of the operator $\mathcal{A}$.

Lemma 3.1. *It holds that*

$$\boldsymbol{W}_h \cap \ker(\operatorname{curl}_h) = \nabla V_h^g.$$

Proof. Since $\operatorname{curl} : (\boldsymbol{x} - \boldsymbol{x}_K) \times \mathbb{P}_1(K;\mathbb{R}^3) \to \mathbb{P}_1(K;\mathbb{R}^3)$ is injective (see [2, 3]), we have

$$\begin{aligned}\boldsymbol{W}_h \cap \ker(\operatorname{curl}_h) &= \{\boldsymbol{v}_h \in \boldsymbol{W}_h : \boldsymbol{v}_h|_K \in \nabla\mathbb{P}_{k+1}(K) \text{ for each } K \in \mathcal{T}_h\},\\ \boldsymbol{V}_h^c \cap \ker(\operatorname{curl}_h) &= \{\boldsymbol{v}_h \in \boldsymbol{V}_h^c : \boldsymbol{v}_h|_K \in \nabla\mathbb{P}_{k+1}(K) \text{ for each } K \in \mathcal{T}_h\}.\end{aligned}$$

Noting that the degrees of freedom (2.2) and (3.2) are the same, we see that

$$\boldsymbol{W}_h \cap \ker(\operatorname{curl}_h) = \boldsymbol{V}_h^c \cap \ker(\operatorname{curl}_h).$$

Thus we finish the proof from (3.3). □

Lemma 3.2. *The nonconforming discrete Stokes complex*

$$\mathbb{R} \xrightarrow{\subset} V_h^g \xrightarrow{\nabla} \boldsymbol{W}_h \xrightarrow{\operatorname{curl}_h} \boldsymbol{V}_h^s \xrightarrow{\operatorname{div}_h} \mathcal{Q}_h \to 0 \tag{3.4}$$

is exact.

Proof. We refer to [6, 19] for $\operatorname{div}_h \boldsymbol{V}_h^s = \mathcal{Q}_h$ and Lemma 3.1 for $\boldsymbol{W}_h \cap \ker(\operatorname{curl}_h) = \nabla V_h^g$. By the definition of $\boldsymbol{W}_h$, apparently we have from (3.1) that

$$\operatorname{curl}_h \boldsymbol{W}_h \subseteq \boldsymbol{V}_h^s \cap \ker(\operatorname{div}_h).$$

Then we prove

$$\operatorname{curl}_h \boldsymbol{W}_h = \boldsymbol{V}_h^s \cap \ker(\operatorname{div}_h)$$

by counting the dimensions of these spaces. Indeed, we have

$$\begin{aligned}\dim \operatorname{curl}_h \boldsymbol{W}_h &= \dim \boldsymbol{W}_h - \dim V_h^g + 1\\ &= (k+1)\#\mathcal{E}_h + 2\#\mathcal{F}_h - \#\mathcal{V}_h - k\#\mathcal{E}_h + 1\\ &= \#\mathcal{E}_h + 2\#\mathcal{F}_h - \#\mathcal{V}_h + 1\end{aligned}$$

and

$$\dim \boldsymbol{V}_h^s \cap \ker(\operatorname{div}_h) = \dim \boldsymbol{V}_h^s - \dim \mathcal{Q}_h = 3\#\mathcal{F}_h - \#\mathcal{T}_h.$$

Finally, apply Euler's formula $\#\mathcal{V}_h - \#\mathcal{E}_h + \#\mathcal{F}_h - \#\mathcal{T}_h = 1$ to end the proof. □

Corollary 3.3. *The nonconforming discrete Stokes complex with the homogeneous boundary condition*

$$0 \xrightarrow{\subset} V_{h0}^g \xrightarrow{\nabla} \boldsymbol{W}_{h0} \xrightarrow{\operatorname{curl}_h} \boldsymbol{V}_{h0}^s \xrightarrow{\operatorname{div}_h} \mathcal{Q}_{h0} \to 0 \tag{3.5}$$

is exact, where $V_{h0}^g := V_h^g \cap H_0^1(\Omega)$*,* $\mathcal{Q}_{h0} := \mathcal{Q}_h \cap L_0^2(\Omega)$ *and*

$$\boldsymbol{W}_{h0} := \{\boldsymbol{v}_h \in \boldsymbol{W}_h : \text{all the degrees of freedom (2.2)–(2.3) on } \partial\Omega \text{ vanish}\},$$

$$\boldsymbol{V}_{h0}^s := \left\{\boldsymbol{v}_h \in \boldsymbol{V}_h^s : \int_F \boldsymbol{v}_h ds = \boldsymbol{0} \text{ for each } F \in \mathcal{F}_h \backslash \mathcal{F}_h^i\right\}.$$

The space $\boldsymbol{V}_{h0}^s$ possesses the norm equivalence (see [7, Subsection 10.6])

$$\|\boldsymbol{v}_h\|_{1,h} \eqsim |\boldsymbol{v}_h|_{1,h}, \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_{h0}^s. \tag{3.6}$$

Equip $\boldsymbol{W}_{h0}$ with the discrete squared norm

$$\|\boldsymbol{v}_h\|_{H_h(\operatorname{gradcurl})}^2 := \|\boldsymbol{v}_h\|_0^2 + \|\operatorname{curl}_h \boldsymbol{v}_h\|_0^2 + |\operatorname{curl}_h \boldsymbol{v}_h|_{1,h}^2.$$

Since $\operatorname{curl}_h \boldsymbol{v}_h \in \boldsymbol{V}_{h0}^s$ for any $\boldsymbol{v}_h \in \boldsymbol{W}_{h0}$, applying (3.6) to $\operatorname{curl}_h \boldsymbol{v}_h$ gives

$$\|\boldsymbol{v}_h\|_{H_h(\operatorname{gradcurl})} \eqsim \|\boldsymbol{v}_h\|_0 + |\operatorname{curl}_h \boldsymbol{v}_h|_{1,h}, \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{W}_{h0}.$$

Next, we focus on the commutative diagrams for the Stokes complexes (3.4) and (3.5). For this, we introduce some interpolation operators. For each $K \in \mathcal{T}_h$, let $I_K^g : H^2(K) \to \mathbb{P}_{k+1}(K)$ be the nodal interpolation operator of the Lagrange element (see [17]), and $\boldsymbol{I}_K^s : \boldsymbol{H}^1(K;\mathbb{R}^3) \to \mathbb{P}_1(K;\mathbb{R}^3)$ be the nodal interpolation operator of the nonconforming linear element (see [7]). We have (see [6])

$$\operatorname{div}(\boldsymbol{I}_K^s \boldsymbol{v}) = Q_K \operatorname{div} \boldsymbol{v}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^1(K;\mathbb{R}^3), \tag{3.7}$$

$$\|\boldsymbol{v} - \boldsymbol{I}_K^s \boldsymbol{v}\|_{0,K} + h_K|\boldsymbol{v} - \boldsymbol{I}_K^s \boldsymbol{v}|_{1,K} \lesssim h_K^j |\boldsymbol{v}|_{j,K}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^j(K;\mathbb{R}^3), \quad j = 1,2. \tag{3.8}$$

Define $\boldsymbol{I}_K^{gc} : \boldsymbol{H}^1(\operatorname{curl}, K) \to \boldsymbol{W}_k(K)$ as the nodal interpolation operator based on the degrees of freedom (2.2)–(2.3). By Lemma 2.1, we get

$$\boldsymbol{I}_K^{gc} \boldsymbol{q} = \boldsymbol{q}, \quad \forall\, \boldsymbol{q} \in \boldsymbol{W}_k(K). \tag{3.9}$$

Lemma 3.4. *It holds that*

$$\|\boldsymbol{v} - \boldsymbol{I}_K^{gc} \boldsymbol{v}\|_{0,K} \lesssim h_K^{k+1} |\boldsymbol{v}|_{k+1,K} + h_K^2 |\boldsymbol{v}|_{2,K}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^2(K;\mathbb{R}^3). \tag{3.10}$$

Proof. Set $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{Q}_K^k \boldsymbol{v}$ for ease of presentation. By the norm equivalence (2.9) and the definition of $\boldsymbol{I}_K^{gc}$, we obtain

$$\begin{aligned}\|\boldsymbol{I}_K^{gc} \boldsymbol{w}\|_{0,K}^2 &\lesssim h_K^2 \sum_{e \in \mathcal{E}(K)} \|Q_e^k((\boldsymbol{I}_K^{gc} \boldsymbol{w}) \cdot \boldsymbol{t}_e)\|_{0,e}^2 + h_K^3 \sum_{F \in \mathcal{F}(K)} \|\boldsymbol{Q}_F^0(\operatorname{curl}(\boldsymbol{I}_K^{gc} \boldsymbol{w}) \times \boldsymbol{n})\|_{0,F}^2\\ &= h_K^2 \sum_{e \in \mathcal{E}(K)} \|Q_e^k(\boldsymbol{w} \cdot \boldsymbol{t}_e)\|_{0,e}^2 + h_K^3 \sum_{F \in \mathcal{F}(K)} \|\boldsymbol{Q}_F^0((\operatorname{curl} \boldsymbol{w}) \times \boldsymbol{n})\|_{0,F}^2\end{aligned}$$

$$\leqslant h_K^2 \sum_{e\in\mathcal{E}(K)} \|\boldsymbol{w}\|_{0,e}^2 + h_K^3 \sum_{F\in\mathcal{F}(K)} \|\operatorname{curl}\boldsymbol{w}\|_{0,F}^2.$$

Then we obtain from (3.9), $\mathbb{P}_k(K;\mathbb{R}^3) \subset \boldsymbol{W}_k(K)$ and the trace inequality that

$$\begin{aligned}\|\boldsymbol{v} - \boldsymbol{I}_K^{gc}\boldsymbol{v}\|_{0,K}^2 &= \|\boldsymbol{w} - \boldsymbol{I}_K^{gc}\boldsymbol{w}\|_{0,K}^2 \leqslant 2\|\boldsymbol{w}\|_{0,K}^2 + 2\|\boldsymbol{I}_K^{gc}\boldsymbol{w}\|_{0,K}^2 \\ &\lesssim \|\boldsymbol{w}\|_{0,K}^2 + h_K^2 \sum_{e\in\mathcal{E}(K)} \|\boldsymbol{w}\|_{0,e}^2 + h_K^3 \sum_{F\in\mathcal{F}(K)} \|\operatorname{curl}\boldsymbol{w}\|_{0,F}^2 \\ &\lesssim \|\boldsymbol{w}\|_{0,K}^2 + h_K \sum_{F\in\mathcal{F}(K)} (\|\boldsymbol{w}\|_{0,F}^2 + h_K^2|\boldsymbol{w}|_{1,F}^2 + h_K^2\|\operatorname{curl}\boldsymbol{w}\|_{0,F}^2) \\ &\lesssim \|\boldsymbol{w}\|_{0,K}^2 + h_K^2|\boldsymbol{w}|_{1,K}^2 + h_K^4|\boldsymbol{w}|_{2,K}^2.\end{aligned}$$

Therefore, the inequality (3.10) holds from the error estimate of $\boldsymbol{Q}_K^k$. □

Lemma 3.5. *The operators I_K^g, $\boldsymbol{I}_K^{gc}$ and $\boldsymbol{I}_K^s$ satisfy the following commuting properties*:

$$\nabla(I_K^g v) = \boldsymbol{I}_K^{gc}(\nabla v), \quad \forall\, v \in H^2(K), \tag{3.11}$$

$$\operatorname{curl}(\boldsymbol{I}_K^{gc}\boldsymbol{v}) = \boldsymbol{I}_K^s(\operatorname{curl}\boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^1(\operatorname{curl}, K). \tag{3.12}$$

Proof. On each edge $e \in \mathcal{E}(K)$, it follows from the definitions of I_K^g and $\boldsymbol{I}_K^{gc}$ that

$$\int_e (\nabla(I_K^g v) - \boldsymbol{I}_K^{gc}(\nabla v)) \cdot \boldsymbol{t}_e q ds = \int_e \partial_{t_e}(I_K^g v - v) q ds = 0, \quad \forall\, q \in \mathbb{P}_k(e).$$

On each face $F \in \mathcal{F}(K)$, we have

$$\int_F \operatorname{curl}(\nabla(I_K^g v) - \boldsymbol{I}_K^{gc}(\nabla v)) \times \boldsymbol{n} ds = \int_F \operatorname{curl}(\nabla(I_K^g v - v)) \times \boldsymbol{n} ds = \boldsymbol{0}.$$

Hence, (3.11) holds from $\nabla(I_K^g v) - \boldsymbol{I}_K^{gc}(\nabla v) \in \boldsymbol{W}_k(K)$.

On the other hand, we obtain from the Stokes formula that

$$\begin{aligned}\int_F (\operatorname{curl}(\boldsymbol{I}_K^{gc}\boldsymbol{v}) - \boldsymbol{I}_K^s(\operatorname{curl}\boldsymbol{v})) \cdot \boldsymbol{n} ds &= \int_F \operatorname{curl}(\boldsymbol{I}_K^{gc}\boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n} ds \\ &= \int_F (\boldsymbol{n} \times \nabla) \cdot (\boldsymbol{I}_K^{gc}\boldsymbol{v} - \boldsymbol{v}) ds \\ &= \int_F \boldsymbol{t}_{F,e} \cdot (\boldsymbol{I}_K^{gc}\boldsymbol{v} - \boldsymbol{v}) ds = 0.\end{aligned}$$

By the definitions of $\boldsymbol{I}_K^{gc}$ and $\boldsymbol{I}_K^s$,

$$\int_F (\operatorname{curl}(\boldsymbol{I}_K^{gc}\boldsymbol{v}) - \boldsymbol{I}_K^s(\operatorname{curl}\boldsymbol{v})) \times \boldsymbol{n} ds = \int_F \operatorname{curl}(\boldsymbol{I}_K^{gc}\boldsymbol{v} - \boldsymbol{v}) \times \boldsymbol{n} ds = \boldsymbol{0}.$$

Therefore, (3.12) follows from the last two identities. □

Now introduce the global version of I_K^g, $\boldsymbol{I}_K^{gc}$, $\boldsymbol{I}_K^s$ and Q_K. Let $I_h^g : H^2(\Omega) \to \boldsymbol{V}_h^g$, $\boldsymbol{I}_h^{gc} : \boldsymbol{H}^1(\operatorname{curl},\Omega) \to \boldsymbol{W}_h$, $\boldsymbol{I}_h^s : \boldsymbol{H}^1(\Omega;\mathbb{R}^3) \to \boldsymbol{V}_h^s$ and $I_h^{L^2} : L^2(\Omega) \to \mathcal{Q}_h$ be defined by $(I_h^g v)|_K := I_K^g(v|_K)$, $(\boldsymbol{I}_h^{gc}\boldsymbol{v})|_K := \boldsymbol{I}_K^{gc}(\boldsymbol{v}|_K)$, $(\boldsymbol{I}_h^s\boldsymbol{v})|_K := \boldsymbol{I}_K^s(\boldsymbol{v}|_K)$ and $(I_h^{L^2}v)|_K := Q_K(v|_K)$ for each $K \in \mathcal{T}_h$, respectively. As the direct result of (3.7), (3.11) and (3.12), we have

$$\nabla(I_h^g v) = \boldsymbol{I}_h^{gc}(\nabla v), \quad \forall\, v \in H^2(\Omega), \tag{3.13}$$

$$\operatorname{curl}_h(\boldsymbol{I}_h^{gc}\boldsymbol{v}) = \boldsymbol{I}_h^s(\operatorname{curl}\boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^1(\operatorname{curl},\Omega), \tag{3.14}$$

$$\operatorname{div}_h(\boldsymbol{I}_h^s\boldsymbol{v}) = I_h^{L^2}\operatorname{div}\boldsymbol{v}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^1(\Omega;\mathbb{R}^3). \tag{3.15}$$

Combining (3.13)–(3.15) and the complex (3.4) yields the commutative diagram

$$
\begin{array}{ccccccccccc}
\mathbb{R} & \xrightarrow{\subset} & H^2(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}^1(\mathrm{curl},\Omega) & \xrightarrow{\mathrm{curl}} & \boldsymbol{H}^1(\Omega;\mathbb{R}^3) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle I_h^g} & & \downarrow{\scriptstyle \boldsymbol{I}_h^{gc}} & & \downarrow{\scriptstyle \boldsymbol{I}_h^s} & & \downarrow{\scriptstyle I_h^{L^2}} & & \\
\mathbb{R} & \xrightarrow{\subset} & V_h^g & \xrightarrow{\nabla} & \boldsymbol{W}_h & \xrightarrow{\mathrm{curl}_h} & \boldsymbol{V}_h^s & \xrightarrow{\mathrm{div}_h} & \mathcal{Q}_h & \longrightarrow & 0
\end{array}
$$

and the commutative diagram with homogeneous boundary conditions

$$
\begin{array}{ccccccccccc}
0 & \xrightarrow{\subset} & H_0^2(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_0^1(\mathrm{curl},\Omega) & \xrightarrow{\mathrm{curl}} & \boldsymbol{H}_0^1(\Omega;\mathbb{R}^3) & \xrightarrow{\mathrm{div}} & L_0^2(\Omega) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle I_h^g} & & \downarrow{\scriptstyle \boldsymbol{I}_h^{gc}} & & \downarrow{\scriptstyle \boldsymbol{I}_h^s} & & \downarrow{\scriptstyle I_h^{L^2}} & & \\
0 & \xrightarrow{\subset} & V_{h0}^g & \xrightarrow{\nabla} & \boldsymbol{W}_{h0} & \xrightarrow{\mathrm{curl}_h} & \boldsymbol{V}_{h0}^s & \xrightarrow{\mathrm{div}_h} & \mathcal{Q}_{h0} & \longrightarrow & 0.
\end{array}
\tag{3.16}
$$

4 Mixed finite element methods for the quad-curl problem

In this section, we advance the mixed finite element method for the quad-curl problem

$$
\begin{cases}
(\mathrm{curl})^4\boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega,\\
\mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Omega,\\
\boldsymbol{u}\times\boldsymbol{n} = (\mathrm{curl}\,\boldsymbol{u})\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \partial\Omega,
\end{cases}
\tag{4.1}
$$

where $\boldsymbol{f} \in \boldsymbol{H}(\mathrm{div},\Omega)$ with $\mathrm{div}\,\boldsymbol{f} = 0$. The quad-curl problem arises in the inverse electromagnetic scattering theory (see [8]) and magnetohydrodynamics (see [48]).

Due to the identity $\mathrm{curl}^2\,\boldsymbol{v} = -\Delta\boldsymbol{v} + \nabla(\mathrm{div}\,\boldsymbol{v})$ and the fact that

$$
(\mathrm{curl}\,\boldsymbol{u})\cdot\boldsymbol{n} = (\boldsymbol{n}\times\nabla)\cdot\boldsymbol{u} = (\boldsymbol{n}\times\nabla)\cdot(\boldsymbol{n}\times\boldsymbol{u}\times\boldsymbol{n}) = 0 \quad \text{on } \partial\Omega,
$$

the quad-curl problem (4.1) is equivalent to

$$
\begin{cases}
-\,\mathrm{curl}\,\Delta\,\mathrm{curl}\,\boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega,\\
\mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Omega,\\
\boldsymbol{u}\times\boldsymbol{n} = \mathrm{curl}\,\boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega.
\end{cases}
\tag{4.2}
$$

Then a mixed formulation of the quad-curl problem (4.1) is to find $\boldsymbol{u} \in \boldsymbol{H}_0(\mathrm{gradcurl},\Omega)$ and $\lambda \in H_0^1(\Omega)$ such that

$$
(\nabla\mathrm{curl}\,\boldsymbol{u}, \nabla\mathrm{curl}\,\boldsymbol{v}) + (\boldsymbol{v}, \nabla\lambda) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall\,\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{gradcurl},\Omega), \tag{4.3}
$$

$$
(\boldsymbol{u}, \nabla\mu) = 0, \quad \forall\,\mu \in H_0^1(\Omega). \tag{4.4}
$$

Replacing $\boldsymbol{v}$ in (4.3) with $\nabla\mu$ for any $\mu \in H_0^1(\Omega)$, we obtain $\lambda = 0$ from the fact that $\mathrm{div}\,\boldsymbol{f} = 0$. Thus it follows from (4.3) that

$$
(\nabla\mathrm{curl}\,\boldsymbol{u}, \nabla\mathrm{curl}\,\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall\,\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{gradcurl},\Omega).
$$

4.1 Mixed finite element methods

Based on the mixed formulation (4.3)–(4.4), we propose the mixed finite element method for the quad-curl problem (4.1) as follows: find $\boldsymbol{u}_h \in \boldsymbol{W}_{h0}$ and $\lambda_h \in V_{h0}^g$ such that

$$
(\nabla_h\,\mathrm{curl}_h\,\boldsymbol{u}_h, \nabla_h\,\mathrm{curl}_h\,\boldsymbol{v}_h) + (\boldsymbol{v}_h, \nabla\lambda_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall\,\boldsymbol{v}_h \in \boldsymbol{W}_{h0}, \tag{4.5}
$$

$$
(\boldsymbol{u}_h, \nabla\mu_h) = 0, \quad \forall\,\mu_h \in V_{h0}^g. \tag{4.6}
$$

Now we show the well-posedness of the mixed finite element method (4.5)–(4.6) and the stability. To this end, we recall the discrete de Rham complex and the corresponding interpolation operators (see [2]). Based on the degrees of freedom (3.2), define $\boldsymbol{I}_K^c : \boldsymbol{H}^2(K;\mathbb{R}^3) \to \boldsymbol{V}_k^c(K)$ for each $K \in \mathcal{T}_h$ by

$$\int_e \boldsymbol{I}_K^c \boldsymbol{v} \cdot \boldsymbol{t}_e q ds = \int_e \boldsymbol{v} \cdot \boldsymbol{t}_e q ds, \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^2(K;\mathbb{R}^3), \quad q \in \mathbb{P}_k(e), \quad e \in \mathcal{E}(K).$$

Then we have

$$\boldsymbol{I}_K^c \boldsymbol{v} = \boldsymbol{v}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_k^c(K), \tag{4.7}$$

$$\|\operatorname{curl}(\boldsymbol{v} - \boldsymbol{I}_K^c \boldsymbol{v})\|_{0,K} \lesssim h_K |\operatorname{curl} \boldsymbol{v}|_{1,K}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^2(K;\mathbb{R}^3), \tag{4.8}$$

$$\|\boldsymbol{I}_K^c \boldsymbol{v}\|_{0,K} \lesssim \|\boldsymbol{v}\|_{0,K}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{W}_k(K). \tag{4.9}$$

Let $\boldsymbol{I}_h^c : \boldsymbol{H}^2(\Omega;\mathbb{R}^3) + \boldsymbol{W}_h \to \boldsymbol{V}_h^c$ be determined by

$$(\boldsymbol{I}_h^c \boldsymbol{v}_h)\,|_K := \boldsymbol{I}_K^c(\boldsymbol{v}_h\,|_K), \quad \forall\, K \in \mathcal{T}_h.$$

The operator $\boldsymbol{I}_h^c$ is well defined, since the degrees of freedom (2.2) for $\boldsymbol{W}_k(K)$ and (3.2) for $\boldsymbol{V}_k^c(K)$ are the same. We have $\boldsymbol{I}_h^c \boldsymbol{v}_h \in \boldsymbol{V}_{h0}^c := \boldsymbol{V}_h^c \cap \boldsymbol{H}_0(\operatorname{curl}, \Omega)$ when $\boldsymbol{v}_h \in \boldsymbol{W}_{h0}^c$.

Let the lowest order Raviart-Thomas element space be (see [37, 40])

$$\boldsymbol{V}_{h0}^d := \{\boldsymbol{v}_h \in \boldsymbol{H}_0(\operatorname{div}, \Omega) : \boldsymbol{v}_h\,|_K \in \mathbb{P}_0(K;\mathbb{R}^3) + \boldsymbol{x}\mathbb{P}_0(K) \text{ for each } K \in \mathcal{T}_h\}.$$

We have the discrete de Rham complex (see [2])

$$0 \xrightarrow{\subset} V_{h0}^g \xrightarrow{\nabla} \boldsymbol{V}_{h0}^c \xrightarrow{\operatorname{curl}} \boldsymbol{V}_{h0}^d \xrightarrow{\operatorname{div}} \mathcal{Q}_{h0} \to 0.$$

Denote by $\boldsymbol{I}_h^d : \boldsymbol{H}_0^1(\Omega;\mathbb{R}^3) + \boldsymbol{V}_{h0}^s \to \boldsymbol{V}_{h0}^d$ the nodal interpolation operator. Then the commutative diagram (3.16) can be extended to the following three-line commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \xrightarrow{\subset} & H_0^2(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_0^1(\operatorname{curl}, \Omega) & \xrightarrow{\operatorname{curl}} & \boldsymbol{H}_0^1(\Omega;\mathbb{R}^3) & \xrightarrow{\operatorname{div}} & L_0^2(\Omega) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle I_h^g} & & \downarrow{\scriptstyle \boldsymbol{I}_h^{gc}} & & \downarrow{\scriptstyle \boldsymbol{I}_h^s} & & \downarrow{\scriptstyle I_h^{L^2}} & & \\
0 & \xrightarrow{\subset} & V_{h0}^g & \xrightarrow{\nabla} & \boldsymbol{W}_{h0} & \xrightarrow{\operatorname{curl}_h} & \boldsymbol{V}_{h0}^s & \xrightarrow{\operatorname{div}_h} & \mathcal{Q}_{h0} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle \boldsymbol{I}_h^c} & & \downarrow{\scriptstyle \boldsymbol{I}_h^d} & & \downarrow{\scriptstyle I} & & \\
0 & \xrightarrow{\subset} & V_{h0}^g & \xrightarrow{\nabla} & \boldsymbol{V}_{h0}^c & \xrightarrow{\operatorname{curl}} & \boldsymbol{V}_{h0}^d & \xrightarrow{\operatorname{div}} & \mathcal{Q}_{h0} & \longrightarrow & 0,
\end{array} \tag{4.10}$$

where I is the identity operator.

Lemma 4.1. *We have*

$$\inf_{q \in \mathbb{P}_{k+1}(K)} \|\boldsymbol{v} - \nabla q\|_{0,K} \lesssim h_K \|\operatorname{curl} \boldsymbol{v}\|_{0,K}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{W}_k(K), \quad K \in \mathcal{T}_h. \tag{4.11}$$

Proof. Due to (2.5), $\boldsymbol{v} - \mathcal{K}_K(\operatorname{curl} \boldsymbol{v}) \in \boldsymbol{W}_k(K) \cap \ker(\operatorname{curl})$, which means that

$$\boldsymbol{v} - \mathcal{K}_K(\operatorname{curl} \boldsymbol{v}) \in \nabla \mathbb{P}_{k+1}(K).$$

Choose $q \in \mathbb{P}_{k+1}(K)$ such that $\boldsymbol{v} - \mathcal{K}_K(\operatorname{curl} \boldsymbol{v}) = \nabla q$. Apply (2.6) to get

$$\|\boldsymbol{v} - \nabla q\|_{0,K} = \|\mathcal{K}_K(\operatorname{curl} \boldsymbol{v})\|_{0,K} \lesssim h_K \|\operatorname{curl} \boldsymbol{v}\|_{0,K},$$

which indicates (4.11). □

Lemma 4.2. *It holds for any $K \in \mathcal{T}_h$ that*

$$\|\boldsymbol{v} - \boldsymbol{I}_K^c \boldsymbol{v}\|_{0,K} \lesssim h_K \|\operatorname{curl} \boldsymbol{v}\|_{0,K}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{W}_k(K). \tag{4.12}$$

Proof. Employing (4.7) and $\nabla\mathbb{P}_{k+1}(K)\subset \boldsymbol{V}_k^c(K)$, it follows that

$$\boldsymbol{v}-\boldsymbol{I}_K^c\boldsymbol{v}=(\boldsymbol{v}-\nabla q)-\boldsymbol{I}_K^c(\boldsymbol{v}-\nabla q),\quad \forall\, q\in\mathbb{P}_{k+1}(K).$$

Then we get from (4.9) that

$$\|\boldsymbol{v}-\boldsymbol{I}_K^c\boldsymbol{v}\|_{0,K}\leqslant\|\boldsymbol{v}-\nabla q\|_{0,K}+\|\boldsymbol{I}_K^c(\boldsymbol{v}-\nabla q)\|_{0,K}\lesssim\|\boldsymbol{v}-\nabla q\|_{0,K},$$

which together with the arbitrariness of $q\in\mathbb{P}_{k+1}(K)$ implies

$$\|\boldsymbol{v}-\boldsymbol{I}_K^c\boldsymbol{v}\|_{0,K}\lesssim\inf_{q\in\mathbb{P}_{k+1}(K)}\|\boldsymbol{v}-\nabla q\|_{0,K}.$$

Thus the inequality (4.12) follows from (4.11). □

Lemma 4.3. *We have the discrete Poincaré inequality*

$$\|\boldsymbol{v}_h\|_0\lesssim\|\mathrm{curl}_h\,\boldsymbol{v}_h\|_0,\quad\forall\,\boldsymbol{v}_h\in\mathcal{K}_h^d,\tag{4.13}$$

where $\mathcal{K}_h^d:=\{\boldsymbol{v}_h\in\boldsymbol{W}_{h0}:(\boldsymbol{v}_h,\nabla q_h)=0$ *for each* $q_h\in V_{h0}^g\}$.

Proof. By the fact that $\boldsymbol{I}_h^c\boldsymbol{v}_h\in\boldsymbol{H}_0(\mathrm{curl},\Omega)$, there exists a $\boldsymbol{\psi}\in\boldsymbol{H}_0^1(\Omega;\mathbb{R}^3)$ such that (see [1, 18, 24])

$$\mathrm{curl}\,\boldsymbol{\psi}=\mathrm{curl}(\boldsymbol{I}_h^c\boldsymbol{v}_h),\quad\|\boldsymbol{\psi}\|_1\lesssim\|\mathrm{curl}(\boldsymbol{I}_h^c\boldsymbol{v}_h)\|_0.\tag{4.14}$$

Let

$$\widetilde{\boldsymbol{I}}_h^c:\boldsymbol{H}_0(\mathrm{curl},\Omega)\to\boldsymbol{V}_{h0}^c$$

and

$$\widetilde{\boldsymbol{I}}_h^d:\boldsymbol{H}_0(\mathrm{div},\Omega)\to\boldsymbol{V}_{h0}^d$$

be the L^2 bounded projection operators devised in [16]. The operators $\widetilde{\boldsymbol{I}}_h^c$ and $\widetilde{\boldsymbol{I}}_h^d$ possess the following properties:

$$\mathrm{curl}(\widetilde{\boldsymbol{I}}_h^c\boldsymbol{v})=\widetilde{\boldsymbol{I}}_h^d(\mathrm{curl}\,\boldsymbol{v}),\quad\|\widetilde{\boldsymbol{I}}_h^c\boldsymbol{v}\|_0\lesssim\|\boldsymbol{v}\|_0,\quad\forall\,\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{curl},\Omega).\tag{4.15}$$

By the commuting properties of $\widetilde{\boldsymbol{I}}_h^c$ and $\widetilde{\boldsymbol{I}}_h^d$, it follows that

$$\mathrm{curl}(\widetilde{\boldsymbol{I}}_h^c\boldsymbol{\psi})=\widetilde{\boldsymbol{I}}_h^d(\mathrm{curl}\,\boldsymbol{\psi})=\widetilde{\boldsymbol{I}}_h^d(\mathrm{curl}(\boldsymbol{I}_h^c\boldsymbol{v}_h))=\mathrm{curl}(\boldsymbol{I}_h^c\boldsymbol{v}_h).$$

By (3.3), there exists a $q_h\in V_{h0}^g$ such that $\boldsymbol{I}_h^c\boldsymbol{v}_h-\widetilde{\boldsymbol{I}}_h^c\boldsymbol{\psi}=\nabla q_h$. Because $(\boldsymbol{v}_h,\nabla q_h)=0$,

$$\begin{aligned}\|\boldsymbol{I}_h^c\boldsymbol{v}_h\|_0^2&=(\boldsymbol{I}_h^c\boldsymbol{v}_h,\boldsymbol{I}_h^c\boldsymbol{v}_h-\widetilde{\boldsymbol{I}}_h^c\boldsymbol{\psi})+(\boldsymbol{I}_h^c\boldsymbol{v}_h,\widetilde{\boldsymbol{I}}_h^c\boldsymbol{\psi})\\&=(\boldsymbol{I}_h^c\boldsymbol{v}_h-\boldsymbol{v}_h,\boldsymbol{I}_h^c\boldsymbol{v}_h-\widetilde{\boldsymbol{I}}_h^c\boldsymbol{\psi})+(\boldsymbol{I}_h^c\boldsymbol{v}_h,\widetilde{\boldsymbol{I}}_h^c\boldsymbol{\psi}).\end{aligned}$$

Due to (4.15) and (4.14), we get

$$\begin{aligned}\|\boldsymbol{I}_h^c\boldsymbol{v}_h\|_0^2&\leqslant\|\boldsymbol{I}_h^c\boldsymbol{v}_h-\boldsymbol{v}_h\|_0\|\boldsymbol{I}_h^c\boldsymbol{v}_h-\widetilde{\boldsymbol{I}}_h^c\boldsymbol{\psi}\|_0+\|\boldsymbol{I}_h^c\boldsymbol{v}_h\|_0\|\widetilde{\boldsymbol{I}}_h^c\boldsymbol{\psi}\|_0\\&\lesssim\|\boldsymbol{I}_h^c\boldsymbol{v}_h-\boldsymbol{v}_h\|_0(\|\boldsymbol{I}_h^c\boldsymbol{v}_h\|_0+\|\boldsymbol{\psi}\|_1)+\|\boldsymbol{I}_h^c\boldsymbol{v}_h\|_0\|\boldsymbol{\psi}\|_1\\&\lesssim\|\boldsymbol{I}_h^c\boldsymbol{v}_h-\boldsymbol{v}_h\|_0(\|\boldsymbol{I}_h^c\boldsymbol{v}_h\|_0+\|\mathrm{curl}(\boldsymbol{I}_h^c\boldsymbol{v}_h)\|_0)+\|\boldsymbol{I}_h^c\boldsymbol{v}_h\|_0\|\|\mathrm{curl}(\boldsymbol{I}_h^c\boldsymbol{v}_h)\|_0\\&=(\|\boldsymbol{v}_h-\boldsymbol{I}_h^c\boldsymbol{v}_h\|_0+\|\mathrm{curl}(\boldsymbol{I}_h^c\boldsymbol{v}_h)\|_0)\|\boldsymbol{I}_h^c\boldsymbol{v}_h\|_0+\|\boldsymbol{v}_h-\boldsymbol{I}_h^c\boldsymbol{v}_h\|_0\|\mathrm{curl}(\boldsymbol{I}_h^c\boldsymbol{v}_h)\|_0.\end{aligned}$$

Thus we have

$$\|\boldsymbol{I}_h^c\boldsymbol{v}_h\|_0\lesssim\|\boldsymbol{v}_h-\boldsymbol{I}_h^c\boldsymbol{v}_h\|_0+\|\mathrm{curl}(\boldsymbol{I}_h^c\boldsymbol{v}_h)\|_0,$$

which indicates

$$\begin{aligned}\|\boldsymbol{v}_h\|_0&\lesssim\|\boldsymbol{v}_h-\boldsymbol{I}_h^c\boldsymbol{v}_h\|_0+\|\mathrm{curl}(\boldsymbol{I}_h^c\boldsymbol{v}_h)\|_0\\&\lesssim\|\boldsymbol{v}_h-\boldsymbol{I}_h^c\boldsymbol{v}_h\|_0+\|\mathrm{curl}(\boldsymbol{v}_h-\boldsymbol{I}_h^c\boldsymbol{v}_h)\|_0+\|\mathrm{curl}_h\,\boldsymbol{v}_h\|_0.\end{aligned}$$

Therefore, (4.13) follows from (4.12), (4.8) and the inverse inequality. □

Lemma 4.4. *We have the discrete stability*

$$\|\widetilde{\boldsymbol{u}}_h\|_{H_h(\mathrm{gradcurl})} + |\widetilde{\lambda}_h|_1 \lesssim \sup_{(\boldsymbol{v}_h,\mu_h)\in \boldsymbol{W}_{h0}\times V_{h0}^g} \frac{(\nabla_h \operatorname{curl}_h \widetilde{\boldsymbol{u}}_h, \nabla_h \operatorname{curl}_h \boldsymbol{v}_h) + (\boldsymbol{v}_h, \nabla\widetilde{\lambda}_h) + (\widetilde{\boldsymbol{u}}_h, \nabla\mu_h)}{\|\boldsymbol{v}_h\|_{H_h(\mathrm{gradcurl})} + |\mu_h|_1} \tag{4.16}$$

for any $\widetilde{\boldsymbol{u}}_h \in \boldsymbol{W}_{h0}$ *and* $\widetilde{\lambda}_h \in V_{h0}^g$.

Proof. For any $\boldsymbol{v}_h \in \mathcal{K}_h^d$, by using (4.13) and (3.6), we derive the coercivity

$$\|\boldsymbol{v}_h\|_{H_h(\mathrm{gradcurl})} \lesssim \|\operatorname{curl}_h \boldsymbol{v}_h\|_{1,h} \lesssim |\operatorname{curl}_h \boldsymbol{v}_h|_{1,h}.$$

Since $\nabla V_{h0}^g \subset \boldsymbol{W}_{h0}$, we have the discrete inf-sup condition

$$|\mu_h|_1 = \sup_{\boldsymbol{v}_h\in\nabla V_{h0}^g} \frac{(\boldsymbol{v}_h, \nabla\mu_h)}{\|\boldsymbol{v}_h\|_0} = \sup_{\boldsymbol{v}_h\in\nabla V_{h0}^g} \frac{(\boldsymbol{v}_h, \nabla\mu_h)}{\|\boldsymbol{v}_h\|_{H_h(\mathrm{gradcurl})}} \leqslant \sup_{\boldsymbol{v}_h\in\boldsymbol{W}_{h0}} \frac{(\boldsymbol{v}_h, \nabla\mu_h)}{\|\boldsymbol{v}_h\|_{H_h(\mathrm{gradcurl})}}.$$

Thus the discrete stability (4.16) follows from the Babuška-Brezzi theory (see [6]). □

Thanks to the discrete stability (4.16), the mixed finite element method (4.5)–(4.6) is well posed. As the continuous case, replacing $\boldsymbol{v}_h$ in (4.5) with $\nabla\mu_h$ for any $\mu_h \in V_{h0}^g$, we obtain $\lambda_h = 0$ from the fact that $\operatorname{div} \boldsymbol{f} = 0$ again. As a result, the solution $\boldsymbol{u}_h \in \boldsymbol{W}_{h0}$ satisfies

$$(\nabla_h \operatorname{curl}_h \boldsymbol{u}_h, \nabla_h \operatorname{curl}_h \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{W}_{h0}. \tag{4.17}$$

4.2 Interpolation operator with lower regularity

In this subsection, we define an interpolation operator on $\boldsymbol{H}_0(\mathrm{gradcurl},\Omega)$. Since the interpolation operator $\boldsymbol{I}_h^{gc}$ is not well defined on $\boldsymbol{H}_0(\mathrm{gradcurl},\Omega)$, we first present a regular decomposition for the space $\boldsymbol{H}_0(\mathrm{gradcurl},\Omega)$.

Lemma 4.5. *We have the stable regular decomposition*

$$\boldsymbol{H}_0(\mathrm{gradcurl},\Omega) = \boldsymbol{H}_0^2(\Omega;\mathbb{R}^3) + \nabla H_0^1(\Omega). \tag{4.18}$$

Specifically, for any $\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{gradcurl},\Omega)$, *let* $\boldsymbol{v}_2 \in \boldsymbol{H}_0^2(\Omega;\mathbb{R}^3)$ *and* $\boldsymbol{\lambda} \in \operatorname{curl} \boldsymbol{H}_0^2(\Omega;\mathbb{R}^3)$ *satisfy*

$$\begin{cases} (\nabla^2\boldsymbol{v}_2, \nabla^2\boldsymbol{\chi}) + (\nabla\operatorname{curl}\boldsymbol{\chi}, \nabla\boldsymbol{\lambda}) = 0, & \forall\, \boldsymbol{\chi} \in \boldsymbol{H}_0^2(\Omega;\mathbb{R}^3), \\ (\nabla\operatorname{curl}\boldsymbol{v}_2, \nabla\boldsymbol{\mu}) = (\nabla\operatorname{curl}\boldsymbol{v}, \nabla\boldsymbol{\mu}), & \forall\, \boldsymbol{\mu} \in \operatorname{curl}\boldsymbol{H}_0^2(\Omega;\mathbb{R}^3). \end{cases} \tag{4.19}$$

Then there exists a $v_1 \in H_0^1(\Omega)$ *such that* $\boldsymbol{v} = \boldsymbol{v}_2 + \nabla v_1$ *and*

$$\|\boldsymbol{v}_2\|_2 \lesssim |\operatorname{curl}\boldsymbol{v}|_1, \quad \|v_1\|_1 \lesssim \|\boldsymbol{v}\|_0 + \|\boldsymbol{v}_2\|_0. \tag{4.20}$$

Proof. Recall the de Rham complex with homogeneous boundary conditions (see [18, the second part of Theorem 1.1])

$$0 \xrightarrow{\subset} H_0^3(\Omega) \xrightarrow{\nabla} \boldsymbol{H}_0^2(\Omega;\mathbb{R}^3) \xrightarrow{\operatorname{curl}} \boldsymbol{H}_0^1(\Omega;\mathbb{R}^3) \xrightarrow{\operatorname{div}} L_0^2(\Omega) \to 0,$$

which is exact for Ω being contractible. For $\boldsymbol{\mu} \in \operatorname{curl}\boldsymbol{H}_0^2(\Omega;\mathbb{R}^3) \subset \boldsymbol{H}_0^1(\Omega;\mathbb{R}^3)$, by this complex, there exists a $\boldsymbol{w} \in \boldsymbol{H}_0^2(\Omega;\mathbb{R}^3)$ satisfying

$$\operatorname{curl}\boldsymbol{w} = \boldsymbol{\mu}, \quad \|\boldsymbol{w}\|_2 \lesssim |\boldsymbol{\mu}|_1.$$

Then we have the inf-sup condition

$$|\boldsymbol{\mu}|_1 = \frac{(\nabla\boldsymbol{\mu}, \nabla\boldsymbol{\mu})}{|\boldsymbol{\mu}|_1} \lesssim \sup_{\boldsymbol{w}\in\boldsymbol{H}_0^2(\Omega;\mathbb{R}^3)} \frac{(\nabla\operatorname{curl}\boldsymbol{w}, \nabla\boldsymbol{\mu})}{\|\boldsymbol{w}\|_2}.$$

By the Babuška-Brezzi theory (see [6]), the problem (4.19) is well posed, and

$$\operatorname{curl}\boldsymbol{v}_2 = \operatorname{curl}\boldsymbol{v}, \quad \|\boldsymbol{v}_2\|_2 + \|\boldsymbol{\lambda}\|_1 \lesssim |\operatorname{curl}\boldsymbol{v}|_1.$$

Finally, we finish the proof by the fact that $\operatorname{curl}(\boldsymbol{v} - \boldsymbol{v}_2) = \boldsymbol{0}$. □

For $\boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{gradcurl}, \Omega)$ satisfying $\operatorname{curl} \boldsymbol{v} = \mathbf{0}$, by (4.20), we have $\boldsymbol{v}_2 = \mathbf{0}$ and $\boldsymbol{v} = \nabla v_1$.

Now we apply the regular decomposition (4.18) to $\boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{gradcurl}, \Omega)$. Let $I_h^{SZ} : H_0^1(\Omega) \to V_{h0}^g$ be the Scott-Zhang interpolation operator (see [41]). Noting that $\boldsymbol{I}_h^{gc}$ can be applied to $\boldsymbol{v}_2$, we define $\boldsymbol{\Pi}_h^{gc} : \boldsymbol{H}_0(\operatorname{gradcurl}, \Omega) \to \boldsymbol{W}_{h0}$ as follows:

$$\boldsymbol{\Pi}_h^{gc} \boldsymbol{v} := \boldsymbol{I}_h^{gc} \boldsymbol{v}_2 + \nabla I_h^{SZ} v_1.$$

Clearly, we have

$$\boldsymbol{\Pi}_h^{gc}(\nabla v) = \nabla(I_h^{SZ} v), \quad \forall\, v \in H_0^1(\Omega). \tag{4.21}$$

We acquire from (3.14) that

$$\operatorname{curl}_h(\boldsymbol{\Pi}_h^{gc} \boldsymbol{v}) = \boldsymbol{I}_h^s(\operatorname{curl} \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{gradcurl}, \Omega). \tag{4.22}$$

Combining (4.21)–(4.22), (3.15) and the complex (3.5) yields the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \xrightarrow{\subset} & H_0^1(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_0(\operatorname{gradcurl}, \Omega) & \xrightarrow{\operatorname{curl}} & \boldsymbol{H}_0^1(\Omega; \mathbb{R}^3) & \xrightarrow{\operatorname{div}} & L_0^2(\Omega) & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle I_h^{SZ}} & & \Big\downarrow {\scriptstyle \boldsymbol{\Pi}_h^{gc}} & & \Big\downarrow {\scriptstyle \boldsymbol{I}_h^{s}} & & \Big\downarrow {\scriptstyle I_h^{L^2}} & & \\
0 & \xrightarrow{\subset} & V_{h0}^g & \xrightarrow{\nabla} & \boldsymbol{W}_{h0} & \xrightarrow{\operatorname{curl}_h} & \boldsymbol{V}_{h0}^s & \xrightarrow{\operatorname{div}_h} & \mathcal{Q}_{h0} & \longrightarrow & 0.
\end{array}$$

Lemma 4.6. *Assume $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega; \mathbb{R}^3)$ and $\operatorname{curl} \boldsymbol{v} \in \boldsymbol{H}^2(\Omega; \mathbb{R}^3)$. Then*

$$\|\boldsymbol{v} - \boldsymbol{\Pi}_h^{gc} \boldsymbol{v}\|_0 \lesssim h(|\boldsymbol{v}|_1 + |\operatorname{curl} \boldsymbol{v}|_1), \tag{4.23}$$

$$\|\operatorname{curl}_h(\boldsymbol{v} - \boldsymbol{\Pi}_h^{gc} \boldsymbol{v})\|_0 + h|\operatorname{curl}_h(\boldsymbol{v} - \boldsymbol{\Pi}_h^{gc} \boldsymbol{v})|_1 \lesssim h^2 |\operatorname{curl} \boldsymbol{v}|_2. \tag{4.24}$$

Proof. Noting that $\nabla v_1 = \boldsymbol{v} - \boldsymbol{v}_2$, we have $|v_1|_2 \lesssim |\boldsymbol{v} - \boldsymbol{v}_2|_1 \lesssim |\boldsymbol{v}|_1 + |\operatorname{curl} \boldsymbol{v}|_1$. Since

$$\boldsymbol{v} - \boldsymbol{\Pi}_h^{gc} \boldsymbol{v} = \boldsymbol{v}_2 - \boldsymbol{I}_h^{gc} \boldsymbol{v}_2 + \nabla(v_1 - I_h^{SZ} v_1),$$

we acquire from (3.10), the error estimate of I_h^{SZ} and (4.20) that

$$\begin{aligned}
\|\boldsymbol{v} - \boldsymbol{\Pi}_h^{gc} \boldsymbol{v}\|_0 &\leqslant \|\boldsymbol{v}_2 - \boldsymbol{I}_h^{gc} \boldsymbol{v}_2\|_0 + |v_1 - I_h^{SZ} v_1|_1 \\
&\lesssim h^{k+1} \|\boldsymbol{v}_2\|_2 + h|v_1|_2 \lesssim h(|\boldsymbol{v}|_1 + |\operatorname{curl} \boldsymbol{v}|_1).
\end{aligned}$$

Employing (3.8), we see from (4.22) that

$$\|\operatorname{curl}_h(\boldsymbol{v} - \boldsymbol{\Pi}_h^{gc} \boldsymbol{v})\|_0 = \|\operatorname{curl} \boldsymbol{v} - \boldsymbol{I}_h^s(\operatorname{curl} \boldsymbol{v})\|_0 \lesssim h^2 |\operatorname{curl} \boldsymbol{v}|_2$$

and

$$|\operatorname{curl}_h(\boldsymbol{v} - \boldsymbol{\Pi}_h^{gc} \boldsymbol{v})|_1 = |\operatorname{curl} \boldsymbol{v} - \boldsymbol{I}_h^s(\operatorname{curl} \boldsymbol{v})|_1 \lesssim h|\operatorname{curl} \boldsymbol{v}|_2.$$

This ends the proof. □

4.3 Error analysis

Hereafter we assume that $\boldsymbol{u} \in \boldsymbol{H}_0(\operatorname{gradcurl}, \Omega)$ possesses the regularity $\operatorname{curl} \boldsymbol{u} \in \boldsymbol{H}^2(\Omega; \mathbb{R}^3)$, which is true for Ω being convex (see Lemma A.1). Applying the integration by parts to the first equation in (4.2), we derive

$$-(\Delta \operatorname{curl} \boldsymbol{u}, \operatorname{curl} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{curl}, \Omega). \tag{4.25}$$

We first present the consistency error of the nonconforming method (4.5)–(4.6).

Lemma 4.7. *We have for any $\boldsymbol{v}_h \in \boldsymbol{W}_{h0}$ that*

$$(\nabla \operatorname{curl} \boldsymbol{u}, \nabla_h \operatorname{curl}_h \boldsymbol{v}_h) + (\Delta \operatorname{curl} \boldsymbol{u}, \operatorname{curl}_h \boldsymbol{v}_h) \lesssim h|\operatorname{curl} \boldsymbol{u}|_2 |\operatorname{curl}_h \boldsymbol{v}_h|_{1,h}. \tag{4.26}$$

Proof. Due to (3.1), we have

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}(\partial_n(\operatorname{curl}\boldsymbol{u}),\operatorname{curl}_h\boldsymbol{v}_h)_{\partial K}\\&\quad=\sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}(K)}(\partial_n(\operatorname{curl}\boldsymbol{u})-\boldsymbol{Q}_F^0\partial_n(\operatorname{curl}\boldsymbol{u}),\operatorname{curl}_h\boldsymbol{v}_h)_F\\&\quad=\sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}(K)}(\partial_n(\operatorname{curl}\boldsymbol{u})-\boldsymbol{Q}_F^0\partial_n(\operatorname{curl}\boldsymbol{u}),\operatorname{curl}_h\boldsymbol{v}_h-\boldsymbol{Q}_F^0\operatorname{curl}_h\boldsymbol{v}_h)_F\\&\quad\lesssim h|\operatorname{curl}\boldsymbol{u}|_2|\operatorname{curl}_h\boldsymbol{v}_h|_{1,h}.\end{aligned}$$

Thus (4.26) follows from the integration by parts. □

Lemma 4.8. *We have for any $\boldsymbol{v}_h\in\boldsymbol{W}_{h0}$ that*

$$(\nabla\operatorname{curl}\boldsymbol{u},\nabla_h\operatorname{curl}_h\boldsymbol{v}_h)-(\boldsymbol{f},\boldsymbol{v}_h)\lesssim h(|\operatorname{curl}\boldsymbol{u}|_2+\|\boldsymbol{f}\|_0)|\operatorname{curl}_h\boldsymbol{v}_h|_{1,h}. \tag{4.27}$$

Proof. We see from (4.25) with $\boldsymbol{v}=\boldsymbol{I}_h^c\boldsymbol{v}_h$ that

$$(\Delta\operatorname{curl}\boldsymbol{u},\operatorname{curl}_h\boldsymbol{v}_h)+(\boldsymbol{f},\boldsymbol{v}_h)=(\Delta\operatorname{curl}\boldsymbol{u},\operatorname{curl}_h(\boldsymbol{v}_h-\boldsymbol{I}_h^c\boldsymbol{v}_h))+(\boldsymbol{f},\boldsymbol{v}_h-\boldsymbol{I}_h^c\boldsymbol{v}_h).$$

Applying (4.8) and (4.12) gives

$$\begin{aligned}-(\Delta\operatorname{curl}\boldsymbol{u},\operatorname{curl}_h\boldsymbol{v}_h)-(\boldsymbol{f},\boldsymbol{v}_h)&\lesssim\|\Delta\operatorname{curl}\boldsymbol{u}\|_0\|\operatorname{curl}_h(\boldsymbol{v}_h-\boldsymbol{I}_h^c\boldsymbol{v}_h)\|_0+\|\boldsymbol{f}\|_0\|\boldsymbol{v}_h-\boldsymbol{I}_h^c\boldsymbol{v}_h\|_0\\&\lesssim h|\operatorname{curl}\boldsymbol{u}|_2|\operatorname{curl}_h\boldsymbol{v}_h|_{1,h}+h\|\boldsymbol{f}\|_0\|\operatorname{curl}_h\boldsymbol{v}_h\|_0.\end{aligned}$$

Together with (4.26), we derive

$$(\nabla\operatorname{curl}\boldsymbol{u},\nabla_h\operatorname{curl}_h\boldsymbol{v}_h)-(\boldsymbol{f},\boldsymbol{v}_h)\lesssim h|\operatorname{curl}\boldsymbol{u}|_2|\operatorname{curl}_h\boldsymbol{v}_h|_{1,h}+h\|\boldsymbol{f}\|_0\|\operatorname{curl}_h\boldsymbol{v}_h\|_0.$$

Hence, (4.27) follows from (3.6). □

Now we can show the *a priori* error estimate.

Theorem 4.9. *Let $\boldsymbol{u}\in\boldsymbol{H}_0(\operatorname{gradcurl},\Omega)$ be the solution of the problem* (4.1)*, and $\boldsymbol{u}_h\in\boldsymbol{W}_{h0}$ be the solution of the mixed finite element method* (4.5)–(4.6)*. Assume $\boldsymbol{u}\in\boldsymbol{H}^1(\Omega;\mathbb{R}^3)$ and $\operatorname{curl}\boldsymbol{u}\in\boldsymbol{H}^2(\Omega;\mathbb{R}^3)$. Then we have*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H_h(\operatorname{gradcurl})}\lesssim h(\|\operatorname{curl}\boldsymbol{u}\|_2+|\boldsymbol{u}|_1+\|\boldsymbol{f}\|_0). \tag{4.28}$$

Proof. It follows from (4.27) that

$$\begin{aligned}&(\nabla_h\operatorname{curl}_h(\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}),\nabla_h\operatorname{curl}_h\boldsymbol{v}_h)-(\boldsymbol{f},\boldsymbol{v}_h)\\&\quad=(\nabla_h\operatorname{curl}_h(\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}-\boldsymbol{u}),\nabla_h\operatorname{curl}_h\boldsymbol{v}_h)+(\nabla\operatorname{curl}\boldsymbol{u},\nabla_h\operatorname{curl}_h\boldsymbol{v}_h)-(\boldsymbol{f},\boldsymbol{v}_h)\\&\quad\lesssim|\operatorname{curl}_h(\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}-\boldsymbol{u})|_{1,h}|\operatorname{curl}_h\boldsymbol{v}_h|_{1,h}+h(|\operatorname{curl}\boldsymbol{u}|_2+\|\boldsymbol{f}\|_0)|\operatorname{curl}_h\boldsymbol{v}_h|_{1,h}.\end{aligned} \tag{4.29}$$

On the other hand, by the discrete stability (4.16) with $\widetilde{\boldsymbol{u}}_h=\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}-\boldsymbol{u}_h$ and $\widetilde{\lambda}_h=0$, we see from (4.17) and the fact $\boldsymbol{u}_h\in\mathcal{K}_h^d$ that

$$\begin{aligned}&\|\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}-\boldsymbol{u}_h\|_{H_h(\operatorname{gradcurl})}\\&\quad\lesssim\sup_{(\boldsymbol{v}_h,\mu_h)\in\boldsymbol{W}_{h0}\times V_{h0}^g}\frac{(\nabla_h\operatorname{curl}_h(\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}-\boldsymbol{u}_h),\nabla_h\operatorname{curl}_h\boldsymbol{v}_h)+(\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}-\boldsymbol{u}_h,\nabla\mu_h)}{\|\boldsymbol{v}_h\|_{H_h(\operatorname{gradcurl})}+|\mu_h|_1}\\&\quad=\sup_{(\boldsymbol{v}_h,\mu_h)\in\boldsymbol{W}_{h0}\times V_{h0}^g}\frac{(\nabla_h\operatorname{curl}_h(\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}),\nabla_h\operatorname{curl}_h\boldsymbol{v}_h)-(\boldsymbol{f},\boldsymbol{v}_h)+(\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}-\boldsymbol{u},\nabla\mu_h)}{\|\boldsymbol{v}_h\|_{H_h(\operatorname{gradcurl})}+|\mu_h|_1}\\&\quad\lesssim\|\boldsymbol{u}-\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}\|_0+\sup_{\boldsymbol{v}_h\in\boldsymbol{W}_{h0}}\frac{(\nabla_h\operatorname{curl}_h(\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}),\nabla_h\operatorname{curl}_h\boldsymbol{v}_h)-(\boldsymbol{f},\boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{H_h(\operatorname{gradcurl})}}.\end{aligned}$$

Hence, we obtain from (4.29) that

$$\|\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}-\boldsymbol{u}_h\|_{H_h(\mathrm{gradcurl})}\lesssim\|\boldsymbol{u}-\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}\|_{H_h(\mathrm{gradcurl})}+h(|\mathrm{curl}\,\boldsymbol{u}|_2+\|\boldsymbol{f}\|_0).$$

Thus,

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H_h(\mathrm{gradcurl})}&\lesssim\|\boldsymbol{u}-\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}\|_{H_h(\mathrm{gradcurl})}+\|\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}-\boldsymbol{u}_h\|_{H_h(\mathrm{gradcurl})}\\&\lesssim\|\boldsymbol{u}-\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}\|_{H_h(\mathrm{gradcurl})}+h(|\mathrm{curl}\,\boldsymbol{u}|_2+\|\boldsymbol{f}\|_0).\end{aligned}$$

Finally, (4.28) follows from (4.23) and (4.24). □

Remark 4.10. As illustrated in [6, Subsection 7.9], the convergence would deteriorate if we use the nonconforming finite element space $\boldsymbol{W}_{h0}$ to the discretized Maxwell equation

$$\begin{cases}\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{u}=\boldsymbol{f} & \text{in } \Omega,\\ \mathrm{div}\,\boldsymbol{u}=0 & \text{in } \Omega,\\ \boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0} & \text{on } \partial\Omega.\end{cases}$$

Next, we estimate $\|\mathrm{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_0$ by the duality argument. To this end, consider the dual problem

$$\begin{cases}-\mathrm{curl}\,\Delta\,\mathrm{curl}\,\widetilde{\boldsymbol{u}}=\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{I}_h^c(\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}-\boldsymbol{u}_h) & \text{in } \Omega,\\ \mathrm{div}\,\widetilde{\boldsymbol{u}}=0 & \text{in } \Omega,\\ \widetilde{\boldsymbol{u}}\times\boldsymbol{n}=(\mathrm{curl}\,\widetilde{\boldsymbol{u}})\times\boldsymbol{n}=\boldsymbol{0} & \text{on } \partial\Omega,\end{cases}\tag{4.30}$$

where $\widetilde{\boldsymbol{u}}\in\boldsymbol{H}_0(\mathrm{gradcurl},\Omega)$. The first equation in the dual problem (4.30) holds in the sense of $\boldsymbol{H}^{-1}(\mathrm{div},\Omega)$, where

$$\boldsymbol{H}^{-1}(\mathrm{div},\Omega):=\{\boldsymbol{v}\in\boldsymbol{H}^{-1}(\Omega;\mathbb{R}^3):\mathrm{div}\,\boldsymbol{v}\in H^{-1}(\Omega)\}$$

is the dual space of $\boldsymbol{H}_0(\mathrm{curl},\Omega)$ (see [11]). Thanks to (4.10) and (4.22), it holds that

$$\mathrm{curl}\,\boldsymbol{I}_h^c(\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}-\boldsymbol{u}_h)=\boldsymbol{I}_h^d\,\mathrm{curl}_h(\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}-\boldsymbol{u}_h)=\boldsymbol{I}_h^d(\boldsymbol{I}_h^s\,\mathrm{curl}\,\boldsymbol{u}-\mathrm{curl}_h\,\boldsymbol{u}_h)=\boldsymbol{I}_h^d\,\mathrm{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h).\tag{4.31}$$

We assume that the dual problem (4.30) possesses the following regularity in this section:

$$\|\widetilde{\boldsymbol{u}}\|_1+\|\mathrm{curl}\,\widetilde{\boldsymbol{u}}\|_2\lesssim\|\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{I}_h^c(\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}-\boldsymbol{u}_h)\|_{-1}\leqslant\|\boldsymbol{I}_h^d\,\mathrm{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_0.\tag{4.32}$$

The regularity (4.32) holds for the domain Ω being convex (see Lemma A.1). Similar to (4.25), it holds from (4.30) that

$$-(\Delta\,\mathrm{curl}\,\widetilde{\boldsymbol{u}},\mathrm{curl}\,\boldsymbol{v})=(\mathrm{curl}\,\boldsymbol{I}_h^c(\boldsymbol{\Pi}_h^{gc}\boldsymbol{u}-\boldsymbol{u}_h),\mathrm{curl}\,\boldsymbol{v}),\quad\forall\,\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{curl},\Omega).\tag{4.33}$$

Theorem 4.11. *Let $\boldsymbol{u}\in\boldsymbol{H}_0(\mathrm{gradcurl},\Omega)$ be the solution of the problem* (4.1)*, and $\boldsymbol{u}_h\in\boldsymbol{W}_{h0}$ be the solution of the mixed finite element method* (4.5)–(4.6)*. Assume that the regularity* (4.32) *holds. We have*

$$\|\mathrm{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_0\lesssim h^{k+1}\|\boldsymbol{f}\|_0+h^2(\|\mathrm{curl}\,\boldsymbol{u}\|_2+|\boldsymbol{u}|_1).\tag{4.34}$$

Proof. It follows from (3.8) that

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}(\partial_n(\mathrm{curl}\,\boldsymbol{u}),\boldsymbol{I}_h^s\,\mathrm{curl}\,\widetilde{\boldsymbol{u}}-\mathrm{curl}\,\widetilde{\boldsymbol{u}})_{\partial K}\\&\quad=\sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}(K)}(\partial_n(\mathrm{curl}\,\boldsymbol{u})-\boldsymbol{Q}_F^0\partial_n(\mathrm{curl}\,\boldsymbol{u}),\boldsymbol{I}_h^s\,\mathrm{curl}\,\widetilde{\boldsymbol{u}}-\mathrm{curl}\,\widetilde{\boldsymbol{u}})_F\\&\quad\lesssim h^2|\mathrm{curl}\,\boldsymbol{u}|_2|\mathrm{curl}\,\widetilde{\boldsymbol{u}}|_2.\end{aligned}$$

Applying (3.8) again, we get

$$(\nabla\mathrm{curl}\,\boldsymbol{u},\nabla_h(\boldsymbol{I}_h^s\,\mathrm{curl}\,\widetilde{\boldsymbol{u}}-\mathrm{curl}\,\widetilde{\boldsymbol{u}}))$$

$$
\begin{aligned}
&= \sum_{K\in\mathcal{T}_h} (\partial_n(\operatorname{curl}\boldsymbol{u}), \boldsymbol{I}_h^s \operatorname{curl}\widetilde{\boldsymbol{u}} - \operatorname{curl}\widetilde{\boldsymbol{u}})_{\partial K} - (\Delta \operatorname{curl}\boldsymbol{u}, \boldsymbol{I}_h^s \operatorname{curl}\widetilde{\boldsymbol{u}} - \operatorname{curl}\widetilde{\boldsymbol{u}}) \\
&\lesssim h^2 |\operatorname{curl}\boldsymbol{u}|_2 |\operatorname{curl}\widetilde{\boldsymbol{u}}|_2.
\end{aligned}
$$

Due to (3.10) and the fact that $\operatorname{div}\boldsymbol{f} = 0$, we have

$$
(\boldsymbol{f}, \widetilde{\boldsymbol{u}} - \boldsymbol{\Pi}_h^{gc}\widetilde{\boldsymbol{u}}) = (\boldsymbol{f}, \widetilde{\boldsymbol{u}}_2 - \boldsymbol{I}_h^{gc}\widetilde{\boldsymbol{u}}_2) \lesssim h^{k+1}\|\boldsymbol{f}\|_0 \|\widetilde{\boldsymbol{u}}_2\|_2 \lesssim h^{k+1}\|\boldsymbol{f}\|_0 |\operatorname{curl}\widetilde{\boldsymbol{u}}|_1.
$$

Combining the last two inequalities, (4.17) and (4.22) implies

$$
\begin{aligned}
(\nabla_h \operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h), \nabla_h \operatorname{curl}_h \boldsymbol{\Pi}_h^{gc}\widetilde{\boldsymbol{u}}) &= (\nabla \operatorname{curl}\boldsymbol{u}, \nabla_h \boldsymbol{I}_h^s \operatorname{curl}\widetilde{\boldsymbol{u}}) - (\boldsymbol{f}, \boldsymbol{\Pi}_h^{gc}\widetilde{\boldsymbol{u}}) \\
&= (\nabla \operatorname{curl}\boldsymbol{u}, \nabla_h(\boldsymbol{I}_h^s \operatorname{curl}\widetilde{\boldsymbol{u}} - \operatorname{curl}\widetilde{\boldsymbol{u}})) + (\boldsymbol{f}, \widetilde{\boldsymbol{u}} - \boldsymbol{\Pi}_h^{gc}\widetilde{\boldsymbol{u}}) \\
&\lesssim h^2 |\operatorname{curl}\boldsymbol{u}|_2 |\operatorname{curl}\widetilde{\boldsymbol{u}}|_2 + h^{k+1}\|\boldsymbol{f}\|_0 |\operatorname{curl}\widetilde{\boldsymbol{u}}|_1.
\end{aligned}
$$

Employing (4.22) and (3.8), we get

$$
\begin{aligned}
(\nabla_h \operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h), \nabla_h \operatorname{curl}_h(\widetilde{\boldsymbol{u}} - \boldsymbol{\Pi}_h^{gc}\widetilde{\boldsymbol{u}})) &= (\nabla_h \operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h), \nabla_h(\operatorname{curl}\widetilde{\boldsymbol{u}} - \boldsymbol{I}_h^s \operatorname{curl}\widetilde{\boldsymbol{u}})) \\
&\leqslant |\operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)|_{1,h} |\operatorname{curl}\widetilde{\boldsymbol{u}} - \boldsymbol{I}_h^s \operatorname{curl}\widetilde{\boldsymbol{u}}|_{1,h} \\
&\lesssim h|\operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)|_{1,h} |\operatorname{curl}\widetilde{\boldsymbol{u}}|_2.
\end{aligned}
$$

It holds from the sum of the last two inequalities that

$$
(\nabla_h \operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h), \nabla \operatorname{curl}\widetilde{\boldsymbol{u}}) \lesssim (h^2|\operatorname{curl}\boldsymbol{u}|_2 + h|\operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)|_{1,h} + h^{k+1}\|\boldsymbol{f}\|_0)\|\operatorname{curl}\widetilde{\boldsymbol{u}}\|_2.
$$

Thanks to (3.1), we obtain

$$
\begin{aligned}
&- \sum_{K\in\mathcal{T}_h} (\operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h), \partial_n \operatorname{curl}\widetilde{\boldsymbol{u}})_{\partial K} \\
&= - \sum_{K\in\mathcal{T}_h} \sum_{F\in\mathcal{F}(K)} ((\boldsymbol{I} - \boldsymbol{Q}_F^0)\operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h), (\boldsymbol{I} - \boldsymbol{Q}_F^0)\partial_n \operatorname{curl}\widetilde{\boldsymbol{u}})_{\partial K} \\
&\lesssim h|\operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)|_{1,h} |\operatorname{curl}\widetilde{\boldsymbol{u}}|_2.
\end{aligned}
$$

Hence, we achieve from the last two inequalities that

$$
-(\operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h), \Delta \operatorname{curl}\widetilde{\boldsymbol{u}}) \lesssim (h^2|\operatorname{curl}\boldsymbol{u}|_2 + h|\operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)|_{1,h} + h^{k+1}\|\boldsymbol{f}\|_0)\|\operatorname{curl}\widetilde{\boldsymbol{u}}\|_2.
$$

On the other hand, it follows from (4.31) and(4.33) that

$$
\begin{aligned}
\|\boldsymbol{I}_h^d \operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_0^2 &= -(\boldsymbol{I}_h^d \operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h), \Delta \operatorname{curl}\widetilde{\boldsymbol{u}}) \\
&= ((\boldsymbol{I} - \boldsymbol{I}_h^d)\operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h), \Delta \operatorname{curl}\widetilde{\boldsymbol{u}}) - (\operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h), \Delta \operatorname{curl}\widetilde{\boldsymbol{u}}) \\
&\lesssim (h^2|\operatorname{curl}\boldsymbol{u}|_2 + h|\operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)|_{1,h} + h^{k+1}\|\boldsymbol{f}\|_0)\|\operatorname{curl}\widetilde{\boldsymbol{u}}\|_2,
\end{aligned}
$$

which together with (4.32) yields

$$
\|\boldsymbol{I}_h^d \operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_0 \lesssim h^2|\operatorname{curl}\boldsymbol{u}|_2 + h|\operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)|_{1,h} + h^{k+1}\|\boldsymbol{f}\|_0.
$$

Hence,

$$
\|\operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_0 \lesssim h^2|\operatorname{curl}\boldsymbol{u}|_2 + h|\operatorname{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)|_{1,h} + h^{k+1}\|\boldsymbol{f}\|_0.
$$

Finally, (4.34) follows from (4.28). □

5 Decoupling of the mixed finite element methods

In this section, we present an equivalent decoupled discretization of the mixed finite element method (4.5)–(4.6) as the decoupled Morley element method for the biharmonic equation in [33], based on which a fast solver is suggested.

5.1 Decoupling

In the continuous level, the mixed formulation (4.3)–(4.4) of the quad-curl problem (4.2) can be decoupled into the following system (see [11, 47]): find $\boldsymbol{w} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega)$, $\lambda \in H_0^1(\Omega)$, $\boldsymbol{\phi} \in \boldsymbol{H}_0^1(\Omega; \mathbb{R}^3)$, $p \in L_0^2(\Omega)$, $\boldsymbol{u} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega)$ and $\sigma \in H_0^1(\Omega)$ such that

$$
\begin{aligned}
&(\mathrm{curl}\,\boldsymbol{w}, \mathrm{curl}\,\boldsymbol{v}) + (\boldsymbol{v}, \nabla\lambda) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega),\\
&(\boldsymbol{w}, \nabla\tau) = 0, \quad \forall\, \tau \in H_0^1(\Omega),\\
&(\boldsymbol{\nabla}\boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\psi}) + (\mathrm{div}\,\boldsymbol{\psi}, p) = (\mathrm{curl}\,\boldsymbol{w}, \boldsymbol{\psi}), \quad \forall\, \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega; \mathbb{R}^3),\\
&(\mathrm{div}\,\boldsymbol{\phi}, q) = 0, \quad \forall\, q \in L_0^2(\Omega),\\
&(\mathrm{curl}\,\boldsymbol{u}, \mathrm{curl}\,\boldsymbol{\chi}) + (\boldsymbol{\chi}, \nabla\sigma) = (\boldsymbol{\phi}, \mathrm{curl}\,\boldsymbol{\chi}), \quad \forall\, \boldsymbol{\chi} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega),\\
&(\boldsymbol{u}, \nabla\mu) = 0, \quad \forall\, \mu \in H_0^1(\Omega).
\end{aligned}
$$

Thanks to the discrete Stokes complex (3.5), the mixed finite element method (4.5)–(4.6) can also be decoupled to find $\boldsymbol{w}_h \in \boldsymbol{W}_{h0}$, $\lambda_h \in V_{h0}^g$, $\boldsymbol{\phi}_h \in \boldsymbol{V}_{h0}^s$, $p_h \in \mathcal{Q}_{h0}$, $\boldsymbol{u}_h \in \boldsymbol{W}_{h0}$ and $\sigma_h \in V_{h0}^g$ such that

$$(\mathrm{curl}_h\,\boldsymbol{w}_h, \mathrm{curl}_h\,\boldsymbol{v}_h) + (\boldsymbol{v}_h, \nabla\lambda_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{W}_{h0}, \tag{5.1}$$

$$(\boldsymbol{w}_h, \nabla\tau_h) = 0, \quad \forall\, \tau_h \in V_{h0}^g, \tag{5.2}$$

$$(\boldsymbol{\nabla}_h\boldsymbol{\phi}_h, \boldsymbol{\nabla}_h\boldsymbol{\psi}_h) + (\mathrm{div}_h\,\boldsymbol{\psi}_h, p_h) = (\mathrm{curl}_h\,\boldsymbol{w}_h, \boldsymbol{\psi}_h), \quad \forall\, \boldsymbol{\psi}_h \in \boldsymbol{V}_{h0}^s, \tag{5.3}$$

$$(\mathrm{div}_h\,\boldsymbol{\phi}_h, q_h) = 0, \quad \forall\, q_h \in \mathcal{Q}_{h0}, \tag{5.4}$$

$$(\mathrm{curl}_h\,\boldsymbol{u}_h, \mathrm{curl}_h\,\boldsymbol{\chi}_h) + (\boldsymbol{\chi}_h, \nabla\sigma_h) = (\boldsymbol{\phi}_h, \mathrm{curl}_h\,\boldsymbol{\chi}_h), \quad \forall\, \boldsymbol{\chi}_h \in \boldsymbol{W}_{h0}, \tag{5.5}$$

$$(\boldsymbol{u}_h, \nabla\mu_h) = 0, \quad \forall\, \mu_h \in V_{h0}^g. \tag{5.6}$$

Both (5.1)–(5.2) and (5.5)–(5.6) are mixed finite element methods for the Maxwell equation. From the discrete Poincaré inequality (4.13) and the fact that $\nabla V_{h0}^g \subset \boldsymbol{W}_{h0}^g$, we have the discrete stability

$$\|\widetilde{\boldsymbol{w}}_h\|_{H_h(\mathrm{curl})} + |\widetilde{\lambda}_h|_1 \lesssim \sup_{(\boldsymbol{v}_h, \tau_h) \in \boldsymbol{W}_{h0} \times V_{h0}^g} \frac{(\mathrm{curl}_h\,\widetilde{\boldsymbol{w}}_h, \mathrm{curl}_h\,\boldsymbol{v}_h) + (\boldsymbol{v}_h, \nabla\widetilde{\lambda}_h) + (\widetilde{\boldsymbol{w}}_h, \nabla\tau_h)}{\|\boldsymbol{v}_h\|_{H_h(\mathrm{curl})} + |\tau_h|_1}$$

for any $\widetilde{\boldsymbol{w}}_h \in \boldsymbol{W}_{h0}$ and $\widetilde{\lambda}_h \in V_{h0}^g$, where the squared norm $\|\boldsymbol{v}_h\|_{H_h(\mathrm{curl})}^2 := \|\boldsymbol{v}_h\|_0^2 + \|\mathrm{curl}_h\,\boldsymbol{v}_h\|_0^2$. Hence both mixed finite element methods (5.1)–(5.2) and (5.5)–(5.6) are well posed. The discrete method (5.3)–(5.4) is exactly the nonconforming P_1-P_0 element method for the Stokes equation.

By replacing $\boldsymbol{v}_h$ in (5.1) with $\nabla\mu_h$ for any $\mu_h \in V_{h0}^g$, we obtain $\lambda_h = 0$ from the fact that $\mathrm{div}\,\boldsymbol{f} = 0$. Similarly, we achieve $\sigma_h = 0$ from (5.5). Then (5.1) and (5.5) will be, respectively, reduced to

$$(\mathrm{curl}_h\,\boldsymbol{w}_h, \mathrm{curl}_h\,\boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{W}_{h0} \tag{5.7}$$

and

$$(\mathrm{curl}_h\,\boldsymbol{u}_h, \mathrm{curl}_h\,\boldsymbol{\chi}_h) = (\boldsymbol{\phi}_h, \mathrm{curl}_h\,\boldsymbol{\chi}_h), \quad \forall\, \boldsymbol{\chi}_h \in \boldsymbol{W}_{h0}. \tag{5.8}$$

Theorem 5.1. *Let*

$$(\boldsymbol{w}_h, 0, \boldsymbol{\phi}_h, p_h, \boldsymbol{u}_h, 0) \in \boldsymbol{W}_{h0} \times V_{h0}^g \times \boldsymbol{V}_{h0}^s \times \mathcal{Q}_{h0} \times \boldsymbol{W}_{h0} \times V_{h0}^g$$

be the solution of the discrete methods (5.1)–(5.6)*. Then* $(\boldsymbol{u}_h, 0)$ *is the solution of the mixed finite element method* (4.5)–(4.6)*.*

Proof. Since (5.6) and (4.6) are the same, we only have to show that $\boldsymbol{u}_h \in \mathcal{K}_h^d$ satisfies (4.17). It follows from (5.4) and the complex (3.5) that there exists a $\widetilde{\boldsymbol{u}}_h \in \mathcal{K}_h^d$ satisfying $\boldsymbol{\phi}_h = \mathrm{curl}_h\,\widetilde{\boldsymbol{u}}_h$, which together with (5.8) yields $(\mathrm{curl}_h(\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_h), \mathrm{curl}_h\,\boldsymbol{\chi}_h) = 0, \forall\, \boldsymbol{\chi}_h \in \boldsymbol{W}_{h0}$. Hence, $\widetilde{\boldsymbol{u}}_h = \boldsymbol{u}_h$ and $\boldsymbol{\phi}_h = \mathrm{curl}_h\,\boldsymbol{u}_h$. Taking $\boldsymbol{\psi}_h = \mathrm{curl}_h\,\boldsymbol{v}_h$ in (5.3) with $\boldsymbol{v}_h \in \boldsymbol{W}_{h0}$, we derive from (5.7) that

$$(\boldsymbol{\nabla}_h\,\mathrm{curl}_h\,\boldsymbol{u}_h, \boldsymbol{\nabla}_h\,\mathrm{curl}_h\,\boldsymbol{v}_h) = (\mathrm{curl}_h\,\boldsymbol{w}_h, \mathrm{curl}_h\,\boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h).$$

Thus the discrete methods (5.1)–(5.6) and the mixed method (4.5)–(4.6) are equivalent. □

5.2 A fast solver

We discuss a fast solver for the mixed method (4.5)–(4.6) in this subsection. The equivalence between the mixed method (4.5)–(4.6) and the mixed methods (5.1)–(5.6) suggests fast solvers for the mixed finite element method (4.5)–(4.6). We can solve the mixed method (5.1)–(5.2), the mixed method (5.3)–(5.4) and the mixed method (5.5)–(5.6) sequentially. The mixed methods (5.1)–(5.2) and (5.5)–(5.6) for the Maxwell equation can be efficiently solved by the solver in [14, Subsection 4.4]. For the mixed method (5.3)–(5.4) of the Stokes equation, we can adopt the block diagonal preconditioner (see [20]) or the approximate block-factorization preconditioner (see [10]).

Finally, we demonstrate the fast solver for the mixed methods (5.1)–(5.2) and (5.5)–(5.6). To this end, define the inner product

$$\langle \lambda_h, \mu_h\rangle := \sum_{i=1}^{n_g} \lambda_i \mu_i \|\psi_i\|_0^2, \quad \text{where } \lambda_h = \sum_{i=1}^{n_g} \lambda_i \psi_i \text{ and } \mu_h = \sum_{i=1}^{n_g} \mu_i \psi_i$$

with $\{\psi_i\}_1^{n_g}$ being the basis functions of V_{h0}^g. The matrix of $\langle \lambda_h, \mu_h\rangle$ is just the diagonal of the mass matrix of (λ_h, μ_h). Then we introduce the following two mixed methods:

$$(\mathrm{curl}_h\, \boldsymbol{w}_h, \mathrm{curl}_h\, \boldsymbol{v}_h) + (\boldsymbol{v}_h, \nabla\lambda_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{W}_{h0}, \tag{5.9}$$

$$(\boldsymbol{w}_h, \nabla\tau_h) - \langle \lambda_h, \tau_h\rangle = 0, \quad \forall\, \tau_h \in V_{h0}^g \tag{5.10}$$

and

$$(\mathrm{curl}_h\, \boldsymbol{u}_h, \mathrm{curl}_h\, \boldsymbol{\chi}_h) + (\boldsymbol{\chi}_h, \nabla\sigma_h) = (\boldsymbol{\phi}_h, \mathrm{curl}_h\, \boldsymbol{\chi}_h), \quad \forall\, \boldsymbol{\chi}_h \in \boldsymbol{W}_{h0}, \tag{5.11}$$

$$(\boldsymbol{u}_h, \nabla\mu_h) - \langle \sigma_h, \mu_h\rangle = 0, \quad \forall\, \mu_h \in V_{h0}^g. \tag{5.12}$$

The well-posedness of the mixed methods (5.9)–(5.10) and (5.11)–(5.12) follows from the stability of the mixed methods (5.1)–(5.2) and (5.5)–(5.6).

Lemma 5.2. *The mixed method* (5.9)–(5.10) *is equivalent to the mixed method* (5.1)–(5.2)*. The mixed method* (5.11)–(5.12) *is equivalent to the mixed method* (5.5)–(5.6)*.*

Proof. Suppose that $(\boldsymbol{w}_h, 0) \in \boldsymbol{W}_{h0} \times V_{h0}^g$ is the solution of the mixed method (5.1)–(5.2). By the fact that $\lambda_h = 0$, apparently $(\boldsymbol{w}_h, 0)$ is also the solution of the mixed method (5.9)–(5.10). The equivalence between the mixed method (5.11)–(5.12) and the mixed method (5.5)–(5.6) follows similarly. □

Such equivalence in the matrix form has been revealed in [14, (77)–(78)]. The matrix form of the mixed finite element method (5.9)–(5.10) is

$$\begin{pmatrix} A & B^{\mathrm{T}} \\ B & -D \end{pmatrix} \begin{pmatrix} \boldsymbol{w}_h \\ \lambda_h \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix}.$$

Here, we still use $\boldsymbol{w}_h$, λ_h and $\boldsymbol{f}$ to represent the vector forms of $\boldsymbol{w}_h$, λ_h and $(\boldsymbol{f}, \boldsymbol{v}_h)$ for ease of presentation. Noting that D is diagonal, we get $(A + B^{\mathrm{T}}D^{-1}B)\boldsymbol{w}_h = \boldsymbol{f}$. The Schur complement $A + B^{\mathrm{T}}D^{-1}B$ corresponds to the symmetric matrix of a discontinuous Galerkin method for the vector Laplacian, which is positive definite and can be solved by the conjugate gradient method with the Hiptmair-Xu (HX) preconditioner in [30].

6 Numerical results

In this section, we perform a numerical experiment to demonstrate the theoretical results of the mixed finite element method (4.5)–(4.6). Let $\Omega = (0,1)^3$. Choose the function $\boldsymbol{f}$ in (4.1) such that the exact solution of (4.1) is

$$\boldsymbol{u} = \mathrm{curl} \begin{pmatrix} 0 \\ 0 \\ \sin^3(\pi x)\sin^3(\pi y)\sin^3(\pi z) \end{pmatrix}.$$

Table 1 Errors $\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$, $\|\mathrm{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_0$ and $|\mathrm{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)|_{1,h}$ for $k=0$ and different h

h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$	Order	$\|\mathrm{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_0$	Order	$\|\mathrm{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_{1,h}$	Order
2^{-1}	1.025E+00	–	1.050E+01	–	1.076E+02	–
2^{-2}	9.687E−01	0.08	5.306E+00	0.98	9.099E+01	0.24
2^{-3}	3.767E−01	1.36	1.618E+00	1.71	5.374E+01	0.76
2^{-4}	1.640E−01	1.20	4.311E−01	1.91	2.820E+01	0.93
2^{-5}	7.828E−02	1.07	1.097E−01	1.97	1.428E+01	0.98

We take uniform triangulations on Ω. Set $k=0$.

Numerical results of errors $\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$, $\|\mathrm{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_0$ and $|\mathrm{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)|_{1,h}$ with respect to h for $k=0$ are shown in Table 1, from which we can see that they all achieve the optimal convergence rates numerically and agree with the theoretical error estimates in (4.28) and (4.34). It is also observed from Table 1 that $\|\mathrm{curl}_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_0 = O(h^2)$ numerically, which is one order higher than the theoretical order in (4.34).

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Appendix A Regularity of the quad-curl problem on convex domains

We prove the regularity of the problem (4.2) under the assumption $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\mathrm{div}, \Omega)$. Similar regularity can be found in [47, Theorem 3.5] when $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega; \mathbb{R}^3)$.

Lemma A.1. *Assume that the domain* Ω *is convex. Let* $\boldsymbol{u} \in \boldsymbol{H}_0(\mathrm{gradcurl}, \Omega)$ *be the solution of the problem* (4.2) *with the divergence-free right-hand side* $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\mathrm{div}, \Omega)$. *Then*

$$\|\boldsymbol{u}\|_1 + \|\mathrm{curl}\, \boldsymbol{u}\|_2 \lesssim \|\boldsymbol{f}\|_{-1}. \tag{A.1}$$

Proof. Due to the framework in [11], the problem (4.2) can be equivalently decoupled into the following system: find $\boldsymbol{w} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega)$, $\lambda \in H_0^1(\Omega)$, $\boldsymbol{\phi} \in \boldsymbol{H}_0^1(\Omega; \mathbb{R}^3)$, $p \in L_0^2(\Omega)$, $\boldsymbol{u} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega)$ and $\sigma \in H_0^1(\Omega)$ such that

$$(\mathrm{curl}\, \boldsymbol{w}, \mathrm{curl}\, \boldsymbol{v}) + (\boldsymbol{v}, \nabla\lambda) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega), \tag{A.2}$$

$$(\boldsymbol{w}, \nabla\tau) = 0, \quad \forall\, \tau \in H_0^1(\Omega), \tag{A.3}$$

$$(\boldsymbol{\nabla}\boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\psi}) + (\mathrm{div}\, \boldsymbol{\psi}, p) = (\mathrm{curl}\, \boldsymbol{w}, \boldsymbol{\psi}), \quad \forall\, \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega; \mathbb{R}^3), \tag{A.4}$$

$$(\mathrm{div}\, \boldsymbol{\phi}, q) = 0, \quad \forall\, q \in L_0^2(\Omega), \tag{A.5}$$

$$(\mathrm{curl}\, \boldsymbol{u}, \mathrm{curl}\, \boldsymbol{\chi}) + (\boldsymbol{\chi}, \nabla\sigma) = (\boldsymbol{\phi}, \mathrm{curl}\, \boldsymbol{\chi}), \quad \forall\, \boldsymbol{\chi} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega), \tag{A.6}$$

$$(\boldsymbol{u}, \nabla\mu) = 0, \quad \forall\, \mu \in H_0^1(\Omega). \tag{A.7}$$

Here, $\langle \cdot, \cdot \rangle$ is the dual pair between $\boldsymbol{H}^{-1}(\mathrm{div}, \Omega)$ and $\boldsymbol{H}_0(\mathrm{curl}, \Omega)$. Since $\boldsymbol{w}, \boldsymbol{u} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$, we have $\boldsymbol{w}, \boldsymbol{u} \in \boldsymbol{H}^1(\Omega; \mathbb{R}^3)$ (see [24, Subsection I.3.4]) and

$$\begin{aligned} &\|\boldsymbol{w}\|_1 \lesssim \|\mathrm{curl}\, \boldsymbol{w}\|_0 \lesssim \|\boldsymbol{f}\|_{-1}, \\ &\|\boldsymbol{u}\|_1 \lesssim \|\mathrm{curl}\, \boldsymbol{u}\|_0 \lesssim \|\boldsymbol{\phi}\|_0. \end{aligned} \tag{A.8}$$

By the regularity of the Stokes problem (A.4)–(A.5) (see [24, Remark I.5.6]), we have

$$\|\boldsymbol{\phi}\|_2 \lesssim \|\mathrm{curl}\, \boldsymbol{w}\|_0 \lesssim \|\boldsymbol{f}\|_{-1}. \tag{A.9}$$

Finally, we conclude (A.1) from (A.8)–(A.9) and the fact that $\boldsymbol{\phi} = \mathrm{curl}\, \boldsymbol{u}$. □