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# New existence of multi-spike solutions for the fractional Schrödinger equations

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**Abstract** We consider the following fractional Schrödinger equation:

$$(-\Delta)^s u + V(y)u = u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N, \tag{0.1}$$

where $s \in (0,1)$, $1 < p < \frac{N+2s}{N-2s}$, and $V(y)$ is a positive potential function and satisfies some expansion condition at infinity. Under the Lyapunov-Schmidt reduction framework, we construct two kinds of multi-spike solutions for (0.1). The first $k$-spike solution $u_k$ is concentrated at the vertices of the regular $k$-polygon in the $(y_1, y_2)$-plane with $k$ and the radius large enough. Then we show that $u_k$ is non-degenerate in our special symmetric workspace, and glue it with an $n$-spike solution, whose centers lie in another circle in the $(y_3, y_4)$-plane, to construct infinitely many multi-spike solutions of new type. The nonlocal property of $(-\Delta)^s$ is in sharp contrast to the classical Schrödinger equations. A striking difference is that although the nonlinear exponent in (0.1) is Sobolev-subcritical, the algebraic (not exponential) decay at infinity of the ground states makes the estimates more subtle and difficult to control. Moreover, due to the non-locality of the fractional operator, we cannot establish the local Pohozaev identities for the solution $u$ directly, but we address its corresponding harmonic extension at the same time. Finally, to construct new solutions we need pointwise estimates of new approximate solutions. To this end, we introduce a special weighted norm, and give the proof in quite a different way.

**Keywords** non-degeneracy, fractional Schrödinger equations, Pohozaev identity, Lyapunov-Schmidt reduction

**MSC(2020)** 35B25, 35J10, 35R11

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## 1 Introduction

In this paper, we consider the following problem involving the fractional Laplacian operator:

$$\begin{cases} (-\Delta)^s u + V(y)u = u^p, & x \in \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

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where $s \in (0,1)$, $1 < p < 2_s^* - 1$, $2_s^* = \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent, and $(-\Delta)^s$ is the fractional Laplacian operator defined as

$$(-\Delta)^s u = c(N,s)\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy,$$

where P.V. is the principal value and $c(N,s) = \pi^{2s+\frac{N}{2}}\Gamma(s+\frac{N}{2})/\Gamma(-s)$ (see [8,11]).

The fractional Laplacian operator, appearing in many areas including biological modeling, physics and mathematical finance, can be regarded as the infinitesimal generator of a stable Lévy process (see [1]). This operator is well defined in $C^{1,1}_{\text{loc}} \cap \mathcal{L}_s$, where

$$\mathcal{L}_s = \left\{ u \in L^1_{\text{loc}} : \int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+2s}} dx < \infty \right\}.$$

For more details on the fractional Laplacian, we refer to [8] and the references therein. Particularly, this nonlocal operator $(-\Delta)^s$ in $\mathbb{R}^N$ can be expressed as a generalized Dirichlet-to-Neumann map for a certain elliptic boundary value problem with local differential operators defined on the upper half-space $\mathbb{R}^{N+1}_+ = \{(y,t) : y \in \mathbb{R}^N, t > 0\}$. More precisely, for any $u \in \dot{H}^s(\mathbb{R}^N)$, set

$$\tilde{u}(y,t) = \mathcal{P}_s[u] = \int_{\mathbb{R}^N} \mathcal{P}_s(y-z,t)u(z)dz, \quad (y,t) \in \mathbb{R}^{N+1}_+,$$

where

$$\mathcal{P}_s(x,t) = \beta(N,s)\frac{t^{2s}}{(|x|^2+t^2)^{\frac{N+2s}{2}}}$$

with a constant $\beta(N,s)$ such that $\int_{\mathbb{R}^N} \mathcal{P}_s(x,1)dx = 1$ (see [7]). Then $\tilde{u} \in L^2(t^{1-2s}, K)$ for any compact set $K$ in $\overline{\mathbb{R}^{N+1}_+}$, $\nabla\tilde{u} \in L^2(t^{1-2s}, \mathbb{R}^{N+1}_+)$ and $\tilde{u} \in C^\infty(\mathbb{R}^{N+1}_+)$. Moreover, $\tilde{u}$ satisfies

$$\begin{cases} \text{div}(t^{1-2s}\nabla\tilde{u}) = 0, & x \in \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to 0} t^{1-2s}\partial_t\tilde{u}(y,t) = \omega_s(-\Delta)^s u(y), \quad \tilde{u}(y,0) = u(y), & x \in \mathbb{R}^N \end{cases}$$

in the distribution sense, where $\omega_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$. Moreover, it holds that

$$\|\tilde{u}\|_{L^2(t^{1-2s},\mathbb{R}^{N+1}_+)} = \omega_s\|u\|_{\dot{H}^s}.$$

Without loss of generality, we may assume $\omega_s = 1$. Problems with fractional Laplacians have been extensively studied recently (see, for example, [2–7,12,14], [13,15,16,19,21,23–25], [26,27,29,30] and the references therein).

Recall the well-known results about the ground state of the following equation:

$$(-\Delta)^s u + u = u^p, \quad u > 0, \quad x \in \mathbb{R}^N, \quad u(0) = \max_{x\in\mathbb{R}^N} u(x). \tag{1.2}$$

Let $N \geqslant 1$, $s \in (0,1)$ and $1 < p < 2_s^* - 1$. Then the following hold (see [14,15]):

(i) (Uniqueness) The ground state solution $U \in H^s(\mathbb{R}^N)$ of (1.2) is unique.

(ii) (Symmetry, regularity and decay) $U(x)$ is radial, positive and strictly decreasing in $|x|$. Moreover, $U \in H^{2s+1}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ and satisfies

$$\frac{C_1}{1+|x|^{N+2s}} \leqslant U(x) \leqslant \frac{C_2}{1+|x|^{N+2s}} \quad \text{for } x \in \mathbb{R}^N \tag{1.3}$$

with some constants $C_2 \geqslant C_1 > 0$.

(iii) (Non-degeneracy) The linearized operator $L_0 = (-\Delta)^s + 1 - pU^{p-1}$ is non-degenerate, i.e., its kernel is given by

$$\ker L_0 = \text{span}\{\partial_{x_1}U, \partial_{x_2}U, \dots, \partial_{x_N}U\}.$$

Moreover, by [15, Lemma C.2], for $j = 1, \dots, N$, $\partial_{x_j} U$ has the decay estimate

$$|\partial_{x_j} U| \leqslant \frac{C}{1+|x|^{N+2s}}.$$

Let $k$ be an integer and consider the vertices of a regular polygon with $k$ edges in the $(y_1, y_2)$-plane given by

$$x_j = \left( r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0 \right), \quad j = 1, \dots, k,$$

where 0 denotes the zero vector in $\mathbb{R}^{N-2}$. For any point $y \in \mathbb{R}^N$, we set $y = (y', y'')$, $y' \in \mathbb{R}^2$ and $y'' \in \mathbb{R}^{N-2}$. Define

$$\bar{H}_s = \bigg\{ u : u \text{ is even in } y_i,\ i = 2, \dots, N, \\ u\left( r\cos\left(\theta + \frac{2\pi j}{k}\right), r\sin\left(\theta + \frac{2\pi j}{k}\right), y'' \right) = u(r\cos\theta, r\sin\theta, y'') \bigg\}.$$

Set

$$W_r(y) = \sum_{j=1}^{k} U_{x_j}(y), \quad U_{x_j}(y) = U(|y - x_j|).$$

In this paper, rather than working in the usual energy space, we construct multi-spike solutions in the weighted $L^\infty$ space, in order to directly obtain the pointwise estimates of the solutions. Precisely, we set

$$\|u\|_* = \sup_{x\in\mathbb{R}^N} \left( \sum_{i=1}^{k} \frac{1}{(1+|x-x_i|)^{N+2s-\theta}} \right)^{-1} |u(x)|, \quad \theta \in \left(0, \frac{N}{2} + 2s\right), \tag{1.4}$$

where $\theta < \frac{N}{2} + 2s$ implies $N + 2s - \theta > \frac{N}{2}$ and thus $\|g\|_* < \infty$ implies $g \in L^2(\mathbb{R}^N)$.

We assume that $V(y) = V(|y|)$ satisfies as $r \to +\infty$,

(V1) $\exists\, a > 0$, $\alpha \in (\frac{N+2s}{N+2s+1}, N + 2s)$ and $\gamma > 0$ such that

$$V(r) = 1 + \frac{a}{r^\alpha} + O\left(\frac{1}{r^{\alpha+\gamma}}\right).$$

Our first existence result is the following theorem.

**Theorem 1.1.** *Suppose that $s \in (0, 1)$, $1 < p < 2_s^* - 1$ and $V(y)$ satisfies* (V1) *further with $\alpha > \frac{1}{p-2}$ when $p > 2$. Then there is an integer $k_0 > 0$, such that for any integer $k \geqslant k_0$, the problem* (1.1) *has a solution $u_k$ of the form*

$$u_k = W_r + \varphi_k,$$

*where $\varphi_k \in \bar{H}_s \cap C(\mathbb{R}^N)$, $r = r_k \in [r_0 k^{\frac{N+2s}{N+2s-\alpha}}, r_1 k^{\frac{N+2s}{N+2s-\alpha}}]$ with some positive constants $r_0 < r_1$, and as $k \to +\infty$,*

$$\|\varphi_k\|_* \leqslant C\left( \frac{1}{r^{\min\{\frac{p}{2},1\}\alpha}} + \frac{1}{r^\theta} \right)$$

*with $\theta \in (\frac{N}{2} + s, \frac{N}{2} + 2s)$.*

**Remark 1.2.** Using the standard finite-dimensional reduction method in the energy space $H^s(\mathbb{R}^N)$, we also construct some multi-spike solutions. In fact, a partner problem was considered in [22], where a similar existence result related to the nonlinear fractional scalar field equation

$$(-\Delta)^s u + u = K(|x|)u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N \tag{1.5}$$

was obtained in $H^s(\mathbb{R}^N)$, where $K(|x|)$ is a positive radial function and $1 < p < \frac{N+2s}{N-2s}$.

However, it is worthwhile to point out that in Theorem 1.1, we propose quite a different new proof of the existence result. Particularly, in the procedure, a pointwise estimate of the solutions is established when we consider the problem in some weighted $L^\infty$ space, which plays an important role in our study on the non-degeneracy result and other relative problems.

Another main goal in this paper is to find a new solution to (1.1), whose shape, at the main order, is

$$u \approx \sum_{j=1}^{k} U_{x_j} + \sum_{j=1}^{n} U_{p_j} \tag{1.6}$$

for large integers $k$ and $n$, where

$$p_j = \left(0, 0, t\cos\frac{2(j-1)\pi}{n}, t\sin\frac{2(j-1)\pi}{n}, \bar{0}\right), \quad j = 1, \dots, n, \quad \bar{0} \in \mathbb{R}^{N-4}$$

and $t \in [t_0 n^{\frac{N+2s}{N+2s-\alpha}}, t_1 n^{\frac{N+2s}{N+2s-\alpha}}]$.

However, it seems hard to obtain a true solution by perturbation arguments. The main reason lies in the observation that if we want to make a small correction to obtain a solution for (1.1) of the form

$$u = \sum_{j=1}^{k} U_{x_j} + \sum_{j=1}^{n} U_{p_j} + \omega_{k,n} \tag{1.7}$$

with $n \gg k$, the estimate of the correction function $\omega_{k,n}$ is dominated by the parameter $k$. In other words, it is hard to see the contribution to the energy from those bumps $U_{p_j}$. Therefore, it is very difficult to use a reduction argument directly to construct solutions of the form (1.7).

To overcome this difficulty, we have to modify the approximate solutions. Following the idea in [17,18], we replace $\sum_{j=1}^{k} U_{x_j}$ by $u_k$ to obtain a better approximate solution $u_k + \sum_{j=1}^{n} U_{p_j}$, where $u_k$ is the $k$-spike solution of (1.1) given by Theorem 1.1. For this purpose, we need that the solution $u_k$ is non-degenerate. By non-degeneracy we mean that the linearized operator

$$L_k\xi = (-\Delta)^s\xi + V(y)\xi - pu_k^{p-1}\xi$$

has a trivial kernel in $\bar{H}_s \cap H^s(\mathbb{R}^N)$, i.e., if $\xi \in \bar{H}_s \cap H^s(\mathbb{R}^N)$ satisfies $L_k\xi = 0$, then $\xi = 0$.

To this end, we impose conditions on $V$ as follows:

$$\begin{aligned} V(r) &= 1 + \frac{a_1}{r^\alpha} + \frac{a_2}{r^{\alpha+1}} + O\left(\frac{1}{r^{\alpha+2}}\right), \\ V'(r) &= -\frac{a_1\alpha}{r^{\alpha+1}} - \frac{a_2(\alpha+1)}{r^{\alpha+2}} + O\left(\frac{1}{r^{\alpha+3}}\right), \end{aligned} \tag{1.8}$$

where $N > \max\{4s, 4-2s\}$, and if $p < 2$,

$$\begin{aligned} &\max\left\{\frac{4(N+2s)}{p(N+2s)-2(1-s)}, \frac{8(N+2s)}{p(3p-1)(N+2s)+2(p+1)(N-2)-8(N+s-1)},\right. \\ &\left.\frac{2(N+2s)}{(p-1)(N+2s)-2s}\right\} < \alpha < \min\left\{N+2s-2, \frac{((3p-1)(N+2s)-8)(N+2s)}{2(2p(N+s-1)-N-4s)}\right\}, \end{aligned} \tag{1.9}$$

while if $p \geqslant 2$,

$$\frac{2(N+2s)}{N} < \alpha < N+2s-2, \tag{1.10}$$

and $a_1 > 0$ and $a_2$ are some constants.

**Remark 1.3.** Note that when $s \geqslant \frac{1}{3}$, $\frac{4(N+2s)}{p(N+2s)-2(1-s)} < \frac{2(N+2s)}{(p-1)(N+2s)-2s}$. From $p < \frac{N+2s}{N-2s}$, we know that if $N > 6s$, $p < 2$, while if $N > 4s$, $p < 3$.

Especially, we further mention that in the case of $p < 2$, $\alpha$ satisfying (1.9) exists. For example, we can directly check that $\alpha$ exists when $s = \frac{1}{2}$, $p = \frac{3}{2}$, $N \geqslant 6$, or $s = \frac{1}{3}$, $p = \frac{3}{2}$, $N \geqslant 6$, or $s = \frac{2}{3}$, $p = \frac{3}{2}$, $N \geqslant 4$, or $s = \frac{3}{4}$, $p = \frac{3}{2}$, $N \geqslant 5$.

**Theorem 1.4.** *Suppose that $V(y)$ satisfies* (1.8). *Then the solution $u_k$ constructed in Theorem* 1.1 *is non-degenerate in $\bar{H}_s \cap H^s(\mathbb{R}^N)$. That is if $\xi \in \bar{H}_s \cap H^s(\mathbb{R}^N)$ satisfies $L_k\xi = 0$, then $\xi = 0$.*

As a consequence of Theorem 1.4, we can construct new multi-spike solutions.

**Theorem 1.5.** *Suppose that $V(y)$ satisfies (1.8), $N \geqslant 4$ and $\alpha < \frac{\min\{p,2\}}{2}N$. Let $u_k$ be a solution in Theorem 1.1 and $k > 0$ is a large even number. Then there is an integer $n_0 > 0$ depending on $k$, such that for any even number $n \geqslant n_0$, (1.1) has a solution of the form (1.6).*

The main difficulty in the proofs of Theorems 1.4 and 1.5 lies in the non-local property of the operator $(-\Delta)^s$. First, although with Sobolev-subcritical nonlinearities, the weak algebraic decay of the ground state of (1.2) makes the estimates become extremely hard to control, in sharp contrast to the classical Schrödinger equations. Moreover, due to the non-local property of the fractional operator, it is almost impossible to build and apply the local Pohozaev identities directly. To overcome it, we also study the corresponding harmonic extension problem at the same time and several kinds of integrals that never appear before have to be handled now. Another difficulty comes from the construction of new multi-spike solutions. The key point is to give the pointwise estimates of the new approximate solution. To this end, we adopt some special weighted norm, reflecting the interaction between the pre-existing solution $u_k$ and the $n$-spikes concentrating at $n$ other new points.

The rest of this paper is organized as follows. We prove Theorem 1.1 and obtain precise pointwise estimates of the multi-spike solutions in Section 2, which are needed in the proof of the non-degeneracy result. In Section 3, we consider the equivalent harmonic extension problem, establish some local Pohozaev identities corresponding to the extension solution and prove the non-degeneracy result. Some sharp estimates involving the integrals in the local Pohozaev identities are also established in this section. The construction of the new solutions will be carried out in Section 4. We put some basic estimates in Appendix A.

## 2 The existence problem

In this section, we give the existence result of multi-spike solutions for the problem (1.1).

Recall

$$\begin{aligned}\bar{H}_s = \bigg\{ & u : u \text{ is even in } y_2, u(y', y'') = u(y', |y''|), \\ & u\bigg(r\cos\bigg(\theta + \frac{2\pi j}{k}\bigg), r\sin\bigg(\theta + \frac{2\pi j}{k}\bigg), y''\bigg) = u(r\cos\theta, r\sin\theta, y'')\bigg\}.\end{aligned}$$

Define

$$Z_j = \frac{\partial U_{x_j}}{\partial r}, \quad x_j = \bigg(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\bigg), \quad j = 1, \ldots, k.$$

Then

$$|x_j - x_1| = 2r\sin\frac{(j-1)\pi}{k}, \quad j = 1, \ldots, k.$$

For any $\lambda > \frac{N+2s-\alpha}{N+2s}$, it holds that (see (A.1) for details)

$$\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{\lambda}} \leqslant C \begin{cases} \bigg(\dfrac{k}{r}\bigg)^{\lambda}, & \lambda > 1, \\ \dfrac{k\ln k}{r^{\lambda}}, & \lambda \leqslant 1. \end{cases} \tag{2.1}$$

Let

$$E_k = \bigg\{ u \in \bar{H}_s \cap C(\mathbb{R}^N), \sum_{j=1}^{k} \int_{\mathbb{R}^N} U_{x_j}^{p-1} Z_j u dy = 0 \bigg\}.$$

Define the linearized operator

$$L\varphi := (-\Delta)^s\varphi + V(y)\varphi - pW_r^{p-1}\varphi, \quad \text{with } W_r = \sum_{j=1}^{k} U_{x_j}, \quad U_{x_j}(y) = U(y - x_j).$$

For $h \in \bar{H}_s \cap C(\mathbb{R}^N)$, we first consider the following linear problem:

$$\begin{cases} Lv = h + b_k \sum_{j=1}^{k} U_{x_j}^{p-1} Z_j, \\ \sum_{j=1}^{k} \int_{\mathbb{R}^N} U_{x_j}^{p-1} Z_j u dy = 0 \end{cases} \tag{2.2}$$

for some $v \in E_k$ and $b_k \in \mathbb{R}$.

**Lemma 2.1.** *Given $\theta \in (0, \frac{N}{2} + 2s)$ and $h = h_k \in \bar{H}_s \cap C(\mathbb{R}^N)$, there exists some solution $(v_k, b_k)$ to* (2.2).

*Proof.* By a standard argument, it suffices to prove the following *a priori* estimate:

$$\|v\|_* \leqslant C\|h\|_*. \tag{2.3}$$

The proof of (2.3) consists of two steps.

**Step 1.** It follows from (2.2) that

$$\begin{aligned} b_k \sum_{j=1}^{k} \int_{\mathbb{R}^N} U_{x_j}^{p-1} Z_j Z_1 dy &= \int_{\mathbb{R}^N} ((-\Delta)^s v + V(y)v - pW_r^{p-1}v - h_k) Z_1 dy \\ &= \int_{\mathbb{R}^N} ((V(y)-1)Z_1 v - p(W_r^{p-1} - U_{x_1}^{p-1})vZ_1 - h_k Z_1) dy. \end{aligned} \tag{2.4}$$

Recall that $Z_1 = \frac{\partial U_{x_1}}{\partial r}$. From the decay of $U$ in (1.3) and the definition of $\|\cdot\|_*$ in (1.4), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} (V(y)-1) Z_1 v dy \\ &\quad\leqslant C \int_{\mathbb{R}^N} \frac{1}{1+|y|^\alpha} \frac{1}{(1+|y-x_1|)^{N+2s}} \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{N+2s-\theta}} \|v\|_* dy \\ &\quad\leqslant C \int_{\mathbb{R}^N} \frac{1}{1+|y|^\alpha} \frac{1}{(1+|y-x_1|)^{2(N+2s)-\theta}} \|v\|_* dy \\ &\qquad + \int_{\mathbb{R}^N} \frac{1}{1+|y|^\alpha} \frac{1}{(1+|y-x_1|)^{N+2s}} \sum_{j=2}^{k} \frac{1}{(1+|y-x_j|)^{N+2s-\theta}} \|v\|_* dy \\ &\quad\leqslant C\left(\frac{1}{r^\alpha} + \frac{1}{r^{N+2s-\theta}}\right)\|v\|_* + C\left(\frac{k}{r}\right)^{N+2s-\theta} \|v\|_* = O\left(\frac{1}{r^\sigma}\|v\|_*\right) \end{aligned} \tag{2.5}$$

with some $\sigma > 0$.

If $p > 2$, we estimate

$$\begin{aligned} \int_{\mathbb{R}^N} (W_r^{p-1} - U_{x_1}^{p-1}) v Z_1 dy &\leqslant C\|v\|_* \int_{\mathbb{R}^N} \left(\left(\sum_{j=2}^{k} U_{x_j}\right)^{p-1} + U_{x_1}^{p-2} \sum_{j=2}^{k} U_{x_j} + \left(\sum_{j=2}^{k} U_{x_j}\right)^{p-2} U_{x_1}\right) \\ &\qquad \times |Z_1| \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{N+2s-\theta}} dy \\ &= C\|v\|_* \sum_{l=1}^{k} \int_{\Omega_l} \left(\left(\sum_{j=2}^{k} U_{x_j}\right)^{p-1} + U_{x_1}^{p-2} \sum_{j=2}^{k} U_{x_j} + \left(\sum_{j=2}^{k} U_{x_j}\right)^{p-2} U_{x_1}\right) \\ &\qquad \times |Z_1| \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{N+2s-\theta}} dy. \end{aligned} \tag{2.6}$$

Next, we compute the items in the sum separately.

For $l=1$, in view of the facts (1.3) and (2.1), we have

$$\begin{aligned}
&\int_{\Omega_1}\bigg(\bigg(\sum_{j=2}^k U_{x_j}\bigg)^{p-1}+U_{x_1}^{p-2}\sum_{j=2}^k U_{x_j}+\bigg(\sum_{j=2}^k U_{x_j}\bigg)^{p-2}U_{x_1}\bigg)|Z_1|\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{N+2s-\theta}}dy\\
&\leqslant C\int_{\Omega_1}\bigg(\bigg(\sum_{j=2}^k U_{x_j}\bigg)^{p-1}+U_{x_1}^{p-2}\sum_{j=2}^k U_{x_j}\bigg)|Z_1|\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{N+2s-\theta}}dy\\
&\leqslant C\int_{\Omega_1}\bigg(\sum_{j=2}^k\frac{1}{(1+|y-x_j|)^{N+2s}}\bigg)^{p-1}\frac{1}{(1+|y-x_1|)^{2N+4s-\theta}}dy\\
&\quad+\int_{\Omega_1}\bigg(\sum_{j=2}^k\frac{1}{(1+|y-x_j|)^{N+2s}}\bigg)^{p-1}\sum_{j=2}^k\frac{1}{(1+|y-x_j|)^{N+2s-\theta}}\frac{1}{(1+|y-x_1|)^{N+2s}}dy\\
&\quad+\int_{\Omega_1}\frac{1}{(1+|y-x_1|)^{p(N+2s)-\theta}}\sum_{j=2}^k\frac{1}{(1+|y-x_j|)^{N+2s}}dy\\
&\quad+\int_{\Omega_1}\frac{1}{(1+|y-x_1|)^{(p-1)(N+2s)}}\sum_{j=2}^k\frac{1}{(1+|y-x_j|)^{N+2s}}\sum_{j=2}^k\frac{1}{(1+|y-x_j|)^{N+2s-\theta}}dy\\
&\leqslant C\bigg(\frac{k}{r}\bigg)^{(p-1)(N+2s)}+C\bigg(\frac{k}{r}\bigg)^{N+2s-\theta}=O\bigg(\frac{1}{r^\sigma}\bigg)
\end{aligned}\tag{2.7}$$

with some $\sigma>0$. In the above, we used the fact that $p-1>\frac{N+2s-\alpha}{\alpha(N+2s)}$ since $N+2s<\alpha(N+2s+1)$, and $N+2s-\theta>\frac{N+2s-\alpha}{\alpha}$ since $\theta<\frac{N}{2}+2s$.

For $l\neq 1$, we have

$$\begin{aligned}
&\int_{\Omega_l}\bigg(\bigg(\sum_{j=2}^k U_{x_j}\bigg)^{p-1}+U_{x_1}^{p-2}\sum_{j=2}^k U_{x_j}+\bigg(\sum_{j=2}^k U_{x_j}\bigg)^{p-2}U_{x_1}\bigg)|Z_1|\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{N+2s-\theta}}dy\\
&\leqslant C\int_{\Omega_l}\bigg(\bigg(\sum_{j=2}^k U_{x_j}\bigg)^{p-1}+\bigg(\sum_{j=2}^k U_{x_j}\bigg)^{p-2}U_{x_1}\bigg)|Z_1|\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{N+2s-\theta}}dy\\
&\leqslant C\int_{\Omega_l}\bigg(\sum_{j=2}^k\frac{1}{(1+|y-x_j|)^{N+2s}}\bigg)^{p-1}\frac{1}{(1+|y-x_1|)^{2N+4s-\theta}}dy\\
&\quad+\int_{\Omega_l}\bigg(\sum_{j=2}^k\frac{1}{(1+|y-x_j|)^{N+2s}}\bigg)^{p-1}\sum_{j=2}^k\frac{1}{(1+|y-x_j|)^{N+2s-\theta}}\frac{1}{(1+|y-x_1|)^{N+2s}}dy\\
&\quad+\int_{\Omega_l}\frac{1}{(1+|y-x_1|)^{2(N+2s)}}\bigg(\sum_{j=2}^k\frac{1}{(1+|y-x_j|)^{N+2s}}\bigg)^{p-2}\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{N+2s-\theta}}dy\\
&\leqslant C\bigg(\frac{k}{r}\bigg)^{(p-2)(N+2s)}+C\bigg(\frac{k}{r}\bigg)^{N+2s-\theta}=O\bigg(\frac{1}{r^\sigma}\bigg),
\end{aligned}\tag{2.8}$$

where the last inequality holds because $\alpha>\frac{1}{p-2}\Rightarrow(p-2)(N+2s)>\frac{N+2s-\alpha}{\alpha}$, and $\sigma>0$.

If $p\leqslant 2$, we can prove it in a similar way.

At last, we compute

$$\begin{aligned}
|\langle h_k,Z_1\rangle|&\leqslant\|h_k\|_*\int_{\mathbb{R}^N}\frac{1}{(1+|y-x_1|)^{N+2s}}\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{N+2s-\theta}}dy\\
&\leqslant\|h_k\|_*\bigg(\int_{\mathbb{R}^N}\frac{1}{(1+|y-x_1|)^{2N+4s-\theta}}dy\\
&\quad+\sum_{l=1}^k\int_{\Omega_l}\frac{1}{(1+|y-x_1|)^{N+2s}}\sum_{j=2}^k\frac{1}{(1+|y-x_j|)^{N+2s-\theta}}dy\bigg)
\end{aligned}$$

$$\leqslant \|h_k\|_*\bigg(\int_{\mathbb{R}^N}\frac{dy}{(1+|y-x_1|)^{2N+4s-\theta}}+\Big(\frac{k}{r}\Big)^{N+2s-\theta}\bigg). \tag{2.9}$$

Combining the above estimates (2.4) and (2.5)–(2.9), we obtain that there exists some constant $\sigma > 0$ satisfying

$$|b_k| \leqslant C\bigg(\frac{\|v\|_*}{r^\sigma}+\|h_k\|_*\bigg).$$

**Step 2.** We show the *a priori* estimate by contradiction. Assume that there exist $h_k$ with $\|h_k\|_* \to 0$, $\|v_k\|_* = 1$ and

$$r = r_k \in [r_0 k^{\frac{N+2s}{N+2s-\alpha}}, r_1 k^{\frac{N+2s}{N+2s-\alpha}}].$$

We first claim that for any $R > 0$,

$$\|v_k\|_{L^\infty(\bigcup_{j=1}^k B_R(x_j))} \to 0. \tag{2.10}$$

Indeed, assume that for a fixed $j$, we know that

$$\|v_k\|_{L^\infty(B_R(x_j))} \geqslant \gamma > 0.$$

Note that $B_R(x_j) \subset \Omega_j$. Let $\bar{v}_k(y) = v_k(y + x_j)$. Then

$$(-\Delta)^s\bar{v}_k + V(y+x_j)\bar{v}_k - p\bigg(\sum_{i=1}^k U(x_j - x_i + y)\bigg)^{p-1}\bar{v}_k = \bar{h}_k$$

with

$$\bar{h}_k(y) = h_k(y+x_j) + b_k\sum_{i=1}^k U(x_j-x_i+y)^{p-1}\partial_r U(x_j-x_i+y).$$

We observe that $\bar{h}_k \to 0$ uniformly on compact sets in view of the assumption $\|h_k\|_* \to 0$ as $k \to +\infty$. From the uniform Hölder estimates, we also obtain equicontinuity of the sequence $\bar{v}_k$. Thus, passing to a subsequence, we may assume that $\bar{v}_k$ converges, uniformly on compact sets, to a bounded function $\bar{v}$ satisfying

$$\|\bar{v}\|_{L^\infty(B_R(0))} \geqslant \gamma > 0.$$

In addition,

$$\|(1+|y|)^{N+2s-\theta}\bar{v}\|_{L^\infty(\mathbb{R}^N)} \leqslant 1$$

and $\bar{v}$ satisfies the equation

$$(-\Delta)^s\bar{v} + \bar{v} - pU^{p-1}\bar{v} = 0 \quad \text{in } \mathbb{R}^N.$$

The non-degeneracy result yields that $\bar{v}$ must be a linear combination of the partial derivatives $\partial_{y_i}U$, $i = 1, \ldots, N$. But the orthogonality conditions and the fact that $\bar{v}$ is even in $y_j$, $j = 2, \ldots, N$ imply that

$$\int_{\mathbb{R}^N} U^{p-1}\partial_{y_1}U\bar{v} = 0,$$

and then $\bar{v} = 0$, which is a contradiction to the assumption $\|v_k\|_{L^\infty(B_R(x_j))} \geqslant \gamma > 0$. So we proved the claim (2.10).

On the other hand, it holds that

$$\|v_k\|_* \leqslant C(\|v_k\|_{L^\infty(\bigcup_{j=1}^k B_R(x_j))} + \|h_k\|_*). \tag{2.11}$$

In fact, (2.2) implies that $v_k$ satisfies that

$$(-\Delta)^s v_k + W(y)v_k = g_k,$$

where $W(y) = V(y) - pW_r^{p-1}(y)$ and $g_k = h_k + b_k \sum_{j=1}^k U_{x_j}^{p-1} Z_j$ satisfy that

$$\inf_{y\in\mathbb{R}^N\setminus\bigcup_{j=1}^k B_R(x_j)} W(y) > 0, \quad \|g_k\|_* < \infty.$$

Following the argument in [9, Lemma 2.5], we obtain (2.11).

Combining (2.10) and (2.11), we indeed obtain that $\|v_k\|_* \to 0$, which again leads to a contradiction and (2.3) is proved.

Finally, applying (2.3), we can use the standard method to obtain the existence result of the linear problem (2.2), for the details of which one can refer to [9]. □

Now we consider the following problem:

$$L\varphi = l_k + R(\varphi) + b_k \sum_{j=1}^k U_{x_j}^{p-1} Z_j \tag{2.12}$$

for $\varphi \in E_k$ and $b_k \in \mathbb{R}$, where

$$l_k = -\sum_{j=1}^k (V(y)-1)U_{x_j} + W_r^p - \sum_{j=1}^k U_{x_j}^p$$

and

$$R(\varphi) = (W_r + \varphi)^p - W_r^p - pW_r^{p-1}\varphi.$$

**Lemma 2.2.** *For $\theta \in (\frac{N}{2}+s, \frac{N}{2}+2s)$, it holds that*

$$\|l_k\|_* \leqslant C\bigg(\frac{1}{r^\theta} + \frac{1}{r^{\alpha\min\{\frac{p}{2},1\}}}\bigg). \tag{2.13}$$

*Proof.* First, it holds that

$$\begin{aligned}\bigg|\sum_{j=1}^k (V(y)-1)U_{x_j}\bigg| &\leqslant \frac{C}{1+|y|^\alpha}\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{N+2s}} \\ &\leqslant C\begin{cases} \dfrac{1}{r^\alpha}\displaystyle\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{N+2s-\theta}}, & |y-x_{j_0}| < \dfrac{r}{2}, \quad \exists\, j_0 \in \{1,\dots,k\}, \\ \dfrac{1}{r^\theta}\displaystyle\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{N+2s-\theta}}, & |y-x_j| \geqslant \dfrac{r}{2}, \quad \forall\, j \in \{1,\dots,k\}. \end{cases}\end{aligned} \tag{2.14}$$

For any $y \in \Omega_1$, $|y-x_j| \geqslant |y-x_1|$ and $|y-x_j| \geqslant C|x_j - x_1|$, $j = 2,\dots,k$, so if $p \geqslant 2$, it holds that

$$\begin{aligned}\bigg|W_r^p - \sum_{j=1}^k U_{x_j}^p\bigg| &\leqslant C\sum_{j=2}^k U_{x_1}^{p-1}U_{x_j} \\ &\leqslant C\sum_{j=2}^k \frac{1}{(1+|y-x_1|)^{(N+2s)(p-1)}}\frac{1}{(1+|y-x_j|)^{N+2s}} \\ &\leqslant C\frac{1}{(1+|y-x_1|)^{N+2s-\theta}}\sum_{j=2}^k \frac{1}{|x_1-x_j|^{N+2s}}.\end{aligned} \tag{2.15}$$

If $p < 2$, since $\theta \in (\frac{N}{2}+s, \frac{N}{2}+2s)$, we have

$$\theta - \frac{2-p}{2}(N+2s) > 0,$$

and hence

$$\begin{aligned}\left|W_r^p-\sum_{j=1}^k U_{x_j}^p\right| &\leqslant C\sum_{j=2}^k U_{x_1}^{\frac{p}{2}}U_{x_j}^{\frac{p}{2}}\\ &\leqslant C\sum_{j=2}^k\frac{1}{(1+|y-x_1|)^{\frac{p}{2}(N+2s)}}\frac{1}{(1+|y-x_j|)^{\frac{p}{2}(N+2s)}}\\ &\leqslant C\frac{1}{(1+|y-x_1|)^{N+2s-\theta}}\sum_{j=2}^k\frac{1}{(1+|y-x_1|)^{\theta-\frac{2-p}{2}(N+2s)}}\frac{1}{|x_1-x_j|^{\frac{p}{2}(N+2s)}}\\ &\leqslant C\frac{1}{(1+|y-x_1|)^{N+2s-\theta}}\sum_{j=2}^k\frac{1}{|x_1-x_j|^{\frac{p}{2}(N+2s)}}. \end{aligned}\tag{2.16}$$

Combining (2.14)–(2.16), we obtain (2.13). □

*Sketch of the proof of Theorem* 1.1. In view of Lemma 2.2, we apply the contraction mapping theorem to prove that there exists some $k_0>0$ such that for any $k\geqslant k_0$, (2.12) has a solution $\varphi_k\in E_k$. In addition,

$$\|\varphi_k\|_*\leqslant C\|l_k\|_*\leqslant C\left(\frac{1}{r^{\min\{\frac{p}{2},1\}\alpha}}+\frac{1}{r^\theta}\right).\tag{2.17}$$

Let

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}u|^2+V(y)u^2)dy-\frac{1}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}dx.$$

By the standard arguments, we have the expansion of the energy functional as follows:

$$I(W_r+\varphi_k)=k\left(A+\frac{aB_1}{r^\alpha}-(B_2+o(1))U(|x_1-x_2|)+O\left(\frac{1}{r^{\alpha+\sigma}}\right)\right),$$

where $\sigma>0$ is some small constant, and $A$, $B_1$ and $B_2$ are some positive constants.

The function

$$\frac{aB_1}{r^\alpha}-(B_2+o(1))U(|x_1-x_2|)=\frac{aB_1}{r^\alpha}-\frac{B_2k^{N+2s}}{r^{N+2s}}$$

has a maximum point in

$$S_k=\left[\left(\frac{B_2(N+2s)}{B_1\alpha}-\delta\right)^{\frac{1}{N+2s-\alpha}}k^{\frac{N+2s}{N+2s-\alpha}},\left(\frac{B_2(N+2s)}{B_1\alpha}+\delta\right)^{\frac{1}{N+2s+\alpha}}k^{\frac{N+2s}{N+2s-\alpha}}\right],$$

where $\delta>0$ is a small constant. Therefore, we can prove that (1.1) has a solution $u_k$ of the form $u_k=W_r+\varphi_k$, with $\varphi_k\in E_k\cap H^s(\mathbb{R}^N)$, $r=r_k\in S_k$ and as $k\to+\infty$,

$$\|\varphi_k\|_*\leqslant C\left(\frac{1}{r^{\min\{\frac{p}{2},1\}\alpha}}+\frac{1}{r^\theta}\right).$$

This completes the proof. □

## 3 The non-degeneracy of the solutions

### 3.1 The Pohozaev identities

We consider the following two equations:

$$(-\Delta)^su+V(y)u=u^p$$

and

$$(-\Delta)^s\xi + V(y)\xi = pu^{p-1}\xi.$$

In order to apply local Pohozaev identities, we quote the extensions of $u$ and $\xi$ to have

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla\tilde{u}) = 0 \quad \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to 0} t^{1-2s}\partial_t\tilde{u}(y,t) = -V(y)u + u^p, \quad \tilde{u}(y,0) = u(y), \quad y \in \mathbb{R}^N \end{cases} \tag{3.1}$$

and

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla\tilde{\xi}) = 0 \quad \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to 0} t^{1-2s}\partial_t\tilde{\xi}(y,t) = -V(y)\xi + pu^{p-1}\xi, \quad \tilde{\xi}(y,0) = \xi(y), \quad y \in \mathbb{R}^N. \end{cases} \tag{3.2}$$

Define

$$\begin{aligned}
&B_R(x_0) = \{y \in \mathbb{R}^N : |y - x_0| \leqslant R\} \subseteq \mathbb{R}^N, \\
&B^+_{R,\rho}(x_0) = \{Y = (y,t) : |y - x_0| \leqslant R, 0 < t \leqslant \rho\} \subseteq \mathbb{R}^{N+1}_+, \\
&\partial' B^+_{R,\rho}(x_0) = \{Y = (y,t) : |y - x_0| \leqslant R, t = 0\} \subseteq \mathbb{R}^N, \\
&\partial'' B^+_{R,\rho}(x_0) = \{Y = (y,t) : |y - x_0| = R, 0 < t \leqslant \rho \text{ or } |y - x_0| \leqslant R, t = \rho\} \subseteq \mathbb{R}^{N+1}_+, \\
&\partial B^+_{R,\rho}(x_0) = \partial' B^+_{R,\rho}(x_0) \cup \partial'' B^+_{R,\rho}(x_0).
\end{aligned} \tag{3.3}$$

Let $\Omega = B_R(x_0)$ and $\Omega^+ = B^+_{R,\rho}(x_0)$. We also denote the integral infinitesimal elements on the surfaces in $R^{NH}_+$ and $R^N$ by $d\sigma'$ and $d\sigma$, respectively. We have the following lemma.

**Lemma 3.1.** *It holds that*

$$\begin{aligned}
&-\int_{\partial''\Omega^+} t^{1-2s}\left(\frac{\partial\tilde{u}}{\partial\nu}\frac{\partial\tilde{\xi}}{\partial y^i} + \frac{\partial\tilde{\xi}}{\partial\nu}\frac{\partial\tilde{u}}{\partial y^i}\right)d\sigma' + \int_{\partial''\Omega^+} t^{1-2s}\langle\nabla\tilde{u}, \nabla\tilde{\xi}\rangle\nu_i d\sigma' \\
&\quad + \int_{\partial\Omega}(V(y)\xi u\nu_i - u^p\xi\nu_i)ds = \int_\Omega \frac{\partial V(y)}{\partial y^i}\xi u dy.
\end{aligned} \tag{3.4}$$

*Proof.* From the first equations in (3.1) and (3.2),

$$\int_{\Omega^+} \operatorname{div}(t^{1-2s}\nabla\tilde{u})\frac{\partial\tilde{\xi}}{\partial y^i}dydt + \int_{\Omega^+} \operatorname{div}(t^{1-2s}\nabla\tilde{\xi})\frac{\partial\tilde{u}}{\partial y^i}dydt = 0,$$

which then gives

$$\begin{aligned}
&\int_{\partial''\Omega^+} t^{1-2s}\frac{\partial\tilde{u}}{\partial\nu}\frac{\partial\tilde{\xi}}{\partial y^i}dS - \int_{\Omega^+} t^{1-2s}\nabla\tilde{u}\cdot\nabla\frac{\partial\tilde{\xi}}{\partial y^i}dydt + \int_{\partial''\Omega^+} t^{1-2s}\frac{\partial\tilde{\xi}}{\partial\nu}\frac{\partial\tilde{u}}{\partial y^i}d\sigma' - \int_{\Omega^+} t^{1-2s}\nabla\tilde{\xi}\cdot\nabla\frac{\partial\tilde{u}}{\partial y^i}dy \\
&\quad - \int_\Omega \lim_{t\to 0} t^{1-2s}\partial_t\tilde{u}\frac{\partial\tilde{\xi}}{\partial y^i}dy - \int_\Omega \lim_{t\to 0} t^{1-2s}\partial_t\tilde{\xi}\frac{\partial\tilde{u}}{\partial y^i}dy = 0.
\end{aligned}$$

Since

$$\begin{aligned}
&-\int_{\Omega^+} t^{1-2s}\nabla\tilde{u}\cdot\nabla\frac{\partial\tilde{\xi}}{\partial y^i}dydt - \int_{\Omega^+} t^{1-2s}\nabla\tilde{\xi}\cdot\nabla\frac{\partial\tilde{u}}{\partial y^i}dydt \\
&\quad = -\int_{\Omega^+} t^{1-2s}\frac{\partial}{\partial y^i}\langle\nabla\tilde{u}, \nabla\tilde{\xi}\rangle dydt = -\int_{\partial''\Omega^+} t^{1-2s}\langle\nabla\tilde{u}, \nabla\tilde{\xi}\rangle\nu_i dS
\end{aligned}$$

and

$$\begin{aligned}
-\int_\Omega \lim_{t\to 0} t^{1-2s}\partial_t\tilde{u}\frac{\partial\tilde{\xi}}{\partial y^i}dy - \int_\Omega \lim_{t\to 0} t^{1-2s}\partial_t\tilde{\xi}\frac{\partial\tilde{u}}{\partial y^i}dy &= \int_\Omega\left((u^p - V(y)u)\frac{\partial\xi}{\partial y^i} + (pu^{p-1}\xi - V(y)\xi)\frac{\partial u}{\partial y^i}\right)dy \\
&= \int_\Omega \frac{\partial V(y)}{\partial y^i}u\xi dy + \int_{\partial\Omega}(u^p\xi - V(y)\xi u)\nu_i ds,
\end{aligned}$$

we obtain (3.4). □

### 3.2 Non-degeneracy

Let $u_k$ be a solution to (1.1) of the form $u_k = W_r + \varphi_k$ with $\varphi_k \in E_k \cap H^s(\mathbb{R}^N)$, $r = r_k \in S_k$ and as $k \to +\infty$,

$$\|\varphi_k\|_* \leqslant C\bigg(\frac{1}{r^{\min\{\frac{p}{2},1\}\alpha}} + \frac{1}{r^\theta}\bigg),$$

where $\theta \in (\frac{N}{2}+s, \frac{N}{2}+2s)$.

Now we prove the non-degeneracy result by contradiction. Suppose that there exists some $k_m \to +\infty$ satisfying $\|\xi_m\|_{L^\infty} = 1$ and $L_{k_m}\xi_m = 0$, where the linear operator is defined by

$$L_{k_m}\xi = (-\Delta)^s\xi + V(y)\xi - pu_{k_m}^{p-1}\xi.$$

Let

$$\bar{\xi}_m(y) = \xi_m(y + x_{k_m,1}).$$

**Lemma 3.2.** *It holds that*

$$\bar{\xi}_m(y) \to b\frac{\partial U}{\partial y^1}, \tag{3.5}$$

*uniformly in $C^1(B_R(0))$ for any $R > 0$, where $b$ is some constant.*

*Proof.* In view of $|\bar{\xi}_m| \leqslant C$, we may assume that $\bar{\xi}_m \to \bar{\xi}$ in $C_{\mathrm{loc}}(\mathbb{R}^N)$. Then $\bar{\xi}$ satisfies that

$$(-\Delta)^s\bar{\xi} + \bar{\xi} = pU^{p-1}\bar{\xi} \quad \text{in } \mathbb{R}^N,$$

which gives that

$$\bar{\xi} = \sum_{j=1}^k b_j \frac{\partial U}{\partial y^j}.$$

Since $\bar{\xi}$ is even in $y_j$, $j = 2, \ldots, N$, it holds that $b_2 = \cdots = b_N = 0$. □

Decompose

$$\xi_m(y) = b_m \sum_{j=1}^{k_m} \frac{\partial U_{x_{k_m,j}}}{\partial r} + \xi_m^*,$$

where $\xi_m^* \in E_{k_m}$. By (3.5), $b_m$ is bounded.

**Lemma 3.3.** *It holds that*

$$\|\xi_m^*\|_* \leqslant C\bigg(\frac{1}{r^\theta} + \frac{1}{r^{\alpha\min\{\frac{p}{2},1\}}}\bigg)^{\min\{\frac{p-1}{2},1\}}. \tag{3.6}$$

*Proof.* It is easy to check that

$$\begin{aligned}
L_{k_m}\xi_m^* &= (-\Delta)^s\xi_m^* + V(y)\xi_m^* - pu_{k_m}^{p-1}\xi_m^* \\
&= ((-\Delta)^s + V(y) - pu_{k_m}^{p-1})\bigg(\xi_m - b_m\sum_{j=1}^k \frac{\partial U_{x_{k_m,j}}}{\partial r}\bigg) \\
&= -b_m \sum_{j=1}^k \bigg((-\Delta)^s \frac{\partial U_{x_{k_m,j}}}{\partial r} + V(y)\frac{\partial U_{x_{k_m,j}}}{\partial r} - pu_{k_m}^{p-1}\frac{\partial U_{x_{k_m,j}}}{\partial r}\bigg) \\
&= -b_m \sum_{j=1}^k \bigg((V(y)-1)\frac{\partial U_{x_{k_m,j}}}{\partial r} - p(u_{k_m}^{p-1} - U_{x_{k_m,j}}^{p-1})\frac{\partial U_{x_{k_m,j}}}{\partial r}\bigg).
\end{aligned}$$

First, we estimate that

$$\begin{aligned}\left|(V(y)-1)\sum_{j=1}^{k}\frac{\partial U_{x_{k_m,j}}}{\partial r}\right| &\leqslant \frac{C}{1+|y|^{\alpha}}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{N+2s}}\\ &\leqslant C\bigg(\frac{1}{r^{\alpha}}+\frac{1}{r^{\theta}}\bigg)\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{N+2s-\theta}},\end{aligned}$$

which gives

$$\left\|(V(y)-1)\sum_{j=1}^{k}\frac{\partial U_{x_{k_m,j}}}{\partial r}\right\|_{*}\leqslant C\bigg(\frac{1}{r^{\alpha}}+\frac{1}{r^{\theta}}\bigg).$$

Second, when $p\geqslant 3$,

$$\begin{aligned}&\left|\sum_{j=1}^{k}(u_{k_m}^{p-1}-U_{x_{k_m,j}}^{p-1})\frac{\partial U_{x_{k_m,j}}}{\partial r}\right|\\ &\quad=\left|\sum_{j=1}^{k}((W_{r_m}+\varphi_{k_m})^{p-1}-U_{x_{k_m,j}}^{p-1})\frac{\partial U_{x_{k_m,j}}}{\partial r}\right|\\ &\quad=\left|\sum_{j=1}^{k}\left(\bigg(U_{x_{k_m,j}}+\sum_{i\neq j}^{k}U_{x_{k_m,i}}+\varphi_{k_m}\bigg)^{p-1}-U_{x_{k_m,j}}^{p-1}\right)\frac{\partial U_{x_{k_m,j}}}{\partial r}\right|\\ &\quad\leqslant C\Bigg(\sum_{j=1}^{k}\bigg(\sum_{i\neq j}^{k}U_{x_{k_m,i}}\bigg)^{p-2}U_{x_{k_m,j}}\frac{\partial U_{x_{k_m,j}}}{\partial r}+\sum_{j=1}^{k}\sum_{i\neq j}^{k}U_{x_{k_m,i}}U_{x_{k_m,j}}^{p-2}\frac{\partial U_{x_{k_m,j}}}{\partial r}\\ &\qquad+\sum_{j=1}^{k}\bigg(\sum_{i\neq j}^{k}U_{x_{k_m,i}}\bigg)^{p-2}\varphi_{k_m}\frac{\partial U_{x_{k_m,j}}}{\partial r}+\sum_{j=1}^{k}U_{x_{k_m,j}}^{p-2}\varphi_{k_m}\frac{\partial U_{x_{k_m,j}}}{\partial r}\Bigg)+O(\varphi_{k_m}^{2})\\ &\quad\leqslant C\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{N+2s-\theta}}\bigg(\sum_{j=2}^{k}\frac{1}{|x_{k_m,1}-x_{k_m,j}|^{N+2s}}+\|\varphi_{k_m}\|_{*}\bigg).\end{aligned}$$

For $p<3$,

$$\begin{aligned}&\left|\sum_{j=1}^{k}(u_{k_m}^{p-1}-U_{x_{k_m,j}}^{p-1})\frac{\partial U_{x_{k_m,j}}}{\partial r}\right|\\ &\quad=\left|\sum_{j=1}^{k}((W_{r_m}+\varphi_{k_m})^{p-1}-U_{x_{k_m,j}}^{p-1})\frac{\partial U_{x_{k_m,j}}}{\partial r}\right|\\ &\quad=\left|\sum_{j=1}^{k}\left(\bigg(U_{x_{k_m,j}}+\sum_{i\neq j}^{k}U_{x_{k_m,i}}+\varphi_{k_m}\bigg)^{p-1}-U_{x_{k_m,j}}^{p-1}\right)\frac{\partial U_{x_{k_m,j}}}{\partial r}\right|\\ &\quad\leqslant C\Bigg(\sum_{j=1}^{k}\bigg(\sum_{i\neq j}^{k}U_{x_{k_m,i}}\bigg)^{\frac{p-1}{2}}U_{x_{k_m,j}}^{\frac{p-1}{2}}\frac{\partial U_{x_{k_m,j}}}{\partial r}+\sum_{j=1}^{k}U_{x_{k_m,j}}^{\frac{p-1}{2}}\varphi_{k_m}^{\frac{p-1}{2}}\frac{\partial U_{x_{k_m,j}}}{\partial r}\Bigg)\\ &\quad\leqslant C\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{N+2s-\theta}}\bigg(\sum_{j=2}^{k}\frac{1}{|x_{k_m,1}-x_{k_m,j}|^{\frac{p-1}{2}(N+2s)}}+\|\varphi_{k_m}\|_{*}^{\frac{p-1}{2}}\bigg).\end{aligned}$$

On the other hand, since $\xi_m^*\in E_{k_m}$, we apply the standard method to prove that

$$\|L_{k_m}\xi_m^*\|_{*}\geqslant c\|\xi_m^*\|_{*},$$

which immediately implies (3.6) and concludes the proof. □

**Lemma 3.4.** *Under the assumption in Theorem* 1.4, *it holds that* $\bar{\xi}_m \to 0$ *uniformly in* $C^1(B_R(0))$ *for any* $R > 0$.

*Proof.* Following a counterpart result in [18], we outline the main steps and drop the subscript $m$ for simplicity.

**Step 1.** The Pohozaev identity (3.4) implies that

$$\mathcal{L}(u,\xi,\Omega) + \int_{\partial\Omega} ((V(y)-1)u\xi\nu_1 - u^p\xi\nu_1)d\sigma = \int_\Omega \frac{\partial V}{\partial y^1}\xi u dy, \tag{3.7}$$

where

$$\Omega = B_{\frac{1}{2}|x_{k,1}-x_{k,2}|}(x_{k,1}), \quad \Omega^+ = B^+_{\frac{1}{2}|x_{k,1}-x_{k,2}|,\rho}(x_{k,1})$$

(see the notation in (3.3)) with some small constant $\rho > 0$, and the bilinear form is defined as

$$\mathcal{L}(u,\xi,\Omega) = -\int_{\partial''\Omega^+} t^{1-2s}\bigg(\frac{\partial\tilde{u}}{\partial\nu}\frac{\partial\tilde{\xi}}{\partial y^1} + \frac{\partial\tilde{\xi}}{\partial\nu}\frac{\partial\tilde{u}}{\partial y^1} - \langle\nabla\tilde{u},\nabla\tilde{\xi}\rangle\nu_i\bigg)d\sigma' + \int_{\partial\Omega} u\xi\nu_1 d\sigma.$$

On the other hand, one can rewrite it as

$$\mathcal{L}(u,\xi,\Omega) = \int_\Omega \bigg(((-\Delta)^s u + u - u^p)\frac{\partial\xi}{\partial y^1} + ((-\Delta)^s\xi + \xi - pu^{p-1}\xi)\frac{\partial u}{\partial y^1}\bigg)dy + \int_{\partial\Omega} u^p\xi\nu_1 d\sigma. \tag{3.8}$$

In fact, from

$$\int_{\Omega^+} \operatorname{div}(t^{1-2s}\nabla\tilde{u})\frac{\partial\tilde{\xi}}{\partial y^1}dydt + \int_{\Omega^+} \operatorname{div}(t^{1-2s}\nabla\tilde{\xi})\frac{\partial\tilde{u}}{\partial y^1}dydt = 0,$$

we have

$$\begin{aligned}&\int_{\partial''\Omega^+} t^{1-2s}\frac{\partial\tilde{u}}{\partial\nu}\frac{\partial\tilde{\xi}}{\partial y^1}d\sigma' + \int_{\partial''\Omega^+} t^{1-2s}\frac{\partial\tilde{\xi}}{\partial\nu}\frac{\partial\tilde{u}}{\partial y^1}d\sigma' - \int_{\partial''\Omega^+} t^{1-2s}\langle\nabla\tilde{u},\nabla\tilde{\xi}\rangle\nu_i d\sigma'\\
&\quad - \int_\Omega \lim_{t\to0} t^{1-2s}\partial_t\tilde{u}\frac{\partial\tilde{\xi}}{\partial y^1}dy - \int_\Omega \lim_{t\to0} t^{1-2s}\partial_t\tilde{\xi}\frac{\partial\tilde{u}}{\partial y^1}dy = 0,\end{aligned}$$

which gives that

$$\begin{aligned}\mathcal{L}(u,\xi,\Omega) &= -\int_\Omega \lim_{t\to0} t^{1-2s}\partial_t\tilde{u}\frac{\partial\tilde{\xi}}{\partial y^1}dy - \int_\Omega \lim_{t\to0} t^{1-2s}\partial_t\tilde{\xi}\frac{\partial\tilde{u}}{\partial y^1}dy + \int_{\partial\Omega} u\xi\nu_1 d\sigma\\
&= \int_\Omega \bigg(((-\Delta)^s u + u)\frac{\partial\xi}{\partial y^1} + ((-\Delta)^s\xi + \xi)\frac{\partial u}{\partial y^1}\bigg)dy\\
&= \int_\Omega \bigg(((-\Delta)^s u + u - u^p)\frac{\partial\xi}{\partial y^1} + ((-\Delta)^s\xi + \xi - pu^{p-1}\xi)\frac{\partial u}{\partial y^1}\bigg)dy + \int_{\partial\Omega} u^p\xi\nu_1 d\sigma,\end{aligned}$$

which implies (3.8).

**Step 2.** The main terms can be calculated directly as follows:

$$\mathcal{L}\bigg(\sum_{j=1}^k U_{x_j}, \sum_{i=1}^k \frac{\partial U_{x_i}}{\partial r}, \Omega\bigg) = (B' + o(1))\frac{1}{r^\alpha k^2} + o\bigg(\frac{1}{kr^{\alpha+1}}\bigg) \tag{3.9}$$

by observing that

$$\frac{\partial U_{x_j}}{\partial r} = -U'(|y-x_j|)\bigg\langle \frac{y-x_j}{|y-x_j|}, \bigg(\cos\frac{2(j-1)\pi}{k}, \sin\frac{2(j-1)\pi}{k}, 0\bigg)\bigg\rangle \tag{3.10}$$

and

$$(-\Delta)^s\frac{\partial U_{x_j}}{\partial r} + \frac{\partial U_{x_j}}{\partial r} - pU_{x_j}^{p-1}\frac{\partial U_{x_j}}{\partial r} = 0.$$

**Step 3.** We calculate the error terms as follows.

Firstly, in order to deal with $\mathcal{L}(\sum_{i=1}^k U_{x_i}, \xi_m^*, \Omega)$, we apply Lemma A.3 with $\beta = N + 2s - \theta < N$ and $t < \rho$, and deduce that on $y \in \partial''\Omega^+$ it holds that

$$\begin{aligned}\tilde{\xi}_m^*(y,t) &\leqslant C\sum_{i=1}^k \int_{\mathbb{R}^N} \frac{t^{2s}}{(|y-z|+t)^{N+2s}} \frac{1}{(1+|z-x_i|)^{N+2s-\theta}} dz \\ &\leqslant C\sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{N+2s-\theta}},\end{aligned}$$

and similarly,

$$\nabla\tilde{\xi}_m^*(y,t) \leqslant C\sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{N+2s-\theta}}.$$

In the following, we use (2.17), Lemma 3.3 and $L^p$-estimates for the fractional Laplacian operator [21]. We assume that if $p \in (1,2)$,

$$\alpha > \max\left\{\frac{8(N+2s)}{(p(p-1)+2)(N+2s)-4(1-s)}, \frac{(8-(p-1)(N+2s))(N+2s)}{2(N+4s-2)}\right\},$$

while if $2 \leqslant p < 3$,

$$\alpha > \max\left\{\frac{4(N+2s)}{p(N+2s)-2(1-s)}, \frac{(8-(p-1)(N+2s))(N+2s)}{2(N+4s-2)}\right\},$$

and if $p \geqslant 3$,

$$\alpha > \frac{4(N+2s)}{3(N+2s)-2(1-s)},$$

which can be satisfied by (1.9) and (1.10). By taking $\tau = 1 \geqslant \frac{N+2s-\alpha}{N+2s}$, we have

$$\begin{aligned}\mathcal{L}\bigg(\sum_{i=1}^k \frac{\partial U_{x_i}}{\partial r}, \xi_m^*, \Omega\bigg) &\leqslant C\|\xi_m^*\|_* \int_{\partial\Omega} \frac{1}{(1+|y-x_1|)^{2N+4s-\theta-2\tau}} d\sigma \\ &\leqslant C\|\xi_m^*\|_* \frac{1}{|x_1-x_2|^{2(N+2s)-\theta-2\tau}} \\ &\leqslant C\bigg(\frac{1}{r^\theta} + \frac{1}{r^{\alpha\min\{\frac{p}{2},1\}}}\bigg)^{\min\{\frac{p-1}{2},1\}} r^{-\alpha(\frac{2(N+2s)-\theta-2\tau}{N+2s} - \frac{N-1}{N+2s})} = o\bigg(\frac{1}{kr^{\alpha+1}}\bigg).\end{aligned}$$

Secondly, under the same condition, it holds similarly that

$$\mathcal{L}\bigg(\varphi_{k_m}, \sum_{i=1}^k \frac{\partial U_{x_i}}{\partial r}, \Omega\bigg) = o\bigg(\frac{1}{kr^{\alpha+1}}\bigg).$$

Thirdly, it holds that

$$\int_{\partial\Omega} (V(y)-1)u_k \xi_m \nu_1 d\sigma = o\bigg(\frac{1}{kr^{\alpha+1}}\bigg).$$

In fact, we assume that if $p \in (1,2)$,

$$\alpha > \max\left\{\frac{8(N+2s)}{(p(p+1)+4)(N+2s)-4(2s+1)}, \frac{(8-(p-1)(N+2s))(N+2s)}{2((p+2)(N+2s)-2(2s+1))}\right\};$$

if $2 \leqslant p < 3$,

$$\alpha > \frac{4(N+2s)}{(p+3)(N+2s)-2(2s+1)};$$

if $p \geqslant 3$,

$$\alpha > \frac{2(N+2s)}{3(N+2s)-(2s+1)},$$

which can also be satisfied by (1.9) and (1.10). Thus,

$$\begin{aligned}
&\int_{\partial\Omega}(V(y)-1)u_k\xi_m\nu_1 d\sigma \\
&\quad\leqslant C\bigg(\frac{1}{r^\theta}+\frac{1}{r^{\alpha\min\{\frac{p}{2},1\}}}\bigg)^{\min\{\frac{p+1}{2},2\}}\int_{\partial\Omega}(V(y)-1)\bigg(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{N+2s-\theta}}\bigg)^2\nu_1 d\sigma \\
&\quad\leqslant C\bigg(\frac{1}{r^\theta}+\frac{1}{r^{\alpha\min\{\frac{p}{2},1\}}}\bigg)^{\min\{\frac{p+1}{2},2\}}\frac{1}{r^\alpha}\int_{\partial\Omega}\frac{1}{(1+|y-x_1|)^{2(N+2s-\theta-\tau)}}\nu_1 d\sigma = o\bigg(\frac{1}{kr^{\alpha+1}}\bigg)
\end{aligned}$$

with $\tau = 1 > \frac{N+2s-\alpha}{N+2s}$.

Finally, it holds that

$$\begin{aligned}
\int_{\partial\Omega}u_k^p\xi_m\nu_1 d\sigma &\leqslant C\bigg(\frac{1}{r^\theta}+\frac{1}{r^{\alpha\min\{\frac{p}{2},1\}}}\bigg)^{\min\{\frac{3p-1}{2},p+1\}}\int_{\partial\Omega}\frac{1}{(1+|y-x_1|)^{(p+1)(N+2s-\theta-\tau)}}\nu_1 d\sigma \\
&= o\bigg(\frac{1}{kr^{\alpha+1}}\bigg).
\end{aligned} \tag{3.11}$$

In fact, the estimate holds if $p \in (1,2)$, $N > \max\{4s, 4-2s\}$ and

$$\begin{aligned}
&\max\bigg\{\frac{8(N+2s)}{p(3p-1)(N+2s)+2(p+1)(N-2)-8(N+s-1)}, \frac{(8-3(p-1)(N+2s))(N+2s)}{2(2p(N+s-1)-N-4s)}\bigg\} \\
&\quad < \alpha < \frac{((3p-1)(N+2s)-8)(N+2s)}{2(2p(N+s-1)-N-4s)},
\end{aligned}$$

while if $p \geqslant 2$,

$$\alpha > \frac{4(N+2s)}{(2p-1)N+3(p-1)s-(p+2)},$$

which again can be satisfied by (1.9) and (1.10).

**Step 4.** We combine (3.7) and (3.9)–(3.11) to obtain that

$$\int_\Omega u_k\xi_m\frac{\partial V}{\partial y_1}dy = b_m(B'+o(1))\frac{1}{r^\alpha k^2}+o\bigg(\frac{1}{kr^{\alpha+1}}\bigg). \tag{3.12}$$

On the other hand, from (2.17) and (3.6), we can also find

$$\int_\Omega u_k\xi_m V'(|y|)|y|^{-1}y^1 dy = b_m\int_{\mathbb{R}^N}U_{x_1}\frac{\partial U_{x_1}}{\partial y^1}V'(|y|)|y|^{-1}y^1 dy + \frac{1}{|x_1|^{\alpha+1}}o\bigg(\frac{1}{r^{\alpha+2}}\bigg).$$

Moreover, by the assumption (1.8),

$$\begin{aligned}
&\int_{\mathbb{R}^N}U_{x_1}\frac{\partial U_{x_1}}{\partial y^1}V'(|y|)|y|^{-1}y^1 dy \\
&\quad = \int_{\mathbb{R}^N+x_1}U\frac{\partial U}{\partial y^1}V'(|y+x_1|)|y+x_1|^{-1}(y^1+x_1^1)dy \\
&\quad = \int_{\mathbb{R}^N}U\frac{\partial U}{\partial y^1}\bigg(-\frac{\alpha a_1}{|y+x_1|^{\alpha+1}}-\frac{(\alpha+1)a_2}{|y+x_1|^{\alpha+2}}+O\bigg(\frac{1}{|y+x_1|^{m+3}}\bigg)\bigg)\frac{y^1+x_1^1}{|y+x_1|}dy \\
&\quad = \frac{\alpha(\alpha+1)a_1}{r^{\alpha+2}}\bigg(\int_{\mathbb{R}^N}UU'(|y|)(y^1)^2dy+o(1)\bigg).
\end{aligned} \tag{3.13}$$

**Step 5.** Combining (3.12) and (3.13), we obtain

$$b_m\frac{\alpha(\alpha+1)a_1}{r^{\alpha+2}}\bigg(\int_{\mathbb{R}^N}UU'(|y|)(y^1)^2dy+o(1)\bigg) = b_m(B'+o(1))\frac{1}{r^\alpha k^2}+o\bigg(\frac{1}{kr^{\alpha+1}}\bigg), \tag{3.14}$$

which gives $b_m \to 0$. □

*Proof of Theorem* 1.4. Firstly, by Lemma 3.4, $\bar{\xi}_m \to 0$ in $C_{\rm loc}$, implying that $\xi_m(y) \to 0$ in $B_R(x_{k,j})$ for any $j = 1, \dots, k$.

On the other hand, if $N \geqslant 2 - 2s + \theta$, which means

$$N + 2s - \theta - 1 > 1$$

and can be satisfied by $N \geqslant 4$, we can prove that (see (A.2))

$$\sum_{i=1}^{k} \frac{1}{(1+|x-x_i|)^{N+2s-\theta}} \leqslant CU(R) \quad \text{in } \mathbb{R}^N \setminus \bigcup_{j=1}^{k} B_R(x_{k,j}).$$

Then by Lemma 3.3, we have

$$|\xi_m(y)| \leqslant C\|\xi_m\|_* \sum_{i=1}^{k} \frac{1}{(1+|x-x_i|)^{N+2s-\theta}} \leqslant CU(R).$$

Hence, if we choose $R \gg 1$ sufficiently large, it holds that

$$|\xi_m(y)| \ll 1 \quad \text{in } \mathbb{R}^N \setminus \bigcup_{j=1}^{k} B_R(x_{k,j}).$$

To sum up we have seen that $\xi_m(y) = o(1)$ in $\mathbb{R}^N$, which is a contradiction to the assumption $\|\xi_m\|_{L^\infty} = 1$. □

## 4 Construction of new solutions

Let $u_k$ be the $k$-spike solutions obtained in Theorem 1.1, and $k$ be a large even integer. Then we know that $u_k$ is even in each $y_i$, $i = 1, \dots, N$. As mentioned before, $u_k$ is radial in $y'' = (y_3, y_4, \dots, y_N)$.

Now we take $n \geqslant k$ as a large even integer, and set

$$p_j = \left(0, 0, t\cos\frac{2(j-1)\pi}{n}, t\sin\frac{2(j-1)\pi}{n}, 0\right), \quad j = 1, \dots, n,$$

where

$$t \in [t_0 n^{\frac{N+2s}{N+2s-\alpha}}, t_1 n^{\frac{N+2s}{N+2s-\alpha}}].$$

Then we define

$$\begin{aligned} X_s = \bigg\{ & u \in H_s : u \text{ is even in } y_l, l = 1, \dots, N, \\ & u(y_1, y_2, t\cos\theta, t\sin\theta, y^*) = u\left(y_1, y_2, t\cos\left(\theta + \frac{2\pi j}{n}\right), t\sin\left(\theta + \frac{2\pi j}{n}\right), y^*\right), y^* = (y_5, \dots, y_N) \bigg\}. \end{aligned}$$

We notice that $u_k$ and $\sum_{j=1}^{n} U_{p_j}$, both of which belong to $X_s$, separate from each other. Let

$$D_j = \left\{ y = (y', y_3, y_4, y^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{N-4} : \left\langle \frac{(0,0,y_3,y_4,0,\dots,0)}{|(y_3,y_4)|}, \frac{p_j}{|p_j|} \right\rangle \geqslant \cos\frac{\pi}{n} \right\}.$$

We aim to construct a solution of the form

$$u = u_k + \sum_{j=1}^{n} U_{p_j} + \omega,$$

where $\omega \in X_s$ is the perturbation term. Define

$$Q_n v = (-\Delta)^s v + V(y)v - p\left(u_k + \sum_{j=1}^{n} U_{p_j}\right)^{p-1} v, \quad \forall\, v \in X_s.$$

Set $Y_j = \frac{\partial U_{p_j}}{\partial t}$, $j = 1, \ldots, n$. For $g_n \in X_s$, we consider the following linear problem:

$$\begin{cases} Q_n \omega_n = g_n + a_n \sum_{j=1}^{n} U_{p_j}^{p-1} Y_j, \\ \omega_n \in X_s, \\ \int_{\mathbb{R}^N} U_{p_j}^{p-1} Y_j \omega_n = 0, \quad j = 1, \ldots, n \end{cases} \tag{4.1}$$

with some constants $a_n$, depending on $\omega_n$.

To obtain a better control of the error term, we introduce the norm

$$\|u\|_{**} = \sup_{x \in \mathbb{R}^N} \left( \frac{1}{(1+|x|)^{N+2s-\theta}} + \sum_{i=1}^{n} \frac{1}{(1+|x-p_i|)^{N+2s-\theta}} \right)^{-1} |u(x)|. \tag{4.2}$$

As in Lemma 2.1, we first solve the linear problem (4.1).

**Lemma 4.1.** *Given $\theta \in (0, \frac{N}{2} + 2s)$, there exist $(\omega_n, a_n)$ to solve (4.1) with some $g_n \in X_s \cap C(\mathbb{R}^N)$.*

*Proof.* It suffices to prove the following *a priori* estimate by three steps:

$$\|\omega\|_{**} \leqslant C \|g_n\|_{**}. \tag{4.3}$$

**Step 1.** It follows from (4.1) that

$$\begin{aligned} a_n \sum_{j=1}^{k} \int_{\mathbb{R}^N} U_{p_j}^{p-1} Y_j Y_1 dy &= \int_{\mathbb{R}^N} ((-\Delta)^s \omega_n + V(y)\omega_n - p(u_k + W_t)^{p-1}\omega_n - g_n) Y_1 dy \\ &= \int_{\mathbb{R}^N} ((V(y) - 1) Y_1 \omega_n - p((u_k + W_t)^{p-1} - U_{p_1}^{p-1})\omega_n Y_1 - g_n Y_1) dy. \end{aligned}$$

Using the similar arguments as in the proof of Lemma 2.1, we can obtain that $a_n = o(1)$. In fact, here we use the following simple fact that

$$|x_i - p_j| \geqslant \max\{r, t\}.$$

**Step 2.** We show the *a priori* estimate by contradiction. Assume that there exist $g_n$ with $\|g_n\|_{**} \to 0$, $\|\omega_n\|_{**} = 1$ and

$$t = t_n \in [t_0 n^{\frac{N+2s}{N+2s-\alpha}}, t_1 n^{\frac{N+2s}{N+2s-\alpha}}].$$

We first claim that for any $R > 0$, it holds that

$$\|\omega_n\|_{L^\infty(B_R(0))} + \|\omega_n\|_{L^\infty(\bigcup_{j=1}^n B_R(p_j))} \to 0. \tag{4.4}$$

Indeed, it is standard to show

$$\|\omega_n\|_{L^\infty(\bigcup_{j=1}^n B_R(p_j))} \to 0.$$

Moreover, using the non-degeneracy theorem 1.4, we also conclude

$$\|\omega_n\|_{L^\infty(B_R(0))} \to 0.$$

In fact, assume that $\|\omega_n\|_{L^\infty(B_R(0))} \geqslant \gamma > 0$. Note that $\omega_n$ satisfies

$$(-\Delta)^s \omega + V(y)\omega - p\bigg(u_k + \sum_{j=1}^{n} U_{p_j}\bigg)^{p-1} \omega = \bar{g}_n, \tag{4.5}$$

where $\bar{g}_n = g_n + a_n \sum_{j=1}^n U_{p_j}^{p-1} Y_j$ uniformly converges to 0 on compact sets.

From the uniform Hölder estimates, we also obtain equicontinuity of the sequence $\omega_n$. Thus, passing to a subsequence, we may assume that $\omega_n$ converges, uniformly on compact sets, to a bounded function $\omega$ satisfying $\|\omega\|_{L^\infty(B_R(0))} \geqslant \gamma > 0$. Moreover, $\omega$ satisfies the equation

$$(-\Delta)^s\omega + V(y)\omega - pu_k^{p-1}\omega = 0.$$

The non-degeneracy of $u_k$ as in Theorem 1.4 yields that $\omega$ must be 0, which is a contradiction to the assumption $\|\omega_n\|_{L^\infty(B_R(0))} \geqslant \gamma > 0$. So we prove the claim (4.4).

**Step 3.** It holds that

$$\|\omega_n\|_{**} \leqslant C(\|\omega_n\|_{L^\infty(B_R(0))} + \|\omega_n\|_{L^\infty(\bigcup_{j=1}^n B_R(p_j))} + \|g_n\|_{**}). \tag{4.6}$$

In fact, (4.5) implies that $\omega_n$ satisfies

$$(-\Delta)^s\omega_n + W(y)\omega_n = \hat{g}_n,$$

where

$$W(y) = V(y) - p\bigg(u_k + \sum_{j=1}^n U_{p_j}\bigg)^{p-1}(y)$$

and

$$\hat{g}_n = g_n + a_n\sum_{j=1}^n U_{p_j}^{p-1}Y_j.$$

Taking any $R > 0$ satisfying $r \ll R \ll t$, and for any $l = 1, \dots, n$ and any $y \in D_l \setminus (B_R(0) \cup B_R(p_l))$, we have

$$\begin{aligned}\bigg(u_k + \sum_{j=1}^n U_{p_j}\bigg)(y) &\leqslant C\bigg(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{N+2s-\theta}} + \sum_{j=1}^n \frac{1}{(1+|y-p_j|)^{N+2s}}\bigg)\\ &\leqslant \frac{Ck}{R^{N+2s-\theta}} + \frac{C}{R^{N+2s}} + \sum_{j=2}^n \frac{C}{|p_1-p_j|^{N+2s}}.\end{aligned}$$

Thus,

$$\inf_{y\in\mathbb{R}^N\setminus\bigcup_{j=0}^n B_R(p_j)} W(y) > 0, \quad \|\hat{g}_n\|_{**} < \infty.$$

Following the argument in [9, Lemma 2.5], we obtain (4.6).

Combining Steps 2 and 3, we indeed obtain that $\|\omega_n\|_{**} \to 0$, which again leads to a contradiction and we complete the proof of (4.3).

Finally, applying (4.3), we can use the standard method to obtain the existence result of the linear problem. For the details, one can refer to [9]. □

Set

$$W_t = \sum_{j=1}^n U_{p_j}.$$

Now we aim to construct a solution $u$ of (1.1) with

$$u = u_k + \sum_{j=1}^n U_{p_j} + \omega,$$

where $\omega \in X_s$ is a small perturbed term satisfying

$$\sum_{j=1}^n U_{p_j}^{p-1}Y_j\omega = 0.$$

Thus, $\omega$ satisfies

$$Q_n\omega = l_n + R_n(\omega),$$

where

$$l_n = -\sum_{j=1}^{k}(V(y)-1)U_{p_j} + (u_k+W_t)^p - u_k^p - \sum_{j=1}^{n} U_{p_j}^p$$

and

$$R_n(\omega) = (u_k+W_t+\omega)^p - (u_k+W_t)^p - p(u_k+W_t)^{p-1}\omega.$$

**Lemma 4.2.** *Suppose that $V$ satisfies (1.8). If further $\alpha < \frac{\min\{p,2\}}{2}N$, then for $\theta \in (\frac{N}{2}+s, \frac{N}{2}+2s)$, it holds that*

$$\|l_n\|_{**} \leqslant C\left(\frac{1}{t^{\frac{\alpha}{2}+\delta}} + \frac{1}{t^{\theta}}\right) \tag{4.7}$$

*with some small $\delta > 0$.*

*Proof.* First, it holds that

$$\begin{aligned}\left|\sum_{j=1}^{n}(V(y)-1)U_{p_j}\right| &\leqslant \frac{C}{1+|y|^{\alpha}}\sum_{j=1}^{n}\frac{1}{(1+|y-p_j|)^{N+2s}} \\ &\leqslant C\begin{cases}\dfrac{1}{t^{\alpha}}\displaystyle\sum_{j=1}^{n}\frac{1}{(1+|y-p_j|)^{N+2s-\theta}}, & |y-p_{j_0}| < \dfrac{t}{2}, \quad \exists\, j_0 \in \{1,\dots,n\}, \\ \dfrac{1}{t^{\theta}}\displaystyle\sum_{j=1}^{n}\frac{1}{(1+|y-p_j|)^{N+2s-\theta}}, & |y-p_j| \geqslant \dfrac{t}{2}, \quad \forall\, j \in \{1,\dots,n\}.\end{cases}\end{aligned} \tag{4.8}$$

If $p \geqslant 2$, then for any $y \in D_1 \bigcap B_R^c(0)$ with some $R \gg r$, we have

$$|p_1 - x_i| = \sqrt{r^2+t^2} > t$$

for any $i = 1,\dots,k$. Thus $|y-x_j| \sim |y|$, and

$$\begin{aligned}&\left|(u_k+W_t)^p - u_k^p - \sum_{j=1}^{n}U_{p_j}^p\right| \\ &\quad\leqslant C\left(\sum_{j=2}^{n}U_{p_1}^{p-1}U_{p_j} + U_{p_1}^{p-1}|u_k| + U_{p_1}|u_k|^{p-1}\right) \\ &\quad\leqslant C\frac{1}{(1+|y-p_1|)^{N+2s-\theta}}\sum_{j=2}^{k}\frac{1}{|p_1-p_j|^{N+2s}} \\ &\qquad + C_k\sum_{j=2}^{k}\frac{1}{|p_1-x_j|^{N+2s-\theta}}\left(\frac{1}{(1+|y-p_1|)^{(N+2s)(p-1)}} + \frac{1}{(1+|y-x_j|)^{(N+2s)(p-1)}}\right) \\ &\quad\leqslant C\frac{1}{(1+|y-p_1|)^{N+2s-\theta}}\sum_{j=2}^{k}\frac{1}{|p_1-p_j|^{N+2s}} \\ &\qquad + C_k\sum_{j=2}^{k}\frac{1}{|p_1-x_j|^{N+2s-\theta}}\left(\frac{1}{(1+|y-p_1|)^{N+2s-\theta}} + \frac{1}{(1+|y|)^{N+2s-\theta}}\right) \\ &\quad\leqslant \frac{C}{t^{\frac{\alpha}{2}+\delta}}\left(\frac{1}{(1+|y-p_1|)^{N+2s-\theta}} + \frac{1}{(1+|y|)^{N+2s-\theta}}\right),\end{aligned} \tag{4.9}$$

where in the last inequality, we used $\frac{\alpha}{2} < N+2s-\theta$.

For any $y \in D_1 \cap B_R(0)$, $|y-p_1| \sim t$, and $|u_k| \leqslant C_0$ implies that

$$|u_k| \leqslant C_0(1+R)^{N+2s-\theta}\frac{1}{(1+|y|)^{N+2s-\theta}}.$$

Hence,

$$\begin{aligned}
&\left|(u_k+W_t)^p - u_k^p - \sum_{j=1}^n U_{p_j}^p\right| \\
&\quad\leqslant C\bigg(\sum_{j=2}^n U_{p_1}^{p-1}U_{p_j} + U_{p_1}^{p-1}|u_k| + U_{p_1}|u_k|^{p-1}\bigg) \\
&\quad\leqslant C\bigg(\sum_{j=2}^n \frac{1}{(1+|y-p_1|)^{(N+2s)(p-1)}}\frac{1}{(1+|y-p_j|)^{N+2s}} + \frac{C_0(1+R)^{N+2s-\theta}}{(1+|y|)^{N+2s-\theta}(1+|y-p_1|)^{N+2s}}\bigg) \\
&\quad\leqslant \frac{C}{t^{\frac{\alpha}{2}+\delta}}\bigg(\frac{1}{(1+|y-p_1|)^{(N+2s)(p-1)}} + \frac{1}{(1+|y|)^{N+2s}}\bigg) \\
&\quad\leqslant \frac{C}{t^{\frac{\alpha}{2}+\delta}}\bigg(\frac{1}{(1+|y-p_1|)^{N+2s-\theta}} + \frac{1}{(1+|y|)^{N+2s-\theta}}\bigg).
\end{aligned} \tag{4.10}$$

Now we consider the case $p<2$.

We take some $R \gg r$. Then for $\theta \in (\frac{N}{2}+s, \frac{N}{2}+2s)$ and $\theta - \frac{2-p}{2}(N+2s) > 0$, it holds for $y \in D_1 \cap B_R^c(0)$ that

$$\begin{aligned}
&\left|(u_k+W_t)^p - u_k^p - \sum_{j=1}^n U_{p_j}^p\right| \\
&\quad\leqslant C\bigg(\sum_{j=2}^n U_{p_1}^{\frac{p}{2}}U_{p_j}^{\frac{p}{2}} + U_{p_1}^{\frac{p}{2}}u_k^{\frac{p}{2}}\bigg) \\
&\quad\leqslant C\frac{1}{(1+|y-p_1|)^{N+2s-\theta}}\sum_{j=2}^n \frac{1}{(1+|y-p_1|)^{\theta-\frac{2-p}{2}(N+2s)}}\frac{1}{|p_1-p_j|^{\frac{p}{2}(N+2s)}} \\
&\qquad + C\sum_{j=1}^k \frac{1}{|p_1-x_j|^{\frac{p}{2}(N+2s-\theta)}}\bigg(\frac{1}{(1+|y-p_1|)^{\frac{p}{2}(N+2s)}} + \frac{1}{(1+|y-x_j|)^{\frac{p}{2}(N+2s)}}\bigg) \\
&\quad\leqslant C\bigg(\frac{1}{(1+|y-p_1|)^{N+2s-\theta}} + \frac{1}{(1+|y|)^{N+2s-\theta}}\bigg)\bigg(\sum_{j=2}^n \frac{1}{|p_1-p_j|^{\frac{p}{2}(N+2s)}} + \frac{1}{t^{\frac{p}{2}(N+2s-\theta)}}\bigg) \\
&\quad\leqslant \frac{C}{t^{\frac{\alpha}{2}+\delta}}\bigg(\frac{1}{(1+|y-p_1|)^{N+2s-\theta}} + \frac{1}{(1+|y|)^{N+2s-\theta}}\bigg),
\end{aligned} \tag{4.11}$$

where in the last inequality, we used

$$\frac{\alpha}{2} < \frac{p}{2}(N+2s-\theta)$$

again.

On the other hand, since $|u_k| \leqslant C_0$, which is independent of $k$, for $y \in D_1 \cap B_R(0)$, $|y-p_1| \sim t$, and $|u_k| \leqslant C_0$ implies that

$$|u_k| \leqslant C_0(1+R)^{N+2s-\theta}\frac{1}{(1+|y|)^{N+2s-\theta}}.$$

Hence, we obtain

$$\left|(u_k+W_t)^p - u_k^p - \sum_{j=1}^n U_{p_j}^p\right|$$

$$
\begin{aligned}
&\leqslant C\bigg(\sum_{j=2}^{n} U_{p_1}^{\frac{p}{2}} U_{p_j}^{\frac{p}{2}} + U_{p_1}^{\frac{p}{2}} u_k^{\frac{p}{2}}\bigg) \\
&\leqslant C\bigg(\sum_{j=2}^{n} \frac{1}{(1+|y-p_1|)^{\frac{p}{2}(N+2s)}} \frac{1}{(1+|y-p_j|)^{\frac{p}{2}(N+2s)}} \\
&\quad + C_0(1+R)^{N+2s-\theta} \frac{1}{(1+|y|)^{N+2s-\theta}} \frac{1}{(1+|y-p_1|)^{\frac{p}{2}(N+2s)}}\bigg) \\
&\leqslant C\bigg(\frac{1}{(1+|y-p_1|)^{N+2s-\theta}} + \frac{1}{(1+|y|)^{N+2s-\theta}}\bigg)\bigg(\sum_{j=2}^{n} \frac{1}{|p_1-p_j|^{\frac{p}{2}(N+2s)}} + \frac{1}{t^{\alpha}}\bigg) \\
&\leqslant \frac{C}{t^{\frac{\alpha}{2}+\delta}}\bigg(\frac{1}{(1+|y-p_1|)^{N+2s-\theta}} + \frac{1}{(1+|y|)^{N+2s-\theta}}\bigg).
\end{aligned} \tag{4.12}
$$

Combining (4.8)–(4.12), we obtain (4.7). □

By direct computation, it is easy to obtain the following estimate for $R_n(\omega)$.

**Lemma 4.3.** *It holds that*

$$\|R_n(\omega)\|_{**} \leqslant C\|\omega\|_{**}^{\min\{p,2\}}.$$

Now we consider the following problem:

$$
\begin{cases}
Q_n\omega_n = l_n + R_n(\omega) + a_n \displaystyle\sum_{j=1}^{n} U_{p_j}^{p-1} Y_j, \\
\omega_n \in X_s, \\
\displaystyle\sum_{j=1}^{n} \int_{\mathbb{R}^N} U_{p_j}^{p-1} Y_j \omega_n = 0.
\end{cases} \tag{4.13}
$$

Applying a standard method, we can prove the following proposition in view of Lemmas 4.1–4.3.

**Proposition 4.4.** *There exists an integer $n_0 > 0$ such that for each $n \geqslant n_0$ and*

$$t \in [t_0 n^{\frac{N+2s}{N+2s-\alpha}}, t_1 n^{\frac{N+2s}{N+2s-\alpha}}],$$

*the problem (4.13) has a solution $\omega_n$ for some constant $a_n$. Moreover, $\omega_n$ is a $C^1$ map from*

$$[t_0 n^{\frac{N+2s}{N+2s-\alpha}}, t_1 n^{\frac{N+2s}{N+2s-\alpha}}]$$

*to $X_s$, and*

$$\|\omega_n\|_{**} \leqslant C\bigg(\frac{1}{t^{\theta}} + \frac{1}{t^{\frac{\alpha}{2}+\delta}}\bigg).$$

Now we are ready to prove Theorem 1.5. Let

$$F(t) = I\bigg(u_k + \sum_{j=1}^{n} U_{p_j} + \omega_n\bigg).$$

Then to obtain a solution of (1.1) of the form

$$u = u_k + \sum_{j=1}^{n} U_{p_j} + \omega_n$$

is reduced to finding a critical point of $F(t)$ in $[t_0 n^{\frac{N+2s}{N+2s-\alpha}}, t_1 n^{\frac{N+2s}{N+2s-\alpha}}]$.

*Proof of Theorem* 1.5. First, we claim

$$F(t)=I\bigg(u_k+\sum_{j=1}^{n}U_{p_j}\bigg)+nO\bigg(\frac{1}{t^{\alpha+\sigma}}\bigg).$$

In fact, since

$$\bigg\langle I'\bigg(u_k+\sum_{j=1}^{n}U_{p_j}+\phi\bigg),\phi\bigg\rangle=0,\quad \forall\,\phi\in E_n$$

with

$$E_n=\bigg\{u\in X_s\cap C(\mathbb{R}^N):\sum_{j=1}^{n}\int_{\mathbb{R}^N}U_{p_j}^{p-1}Y_j u dy=0\bigg\},$$

we have

$$\begin{aligned}I\bigg(u_k+\sum_{j=1}^{n}U_{p_j}+\omega_n\bigg)&=I\bigg(u_k+\sum_{j=1}^{n}U_{p_j}\bigg)-\frac{1}{2}D^2I\bigg(u_k+\sum_{j=1}^{n}U_{p_j}+\zeta\omega_n\bigg)(\omega_n,\omega_n)\\&=I\bigg(u_k+\sum_{j=1}^{n}U_{p_j}\bigg)+O\bigg(\int_{\mathbb{R}^N}(|\omega_n|^{p+1}+|\omega_n|^3+|l_n\omega_n|+|R_n(\omega_n)\omega_n|)dy\bigg)\\&=I\bigg(u_k+\sum_{j=1}^{n}U_{p_j}\bigg)+O\bigg(\frac{n}{t^{\alpha+\sigma}}\bigg).\end{aligned}$$

Then we compute

$$\begin{aligned}I\bigg(u_k+\sum_{j=1}^{n}U_{p_j}\bigg)&=I\bigg(\sum_{j=1}^{n}U_{p_j}\bigg)+I(u_k)+\sum_{j=1}^{n}\int_{\mathbb{R}^N}u_k^pU_{p_j}dy\\&\quad-\frac{1}{p+1}\int_{\mathbb{R}^N}\bigg(\bigg(u_k+\sum_{j=1}^{n}U_{p_j}\bigg)^{p+1}-\bigg(\sum_{j=1}^{n}U_{p_j}\bigg)^{p+1}-(u_k)^{p+1}\bigg)dy.\end{aligned}$$

Since $\alpha<\min\{\frac{Np}{2},N+2s\}<\min\{p(N+2s-\theta),N+2s\}$, we have

$$\begin{aligned}\sum_{j=1}^{n}\int_{\mathbb{R}^N}u_k^pU_{p_j}dy&\leqslant C\sum_{j=1}^{n}\int_{\mathbb{R}^N}\bigg(\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{N+2s-\theta}}\bigg)^p\frac{1}{(1+|y-p_j|)^{N+2s}}dy\\&=O\bigg(\frac{n}{t^{\alpha+\sigma}}\bigg),\end{aligned}$$

and for some small $\tau>0$,

$$\begin{aligned}\bigg|\int_{\mathbb{R}^N}(u_k^pW_t+u_kW_t^p)dy\bigg|&=n\bigg|\int_{D_1}(u_k^pW_t+u_kW_t^p)dy\bigg|\\&\leqslant Cn\int_{D_1}\bigg(\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{N+2s-\theta}}\bigg)^p\frac{1}{(1+|y-p_1|)^{N+2s-\tau}}dy\\&\quad+Cn\int_{D_1}\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{N+2s-\theta}}\frac{1}{(1+|y-p_1|)^{p(N+2s-\tau)}}dy\\&=O\bigg(\frac{n}{t^{\alpha+\sigma}}\bigg).\end{aligned}$$

Then we see that

$$\sum_{j=1}^{n}\int_{\mathbb{R}^N}u_k^pU_{p_j}dy-\frac{1}{p+1}\int_{\mathbb{R}^N}\bigg(\bigg(u_k+\sum_{j=1}^{n}U_{p_j}\bigg)^{p+1}-\bigg(\sum_{j=1}^{n}U_{p_j}\bigg)^{p+1}-(u_k)^{p+1}\bigg)dy=O\bigg(\frac{n}{t^{\alpha+\sigma}}\bigg),$$

which gives

$$I\bigg(u_k+\sum_{j=1}^{n}U_{p_j}\bigg)=I\bigg(\sum_{j=1}^{n}U_{p_j}\bigg)+I(u_k)+O\bigg(\frac{n}{t^{\alpha+\sigma}}\bigg).$$

Hence, we find that

$$\begin{aligned}F(t)&=I\bigg(\sum_{j=1}^{n}U_{p_j}\bigg)+I(u_k)+O\bigg(\frac{n}{t^{\alpha+\sigma}}\bigg)\\&=I(u_k)+nA+n\bigg(\frac{B}{t^{\alpha}}-\frac{Dn^{N+2s}}{t^{N+2s}}\bigg)+O\bigg(\frac{n}{t^{\alpha+\sigma}}\bigg),\end{aligned}$$

where $A$, $B$ and $D$ are positive constants. Finally, the critical point for $F(t)$ can be obtained by using the method similar to that in [22]. □

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## Appendix A Some essential estimates

Recalling that $|x_j - x_1| = 2r\sin\frac{(j-1)\pi}{k}$, $j=1,\ldots,k$ for any $\lambda > \frac{N+2s-\alpha}{N+2s}$, we have

$$
\begin{aligned}
\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\lambda}} &= \frac{1}{(2r)^{\lambda}}\sum_{j=2}^{k}\frac{1}{\sin^{\lambda}\frac{(j-1)\pi}{k}}\\
&= \begin{cases} \dfrac{2}{(2r)^{\lambda}}\displaystyle\sum_{j=2}^{\frac{k}{2}}\frac{1}{\sin^{\lambda}\frac{(j-1)\pi}{k}} + \frac{1}{(2r)^{\lambda}}, & \text{if } k \text{ is even},\\ \dfrac{2}{(2r)^{\lambda}}\displaystyle\sum_{j=2}^{[\frac{k}{2}]}\frac{1}{\sin^{\lambda}\frac{(j-1)\pi}{k}}, & \text{if } k \text{ is odd} \end{cases}\\
&\leqslant C\left(\frac{k}{r}\right)^{\lambda}\sum_{j=1}^{k}\frac{1}{j^{\lambda}} \leqslant C\begin{cases} \left(\dfrac{k}{r}\right)^{\lambda}, & \lambda > 1,\\ \dfrac{k\ln k}{r^{\lambda}}, & \lambda \leqslant 1. \end{cases}
\end{aligned} \tag{A.1}
$$

**Lemma A.1** (See [28]). *For any positive constant $\sigma \leqslant \min\{\alpha,\beta\}$, there exists some constant $C>0$ such that*

$$
\frac{1}{(1+|y-x_i|)^{\alpha}}\frac{1}{(1+|y-x_j|)^{\beta}} \leqslant \frac{C}{|x_i-x_j|^{\sigma}}\left(\frac{1}{(1+|y-x_i|)^{\alpha+\beta-\sigma}} + \frac{1}{(1+|y-x_j|)^{\alpha+\beta-\sigma}}\right).
$$

**Lemma A.2** (See [19]). *For any $x \in \Omega_1$ and $\eta \in (1, N+2s]$, there is a constant $C>0$ such that*

$$
\sum_{i=2}^{k} U_{x_i} \leqslant C\frac{1}{(1+|y-x_1|)^{N+2s-\eta}}\frac{k^{\eta}}{|x_1|^{\eta}}.
$$

**Lemma A.3** (See [20]). *Let $\rho > \theta > 0$ be two constants. Suppose $(y-x)^2+t^2 \geqslant \rho^2, t>0$ and $\alpha > N$. Then when $0<\beta<N$, it holds that*

$$
\int_{\mathbb{R}^N}\frac{1}{(t+|z|)^{\alpha}(1+|y-z-x|)^{\beta}}dz \leqslant C\left(\frac{1}{(1+|y-x|)^{\beta}}\frac{1}{t^{\alpha-N}} + \frac{1}{(1+|y-x|)^{\alpha+\beta-N}}\right)
$$

*and*

$$\int_{\mathbb{R}^N\setminus B_\theta(0)} \frac{dz}{(t+|z|)^\alpha(1+|y-z-x|)^\beta} \leqslant \frac{C}{(1+|y-x|)^\beta}\frac{1}{\theta^{\alpha-N}};$$

*when* $N<\beta$, *it holds that*

$$\int_{\mathbb{R}^N\setminus B_\theta(y-x)} \frac{dz}{(t+|z|)^\alpha(1+|y-z-x|)^\beta} \leqslant C\left(\frac{1}{(1+|y-x|)^\beta}\frac{1}{t^{\alpha-N}}+\frac{1}{(1+|y-x|)^\beta}\frac{1}{\theta^{\beta-N}}\right),$$

*where* $C>0$ *is a constant independent of* $\theta$.

Some technical difficulties arise when the number of spikes goes to infinity. To deal with these difficulties, we use the following lemma.

**Lemma A.4** (See [10]). *For all* $y\in\mathbb{R}^N$ *and all* $l\in\mathbb{N}$, *it holds that*

$$\sharp\left\{x_j : \frac{l\rho}{2}\leqslant|x_j-y|\leqslant\frac{(l+1)\rho}{2}\right\}\leqslant 6(l+1),$$

*where* $\rho=|x_1-x_2|$.

Applying this lemma, we estimate that for $y\in\mathbb{R}^N\setminus\bigcup_{j=1}^k B_R(x_j)$,

$$\begin{aligned}
&\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{N+2s-\theta}}\\
&\quad\leqslant \sum_{\{j:|x_j-y|<\frac{\rho}{2}\}}\frac{1}{(1+|y-x_j|)^{N+2s-\theta}}+\sum_{l=1}^\infty\sum_{\{j:\frac{l\rho}{2}\leqslant|x_j-y|<\frac{(l+1)\rho}{2}\}}\frac{1}{(1+|y-x_j|)^{N+2s-\theta}}\\
&\quad\leqslant CU(R)+C\sum_{l=1}^\infty(l+1)\frac{1}{(\rho l)^{N+2s-\theta}}\leqslant CU(R).
\end{aligned}\tag{A.2}$$