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A stabilizer-free C^0 weak Galerkin method for the biharmonic equations

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Abstract In this article, we present and analyze a stabilizer-free C^0 weak Galerkin (SF-C0WG) method for solving the biharmonic problem. The SF-C0WG method is formulated in terms of cell unknowns which are C^0 continuous piecewise polynomials of degree $k+2$ with $k \geqslant 0$ and in terms of face unknowns which are discontinuous piecewise polynomials of degree $k+1$. The formulation of this SF-C0WG method is without the stabilized or penalty term and is as simple as the C^1 conforming finite element scheme of the biharmonic problem. Optimal order error estimates in a discrete H^2-like norm and the H^1 norm for $k \geqslant 0$ are established for the corresponding WG finite element solutions. Error estimates in the L^2 norm are also derived with an optimal order of convergence for $k > 0$ and sub-optimal order of convergence for $k = 0$. Numerical experiments are shown to confirm the theoretical results.

Keywords weak Galerkin, finite element method, weak Laplacian, biharmonic equations

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1 Introduction

We consider the biharmonic equation of the form

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{1.1a}$$

$$u = g_D \quad \text{on } \Gamma, \tag{1.1b}$$

$$\frac{\partial u}{\partial \boldsymbol{n}} = g_N \quad \text{on } \Gamma, \tag{1.1c}$$

where Ω is a bounded polytopal domain in $\mathbb{R}^2$ and $\Gamma = \partial\Omega$.

In the case of homogeneous boundary conditions $g_D = g_N = 0$, the variational form of the problem (1.1a)–(1.1c) reads as: find $u \in H_0^2(\Omega)$ such that

$$(\Delta u, \Delta v) = (f, v), \quad \forall\, v \in H_0^2(\Omega), \tag{1.2}$$

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where $H_0^2(\Omega)$ is the subspace of $H^2(\Omega)$ consisting of functions with the vanishing value and the normal derivative on $\partial\Omega$.

For the case of nonhomogeneous boundary conditions, assume that g_D and g_N are the Dirichlet boundary data of some function in $H^2(\Omega)$, i.e., there exists $\psi \in H^2(\Omega)$ such that

$$\begin{aligned}
&\Delta^2\psi = 0 \quad \text{in } \Omega,\\
&\psi = g_D \quad \text{on } \Gamma,\\
&\frac{\partial\psi}{\partial \boldsymbol{n}} = g_N \quad \text{on } \Gamma.
\end{aligned}$$

Then by setting $\widetilde{u} = u - \psi$, we arrive at the weak form (1.2) for $\widetilde{u}$. Therefore for brevity, but without loss of generality, we assume homogeneous boundary conditions in the remainder of this paper.

It is well known that H^2-conforming finite element methods for the problem (1.1a)–(1.1c) involve C^1 finite elements, which are of complex implementation and contain high order polynomials even in two dimensions. For example, Argyris and Bell finite elements have 21 and 18 degrees of freedom per triangle, respectively.

In order to avoid the use of such C^1 elements, nonconforming finite elements have been used to solve biharmonic problems. Morley element [12] is one of the most popular nonconforming finite elements for the biharmonic equations, which only uses quadratic piecewise polynomials on triangle elements in two-dimensional domains and does not need any stabilization along mesh interfaces. However, it cannot be generalized to arbitrarily high order polynomials.

Discontinuous Galerkin (DG) approaches can also be applied to the biharmonic problems. The first discontinuous Galerkin method—the interior penalty method for the fourth order PDE was presented in [2], which uses fully discontinuous piecewise polynomials as basis functions. A nonsymmetric version of the interior penalty method was proposed and analyzed in [13]. Although the DG methods have the advantage of using arbitrarily high order elements, they also have some disadvantages. The weak forms are more complicated than those used for conforming and nonconforming finite element methods. The discrete linear system of the DG method is large because it has a large number of degrees of freedom. To reduce the degrees of freedom of DG methods, C^0 interior penalty (C0IP) methods have been proposed for the fourth order PDEs first in [6] and then analyzed in [4], where the simple Lagrange elements are used and the continuity of the function derivatives are weakly enforced by stabilization terms on interior edges. However, the C0IP methods still have the disadvantage of the complex weak form and the need for the penalty parameters.

Another approach to avoid the use of C^1 elements is the mixed methods [1, 7, 11], which reduces the biharmonic problem to a system of two second order elliptic problems. One of the main drawbacks of the mixed formulation is that the mixed method leads to the saddle-point linear system, which causes difficulty in efficiently solving the linear algebra system.

The weak Galerkin (WG) finite element method was first introduced for the second order elliptic problems in [22]. One of its main characteristics is the use of the concept of weak functions and its weak derivatives. The classical differential operators, such as the gradient and the Laplacian, are approximated by the weak differential operator defined as distributions, which are further approximated by piecewise polynomials. These weakly defined functions and differential operators make the WG methods highly flexible in choosing finite element spaces and using polytopal meshes. In recent years, the WG method has been a focus of great interest in the scientific community. Several WG methods have been developed to solve a wide variety of partial differential equations (see, e.g., [8–10, 15, 17, 18, 23]). Especially, there are some works [14, 16, 19–21, 26, 27] for biharmonic equations. Compared with the DG methods, there is no penalty parameters needed to tune in the formulation of WG methods. Similar to the DG methods, the WG methods also involve stabilization along mesh skeleton, which makes the implementation of DG and WG methods more complex than the ones of conforming and nonconforming finite element methods.

Most recently, a new WG method without the stabilizer term was presented for the second order elliptic problems in [24], where we can remove the stabilization and pay the price in the form of using high enough degree of polynomials in the definition of the weak gradient. The resulting numerical scheme is as simple as the conforming finite element scheme and it is easy to implement. The idea has been extended to the biharmonic equations in [25], where a stabilizer-free WG (SFWG) method has been proposed which uses full discontinuous piecewise polynomials of degrees $k+2$, $k+2$ and $k+1$ with $k \geqslant 0$, respectively, for discretization of the unknown solution u, the trace of u and the trace of the normal derivative $\frac{\partial u}{\partial n}$ on the skeleton of the mesh. For the triangular mesh, the minimum degree of polynomials used for the computation of the weak Laplacian is $k+7$ in theory and is $k+4$ in practical computation. As it is pointed out in [25], it is a challenging task to compute the weak Laplacian and its numerical integration when the degree of polynomials used in the computation of weak Laplacian is very high.

In this paper, we present and analyze a stabilizer-free C^0 weak Galerkin method to approximate the solutions of the biharmonic problem (1.1a)–(1.1c). The method is formulated in terms of face unknowns which are discontinuous piecewise polynomials of degree $k+1$ with $k \geqslant 0$ and in terms of cell unknowns which are C^0 continuous piecewise polynomials of degree $k+2$. We have proved that for the triangular mesh, it is enough to take $k+3$ as the degree of polynomials used in the computation of weak Laplacian. In comparison with the SFWG method [25], the SF-C0WG methods in this paper involve fewer degrees of freedom because nodal values are shared on inter-element boundaries.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and the formulation of our SF-C0WG method and the related methods. Two energy-like norms and their equivalence and the well-posedness of the SF-C0WG method are discussed in Section 3. Then, in Section 4, we derive an error equation which plays an important role in our error estimates. The error analysis of our SF-C0WG method for the H^2-like norm and the L^2 and H^1 norms are established in Sections 5 and 6, respectively. Finally, in Section 7, we report some numerical experiment results to confirm the theoretical analysis developed.

2 Weak Galerkin finite element methods

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of the domain Ω. Denote by $\mathcal{E}_h$ the set of all the edges in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \Gamma$ be the set of all the interior edges.

For convenience, we adopt the following notations:

$$\begin{aligned}(v,w)_{\mathcal{T}_h} &= \sum_{K\in\mathcal{T}_h}(v,w)_K = \sum_{K\in\mathcal{T}_h}\int_K vw d\boldsymbol{x},\\ \langle v,w\rangle_{\partial\mathcal{T}_h} &= \sum_{K\in\mathcal{T}_h}\langle v,w\rangle_{\partial K} = \sum_{K\in\mathcal{T}_h}\int_{\partial K} vw ds.\end{aligned}$$

For any nonnegative integer m, let $\mathbb{P}_m(D)$ denote the set of polynomials defined on D with degree no more than m, where D may be an element K of $\mathcal{T}_h$ or an edge e of $\mathcal{E}_h$. In what follows, we often consider the broken polynomial spaces

$$\mathbb{P}_m(\mathcal{T}_h) := \{v \in L^2(\Omega) : v\,|_K \in \mathbb{P}_m(K), \forall\, K \in \mathcal{T}_h\}$$

and

$$\mathbb{P}_m(\mathcal{E}_h) := \{v \in L^2(\mathcal{E}_h) : v\,|_e \in \mathbb{P}_m(e), \forall\, e \in \mathcal{E}_h\}.$$

First of all, we introduce a set of normal directions on $\mathcal{E}_h$ as follows:

$$\mathcal{D}_h = \{\boldsymbol{n}_e : \boldsymbol{n}_e \text{ is unit and normal to } e,\ e \in \mathcal{E}_h\}. \tag{2.1}$$

Then a weak Galerkin finite element space V_h for $k \geqslant 0$ is defined by

$$V_h = \{v = \{v_0, v_n\boldsymbol{n}_e\} : v_0 \in S_h, v_n \in \mathbb{P}_{k+1}(\mathcal{E}_h)\} \tag{2.2}$$

with

$$S_h = \{w \in H_0^1(\Omega) : w\,|_K \in \mathbb{P}_{k+2}(K), \forall\, K \in \mathcal{T}_h\}, \tag{2.3}$$

where v_n can be viewed as an approximation of $\frac{\partial v_0}{\partial \boldsymbol{n}_e} := \nabla v_0 \cdot \boldsymbol{n}_e$.

Denote by V_h^0 a subspace of V_h with vanishing traces, i.e.,

$$V_h^0 = \{v = \{v_0, v_n \boldsymbol{n}_e\} \in V_h,\, v_n|_e = 0,\, e \subset \partial K \cap \Gamma\}. \tag{2.4}$$

Definition 2.1 (Weak Laplacian). For any function $v = \{v_0, v_n \boldsymbol{n}_e\} \in V_h$, its weak Laplacian $\Delta_{w,m} v$ is piecewisely defined as the unique polynomial $(\Delta_{w,m} v)|_K \in \mathbb{P}_m(K)$ such that

$$(\Delta_{w,m} v, \varphi)_K = -(\nabla v_0, \nabla \varphi)_K + \langle v_n \boldsymbol{n}_e \cdot \boldsymbol{n}, \varphi \rangle_{\partial K}, \quad \forall\, \varphi \in \mathbb{P}_m(K) \tag{2.5}$$

for any $K \in \mathcal{T}_h$.

Now, we are ready to present our stabilizer-free C^0 weak Galerkin finite element method for the biharmonic problem (1.1a)–(1.1c).

Method 1 (SF-C0WG method). The stabilizer-free C^0 weak Galerkin finite element scheme for solving the problem (1.1a)–(1.1c) is defined as follows: find $u_h = \{u_0, u_n \boldsymbol{n}_e\} \in V_h^0$ such that

$$\mathcal{A}_h(u_h, v_h) = (f, v_0), \quad \forall\, v_h = \{v_0, v_n \boldsymbol{n}_e\} \in V_h^0, \tag{2.6}$$

where the bilinear form $a_h(\cdot,\cdot)$ is defined by

$$\mathcal{A}_h(v, w) := (\Delta_{w,k+3} v, \Delta_{w,k+3} w)_{\mathcal{T}_h}, \quad \forall\, v, w \in V_h.$$

Remark 2.2. By using the same WG finite element space V_h^0 defined by (2.4), a C^0 weak Galerkin finite element method has been presented in [16], which is stated as follows:

Method 2 (C0WG method). The C^0 weak Galerkin finite element scheme for solving the problem (1.1a)–(1.1c) is defined as follows: find $u_h = \{u_0, u_n \boldsymbol{n}_e\} \in V_h^0$ such that

$$\mathcal{A}_{wg}(u_h, v_h) = (f, v_0), \quad \forall\, v_h = \{v_0, v_n \boldsymbol{n}_e\} \in V_h^0, \tag{2.7}$$

where the bilinear form $a_h(\cdot,\cdot)$ is defined by

$$\mathcal{A}_{wg}(v, w) := (\Delta_{w,k} v, \Delta_{w,k} w)_{\mathcal{T}_h} + s_h(v, w), \quad \forall\, v, w \in V_h$$

with the stabilizer term

$$s_h(v, w) = \sum_{K \in \mathcal{T}_h} h_K^{-1} \left\langle \frac{\partial v_0}{\partial \boldsymbol{n}_e} - v_n, \frac{\partial w_0}{\partial \boldsymbol{n}_e} - w_n \right\rangle_{\partial K}, \quad \forall\, v, w \in V_h.$$

From the formulation of the SF-C0WG method (2.6) and the C0WG method (2.7), we can see that the SF-C0WG method is obtained by removing the stabilizer $s_h(\cdot,\cdot)$ in the C0WG method via raising the degree of polynomials used in the definition of the weak Laplacian from k to $k+3$. A comparison of numerical performance of both WG methods is discussed in Section 7.

Remark 2.3. By using the C^0 conforming finite element space S_h defined by (2.3), a C^0 interior penalty method has been presented in [4, 6], which is stated as follows:

Method 3 (C0IP method). The C^0 interior penalty method for solving the problem (1.1a)–(1.1c) is defined as follows: find $u_h \in S_h$ such that

$$\mathcal{A}_{dg}(u_h, v_h) = (f, v_h), \quad \forall\, v_h \in S_h, \tag{2.8}$$

where the bilinear form $\mathcal{A}_{dg}(\cdot,\cdot)$ is defined as follows: for any $v, w \in S_h$,

$$\mathcal{A}_{dg}(v, w) := (D^2 v, D^2 w)_{\mathcal{T}_h} - \left\langle [\![\nabla v]\!], \left\{\!\!\left\{ \frac{\partial^2 w}{\partial \boldsymbol{n}_e^2} \right\}\!\!\right\} \right\rangle_{\mathcal{E}_h} - \left\langle [\![\nabla w]\!], \left\{\!\!\left\{ \frac{\partial^2 v}{\partial \boldsymbol{n}_e^2} \right\}\!\!\right\} \right\rangle_{\mathcal{E}_h} + j_h(v, w)$$

with the stabilizer term

$$j_h(v, w) = \sum_{e \in \mathcal{E}_h} \eta h_e^{-1} \langle [\![\nabla v]\!], [\![\nabla w]\!] \rangle_e, \quad \forall\, v, w \in S_h.$$

Here, the penalty parameter η is a positive constant.

For any $v \in H^2(\mathcal{T}_h)$, the jump $[\![\nabla v]\!]$ and the average $\{\!\!\{\frac{\partial^2 v}{\partial \boldsymbol{n}_e^2}\}\!\!\}$ are defined as follows.

Let $e \in \mathcal{E}_h^0$ be the common edge of K_1 and K_2 of $\mathcal{T}_h$ and denote by $\boldsymbol{n}_i$ $(i = 1, 2)$ the outward unit normal vector of the boundary ∂K_i $(i = 1, 2)$. We define on the edge e:

$$\left\{\!\!\left\{\frac{\partial^2 v}{\partial \boldsymbol{n}_e^2}\right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial^2 v_1}{\partial \boldsymbol{n}_e^2} + \frac{\partial^2 v_2}{\partial \boldsymbol{n}_e^2}\right) \quad \text{and} \quad [\![\nabla v]\!] = \nabla v_1 \cdot \boldsymbol{n}_1 + \nabla v_2 \cdot \boldsymbol{n}_2,$$

where $v_i = v\,|_{K_i}$ $(i = 1, 2)$. On a boundary edge $e \subset \partial\Omega$, we simply take $\{\!\!\{\frac{\partial^2 v}{\partial \boldsymbol{n}_e^2}\}\!\!\} = \frac{\partial^2 v}{\partial \boldsymbol{n}_e^2}$ and $[\![\nabla v]\!] = \nabla v \cdot \boldsymbol{n}$.

Compared with the C0IP method (2.8), our SF-C0WG method (2.6) has a simple formulation without any integration term on the edges of $\mathcal{E}_h$, which will simplify the implementation of the corresponding numerical scheme and reduce the assembling time of the stiffness matrix. Although the SF-C0WG method (2.6) has more degrees of freedom than the C0IP method (2.8), numerical experiments in Section 7 indicate that its total computational time is less than that of the C0IP method (2.8).

3 Well-posedness

For simplicity of notation, from now on we shall drop the subscript $k + 3$ in the notation $\Delta_{w,k+3}$ for the discrete weak Laplacian.

In order to analyze the SF-C0WG method (2.6), we introduce two H^2-like norms $|\!|\!| \cdot |\!|\!|$ and $\| \cdot \|_{2,h}$ over V_h^0 by

$$|\!|\!| v |\!|\!| = \left[\sum_{K \in \mathcal{T}_h} \|\Delta_w v\|^2_{L^2(K)}\right]^{1/2} \tag{3.1}$$

and

$$\|v\|_{2,h} = \left[\sum_{K \in \mathcal{T}_h} \left(\|\Delta v_0\|^2_{L^2(K)} + h_K^{-1}\left\|\frac{\partial v_0}{\partial \boldsymbol{n}_e} - v_n\right\|^2_{L^2(\partial K)}\right)\right]^{1/2} \tag{3.2}$$

for all $v \in V_h^0$. Obviously, $\| \cdot \|_{2,h}$ is indeed a norm on V_h^0. We show that $|\!|\!| \cdot |\!|\!|$ is also a norm by proving that the norms $\| \cdot \|_{2,h}$ and $|\!|\!| \cdot |\!|\!|$ are equivalent on the finite element space V_h^0 in Lemma 3.2.

In what follows, the trace inequality is a frequently used analysis tool, which states as [23]: for any function $\phi \in H^1(K)$, it holds that

$$\|\phi\|^2_{L^2(\partial K)} \leqslant C(h_K^{-1}\|\phi\|^2_{L^2(K)} + h_K\|\nabla\phi\|^2_{L^2(K)}). \tag{3.3}$$

The following lemma plays a key role in the proof of Lemma 3.2.

Lemma 3.1. *For any $v = \{v_0, v_n\boldsymbol{n}_e\} \in V_h$ and $K \in \mathcal{T}_h$, there exists a polynomial $\varphi \in \mathbb{P}_{k+3}(K)$ such that*

$$(\Delta v_0, \varphi)_K = 0, \quad \langle(\nabla v_0 - v_n\boldsymbol{n}_e) \cdot \boldsymbol{n}, \varphi\rangle_{\partial K} = \|(\nabla v_0 - v_n\boldsymbol{n}_e) \cdot \boldsymbol{n}\|^2_{L^2(\partial K)}$$

and

$$\|\varphi\|_{L^2(K)} \leqslant Ch_K^{1/2}\|(\nabla v_0 - v_n\boldsymbol{n}_e) \cdot \boldsymbol{n}\|_{L^2(\partial K)}. \tag{3.4}$$

Proof. For any $K \in \mathcal{T}_h$, let e_i $(i = 1, 2, 3)$ be the three edges of K, and λ_i's be the barycentric coordinates of K. Then we define a polynomial $\varphi_i \in \mathbb{P}_{k+3}(K)$ for $i = 1, 2, 3$, respectively, by requiring that

$$\varphi_i = \prod_{j=1, j\neq i}^{3} \lambda_j q \tag{3.5}$$

with $q \in \mathbb{P}_{k+1}(K)$ and such that

$$\langle\varphi_i, \tau\rangle_{e_i} = \langle(\nabla v_0 - v_n\boldsymbol{n}_e) \cdot \boldsymbol{n}, \tau\rangle_{e_i}, \quad \forall\, \tau \in \mathbb{P}_{k+1}(e_i), \tag{3.6a}$$

$$(\varphi_i, \tau)_K = 0, \quad \forall\, \tau \in \mathbb{P}_k(K). \tag{3.6b}$$

Since there are

$$(k+2) + \frac{1}{2}(k+1)(k+2) = \frac{1}{2}(k+2)(k+3)$$

equations and the same number of unknowns in the linear system (3.6a)–(3.6b), the existence and uniqueness of φ_i are equivalent.

Assume that both φ_i and $\widehat{\varphi}_i$ satisfy the linear system (3.6a)–(3.6b). We prove their difference $d_i = \varphi_i - \widehat{\varphi}_i$ vanishes on K. From (3.5)–(3.6b), we know that d_i can be expressed as

$$d_i = \prod_{j=1, j\neq i}^{3} \lambda_j \widetilde{q} \tag{3.7}$$

with $\widetilde{q} \in \mathbb{P}_{k+1}(K)$ and satisfies the following conditions:

$$\langle d_i, \tau\rangle_{e_i} = 0, \quad \forall\, \tau \in \mathbb{P}_{k+1}(e_i), \tag{3.8a}$$

$$(d_i, \tau)_K = 0, \quad \forall\, \tau \in \mathbb{P}_k(K). \tag{3.8b}$$

It follows from (3.8a) that $d_i = 0$ on e_i, which together with (3.7) implies that $\widetilde{q}$ in (3.7) can be written as $\widetilde{q} = \lambda_i \omega$ with $\omega \in \mathbb{P}_k(K)$. Therefore, we have

$$d_i = \prod_{j=1}^{3} \lambda_j \omega \quad \text{with } \omega \in \mathbb{P}_k(K),$$

which combining (3.8b) implies $d_i = 0$ on K.

Hence, the linear system (3.6a)–(3.6b) has a unique solution φ_i in the form of (3.5), which belongs to $\mathbb{P}_{k+3}(K)$.

Then by a scaling argument, we have

$$\|\varphi_i\|_{L^2(K)} \leqslant C h_K^{1/2} \|\varphi_i\|_{L^2(\partial K)}.$$

Thanks to (3.5), it is known that $\varphi_i = 0$ on e_j for $j \neq i$. Then $\|\varphi_i\|_{L^2(\partial K)} = \|\varphi_i\|_{L^2(e_i)}$. Therefore,

$$\|\varphi_i\|_{L^2(K)} \leqslant C h_K^{1/2} \|\varphi_i\|_{L^2(e_i)}. \tag{3.9}$$

Let $\theta_i(x) = \prod_{j=1, j\neq i}^{3} \lambda_j(x)$. Using (3.5) and (3.6a), we have

$$\langle \theta_i q, \tau\rangle_{e_i} = \langle \varphi_i, \tau\rangle_{e_i} \leqslant \|(\nabla v_0 - v_n \boldsymbol{n}_e)\cdot \boldsymbol{n}\|_{L^2(e_i)} \|\tau\|_{L^2(e_i)}.$$

By taking $\tau = q$ in the above inequality, and by the second mean value theorem of integrals, we have that there exists a point $\varepsilon_1 \in e_i$ such that

$$\theta_i(\varepsilon_1)\|q\|^2_{L^2(e_i)} = \langle \theta_i q, q\rangle_{e_i} \leqslant \|(\nabla v_0 - v_n \boldsymbol{n}_e)\cdot \boldsymbol{n}\|_{L^2(e_i)} \|q\|_{L^2(e_i)}.$$

Then after cancelling $\|q\|_{L^2(e_i)}$, we obtain

$$\|q\|_{L^2(e_i)} \leqslant \theta_i^{-1}(\varepsilon_1)\|(\nabla v_0 - v_n \boldsymbol{n}_e)\cdot \boldsymbol{n}\|_{L^2(e_i)}. \tag{3.10}$$

Therefore, by using (3.5) and the second mean value theorem of integrals again, we see that there exists a point $\varepsilon_2 \in e_i$ such that

$$\|\varphi_i\|_{L^2(e_i)} = \sqrt{\langle \theta_i^2, q^2\rangle_{e_i}} = \theta_i(\varepsilon_2)\|q\|_{L^2(e_i)},$$

which together with (3.10) leads to

$$\|\varphi_i\|_{L^2(e_i)} \leqslant \theta_i(\varepsilon_2)\theta_i^{-1}(\varepsilon_1)\|(\nabla v_0 - v_n \boldsymbol{n}_e)\cdot \boldsymbol{n}\|_{L^2(e_i)}.$$

Thus, from (3.9) and the above inequality, we obtain

$$\|\varphi_i\|_{L^2(K)} \leqslant C h_K^{1/2}\|(\nabla v_0 - v_n \boldsymbol{n}_e)\cdot \boldsymbol{n}\|_{L^2(e_i)}.$$

Finally, choosing $\varphi = \sum_{i=1}^{3} \varphi_i$ ends the proof. □

Lemma 3.2. *There exist two positive constants C_1 and C_2 such that for any $v = \{v_0, v_n\boldsymbol{n}_e\} \in V_h$, we have*

$$C_1\|v\|_{2,h} \leqslant |\!|\!|v|\!|\!| \leqslant C_2\|v\|_{2,h}.$$

Proof. For any $v = \{v_0, v_n\boldsymbol{n}_e\} \in V_h$ and $\varphi \in \mathbb{P}_{k+3}(K)$, it follows from the definition of the weak Laplacian (2.5) and integration by parts that

$$\begin{aligned}(\Delta_w v, \varphi)_K &= -(\nabla v_0, \nabla\varphi)_K + \langle v_n\boldsymbol{n}_e \cdot \boldsymbol{n}, \varphi\rangle_{\partial K}\\ &= (\Delta v_0, \varphi)_K + \langle (v_n\boldsymbol{n}_e - \nabla v_0)\cdot\boldsymbol{n}, \varphi\rangle_{\partial K}.\end{aligned} \tag{3.11}$$

By letting $\varphi = \Delta_w v$ in (3.11), we arrive at

$$\|\Delta_w v\|^2_{L^2(K)} = (\Delta v_0, \Delta_w v)_K + \langle (v_n\boldsymbol{n}_e - \nabla v_0)\cdot\boldsymbol{n}, \Delta_w v\rangle_{\partial K}.$$

From the trace inequality (3.3) and the inverse inequality, we have

$$\begin{aligned}\|\Delta_w v\|^2_{L^2(K)} &\leqslant \|\Delta v_0\|_{L^2(K)}\|\Delta_w v\|_{L^2(K)} + \|(v_n\boldsymbol{n}_e - \nabla v_0)\cdot\boldsymbol{n}\|_{L^2(\partial K)}\|\Delta_w v\|_{L^2(\partial K)}\\ &\leqslant C(\|\Delta v_0\|_{L^2(K)} + h_K^{-1/2}\|(v_n\boldsymbol{n}_e - \nabla v_0)\cdot\boldsymbol{n}\|_{L^2(\partial K)})\|\Delta_w v\|_{L^2(K)},\end{aligned}$$

which implies

$$\|\Delta_w v\|_{L^2(K)} \leqslant C(\|\Delta v_0\|_{L^2(K)} + h_K^{-1/2}\|(v_n\boldsymbol{n}_e - \nabla v_0)\cdot\boldsymbol{n}\|_{L^2(\partial K)})$$

and consequently,

$$|\!|\!|v|\!|\!| \leqslant C_2\|v\|_{2,h}.$$

Next, we prove

$$\sum_{K\in\mathcal{T}_h} h_K^{-1}\|(\nabla v_0 - v_n\boldsymbol{n}_e)\cdot\boldsymbol{n}\|^2_{L^2(\partial K)} \leqslant C|\!|\!|v|\!|\!|^2. \tag{3.12}$$

Let φ_0 be obtained from Lemma 3.1. Taking $\varphi = \varphi_0$ in (3.11) yields

$$\begin{aligned}\|(v_n\boldsymbol{n}_e - \nabla v_0)\cdot\boldsymbol{n}\|^2_{L^2(\partial K)} &= (\Delta_w v, \varphi_0)_K \leqslant \|\Delta_w v\|_{L^2(K)}\|\varphi_0\|_{L^2(K)}\\ &\leqslant Ch_K^{1/2}\|\Delta_w v\|_{L^2(K)}\|(v_n\boldsymbol{n}_e - \nabla v_0)\cdot\boldsymbol{n}\|_{L^2(\partial K)},\end{aligned} \tag{3.13}$$

which implies (3.12).

Finally, by letting $\varphi = \Delta v_0$ in (3.11), we arrive at

$$\|\Delta v_0\|^2_{L^2(K)} = (\Delta v_0, \Delta_w v)_K - \langle (v_n\boldsymbol{n}_e - \nabla v_0)\cdot\boldsymbol{n}, \Delta_w v\rangle_{\partial K}.$$

Using the trace inequality (3.3), the inverse inequality and (3.12), one has

$$\|\Delta v_0\|^2_{L^2(K)} \leqslant C\|\Delta_w v\|_{L^2(K)}\|\Delta v_0\|_{L^2(K)},$$

which gives

$$\sum_{K\in\mathcal{T}_h} \|\Delta v_0\|^2_{L^2(K)} \leqslant C|\!|\!|v|\!|\!|^2, \tag{3.14}$$

which together with (3.12) yields

$$|\!|\!|v|\!|\!| \geqslant C_1\|v\|_{2,h}.$$

The proof is completed. □

In the following lemma, we prove the well-posedness of the SF-C0WG method (2.6).

Lemma 3.3. *The SF-C0WG finite element scheme* (2.6) *has a unique solution.*

Proof. To show the well-posedness of (2.6), assume that $f = g_D = g_N = 0$. We show that u_h vanishes. Take $v = u_h$ in (2.6). It follows that

$$(\Delta_w u_h, \Delta_w u_h)_{\mathcal{T}_h} = 0.$$

Then Lemma 3.2 implies that $\|u_h\|_{2,h} = 0$. Consequently, we have $\Delta u_0 = 0$ and $\nabla u_0 \cdot \boldsymbol{n}_e = u_n$ on ∂K. Thus, u_0 is the solution of (1.1a)–(1.1c) with $f = g_D = g_N = 0$. We have $u_0 = 0$, and then $u_n = 0$, which ends the proof. □

4 An error equation

Let $Q_0 : H^2(\Omega) \to S_h$ be the Scott-Zhang interpolation operator introduced in [16], which has the following properties:

(a) (See [16, p. 493]) Q_0 preserves the polynomial of degree up to $k+2$, i.e., $Q_0 v = v \in \mathbb{P}_{k+2}(\mathcal{T}_h)$.

(b) (See [16, Lemma 8.2]) Q_0 preserves the face mass of order k, i.e.,

$$\langle v - Q_0 v, p\rangle_e = 0, \quad \forall\, p \in \mathbb{P}_k(e), \quad e \in \mathcal{E}_h. \tag{4.1}$$

(c) (See [16, Theorem 8.1]) For any $v \in H^\gamma(\Omega)$ with $\gamma \geqslant 2$, it holds that

$$\left[\sum_{K\in\mathcal{T}_h} h^{2s}|v - Q_0 v|^2_{H^s(K)}\right]^{1/2} \leqslant Ch^{\min\{k+3,\gamma\}}|v|_{H^\gamma(\Omega)}, \quad 0 \leqslant s \leqslant 2. \tag{4.2}$$

Now for the true solution u of (1.1a)–(1.1c), we introduce an interpolation operator $Q_h : H^2(\Omega) \to V_h$ such that on each element $K \in \mathcal{T}_h$,

$$Q_h u = \left\{Q_0 u, Q_n\left(\frac{\partial u}{\partial \boldsymbol{n}_e}\right)\boldsymbol{n}_e\right\},$$

where Q_n denotes the element-wise defined L^2 projections from $L^2(e)$ onto $\mathbb{P}_{k+1}(e)$ for each $e \subset \partial K$.

Define the error between the WG solution $u_h = \{u_0, u_n\boldsymbol{n}_e\}$ and the projection $Q_h u = \{Q_0 u, Q_n(\frac{\partial u}{\partial \boldsymbol{n}_e})\boldsymbol{n}_e\}$ of the exact solution u as

$$e_h = Q_h u - u_h := \{e_0, e_n\boldsymbol{n}_e\}$$

with

$$e_0 = Q_0 u - u_0, \quad e_n = Q_n\left(\frac{\partial u}{\partial \boldsymbol{n}_e}\right) - u_n.$$

The aim of this section is to obtain an error equation that e_h satisfies.

Lemma 4.1. *Let π_h be an element-wise defined L^2 projections onto $\mathbb{P}_{k+3}(K)$ on each element $K \in \mathcal{T}_h$. For any $K \in \mathcal{T}_h$ and $w \in H^2(\Omega)$, we have*

$$(\Delta_w(Q_h w), v)_K = (\Delta Q_0 w, v)_K + \left\langle Q_n\left(\frac{\partial w}{\partial \boldsymbol{n}}\right) - \frac{\partial}{\partial \boldsymbol{n}}(Q_0 w), v\right\rangle_{\partial K} \tag{4.3}$$

for any $v \in \mathbb{P}_{k+3}(K)$.

Proof. From the definition (2.5) of weak Laplacian, it follows that

$$(\Delta_w(Q_h w), v)_K = -(\nabla Q_0 w, \nabla v)_K + \left\langle Q_n\left(\frac{\partial w}{\partial \boldsymbol{n}_e}\right)\boldsymbol{n}_e\cdot\boldsymbol{n}, v\right\rangle_{\partial K} \tag{4.4}$$

for any $v \in \mathbb{P}_{k+3}(K)$.

Using integration by parts, we get

$$-(\nabla Q_0 w, \nabla v)_K = (\Delta Q_0 w, v)_K - \langle \nabla Q_0 w \cdot \boldsymbol{n}, w\rangle_{\partial K}. \tag{4.5}$$

By plugging (4.5) into (4.4), and recalling that

$$Q_n\left(\frac{\partial w}{\partial \boldsymbol{n}_e}\right)\boldsymbol{n}_e\cdot\boldsymbol{n} = Q_n\left(\frac{\partial w}{\partial \boldsymbol{n}}\right)$$

yields (4.3), we complete the proof. □

Lemma 4.2 (Error equation). *Let u and u_h be the solutions of the problem* (1.1a)–(1.1c) *and the SF-C0WG scheme* (2.6)*, respectively. For any $v \in V_h^0$, we have*

$$\mathcal{A}_h(e_h, v) = \ell(u, v), \tag{4.6}$$

where $\ell(u,v) := \sum_{i=1}^{2} \ell_i(u,v)$ *with*

$$\ell_1(u,v) := (\Delta_w(Q_h u) - \pi_h \Delta u, \Delta_w v)_{\mathcal{T}_h}, \tag{4.7a}$$

$$\ell_2(u,v) := \langle \Delta u - \pi_h \Delta u, (\nabla v_0 - v_n \boldsymbol{n}_e) \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h}. \tag{4.7b}$$

Proof. For $v = \{v_0, v_n \boldsymbol{n}_e\} \in V_h^0$, testing (1.1a) by v_0 and using the fact that

$$\sum_{K \in \mathcal{T}_h} \langle \Delta u, v_n \boldsymbol{n}_e \cdot \boldsymbol{n} \rangle_{\partial K} = 0$$

and integration by parts, we arrive at

$$\begin{aligned} (f, v_0) &= (\Delta^2 u, v_0)_{\mathcal{T}_h} \\ &= (\Delta u, \Delta v_0)_{\mathcal{T}_h} - \langle \Delta u, \nabla v_0 \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} + \langle \nabla(\Delta u) \cdot \boldsymbol{n}, v_0 \rangle_{\partial \mathcal{T}_h} \\ &= (\Delta u, \Delta v_0)_{\mathcal{T}_h} - \langle \Delta u, (\nabla v_0 - v_n \boldsymbol{n}_e) \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h}. \end{aligned} \tag{4.8}$$

Next, we investigate the term $(\Delta u, \Delta v_0)_{\mathcal{T}_h}$ in the above equation. Using (4.3), integration by parts and the definition of weak Laplacian, we have

$$\begin{aligned} (\Delta u, \Delta v_0)_{\mathcal{T}_h} &= (\pi_h \Delta u, \Delta v_0)_{\mathcal{T}_h} \\ &= -(\nabla v_0, \nabla(\pi_h \Delta u))_{\mathcal{T}_h} + \langle \nabla v_0 \cdot \boldsymbol{n}, \pi_h \Delta u \rangle_{\partial \mathcal{T}_h} \\ &= (\Delta_w v,\ \pi_h \Delta u)_{\mathcal{T}_h} + \langle (\nabla v_0 - v_n \boldsymbol{n}_e) \cdot \boldsymbol{n}, \pi_h \Delta u \rangle_{\partial \mathcal{T}_h} \\ &= (\Delta_w(Q_h u),\ \Delta_w v)_{\mathcal{T}_h} - \ell_1(u,v) + \langle (\nabla v_0 - v_n \boldsymbol{n}_e) \cdot \boldsymbol{n}, \pi_h \Delta u \rangle_{\partial \mathcal{T}_h}, \end{aligned}$$

which together with (4.8) yields

$$(f, v_0) = \mathcal{A}_h(Q_h u, v) - \ell_1(u,v) - \langle (\nabla v_0 - v_n \boldsymbol{n}_e) \cdot \boldsymbol{n}, \Delta u - \pi_h \Delta u \rangle_{\partial \mathcal{T}_h}, \tag{4.9}$$

which implies that

$$\mathcal{A}_h(Q_h u, v) = (f, v_0) + \sum_{i=1}^{2} \ell_i(u,v).$$

Subtracting (2.6) from the above equation ends the proof. □

5 An error estimate in the H^2-like norm

We obtain the optimal convergence rate for the solution u_h of the SF-C0WG method (2.6) in a discrete H^2 norm.

Lemma 5.1. *Assume that* $w \in H^{\gamma+2}(\Omega)$ *with* $\gamma > 0$. *There exists a constant* C *such that the following estimates hold true*:

$$\left(\sum_{K \in \mathcal{T}_h} h_K \|\Delta w - \pi_h \Delta w\|_{L^2(\partial K)}^2 \right)^{1/2} \leqslant C h^{\min\{k+4,\gamma\}} |w|_{H^{\gamma+2}(\Omega)}, \tag{5.1}$$

$$\left(\sum_{K \in \mathcal{T}_h} h_K^{-1} \left\| \frac{\partial}{\partial \boldsymbol{n}}(Q_0 w) - Q_n\left(\frac{\partial w}{\partial \boldsymbol{n}}\right) \right\|_{L^2(\partial K)}^2 \right)^{1/2} \leqslant C h^{\min\{k+1,\gamma\}} |w|_{H^{\gamma+2}(\Omega)}, \tag{5.2}$$

$$\|\Delta_w(Q_h w) - \pi_h \Delta w\|_{L^2(\mathcal{T}_h)} \leqslant C h^{\min\{k+1,\gamma\}} |w|_{H^{\gamma+2}(\Omega)}. \tag{5.3}$$

Proof. By the trace inequality (3.3) and the approximation property of the L^2 orthogonal projection π_h, we have

$$h_K \|\Delta w - \pi_h \Delta w\|_{L^2(\partial K)}^2 \leqslant C(\|\Delta w - \pi_h \Delta w\|_{L^2(K)}^2 + h_K^2 \|\nabla(\Delta w - \pi_h \Delta w)\|_{L^2(K)}^2)$$

$$
\begin{aligned}
&\leqslant Ch_K^{2\min\{k+4,\gamma\}}|\Delta w|^2_{H^\gamma(K)}\\
&\leqslant Ch_K^{2\min\{k+4,\gamma\}}|w|^2_{H^{\gamma+2}(K)}.
\end{aligned}
$$

Taking the summation of the above inequalities over all $K\in\mathcal{T}_h$, we complete the proof of (5.1).

Next, we turn to the estimate (5.2). It follows from the definitions of Q_0 and Q_n that

$$
\begin{aligned}
&\left\|\frac{\partial}{\partial\boldsymbol{n}}(Q_0w)-Q_n\left(\frac{\partial w}{\partial\boldsymbol{n}}\right)\right\|_{L^2(\partial K)}\\
&\quad\leqslant\left\|\frac{\partial}{\partial\boldsymbol{n}}(Q_0w-w)\right\|_{L^2(\partial K)}+\left\|\frac{\partial w}{\partial\boldsymbol{n}}-Q_n\left(\frac{\partial w}{\partial\boldsymbol{n}}\right)\right\|_{L^2(\partial K)}\\
&\quad\leqslant 2\left\|\frac{\partial}{\partial\boldsymbol{n}}(Q_0w-w)\right\|_{L^2(\partial K)}.
\end{aligned}\tag{5.4}
$$

Furthermore, using the trace inequality (3.3) and the approximation property (4.2) of Q_0, we obtain

$$
\begin{aligned}
&\left\|\frac{\partial}{\partial\boldsymbol{n}}(Q_0w-w)\right\|^2_{L^2(\partial K)}\\
&\quad\leqslant C(h_K^{-1}\|\nabla(Q_0w-w)\|^2_{L^2(K)}+h_K\|\nabla^2(Q_0w-w)\|^2_{L^2(K)})\\
&\quad\leqslant Ch_K^{\min\{2k+3,2\gamma+1\}}|w|^2_{H^{\gamma+2}(K)},
\end{aligned}
$$

which together with (5.4) yields

$$
\sum_{K\in\mathcal{T}_h}h_K^{-1}\left\|\frac{\partial}{\partial\boldsymbol{n}}(Q_0w)-Q_n\left(\frac{\partial w}{\partial\boldsymbol{n}}\right)\right\|^2_{L^2(\partial K)}\leqslant Ch^{\min\{2k+2,2\gamma\}}|w|^2_{H^{\gamma+2}(\Omega)},
$$

which ends the proof of (5.2).

Now we consider the estimate (5.3). For any $v\in\mathbb{P}_{k+3}(\mathcal{T}_h)$, from (4.3) and the orthogonal property of the L^2 projection π_h, it follows that

$$
\begin{aligned}
&(\Delta_w(Q_hw)-\pi_hw,v)_{\mathcal{T}_h}\\
&\quad=(\Delta(Q_0w-w),v)_{\mathcal{T}_h}+\left\langle Q_n\left(\frac{\partial u}{\partial\boldsymbol{n}}\right)-\frac{\partial}{\partial\boldsymbol{n}}(Q_0u),v\right\rangle_{\partial\mathcal{T}_h}\\
&\quad=:I_1+I_2.
\end{aligned}\tag{5.5}
$$

From the Cauchy-Schwarz inequality and the approximation property (4.2) of Q_0, one has

$$
\begin{aligned}
|I_1|&\leqslant\sum_{K\in\mathcal{T}_h}\|\Delta(Q_0w-w)\|_{L^2(K)}\|v\|_{L^2(K)}\\
&\leqslant\left(\sum_{K\in\mathcal{T}_h}|Q_0w-w|^2_{H^2(K)}\right)^{1/2}\|v\|_{L^2(\mathcal{T}_h)}\\
&\leqslant Ch^{\min\{k+1,\gamma\}}|w|_{H^{\gamma+2}(\Omega)}\|v\|_{L^2(\mathcal{T}_h)}.
\end{aligned}\tag{5.6}
$$

Using the Cauchy-Schwarz inequality, (5.2) and the inverse inequality, we arrive at

$$
\begin{aligned}
|I_2|&\leqslant\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\left\|Q_n\left(\frac{\partial u}{\partial\boldsymbol{n}}\right)-\frac{\partial}{\partial\boldsymbol{n}}(Q_0u)\right\|^2_{L^2(\partial K)}\right)^{1/2}\left(\sum_{K\in\mathcal{T}_h}h_K\|v\|^2_{L^2(\partial K)}\right)^{1/2}\\
&\leqslant Ch^{\min\{k+1,\gamma\}}|w|_{H^{\gamma+2}(\Omega)}\|v\|_{L^2(\mathcal{T}_h)},
\end{aligned}
$$

which together with (5.5) and (5.6) yields

$$
|(\Delta_w(Q_hw)-\pi_hw,v)_{\mathcal{T}_h}|\leqslant Ch^{\min\{k+1,\gamma\}}|w|_{H^{\gamma+2}(\Omega)}\|v\|_{L^2(\mathcal{T}_h)}.
$$

Taking $v=\Delta_w(Q_hw)-\pi_hw$ in the above inequality ends the proof of (5.3). □

Lemma 5.2. *Assume that $w \in H^{\gamma+2}(\Omega)$ with $\gamma > 0$. There exists a constant C such that the following estimates hold true*:

$$|\ell_1(w, v)| \leqslant Ch^{\min\{k+1,\gamma\}}|w|_{H^{\gamma+2}(\Omega)}|||v|||, \tag{5.7}$$

$$|\ell_2(w, v)| \leqslant Ch^{\min\{k+4,\gamma\}}|w|_{H^{\gamma+2}(\Omega)}|||v||| \tag{5.8}$$

for any $v \in V_h^0$.

Proof. Using the Cauchy-Schwarz inequality and (5.3) of Lemma 5.1, we have

$$\begin{aligned}|\ell_1(w, v)| &= |(\Delta_w(Q_h w) - \pi_h w, \Delta_w v)_{\mathcal{T}_h}| \\ &\leqslant \|\Delta_w(Q_h w) - \pi_h w\|_{L^2(\mathcal{T}_h)}\|\Delta_w v\|_{L^2(\mathcal{T}_h)} \\ &\leqslant Ch^{\min\{k+1,\gamma\}}|w|_{H^{\gamma+2}(\Omega)}|||v|||.\end{aligned}$$

It follows from the Cauchy-Schwarz inequality, (5.1) and Lemma 3.2 that

$$\begin{aligned}|\ell_2(w, v)| &= \left|\sum_{K\in\mathcal{T}_h}\langle \Delta w - \pi_h\Delta w, (\nabla v_0 - v_n\boldsymbol{n}_e)\cdot\boldsymbol{n}\rangle_{\partial K}\right| \\ &\leqslant \left(\sum_{K\in\mathcal{T}_h} h_K\|\Delta w - \pi_h\Delta w\|^2_{L^2(\partial K)}\right)^{1/2} \\ &\quad\times\left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\|(\nabla v_0 - v_n\boldsymbol{n}_e)\cdot\boldsymbol{n}\|^2_{L^2(\partial K)}\right)^{1/2} \\ &\leqslant Ch^{\min\{k+4,\gamma\}}|w|_{H^{\gamma+2}(\Omega)}\|v\|_{2,h} \\ &\leqslant Ch^{\min\{k+4,\gamma\}}|w|_{H^{\gamma+2}(\Omega)}|||v|||.\end{aligned}$$

We have completed the proof. □

Theorem 5.3. *Let $u_h \in V_h$ be the solution arising from the SF-C0WG scheme* (2.6)*. Assume that the exact solution $u \in H^{k+3}(\Omega)$. Then there exists a constant C such that*

$$|||Q_h u - u_h||| \leqslant Ch^{k+1}|u|_{H^{k+3}(\Omega)}. \tag{5.9}$$

Proof. Taking $v = e_h$ in the error equation (4.6) and using Lemma 5.2 with $\gamma = k + 1$, we arrive at

$$|||e_h|||^2 = \ell(u, e_h) \leqslant Ch^{k+1}|u|_{H^{k+3}(\Omega)}|||e_h|||,$$

which completes the proof. □

6 Error estimates in the L^2 norm and the H^1 norm

In this section, we provide estimates for the L^2 norm and the H^1 norm of the error between the exact solution u and its corresponding WG finite element solution u_h.

Firstly, let us introduce the following dual problem:

$$\Delta^2\phi = \chi \quad \text{in } \Omega, \tag{6.1}$$

$$\phi = 0 \quad \text{on } \Gamma, \tag{6.2}$$

$$\nabla\phi\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma. \tag{6.3}$$

Assume that the dual problem has the $H^{\alpha+2}$-regularity in the sense that there exists a constant C such that

$$\|\phi\|_{H^{\alpha+2}(\Omega)} \leqslant C\|\chi\|_{H^{\alpha-2}(\Omega)} \quad \text{for } \alpha = 1, 2. \tag{6.4}$$

For $\chi \in H^{\alpha-2}(\Omega)$ with $\alpha > 0$, the $H^{\alpha+2}$-regularity has been proved for smooth domains in any dimension [5]. The H^4-regularity has been proved by Blum and Rannacher [3] for the two-dimensional convex polygonal domains with inner angles less than $126.28\cdots^\circ$.

Lemma 6.1. *Let $\phi \in H^{\alpha+2}(\Omega)$ with $\alpha = 1, 2$. Then it holds that*

$$|\Delta_w(Q_h\phi)|_{H^\alpha(\mathcal{T}_h)} \leqslant Ch^{\min\{k+1-\alpha,0\}}|\phi|_{H^{\alpha+2}(\Omega)}. \tag{6.5}$$

Proof. The proof is given in Appendix A. $\square$

Lemma 6.2. *Assume that $u \in H^{k+3}(\Omega)$ and $\phi \in H^{\alpha+2}(\Omega)$ with $\alpha = 1, 2$. Then for $k \geqslant 0$, it holds that*

$$|\ell_1(u, Q_h\phi)| \leqslant Ch^{\min\{2k+2,k+1+\alpha\}}|u|_{H^{k+3}(\Omega)}|\phi|_{H^{\alpha+2}(\Omega)}, \tag{6.6}$$

$$|\ell_2(u, Q_h\phi)| \leqslant Ch^{\min\{2k+2,k+1+\alpha\}}|u|_{H^{k+3}(\Omega)}|\phi|_{H^{\alpha+2}(\Omega)}. \tag{6.7}$$

Proof. Let $\mathcal{P}_h^{\alpha-1}$ be the L^2 orthogonal projection onto the piecewise polynomial space $\mathbb{P}_{\alpha-1}(\mathcal{T}_h)$. For simplicity, define $\phi_h = \Delta_w(Q_h\phi)$ and $\widehat{\phi}_h = \mathcal{P}_h^{\alpha-1}(\phi_h)$. Then

$$\begin{aligned}\ell_1(u, Q_h\phi) &= (\Delta_w(Q_hu) - \pi_h\Delta u, \phi_h)_{\mathcal{T}_h}\\ &= (\Delta_w(Q_hu) - \pi_h\Delta u, \phi_h - \widehat{\phi}_h)_{\mathcal{T}_h} + (\Delta_w(Q_hu) - \pi_h\Delta u, \widehat{\phi}_h)_{\mathcal{T}_h}\\ &=: T_1 + T_2.\end{aligned} \tag{6.8}$$

Using the Cauchy-Schwarz inequality, (5.3) of Lemma 5.1 and (6.5), one has

$$\begin{aligned}|T_1| &= |(\Delta_w(Q_hu) - \pi_h\Delta u, \phi_h - \widehat{\phi}_h)_{\mathcal{T}_h}|\\ &\leqslant \|\Delta_w(Q_hu) - \pi_h\Delta u\|_{L^2(\mathcal{T}_h)}\|\phi_h - \widehat{\phi}_h\|_{L^2(\mathcal{T}_h)}\\ &\leqslant Ch^{k+1}|u|_{H^{k+3}(\Omega)} \cdot h^\alpha|\phi_h|_{H^\alpha(\Omega)}\\ &\leqslant Ch^{k+1+\alpha}|u|_{H^{k+3}(\Omega)} \cdot h^{\min\{k+1-\alpha,0\}}|\phi|_{H^{\alpha+2}(\Omega)}\\ &\leqslant Ch^{\min\{2k+2,k+1+\alpha\}}|u|_{H^{k+3}(\Omega)}|\phi|_{H^{\alpha+2}(\Omega)}.\end{aligned} \tag{6.9}$$

Now we turn to the estimate of the term T_2. Firstly, we rewrite T_2 as follows:

$$\begin{aligned}T_2 &= (\Delta_w(Q_hu) - \pi_h\Delta u, \widehat{\phi}_h)_{\mathcal{T}_h}\\ &= (\Delta(Q_0u - u), \widehat{\phi}_h)_{\mathcal{T}_h} + \left\langle Q_n\left(\frac{\partial u}{\partial \boldsymbol{n}}\right) - \frac{\partial}{\partial \boldsymbol{n}}(Q_0u), \widehat{\phi}_h\right\rangle_{\partial\mathcal{T}_h}\\ &= -(\nabla(Q_0u - u), \nabla\widehat{\phi}_h)_{\mathcal{T}_h} + \left\langle Q_n\left(\frac{\partial u}{\partial \boldsymbol{n}}\right) - \frac{\partial u}{\partial \boldsymbol{n}}, \widehat{\phi}_h\right\rangle_{\partial\mathcal{T}_h}\\ &=: J_1 + J_2.\end{aligned} \tag{6.10}$$

For the first term J_1, we discuss it in the following two cases:

- In the case of $\alpha = 1$, $\nabla\widehat{\phi}_h = 0$ since $\widehat{\phi}_h = \mathcal{P}_h^0(\phi_h) \in \mathbb{P}_0(\mathcal{T}_h)$. Therefore, $J_1 = 0$.
- In the case of $\alpha = 2$, $\nabla\widehat{\phi}_h$ is a piecewise constant vector due to $\widehat{\phi}_h = \mathcal{P}_h^1(\phi_h) \in \mathbb{P}_1(\mathcal{T}_h)$. Then by Green's formula and (4.1), we get

$$J_1 = \sum_{K\in\mathcal{T}_h} -\langle Q_0u - u, \nabla\widehat{\phi}_h \cdot \boldsymbol{n}\rangle_{\partial K} = 0.$$

Thus, in both cases where $\alpha = 1$ and $\alpha = 2$, we have

$$J_1 = 0. \tag{6.11}$$

As to the second term J_2, recalling the fact that

$$\left\langle Q_n\left(\frac{\partial u}{\partial \boldsymbol{n}}\right) - \frac{\partial u}{\partial \boldsymbol{n}}, \Delta\phi\right\rangle_{\partial\mathcal{T}_h} = 0,$$

we split J_2 into the following two terms:

$$
\begin{aligned}
J_2 &= \left\langle Q_n\left(\frac{\partial u}{\partial \boldsymbol{n}}\right) - \frac{\partial u}{\partial \boldsymbol{n}}, \widehat{\phi}_h \right\rangle_{\partial \mathcal{T}_h} \\
&= \left\langle Q_n\left(\frac{\partial u}{\partial \boldsymbol{n}}\right) - \frac{\partial u}{\partial \boldsymbol{n}}, \mathcal{P}_h^{\alpha-1}(\Delta_w(Q_h\phi) - \Delta\phi) \right\rangle_{\partial \mathcal{T}_h} \\
&\quad + \left\langle Q_n\left(\frac{\partial u}{\partial \boldsymbol{n}}\right) - \frac{\partial u}{\partial \boldsymbol{n}}, \mathcal{P}_h^{\alpha-1}(\Delta\phi) - \Delta\phi \right\rangle_{\partial \mathcal{T}_h}.
\end{aligned}
$$

Then by the Cauchy-Schwarz inequality and (5.2) of Lemma 5.1 with $\gamma = k+1$, we get

$$
\begin{aligned}
|J_2| &\leqslant \left(\sum_{K\in\mathcal{T}_h} h_K^{-1} \left\| Q_n\left(\frac{\partial u}{\partial \boldsymbol{n}}\right) - \frac{\partial u}{\partial \boldsymbol{n}} \right\|_{L^2(\partial K)}^2 \right)^{1/2} (\Theta_1^{1/2} + \Theta_2^{1/2}) \\
&\leqslant Ch^{k+1}|u|_{H^{k+3}(\Omega)}(\Theta_1^{1/2} + \Theta_2^{1/2}),
\end{aligned} \tag{6.12}
$$

where

$$
\begin{aligned}
\Theta_1 &:= \sum_{K\in\mathcal{T}_h} h_K \|\mathcal{P}_h^{\alpha-1}(\Delta_w(Q_h\phi) - \Delta\phi)\|_{L^2(\partial K)}^2, \\
\Theta_2 &:= \sum_{K\in\mathcal{T}_h} h_K \|\mathcal{P}_h^{\alpha-1}(\Delta\phi) - \Delta\phi\|_{L^2(\partial K)}^2.
\end{aligned}
$$

From the trace inequality and the stability of the L^2 projection $\mathcal{P}_h^{\alpha-1}$, it follows that

$$
\Theta_1 \leqslant C \sum_{K\in\mathcal{T}_h} \|\mathcal{P}_h^{\alpha-1}(\Delta_w(Q_h\phi) - \Delta\phi)\|_{L^2(K)}^2 \leqslant C\|\Delta_w(Q_h\phi) - \Delta\phi\|_{L^2(\mathcal{T}_h)}^2.
$$

Then by the triangle inequality and (5.3) of Lemma 5.1, we arrive at

$$
\begin{aligned}
\Theta_1 &\leqslant C(\|\Delta_w(Q_h\phi) - \pi_h\Delta\phi\|_{L^2(\mathcal{T}_h)}^2 + \|\pi_h\Delta\phi - \Delta\phi\|_{L^2(\mathcal{T}_h)}^2) \\
&\leqslant Ch^{2\min\{k+1,\alpha\}}|\phi|_{H^{\alpha+2}(\Omega)}^2.
\end{aligned} \tag{6.13}
$$

It follows from the trace inequality and the approximation property of the L^2 projection $\mathcal{P}_h^{\alpha-1}$ that

$$
\begin{aligned}
\Theta_2 &\leqslant \sum_{K\in\mathcal{T}_h} (h_K\|\mathcal{P}_h^{\alpha-1}(\Delta\phi) - \Delta\phi\|_{L^2(K)}^2 + h_K|\mathcal{P}_h^{\alpha-1}(\Delta\phi) - \Delta\phi|_{H^1(K)}^2) \\
&\leqslant Ch^{2\alpha}|\phi|_{H^{\alpha+2}(\Omega)}^2,
\end{aligned}
$$

which together with (6.13) and (6.12) leads to

$$
|J_2| \leqslant Ch^{\min\{2k+2,k+1+\alpha\}}|u|_{H^{k+3}(\Omega)}|\phi|_{H^{\alpha+2}(\Omega)}. \tag{6.14}
$$

Collecting (6.10), (6.11) and (6.14) yields

$$
|T_2| \leqslant Ch^{\min\{2k+2,k+1+\alpha\}}|u|_{H^{k+3}(\Omega)}|\phi|_{H^{\alpha+2}(\Omega)},
$$

which combining (6.9) and (6.8) completes the proof of (6.6).

As to the proof of (6.7), from the Cauchy-Schwarz inequality and Lemma 5.1 with $\gamma = k+1$, it follows that

$$
\begin{aligned}
|\ell_2(u, Q_h\phi)| &= \left| \sum_{T\in\mathcal{T}_h} \left\langle \Delta u - \pi_h\Delta u, \frac{\partial}{\partial \boldsymbol{n}}(Q_0\phi) - Q_n\left(\frac{\partial \phi}{\partial \boldsymbol{n}_e}\right)\boldsymbol{n}_e\cdot\boldsymbol{n} \right\rangle_{\partial K} \right| \\
&\leqslant \left(\sum_{K\in\mathcal{T}_h} h_K\|\Delta u - \pi_h\Delta u\|_{L^2(\partial K)}^2 \right)^{1/2}
\end{aligned}
$$

$$\times\left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\left\|\frac{\partial}{\partial \boldsymbol{n}}(Q_0\phi)-Q_n\left(\frac{\partial\phi}{\partial\boldsymbol{n}}\right)\right\|_{L^2(\partial K)}^2\right)^{1/2}$$
$$\leqslant Ch^{k+1}|u|_{H^{k+3}(\Omega)}\cdot h^{\min\{k+1,\alpha\}}|\phi|_{H^{\alpha+2}(\Omega)}$$
$$\leqslant Ch^{\min\{2k+2,k+1+\alpha\}}|u|_{H^{k+3}(\Omega)}|\phi|_{H^{\alpha+2}(\Omega)}.$$

The proof is completed. □

Theorem 6.3. *Let $u_h=\{u_0,u_n\boldsymbol{n}_e\}\in V_h$ be the solution of the SF-C0WG scheme (2.6). Assume that the exact solution $u\in H^{k+3}(\Omega)$ and the regularity assumption (6.4) holds true. Then there exists a constant C such that*

$$\|Q_0u-u_0\|_{L^2(\Omega)}\leqslant Ch^{k+3-\delta_{k,0}}|u|_{H^{k+3}(\Omega)} \tag{6.15}$$

and

$$\|\nabla(Q_0u-u_0)\|_{L^2(\Omega)}\leqslant Ch^{k+2}|u|_{H^{k+3}(\Omega)}. \tag{6.16}$$

Here, $\delta_{i,j}$ is the usual Kronecker's delta with the value 1 when $i=j$ and the value 0 otherwise.

Proof. Testing (6.1) by the error function e_0 and then using a similar procedure to that in the proof of the equation (4.9), we obtain

$$(\chi,e_0)=(\Delta^2\phi,e_0)_{\mathcal{T}_h}=\mathcal{A}_h(e_h,Q_h\phi)-\ell(\phi,e_h). \tag{6.17}$$

The error equation (4.6) gives

$$\mathcal{A}_h(e_h,Q_h\phi)=\ell(u,Q_h\phi),$$

which combining (6.17) leads to

$$(\chi,e_0)=\ell(u,Q_h\phi)-\ell(\phi,e_h). \tag{6.18}$$

In view of Lemma 6.2, we infer that

$$|\ell(u,Q_h\phi)|\leqslant Ch^{\min\{2k+2,k+1+\alpha\}}|u|_{H^{k+3}(\Omega)}|\phi|_{H^{\alpha+2}(\Omega)}. \tag{6.19}$$

Using Lemma 5.2 with $\gamma=\alpha$ and Theorem 5.3, we have

$$|\ell(\phi,e_h)|\leqslant Ch^{\min\{k+1,\alpha\}}|\phi|_{H^{\alpha+2}(\Omega)}|\!|\!|e_h|\!|\!|$$
$$\leqslant Ch^{\min\{2k+2,k+1+\alpha\}}|u|_{H^{k+3}(\Omega)}|\phi|_{H^{\alpha+2}(\Omega)},$$

which combining (6.18) and (6.20) leads to

$$|(\chi,e_0)|\leqslant Ch^{\min\{2k+2,k+1+\alpha\}}|u|_{H^{k+3}(\Omega)}|\phi|_{H^{\alpha+2}(\Omega)}. \tag{6.20}$$

For the L^2-norm estimate of e_0, taking $\chi=e_0$ in the dual problem (6.1)–(6.3), and then using the estimate of (6.20) with the H^4-regularity, we find

$$\|e_0\|_{L^2(\Omega)}^2\leqslant Ch^{\min\{2k+2,k+3\}}|u|_{H^{k+3}(\Omega)}|\phi|_{H^4(\Omega)},$$

which together with the assumption (6.4) with $\alpha=2$ and $\|\phi\|_{H^4(\Omega)}\leqslant C\|e_0\|_{L^2(\Omega)}$ completes the proof of (6.15).

Then using the estimate of (6.20) with the H^3-regularity yields

$$|(\chi,e_0)|\leqslant Ch^{k+2}|u|_{H^{k+3}(\Omega)}|\phi|_{H^3(\Omega)},$$

which together with the assumption (6.4) with $\alpha=1$ and $\|\phi\|_{H^3(\Omega)}\leqslant C\|e_0\|_{H^{-1}(\Omega)}$ leads to

$$\|\nabla e_0\|_{L^2(\Omega)}=\sup_{\chi\in H^{-1}(\Omega)}\frac{(\chi,e_0)}{\|\chi\|_{H^{-1}(\Omega)}}\leqslant Ch^{k+2}|u|_{H^{k+3}(\Omega)}, \tag{6.21}$$

which ends the proof of (6.16). □

By the triangle inequality, from Theorem 6.3 and (4.2), we immediately obtain the L^2 norm and H^1 norm error estimates between the exact solution u and its WG finite element approximation u_0 as follows.

Corollary 6.4. *Let $u_h = \{u_0, u_n \boldsymbol{n}_e\} \in V_h$ be the solution of the SF-C0WG scheme* (2.6)*. Assume that the exact solution $u \in H^{k+3}(\Omega)$ and the regularity assumption* (6.4) *holds true. Then there exists a constant C such that*

$$\|u - u_0\|_{L^2(\Omega)} \leqslant Ch^{k+3-\delta_{k,0}} |u|_{H^{k+3}(\Omega)} \tag{6.22}$$

and

$$\|\nabla(u - u_0)\|_{L^2(\Omega)} \leqslant Ch^{k+2} |u|_{H^{k+3}(\Omega)}. \tag{6.23}$$

Here, $\delta_{i,j}$ is the usual Kronecker's delta with the value 1 *when $i = j$ and the value* 0 *otherwise.*

7 Numerical experiments

In this section, we conduct some numerical experiments to verify the theoretical predication on the SF-C0WG method (2.6) and also to compare its numerical performance with the C0WG method (2.7) and the C0IP method (2.8).

Example 7.1. Consider the model problem (1.1a)–(1.1c) with $\Omega = (0,1)^2$. The source data f and the boundaries data g_D and g_N are chosen so that the exact solution is

$$u = \sin(\pi x)\sin(\pi y).$$

The initial mesh in our computation is shown in Figure 1, which is generated by MATLAB function initmesh. The next level of the mesh is derived by uniformly refining the previous level of the mesh. The errors and the orders of convergence for the SF-C0WG method (2.6) with $k = 0$ and $k = 1$ are reported in Table 1, which confirm the theoretical predication in Theorems 5.3 and 6.3.

Table 2 lists the errors and the rates of convergence for the C0WG method (2.7). The results in Tables 1 and 2 show that both the SF-C0WG method and the C0WG method converge with the same rates, but the accuracy reached on a given mesh with a given polynomial degree is significantly different. The SF-C0WG method is more accurate than the C0WG method.

Table 3 shows the errors and the rates of convergence for the C0IP method (2.8). The errors in the first column of Table 3 is measured in the following H^2-like norm tailored for the C0IP method:

$$\|v\|_{dg} := \left[\sum_{K \in \mathcal{T}_h} |v|^2_{H^2(K)} + \sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\![\nabla v]\!]\|^2_{L^2(e)} \right]^{1/2}.$$

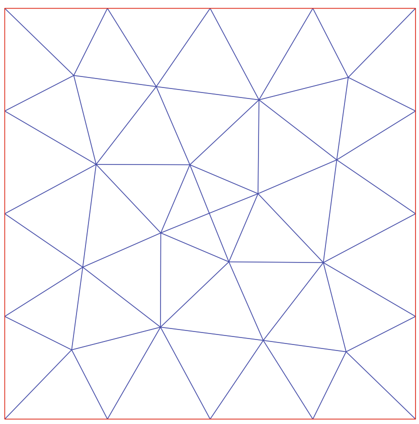

Figure 1 (Color online) The initial mesh

Table 1 Error profiles and convergence rates of the SF-C0WG method

k	Level	$\|\!\|\!\|Q_h u - u_h\|\!\|\!\|$	Rate	$\|\nabla(u - u_0)\|$	Rate	$\|u - u_0\|$	Rate
0	1	3.34E+00	–	8.06E−02	–	8.15E−03	–
	2	1.66E+00	1.0076	2.03E−02	1.9899	2.00E−03	2.0305
	3	8.25E−01	1.0095	5.20E−03	1.9653	5.09E−04	1.9710
	4	4.11E−01	1.0059	1.32E−03	1.9787	1.29E−04	1.9760
	5	2.05E−01	1.0031	3.32E−04	1.9915	3.26E−05	1.9893
1	1	3.61E−01	–	5.79E−03	–	2.97E−04	–
	2	9.12E−02	1.9853	7.26E−04	2.9952	2.16E−05	3.7835
	3	2.28E−02	1.9975	8.99E−05	3.0144	1.41E−06	3.9374
	4	5.71E−03	2.0001	1.12E−05	3.0096	8.91E−08	3.9820
	5	1.43E−03	2.0004	1.39E−06	3.0048	6.29E−09	3.8238

Table 2 Error profiles and convergence rates of the C0WG method

k	Level	$\|\!\|\!\|Q_h u - u_h\|\!\|\!\|$	Rate	$\|\nabla(u - u_0)\|$	Rate	$\|u - u_0\|$	Rate
0	1	4.73E+00	–	6.35E−01	–	1.36E−01	–
	2	2.33E+00	1.0189	1.52E−01	2.0620	3.35E−02	2.0236
	3	1.16E+00	1.0046	3.77E−02	2.0109	8.35E−03	2.0033
	4	5.81E−01	1.0018	9.41E−03	2.0047	2.08E−03	2.0019
	5	2.90E−01	1.0008	2.35E−03	2.0022	5.21E−04	2.0011
1	1	7.14E−01	–	7.91E−02	–	4.46E−03	–
	2	1.91E−01	1.9043	1.04E−02	2.9287	3.04E−04	3.8743
	3	4.89E−02	1.9629	1.33E−03	2.9627	1.96E−05	3.9522
	4	1.24E−02	1.9825	1.70E−04	2.9743	1.25E−06	3.9737
	5	3.11E−03	1.9914	2.14E−05	2.9849	7.89E−08	3.9852

Table 3 Error profiles and convergence rates of the C0IP method

k	Level	$\|u - u_h\|_{dg}$	Rate	$\|\nabla(u - u_h)\|$	Rate	$\|u - u_h\|$	Rate
0	1	1.33E+00	–	8.47E−02	–	1.02E−02	–
	2	6.40E−01	1.0595	2.24E−02	1.9150	2.91E−03	1.8035
	3	3.20E−01	1.0009	5.92E−03	1.9222	7.99E−04	1.8635
	4	1.60E−01	0.9955	1.52E−03	1.9584	2.10E−04	1.9310
	5	8.03E−02	0.9983	3.86E−04	1.9824	5.35E−05	1.9689
1	1	2.00E−01	–	6.24E−03	–	3.47E−04	–
	2	5.15E−02	1.9533	7.83E−04	2.9950	2.58E−05	3.7489
	3	1.31E−02	1.9780	9.62E−05	3.0238	1.70E−06	3.9194
	4	3.30E−03	1.9870	1.19E−05	3.0157	1.09E−07	3.9712
	5	8.29E−04	1.9930	1.48E−06	3.0076	6.53E−09	4.0564

The results in Tables 1 and 3 show that both the SF-C0WG method and the C0IP method converge with the same rate and the accuracies are also similar when the errors are measured in the H^1 semi-norm and the L^2 norm.

A comparison of the assembling time and solving time for both the C0WG method and the SF-C0WG method is displayed in Table 4. It can be observed that the assembling time and solving time for the SF-C0WG method are always smaller than that for the C0WG method.

The assembling time, solving time and total time (the sum of the assembling time and solving time) for both the C0IP method and the SF-C0WG method are illustrated in Table 5. As can be seen, although the solving time of the C0IP method is less than the SF-C0WG method, the assembling time and total time for the SF-C0WG method are always smaller than that for the C0IP method.

Table 4 Comparison of assembling time and solving time for the C0WG method and the SF-C0WG method

k	Level	C0WG method		SF-C0WG method	
		Assembling time	Solving time	Assembling time	Solving time
0	1	0.052233	0.001690	0.065710	0.003116
	2	0.175486	0.007218	0.157510	0.006127
	3	0.734831	0.039118	0.546378	0.030866
	4	2.602549	0.171534	2.240196	0.133192
	5	10.678900	0.874419	8.766250	0.628217
1	1	0.347160	0.027630	0.057160	0.016028
	2	0.184210	0.020082	0.201938	0.017132
	3	0.864096	0.072827	0.793079	0.061735
	4	3.537430	0.458761	2.800155	0.305041
	5	23.917670	2.630822	12.141470	1.752295

Table 5 Comparison of assembling, solving and total times for the C0IP method and the SF-C0WG method

Time (second)	Level	$k=0$		$k=1$	
		C0IP	SF-C0WG	C0IP	SF-C0WG
Assembling time	1	0.073452	0.065710	0.091383	0.057160
	2	0.163071	0.157510	0.252789	0.201938
	3	0.711536	0.546378	1.004696	0.793079
	4	2.308666	2.240196	3.720932	2.800155
	5	9.169572	8.766250	14.909280	12.141470
Solving time	1	0.007031	0.003116	0.009475	0.016028
	2	0.004608	0.006127	0.015663	0.017132
	3	0.021312	0.030866	0.060428	0.061735
	4	0.079957	0.133192	0.252089	0.305041
	5	0.385537	0.628217	1.610523	1.752295
Total time	1	0.080483	0.068825	0.100858	0.073188
	2	0.167680	0.163636	0.268452	0.219071
	3	0.732848	0.577244	1.065125	0.854814
	4	2.388623	2.373388	3.973021	3.105196
	5	9.555110	9.394467	16.519810	13.893770

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Appendix A

In this appendix, we introduce some technical tools which are useful in the L^2 norm and H^1 norm error analysis.

In order to prove Lemma 6.1, we introduce the following two lemmas.

Lemma A.1. *For any $K \in \mathcal{T}_h$, it holds that*

$$\Delta_w(Q_h w) = \Delta w, \quad \forall\, w \in \mathbb{P}_{k+2}(K). \tag{A.1}$$

Proof. For any $w \in \mathbb{P}_{k+2}(K)$, from the definitions of Q_0 and Q_n, we have $Q_0 w = w$ and $Q_n(\frac{\partial w}{\partial \boldsymbol{n}_e}) = \frac{\partial w}{\partial \boldsymbol{n}_e}$. Then for any $K \in \mathcal{T}_h$ and $v \in \mathbb{P}_{k+3}(K)$, from the definition (2.5) of the weak Laplacian, it follows that

$$\begin{aligned}(\Delta_w Q_h w, v)_K &= -(\nabla Q_0 w, \nabla v)_K + \left\langle Q_n\left(\frac{\partial w}{\partial \boldsymbol{n}_e}\right)\boldsymbol{n}_e \cdot \boldsymbol{n}, v\right\rangle_{\partial K} \\ &= -(\nabla w, \nabla v)_K + \left\langle \frac{\partial w}{\partial \boldsymbol{n}}, v\right\rangle_{\partial K} \\ &= (\Delta w, v)_K,\end{aligned}$$

which completes the proof. □

Let $\mathcal{P}_h^{k+2}: L^2(\mathcal{T}_h) \to \mathbb{P}_{k+2}(\mathcal{T}_h)$ be the element-wise defined L^2 orthogonal projection.

Lemma A.2. *Assume that* $\phi \in H^{\alpha+2}(\Omega)$ *with* $\alpha = 1, 2$. *Then it holds that*

$$\|\Delta_w Q_h(\phi - \mathcal{P}_h^{k+2}\phi)\|_{L^2(\mathcal{T}_h)} \leqslant Ch^{\min\{k+1,\alpha\}}|\phi|_{H^{\alpha+2}(\Omega)}. \tag{A.2}$$

Proof. For simplicity, define $w = \phi - \mathcal{P}_h^{k+2}\phi$. It follows from (4.3) and the Cauchy-Schwarz inequality that

$$\begin{aligned}
(\Delta_w Q_h w, v)_{\mathcal{T}_h} &= (\Delta Q_0 w, v)_{\mathcal{T}_h} + \left\langle Q_n\left(\frac{\partial w}{\partial \boldsymbol{n}}\right) - \frac{\partial}{\partial \boldsymbol{n}}(Q_0 w), v\right\rangle_{\partial\mathcal{T}_h} \\
&\leqslant \|\Delta Q_0 w\|_{L^2(\mathcal{T}_h)}\|v\|_{L^2(\mathcal{T}_h)} \\
&\quad + \left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\left\|Q_n\left(\frac{\partial w}{\partial \boldsymbol{n}}\right) - \frac{\partial}{\partial \boldsymbol{n}}(Q_0 w)\right\|_{\partial K}^2\right)^{1/2}\left(\sum_{K\in\mathcal{T}_h} h_K\|v\|_{\partial K}^2\right)^{1/2} \\
&\leqslant C\left(\|\Delta Q_0 w\|_{L^2(\mathcal{T}_h)} + \left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\left\|Q_n\left(\frac{\partial w}{\partial \boldsymbol{n}}\right) - \frac{\partial}{\partial \boldsymbol{n}}(Q_0 w)\right\|_{\partial K}^2\right)^{1/2}\right)\|v\|_{L^2(\mathcal{T}_h)}
\end{aligned} \tag{A.3}$$

for any $v \in \mathbb{P}_{k+3}(\mathcal{T}_h)$.

Letting $v = \Delta_w Q_h w$ in (A.3), and then cancelling out $\|\Delta_w Q_h w\|_{L^2(\mathcal{T}_h)}$ from both sides, we have

$$\|\Delta_w Q_h w\|_{\mathcal{T}_h} \leqslant C\left[\|\Delta Q_0 w\|_{L^2(\mathcal{T}_h)} + \left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\left\|Q_n\left(\frac{\partial w}{\partial \boldsymbol{n}}\right) - \frac{\partial}{\partial \boldsymbol{n}}(Q_0 w)\right\|_{\partial K}^2\right)^{1/2}\right]. \tag{A.4}$$

Since the interpolant Q_0 preserves polynomials of degree up to $k+2$, it is easy to know

$$Q_0(\mathcal{P}_h^{k+2}\phi) = \mathcal{P}_h^{k+2}\phi.$$

Then by the triangle inequality, we have

$$\begin{aligned}
\|\Delta Q_0 w\|_{L^2(\mathcal{T}_h)} &= \|\Delta Q_0(\phi - \mathcal{P}_h^{k+2}\phi)\|_{L^2(\mathcal{T}_h)} \\
&\leqslant \|\Delta(Q_0\phi - \phi)\|_{L^2(\mathcal{T}_h)} + \|\Delta(\phi - \mathcal{P}_h^{k+2}\phi)\|_{L^2(\mathcal{T}_h)} \\
&\leqslant Ch^{\min\{k+1,\alpha\}}|\phi|_{H^{\alpha+2}(\Omega)}.
\end{aligned} \tag{A.5}$$

Since Q_n and Q_0 preserve the polynomials of orders $k+1$ and $k+2$, respectively, it holds that

$$\begin{aligned}
Q_n\left(\frac{\partial w}{\partial \boldsymbol{n}}\right) - \frac{\partial}{\partial \boldsymbol{n}}(Q_0 w) &= Q_n\left(\frac{\partial}{\partial \boldsymbol{n}}(\phi - \mathcal{P}_h^{k+2}\phi)\right) - \frac{\partial}{\partial \boldsymbol{n}}(Q_0(\phi - \mathcal{P}_h^{k+2}\phi)) \\
&= Q_n\left(\frac{\partial \phi}{\partial \boldsymbol{n}}\right) - \frac{\partial}{\partial \boldsymbol{n}}(Q_0\phi),
\end{aligned}$$

which together with (5.2) of Lemma 5.1 leads to

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h} h_K^{-1}\left\|Q_n\left(\frac{\partial w}{\partial \boldsymbol{n}}\right) - \frac{\partial}{\partial \boldsymbol{n}}(Q_0 w)\right\|_{\partial K}^2 &= \sum_{K\in\mathcal{T}_h} h_K^{-1}\left\|Q_n\left(\frac{\partial \phi}{\partial \boldsymbol{n}}\right) - \frac{\partial}{\partial \boldsymbol{n}}(Q_0\phi)\right\|_{\partial K}^2 \\
&\leqslant Ch^{2\min\{k+1,\alpha\}}|\phi|_{H^{\alpha+2}(\Omega)}^2.
\end{aligned} \tag{A.6}$$

Combining the estimates of (A.4)–(A.6) completes the proof of (A.2). □

Now, we are ready to give the proof of Lemma 6.1 below.

Proof of Lemma 6.1. In view of (A.1) of Lemma A.1, we have

$$\Delta_w Q_h(\mathcal{P}_h^{k+2}\phi) = \Delta(\mathcal{P}_h^{k+2}\phi)$$

on each element K of $\mathcal{T}_h$.

If $\alpha > k$, we have $|\Delta(\mathcal{P}_h^{k+2}\phi)|_{H^\alpha(\mathcal{T}_h)} = 0$ since $\Delta(\mathcal{P}_h^{k+2}\phi) \in \mathbb{P}_k(\mathcal{T}_h)$. Therefore, by the triangle inequality, we have

$$\begin{aligned}|\Delta_w(Q_h\phi)|_{H^\alpha(\mathcal{T}_h)} &= |\Delta_w Q_h(\phi - \mathcal{P}_h^{k+2}\phi) + \Delta(\mathcal{P}_h^{k+2}\phi)|_{H^\alpha(\mathcal{T}_h)} \\ &\leqslant |\Delta_w Q_h(\phi - \mathcal{P}_h^{k+2}\phi)|_{H^\alpha(\mathcal{T}_h)} + |\Delta(\mathcal{P}_h^{k+2}\phi)|_{H^\alpha(\mathcal{T}_h)} \\ &= |\Delta_w Q_h(\phi - \mathcal{P}_h^{k+2}\phi)|_{H^\alpha(\mathcal{T}_h)}.\end{aligned} \tag{A.7}$$

Then from the inverse inequality, (A.7) and (A.2) of Lemma A.2, it follows that

$$\begin{aligned}|\Delta_w(Q_h\phi)|_{H^\alpha(\mathcal{T}_h)} &\leqslant Ch^{-\alpha}\|\Delta_w Q_h(\phi - \mathcal{P}_h^{k+2}\phi)\|_{L^2(\mathcal{T}_h)} \\ &\leqslant Ch^{\min\{k+1-\alpha,0\}}|\phi|_{H^{\alpha+2}(\Omega)}.\end{aligned}$$

If $\alpha \leqslant k$, from the triangle inequality, the inverse inequality and (A.2) of Lemma A.2, we can infer that

$$\begin{aligned}|\Delta_w(Q_h\phi)|_{H^\alpha(\mathcal{T}_h)} &\leqslant |\Delta_w Q_h(\phi - \mathcal{P}_h^{k+2}\phi)|_{H^\alpha(\mathcal{T}_h)} + |\Delta(\phi - \mathcal{P}_h^{k+2}\phi)|_{H^\alpha(\mathcal{T}_h)} + |\Delta\phi|_{H^\alpha(\mathcal{T}_h)} \\ &\leqslant Ch^{-\alpha}\|\Delta_w Q_h(\phi - \mathcal{P}_h^{k+2}\phi)\|_{L^2(\mathcal{T}_h)} + Ch^{\min\{k+1-\alpha,0\}}|\phi|_{H^{\alpha+2}(\mathcal{T}_h)} \\ &\leqslant Ch^{\min\{k+1-\alpha,0\}}|\phi|_{H^{\alpha+2}(\mathcal{T}_h)}.\end{aligned}$$

Therefore, in all the cases, we have

$$|\Delta_w(Q_h\phi)|_{H^\alpha(\mathcal{T}_h)} \leqslant Ch^{\min\{k+1-\alpha,0\}}|\phi|_{H^{\alpha+2}(\mathcal{T}_h)},$$

as desired. $\square$