Non-weight modules over the mirror Heisenberg-Virasoro algebra

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Abstract In this paper, we study irreducible non-weight modules over the mirror Heisenberg-Virasoro algebra $\mathcal{D}$, including Whittaker modules, $\mathcal{U}(\mathbb{C}d_0)$-free modules and their tensor products. More precisely, we give the necessary and sufficient conditions for the Whittaker modules to be irreducible. We determine all the $\mathcal{D}$-module structures on $\mathcal{U}(\mathbb{C}d_0)$, and find the necessary and sufficient conditions for these modules to be irreducible. At last, we determine the necessary and sufficient conditions for the tensor products of Whittaker modules and $\mathcal{U}(\mathbb{C}d_0)$-free modules to be irreducible, and obtain that any two such tensor products are isomorphic if and only if the corresponding Whittaker modules and $\mathcal{U}(\mathbb{C}d_0)$-free modules are isomorphic. These lead to many new irreducible non-weight modules over $\mathcal{D}$.

Keywords mirror Heisenberg-Virasoro algebra, tensor product, Whittaker module, $U(\mathbb{C}d_0)$-free module, irreducible module

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1 Introduction

It is well known that the Virasoro algebra $\mathcal{V}$ is one of the most important Lie algebras in mathematics and in mathematical physics because of its widespread applications in quantum physics (see [21]), conformal field theory (see [15]), vertex operator algebras (see [16, 19]), and so on. Many other interesting and important algebras are closely related to the Virasoro algebra, such as the Schrödinger-Virasoro algebra (see [23, 24]), the twisted Heisenberg-Virasoro algebra (see [3, 25, 30]) and the mirror Heisenberg-Virasoro algebra $\mathcal{D}$ (see [6, 20, 28]) which is the even part of the mirror $N = 2$ superconformal algebra (see [6]). The mirror Heisenberg-Virasoro algebra $\mathcal{D}$ has a nice structure (see Definition 2.1) which is similar to the twisted Heisenberg-Virasoro algebra. This algebra is the main object we concern in this paper.

Harish-Chandra modules and weight modules with infinite-dimensional weight spaces have been the most popular modules in the representation theory for many Lie algebras with the triangular

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decomposition $\mathcal{G} = \mathcal{G}_+ \oplus \mathfrak{h} \oplus \mathcal{G}_-$. To some extent, Harish-Chandra modules are well understood for many infinite-dimensional Lie algebras, for example, the affine Kac-Moody algebras in [11, 22], the Virasoro algebra in [5, 17, 33], the twisted Heisenberg-Virasoro algebra in [3, 30], the Schrödinger-Virasoro algebra in [27, 38] and the mirror Heisenberg-Virasoro algebra in [28]. There are also some researches about weight modules with infinite-dimensional weight spaces (see [8, 13, 20, 32]).

Recently, non-weight modules over $\mathcal{G}$ attract much attention from mathematicians. In particular, Whittaker modules and $\mathcal{U}(\mathfrak{h})$-free $\mathcal{G}$-modules have been widely studied for many Lie algebras. Whittaker modules for $\mathrm{sl}_2(\mathbb{C})$ were constructed by Arnal and Pinzcon [4]. Whittaker modules for the arbitrary finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ were introduced and systematically studied by Kostant [26], where he proved that these modules with a fixed regular Whittaker function (Lie homomorphism) on a nilpotent radical are (up to isomorphism) in bijective correspondence with central characters of $\mathcal{U}(\mathcal{G})$. In recent years, Whittaker modules for many other Lie algebras have been investigated (see [1, 2, 7, 9, 14, 34, 35]). The notation of $\mathcal{U}(\mathfrak{h})$-free modules was first introduced by Nilsson [36] for the simple Lie algebra $\mathrm{sl}_{n+1}(\mathbb{C})$. At the same time, these modules were introduced in a very different approach in the paper [37]. Later, $\mathcal{U}(\mathfrak{h})$-free modules for many infinite-dimensional Lie algebras are determined, for example, the Kac-Moody algebras in [10], the Virasoro algebra in [31, 37], the Witt algebra in [37], the twisted Heisenberg-Virasoro algebra and $W(2, 2)$ algebra in [12], and so on. In the present paper, we study the Whittaker modules and $\mathcal{U}(\mathbb{C}d_0)$-free modules over $\mathcal{D}$. Also, we study the tensor products of Whittaker modules and $\mathcal{U}(\mathbb{C}d_0)$-free modules.

The rest of the paper is organized as follows. In Section 2, we recall notations related to the mirror Heisenberg-Virasoro algebra $\mathcal{D}$ and collect some known results on Whittaker modules and $\mathcal{U}(\mathbb{C}d_0)$-free modules over $\mathcal{V}$, including also two important lemmas on tensor product modules for later use. In Section 3, we give the necessary and sufficient conditions for the Whittaker modules W_{φ_m} over $\mathcal{D}$ to be irreducible (see Theorem 3.4). In Section 4, we determine all the $\mathcal{D}$-module structures on $\mathcal{U}(\mathbb{C}d_0)$, and find the necessary and sufficient conditions for these modules to be irreducible (see Theorem 4.3). In Section 5, we give the necessary and sufficient conditions for the tensor product of a Whittaker module W_{φ_m} and a $\mathcal{U}(\mathbb{C}d_0)$-free $\mathcal{D}$-module $\Omega(\lambda, \alpha, \beta)$ to be irreducible (see Theorem 5.3). Furthermore, we show that two such tensor product modules are isomorphic if and only if the corresponding Whittaker modules and $\mathcal{U}(\mathbb{C}d_0)$-free $\mathcal{D}$-modules are isomorphic (see Theorem 5.4). Consequently, we obtain a lot of new irreducible non-weight modules over $\mathcal{D}$.

Throughout this paper, we denote by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{C}$ and $\mathbb{C}^*$ the sets of integers, positive integers, non-negative integers, complex numbers and nonzero complex numbers, respectively. All the vector spaces and Lie algebras are over $\mathbb{C}$. We denote by $\mathcal{U}(\mathcal{G})$ the universal enveloping algebra for a Lie algebra $\mathcal{G}$.

2 Notations and preliminaries

For convenience, in this section we recall some notations and collect some known results.

Definition 2.1. The *mirror Heisenberg-Virasoro algebra* $\mathcal{D}$ is a Lie algebra with the basis

$$\left\{ d_m, h_r, \boldsymbol{c}, \boldsymbol{l} \,\middle|\, m \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z} \right\}$$

subject to the following commutation relations:

$$
\begin{aligned}
& [d_m, d_n] = (m - n)d_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}\boldsymbol{c}, \\
& [d_m, h_r] = -rh_{m+r}, \\
& [h_r, h_s] = r\delta_{r+s,0}\boldsymbol{l}, \\
& [\boldsymbol{c}, \mathcal{D}] = [\boldsymbol{l}, \mathcal{D}] = 0
\end{aligned}
$$

for $m, n \in \mathbb{Z}$ and $r, s \in \frac{1}{2} + \mathbb{Z}$.

The Lie subalgebra spanned by $\{d_m, \boldsymbol{c} \mid m \in \mathbb{Z}\}$ is the Virasoro algebra $\mathcal{V}$, and the Lie subalgebra spanned by $\{h_r, \boldsymbol{l} \mid r \in \frac{1}{2} + \mathbb{Z}\}$ is the twisted Heisenberg algebra $\mathcal{H}$. Moreover, $\mathcal{H}$ is an ideal of $\mathcal{D}$ and $\mathcal{D}$ is the semi-direct product of $\mathcal{V}$ and $\mathcal{H}$. It is clear that $\mathcal{D}$, $\mathcal{V}$ and $\mathcal{H}$ are all $\mathbb{Z}$-graded Lie algebras.

Definition 2.2.   Let $\mathcal{G} = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}_i$ (resp. $\mathcal{G} = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \mathcal{G}_i$) be a $\mathbb{Z}$ (resp. $\frac{1}{2}\mathbb{Z}$)-graded Lie algebra. A $\mathcal{G}$-module V is called the *restricted* module if for any $v \in V$, there exists an $n \in \mathbb{N}$ such that $\mathcal{G}_i v = 0$ for $i > n$ (resp. $i > \frac{1}{2}n$). The category of restricted modules over $\mathcal{G}$ will be denoted by $\mathcal{R}_\mathcal{G}$.

Note that if V is a $\mathcal{V}$-module, then V can be easily viewed as a $\mathcal{D}$-module by defining $\mathcal{H}V = 0$, and the resulting module is denoted by $V^\mathcal{D}$. Generalizing the result in [18] from highest weight modules to restricted modules over $\mathcal{H}$, for any $H \in \mathcal{R}_\mathcal{H}$ with the action of $\boldsymbol{l}$ as a nonzero scalar l, we can give H a $\mathcal{D}$-module structure denoted by $H^\mathcal{D}$ via the following map:

$$d_n \mapsto \frac{1}{2l} \sum_{k \in \mathbb{Z} + \frac{1}{2}} h_{n-k} h_k, \quad \forall\, n \in \mathbb{Z}, \quad n \neq 0, \tag{2.1}$$

$$d_0 \mapsto \frac{1}{2l} \sum_{k \in \mathbb{Z} + \frac{1}{2}} h_{-|k|} h_{|k|} + \frac{1}{16}, \tag{2.2}$$

$$h_r \mapsto h_r, \quad \forall\, r \in \frac{1}{2} + \mathbb{Z}, \quad \boldsymbol{c} \mapsto 1, \quad \boldsymbol{l} \mapsto l. \tag{2.3}$$

Definition 2.3.   Let $\mathcal{G}$ be a Lie algebra. Suppose that f is an automorphism of $\mathcal{G}$ and V is a $\mathcal{G}$-module. The following actions:

$$x \cdot v = f(x)v, \quad \forall\, x \in \mathcal{G}, v \in V$$

give V a new $\mathcal{G}$-module structure, denoted by V'. Then V and V' are called *equivalent* $\mathcal{G}$-modules.

Remark 2.4.   It is easy to see that equivalent modules over the Lie algebra have the same irreducibility.

For convenience, we define the following subalgebras: for any $m, n \in \mathbb{Z}_+$, set

$$\mathcal{D}^{(m,-n)} = \sum_{i \geqslant m} \mathbb{C}d_i \oplus \sum_{i \in \mathbb{Z}_+} \mathbb{C}h_{-n+i+\frac{1}{2}} \oplus \mathbb{C}\boldsymbol{c} \oplus \mathbb{C}\boldsymbol{l},$$

$$\mathcal{D}^{(m,-\infty)} = \sum_{i \in \mathbb{Z}_+} \mathbb{C}d_{m+i} \oplus \sum_{i \in \mathbb{Z}} \mathbb{C}h_{i+\frac{1}{2}} + \mathbb{C}\boldsymbol{c} + \mathbb{C}\boldsymbol{l},$$

$$\mathcal{V}^{(m)} = \sum_{i \in \mathbb{Z}_+} \mathbb{C}d_{m+i} + \mathbb{C}\boldsymbol{c},$$

$$\mathcal{H}^{(m)} = \sum_{i \in \mathbb{Z}_+} \mathbb{C}h_{m+i+\frac{1}{2}} + \mathbb{C}\boldsymbol{l}.$$

Lemma 2.5 (See [28, Theorem 6.2]).   *Let V be an irreducible $\mathcal{D}^{(0,-q)}$-module for some $q \in \mathbb{Z}_+$ such that the action of $\boldsymbol{c}$ and $\boldsymbol{l}$ on V are the scalars c and 0, respectively. Assume that there exists an integer $t \geqslant -q$ satisfying the following two conditions*:
  (a) *The action of $h_{t+\frac{1}{2}}$ on V is bijective.*
  (b) *$h_{n+\frac{1}{2}}V = 0 = d_{n+q}V$ for all $n > t$.*
Then the induced $\mathcal{D}$-module $\mathrm{Ind}_{\mathcal{D}^{(0,-q)}}^{\mathcal{D}}(V)$ is irreducible.

Recall that for any $m \in \mathbb{Z}_+$, let $\psi_m : \mathcal{V}^{(m)} \to \mathbb{C}$ be a Whittaker function, i.e., a Lie algebra homomorphism. Then we have the one-dimensional module $\mathbb{C}w_{\psi_m}$ over $\mathcal{V}^{(m)}$ with $x \cdot w_{\psi_m} = \psi_m(x)w_{\psi_m}$ for $x \in \mathcal{V}^{(m)}$. The induced $\mathcal{V}$-module

$$W_{\psi_m} = \mathrm{Ind}_{\mathcal{V}^{(m)}}^{\mathcal{V}} \mathbb{C}w_{\psi_m} \tag{2.4}$$

is called the *Whittaker module* over $\mathcal{V}$ with respect to ψ_m.

Lemma 2.6 (See [29, Theorem 7]).   *For any $m \in \mathbb{N}$ and any Whittaker function $\psi_m : \mathcal{V}^{(m)} \to \mathbb{C}$, the Whittaker module W_{ψ_m} is irreducible if and only if $(\psi(d_{2m}), \psi(d_{2m-1})) \neq (0,0)$.*

For $\lambda \in \mathbb{C}^*$ and $\alpha \in \mathbb{C}$, denote by $\Omega(\lambda, \alpha) = \mathbb{C}[t]$ the polynomial algebra over $\mathbb{C}$. It is well known that $\Omega(\lambda, \alpha)$ is a $\mathcal{V}$-module with the actions

$$\boldsymbol{c}(f(t)) = 0, \quad d_m(f(t)) = \lambda^m(t + m\alpha)f(t + m), \quad \forall m \in \mathbb{Z}.$$

Thanks to [31], we know that $\Omega(\lambda, \alpha)$ is irreducible if and only if $\alpha \neq 0$, and $\Omega(\lambda, 0)$ has an irreducible submodule $t\Omega(\lambda, \alpha)$. Moreover, we have the following lemma.

Lemma 2.7 (See [37, Theorem 3]). *Let V be a $\mathcal{V}$-module. Assume that V viewed as a $\mathcal{U}(\mathbb{C}d_0)$-module is free of rank 1. Then $V \cong \Omega(\lambda, \alpha)$ for some $\lambda \in \mathbb{C}^*$ and $\alpha \in \mathbb{C}$.*

We conclude this section by recalling two results about the tensor product for later use.

Lemma 2.8 (See [32, Lemma 8]). *Let V be a module over a Lie algebra $\mathcal{G}$, $\mathfrak{g}$ be a subalgebra of $\mathcal{G}$, and W be a $\mathfrak{g}$-module. Then the $\mathcal{G}$-module homomorphism $\tau : \mathrm{Ind}_{\mathfrak{g}}^{\mathcal{G}}(V \otimes W) \to V \otimes \mathrm{Ind}_{\mathfrak{g}}^{\mathcal{G}}(W)$ induced from the inclusion map $V \otimes W \to V \otimes \mathrm{Ind}_{\mathfrak{g}}^{\mathcal{G}}(W)$ is a $\mathcal{G}$-module isomorphism.*

Lemma 2.9 (See [20, Lemma 3.1]). *Suppose that V and W are $\mathcal{V}$-modules, and $H, K \in \mathcal{R}_{\mathcal{H}}$ are irreducible with nonzero action of $\boldsymbol{l}$. Then*

(1) any $\mathcal{D}$-submodule of $V^{\mathcal{D}} \otimes H^{\mathcal{D}}$ is of the form $(V')^{\mathcal{D}} \otimes H^{\mathcal{D}}$ for some $\mathcal{V}$-submodule V' of V; in particular, $V^{\mathcal{D}} \otimes H^{\mathcal{D}}$ is an irreducible $\mathcal{D}$-module if and only if V is an irreducible $\mathcal{V}$-module;

(2) $V^{\mathcal{D}} \otimes H^{\mathcal{D}} \cong W^{\mathcal{D}} \otimes K^{\mathcal{D}}$ if and only if $V \cong W$ and $H \cong K$.

3 Whittaker modules over $\mathcal{D}$

In this section, we determine the necessary and sufficient conditions for the Whittaker modules over $\mathcal{D}$ to be irreducible.

Assume that $\phi : \mathcal{H}^{(0)} \to \mathbb{C}$ is a Whittaker function, and that $\mathbb{C}w_\phi$ is the one-dimensional module over $\mathcal{H}^{(0)}$ defined by $x \cdot w_\phi = \phi(x)w_\phi$ for $x \in \mathcal{H}^{(0)}$. Then for the Whittaker $\mathcal{H}$-module

$$W_\phi = \mathrm{Ind}_{\mathcal{H}^{(0)}}^{\mathcal{H}}\mathbb{C}w_\phi,$$

we have the following result.

Lemma 3.1. *The Whittaker $\mathcal{H}$-module W_ϕ is irreducible if and only if $\phi(\boldsymbol{l}) \neq 0$.*

Proof. We first assume that $\phi(\boldsymbol{l}) \neq 0$. For any nonzero $v \in W_\phi$, v is a linear combination of vectors in the form

$$h_{-r-\frac{1}{2}}^{i_r} \cdots h_{-1-\frac{1}{2}}^{i_1} h_{-\frac{1}{2}}^{i_0} w_\phi,$$

where $i_0, i_1, \ldots, i_r \in \mathbb{Z}_+$ by the PBW theorem. Let $\langle v \rangle$ denote the submodule generated by v of W_ϕ. It is not hard to see $w_\phi \in \langle v \rangle$. Thus $\langle v \rangle = W_\phi$, which implies that W_ϕ is irreducible.

Now we assume that $\phi(\boldsymbol{l}) = 0$. Then it is easy to see that $h_{-\frac{1}{2}}w_\phi$ is a Whittaker vector, which implies that $h_{-\frac{1}{2}}w_\phi$ generates a nonzero proper submodule of W_ϕ. So W_ϕ is not irreducible. $\square$

Now for any $m \in \mathbb{Z}_+$, let $\varphi_m : \mathcal{D}^{(m,0)} \to \mathbb{C}$ be a Whittaker function, and $\mathbb{C}w_{\varphi_m}$ be the one-dimensional module over $\mathcal{D}^{(m,0)}$ defined by $x \cdot w_{\varphi_m} = \varphi_m(x)w_{\varphi_m}$ for $x \in \mathcal{D}^{(m,0)}$. In the rest of this section, we determine the irreducibility of the Whittaker $\mathcal{D}$-module $W_{\varphi_m} = \mathrm{Ind}_{\mathcal{D}^{(m,0)}}^{\mathcal{D}}\mathbb{C}w_{\varphi_m}$. Each irreducible quotient of W_{φ_m} (if it exists) is called the irreducible Whittaker $\mathcal{D}$-module with respect to φ_m. It is clear that

$$\varphi_m(d_{2m+j}) = \varphi_m(h_{m+j-\frac{1}{2}}) = 0, \quad \forall j \in \mathbb{N},$$

since $\varphi_m([\mathcal{D}^{(m,0)}, \mathcal{D}^{(m,0)}]) = 0$. In particular, if $m = 0$, then W_{φ_m} is a Verma module, which has been studied in [28].

First, we consider W_{φ_m} with $\varphi_m(\boldsymbol{l}) = l \neq 0$. We define a new Whittaker function $\varphi_m' : \mathcal{V}^{(m)} \to \mathbb{C}$ as follows:

$$\varphi_m'(\boldsymbol{c}) = \varphi_m(\boldsymbol{c}) - 1,$$

$$\varphi_m'(d_k) = 0, \quad \forall\, k \geqslant 2m+1,$$

$$\varphi_m'(d_k) = \varphi_m(d_k) - \frac{1}{2l} \sum_{i=0}^{m-1} \varphi_m(h_{i+\frac{1}{2}}) \varphi_m(h_{k-i-\frac{1}{2}}) - \delta_{0,k} \frac{1}{16}, \quad \forall\, m \leqslant k \leqslant 2m.$$

Then we have the Whittaker $\mathcal{V}$-module

$$W_{\varphi_m'} = \mathrm{Ind}_{\mathcal{V}^{(m)}}^{\mathcal{V}} \mathbb{C} w_{\varphi_m'},$$

where $\mathbb{C} w_{\varphi_m'}$ is the one-dimensional module over $\mathcal{V}^{(m)}$ defined by $x \cdot w_{\varphi_m'} = \varphi_m'(x) w_{\varphi_m'}$ for $x \in \mathcal{V}^{(m)}$.

Proposition 3.2. *Suppose that $m \in \mathbb{Z}_+$, and φ_m and φ_m' are given as above with $\varphi_m(l) \neq 0$. Let $H = U(\mathcal{H}) w_{\varphi_m}$ be in W_{φ_m}. The following observations hold:*

(1) $W_{\varphi_m} \cong H^{\mathcal{D}} \otimes W_{\varphi_m'}^{\mathcal{D}}$. In particular, this agrees with [28, Proposition 5.1] if $m = 0$.

(2) Suppose $m \in \mathbb{N}$. Then W_{φ_m} is an irreducible $\mathcal{D}$-module if and only if

$$\varphi_m(d_{2m}) \neq 0 \quad or \quad 2\varphi_m(l)\varphi_m(d_{2m-1}) - \varphi_m(h_{m-\frac{1}{2}})^2 \neq 0.$$

(3) Each irreducible Whittaker module over $\mathcal{D}$ with respect to φ_m is isomorphic to $H^{\mathcal{D}} \otimes Q^{\mathcal{D}}$, where Q is an irreducible quotient of $\mathcal{V}$-module $W_{\varphi_m'}$.

Proof. (1) For $\mathcal{V}^{(m)}$-module $\mathbb{C} w_{\varphi_m'}$, we can easily give $\mathbb{C} w_{\varphi_m'}$ a $\mathcal{D}^{(m,-\infty)}$-module structure by defining $\mathcal{H} w_{\varphi_m'} = 0$. The resulting module is denoted by $\mathbb{C}_{\varphi_m'}^{\mathcal{D}^{(m,-\infty)}}$. By the PBW theorem, we can define the linear map

$$\pi : \mathrm{Ind}_{\mathcal{D}^{(m,0)}}^{\mathcal{D}^{(m,-\infty)}} \mathbb{C} w_{\varphi_m} \to H^{\mathcal{D}} \otimes \mathbb{C}_{\varphi_m'}^{\mathcal{D}^{(m,-\infty)}},$$

$$h_{-r-\frac{1}{2}}^{q_r} \cdots h_{-1-\frac{1}{2}}^{q_1} h_{-\frac{1}{2}}^{q_0} w_{\varphi_m} \mapsto h_{-r-\frac{1}{2}}^{q_r} \cdots h_{-1-\frac{1}{2}}^{q_1} h_{-\frac{1}{2}}^{q_0} w_{\varphi_m} \otimes w_{\varphi_m'}.$$

It is clear that π is a bijection.

Claim 1. $\pi(h w_{\varphi_m}) = h w_{\varphi_m} \otimes w_{\varphi_m'}$ for $h \in \mathcal{U}(\mathcal{H})$.

For $h \in \mathcal{U}(\mathcal{H})$, h can be written as a linear combination of vectors in the form

$$h_{-r_1-\frac{1}{2}}^{q_{r_1}} \cdots h_{-1-\frac{1}{2}}^{q_1} h_{-\frac{1}{2}}^{q_0} l^t h_{\frac{1}{2}}^{p_0} h_{1+\frac{1}{2}}^{p_1} \cdots h_{s_1+\frac{1}{2}}^{p_{s_1}}$$

by the PBW theorem. Without loss of generality, we can assume

$$h = h_{-r_1-\frac{1}{2}}^{q_{r_1}} \cdots h_{-1-\frac{1}{2}}^{q_1} h_{-\frac{1}{2}}^{q_0} l^t h_{\frac{1}{2}}^{p_0} h_{1+\frac{1}{2}}^{p_1} \cdots h_{s_1+\frac{1}{2}}^{p_{s_1}},$$

where $q_i, p_j, t, r_1, s_1 \in \mathbb{Z}_+$ for $0 \leqslant i \leqslant r_1$ and $0 \leqslant j \leqslant s_1$. We compute

$$
\begin{aligned}
\pi(h w_{\varphi_m}) &= \pi(h_{-r_1-\frac{1}{2}}^{q_{r_1}} \cdots h_{-1-\frac{1}{2}}^{q_1} h_{-\frac{1}{2}}^{q_0} l^t h_{\frac{1}{2}}^{p_0} h_{1+\frac{1}{2}}^{p_1} \cdots h_{s_1+\frac{1}{2}}^{p_{s_1}} w_{\varphi_m}) \\
&= \varphi_m(l)^t \varphi_m(h_{\frac{1}{2}})^{p_0} \varphi_m(h_{1+\frac{1}{2}})^{p_1} \cdots \varphi_m(h_{s_1+\frac{1}{2}})^{p_{s_1}} \pi(h_{-r_1-\frac{1}{2}}^{q_{r_1}} \cdots h_{-1-\frac{1}{2}}^{q_1} h_{-\frac{1}{2}}^{q_0} w_{\varphi_m}) \\
&= \varphi_m(l)^t \varphi_m(h_{\frac{1}{2}})^{p_0} \varphi_m(h_{1+\frac{1}{2}})^{p_1} \cdots \varphi_m(h_{s_1+\frac{1}{2}})^{p_{s_1}} h_{-r_1-\frac{1}{2}}^{q_{r_1}} \cdots h_{-1-\frac{1}{2}}^{q_1} h_{-\frac{1}{2}}^{q_0} w_{\varphi_m} \otimes w_{\varphi_m'} \\
&= h_{-r_1-\frac{1}{2}}^{q_{r_1}} \cdots h_{-1-\frac{1}{2}}^{q_1} h_{-\frac{1}{2}}^{q_0} \varphi_m(l)^t \varphi_m(h_{\frac{1}{2}})^{p_0} \varphi_m(h_{1+\frac{1}{2}})^{p_1} \cdots \varphi_m(h_{s_1+\frac{1}{2}})^{p_{s_1}} w_{\varphi_m} \otimes w_{\varphi_m'} \\
&= h_{-r_1-\frac{1}{2}}^{q_{r_1}} \cdots h_{-1-\frac{1}{2}}^{q_1} h_{-\frac{1}{2}}^{q_0} l^t h_{\frac{1}{2}}^{p_0} h_{1+\frac{1}{2}}^{p_1} \cdots h_{s_1+\frac{1}{2}}^{p_{s_1}} w_{\varphi_m} \otimes w_{\varphi_m'} \\
&= h w_{\varphi_m} \otimes w_{\varphi_m'}.
\end{aligned}
$$

This proves Claim 1.

Next, we show that π is a $\mathcal{D}^{(m,-\infty)}$-module homomorphism. Notice that for any

$$v = h_{-k-\frac{1}{2}}^{q_k} \cdots h_{-1-\frac{1}{2}}^{q_1} h_{-\frac{1}{2}}^{q_0} w_{\varphi_m} \in \mathrm{Ind}_{\mathcal{D}^{(m,0)}}^{\mathcal{D}^{(m,-\infty)}} \mathbb{C} w_{\varphi_m},$$

where $q_i, k \in \mathbb{Z}_+$ and $0 \leqslant i \leqslant k$, we have the following claim.

Claim 2. $\pi(h_r v) = h_r \pi(v)$ and $\pi(d_n v) = d_n \pi(v)$ for any $r \in \mathbb{Z} + \frac{1}{2}$ and $n \in \mathbb{Z}$ with $n \geqslant m$.

By Claim 1 and the definition of π, we deduce

$$
\begin{aligned}
\pi(h_r v) &= \pi(h_r h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}} w_{\varphi_m}) = h_r h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}} w_{\varphi_m} \otimes w_{\varphi'_m} \\
&= h_r(h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}} w_{\varphi_m} \otimes w_{\varphi'_m}) = h_r \pi(h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}} w_{\varphi_m}) \\
&= h_r \pi(v)
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
\pi(d_n h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}} w_{\varphi_m}) &= \pi([d_n, h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}}] w_{\varphi_m} + h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}} d_n w_{\varphi_m}) \\
&= [d_n, h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}}] w_{\varphi_m} \otimes w_{\varphi'_m} \\
&\quad + \varphi_m(d_n) h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}} w_{\varphi_m} \otimes w_{\varphi'_m}.
\end{aligned}
\tag{3.2}
$$

Using the definitions of π and φ'_m and the equalities (2.1) and (2.2), we see

$$
\begin{aligned}
&d_n \pi(h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}} w_{\varphi_m}) \\
&= d_n(h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}} w_{\varphi_m} \otimes w_{\varphi'_m}) \\
&= d_n h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}} w_{\varphi_m} \otimes w_{\varphi'_m} + h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}} w_{\varphi_m} \otimes d_n w_{\varphi'_m} \\
&= [d_n, h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}}] w_{\varphi_m} \otimes w_{\varphi'_m} + h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}} d_n w_{\varphi_m} \otimes w_{\varphi'_m} \\
&\quad + h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}} w_{\varphi_m} \otimes \varphi'_m(d_n) w_{\varphi'_m} \\
&= [d_n, h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}}] w_{\varphi_m} \otimes w_{\varphi'_m} + \varphi_m(d_n) h^{q_k}_{-k-\frac{1}{2}} \cdots h^{q_1}_{-1-\frac{1}{2}} h^{q_0}_{-\frac{1}{2}} w_{\varphi_m} \otimes w_{\varphi'_m}.
\end{aligned}
\tag{3.3}
$$

Combining the equalities (3.1)–(3.3), we have proven Claim 2. Furthermore,

$$
\pi(\boldsymbol{c}v) = \boldsymbol{c}\pi(v), \quad \pi(\boldsymbol{l}v) = \boldsymbol{l}\pi(v), \quad \forall\, v \in \mathrm{Ind}_{\mathcal{D}^{(m,0)}}^{\mathcal{D}^{(m,-\infty)}} \mathbb{C} w_{\varphi_m}
$$

are clear. So $\mathrm{Ind}_{\mathcal{D}^{(m,0)}}^{\mathcal{D}^{(m,-\infty)}} \mathbb{C} w_{\varphi_m} \cong H^{\mathcal{D}} \otimes \mathbb{C}_{\varphi'_m}^{\mathcal{D}^{(m,-\infty)}}$ as $\mathcal{D}^{(m,-\infty)}$-modules. Then by the property of induced modules and Lemma 2.8, we have

$$
\begin{aligned}
W_{\varphi_m} = \mathrm{Ind}_{\mathcal{D}^{(m,0)}}^{\mathcal{D}} \mathbb{C} w_{\varphi_m} &\cong \mathrm{Ind}_{\mathcal{D}^{(m,-\infty)}}^{\mathcal{D}} (\mathrm{Ind}_{\mathcal{D}^{(m,0)}}^{\mathcal{D}^{(m,-\infty)}} \mathbb{C} w_{\varphi_m}) \cong \mathrm{Ind}_{\mathcal{D}^{(m,-\infty)}}^{\mathcal{D}} (H^{\mathcal{D}} \otimes \mathbb{C}_{\varphi'_m}^{\mathcal{D}^{(m,-\infty)}}) \\
&\cong H^{\mathcal{D}} \otimes \mathrm{Ind}_{\mathcal{D}^{(m,-\infty)}}^{\mathcal{D}} (\mathbb{C}_{\varphi'_m}^{\mathcal{D}^{(m,-\infty)}}) \cong H^{\mathcal{D}} \otimes W_{\varphi'_m}^{\mathcal{D}}.
\end{aligned}
$$

(2) From Lemma 3.1, we know that $H^{\mathcal{D}}$ is an irreducible $\mathcal{D}$-module. Then from (1), Lemmas 2.6 and 2.9, we obtain the statement in (2).

(3) follows from (1) and Lemmas 2.9 and 3.1. $\qquad\square$

Now we consider the Whittaker $\mathcal{D}$-module W_{φ_m} with $\varphi_m(\boldsymbol{l}) = 0$. Let $\alpha = \sum_{i \in \mathbb{Z}} \frac{2a_i}{2i-1} h_{i-\frac{1}{2}}$, where $a_i \in \mathbb{C}$ and only finitely many of a_i's are nonzero. Then we have the automorphism $\theta_\alpha = \exp(\mathrm{ad}\alpha)$ of $\mathcal{D}$ such that

$$
\begin{aligned}
\theta_\alpha(d_n) &= d_n + \sum_{i \in \mathbb{Z}} a_i h_{n+i-\frac{1}{2}} + \frac{1}{2} \sum_{i \in \mathbb{Z}} a_i a_{-n-i+1} \boldsymbol{l}, \\
\theta_\alpha(h_r) &= h_r + a_{-r+\frac{1}{2}} \boldsymbol{l}, \\
\theta_\alpha(\boldsymbol{c}) &= \boldsymbol{c}, \quad \theta_\alpha(\boldsymbol{l}) = \boldsymbol{l}
\end{aligned}
$$

for $n \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$.

Proposition 3.3. *Suppose that $m \geqslant 1$ and $\varphi_m : \mathcal{D}^{(m,0)} \to \mathbb{C}$ is a Whittaker function with $\varphi_m(\boldsymbol{l}) = 0$. Then the Whittaker $\mathcal{D}$-module W_{φ_m} is irreducible if and only if $\varphi_m(h_{m-\frac{1}{2}}) \neq 0$.*

Proof.　Sufficiency. Since $\varphi_m(h_{m-\frac{1}{2}}) \neq 0$, we may choose $a_0, a_{-1}, \ldots, a_{-m} \in \mathbb{C}$ such that

$$
\begin{pmatrix} \varphi_m(d_m) \\ \varphi_m(d_{m+1}) \\ \vdots \\ \varphi_m(d_{2m}) \end{pmatrix} = \begin{pmatrix} \varphi_m(h_{m-\frac{1}{2}}) & \varphi_m(h_{m-\frac{3}{2}}) & \cdots & \varphi_m(h_{-\frac{1}{2}}) \\ 0 & \varphi_m(h_{m-\frac{1}{2}}) & \cdots & \varphi_m(h_{\frac{1}{2}}) \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi_m(h_{m-\frac{1}{2}}) \end{pmatrix} \begin{pmatrix} a_0 \\ a_{-1} \\ \vdots \\ a_{-m} \end{pmatrix},
$$

where we define $\varphi_m(h_{-\frac{1}{2}}) = 0$. Then

$$
0 = \varphi_m(d_n) + \sum_{i=-m}^{0} (-a_i)\varphi_m(h_{n+i-\frac{1}{2}}) + \frac{1}{2}\sum_{i\in\mathbb{Z}} a_i a_{-n-i+1}\varphi_m(\boldsymbol{l}), \quad \forall n \geqslant m, \tag{3.4}
$$

since $\varphi_m(\boldsymbol{l}) = 0$ and $\varphi_m(d_{2m+j}) = \varphi_m(h_{m-\frac{1}{2}+j}) = 0$ for all $j \in \mathbb{N}$. Define $\alpha = -\sum_{i=-m}^{0} \frac{2a_i}{2i-1} h_{i-\frac{1}{2}}$. Then we have the Lie algebra automorphism of $\theta_\alpha : \mathcal{D} \to \mathcal{D}$. Let $W_{\varphi_m}^{\theta_\alpha}$ be the new Whittaker module with the action $x \circ w_{\varphi_m} = \theta_\alpha(x)w_{\varphi_m}$ for any $x \in \mathcal{D}$, which is equivalent to W_{φ_m}. By the definition of θ_α and the equality (3.4), it is easy to see that $d_k \circ w_{\varphi_m} = 0$ for $k > m - 1$, and the actions of $\mathcal{H}$ and $\boldsymbol{c}$ are unchanged. Therefore, without loss of generality, we may assume that

$$
\varphi_m(d_k) = 0, \quad \forall k > m - 1. \tag{3.5}
$$

Define $W_0 = \mathrm{Ind}_{\mathcal{D}^{(m,0)}}^{\mathcal{D}^{(0,0)}} \mathbb{C}w_{\varphi_m}$. Then $\{d_{m-1}^{i_{m-1}} \cdots d_0^{i_0} w_{\varphi_m} \mid (i_{m-1}, \ldots, i_0) \in \mathbb{Z}_+^m\}$ is a basis of W_0 by the PBW theorem. It is not hard to see that $h_{m-\frac{1}{2}}$ acts injectively on W_0.

Claim 3.　W_0 is an irreducible $\mathcal{D}^{(0,0)}$-module.

By the PBW theorem, for any $v \in W_0$, we may write v in the form of

$$
v = \sum_{\boldsymbol{i}\in\mathbb{Z}_+^m} a_{\boldsymbol{i}} d^{\boldsymbol{i}} w_{\varphi_m},
$$

where $d^{\boldsymbol{i}} = d_{m-1}^{i_{m-1}} \cdots d_0^{i_0}$, and only finitely many $a_{\boldsymbol{i}} \in \mathbb{C}$ are nonzero. Denote by $\mathrm{supp}(v)$ the set of all $\boldsymbol{i} \in \mathbb{Z}_+^m$ such that $a_{\boldsymbol{i}} \neq 0$. Let $\succ$ denote the **reverse lexicographical** total order on $\mathbb{Z}_+^m$, i.e., for any $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}_+^m$,

$$\boldsymbol{j} \succ \boldsymbol{i} \Leftrightarrow \text{there exists } 0 \leqslant k \leqslant m - 1 \text{ such that } (j_s = i_s, \forall 0 \leqslant s < k) \text{ and } j_k > i_k.$$

Let $\deg(v)$ be the maximal element in $\mathrm{supp}(v)$ with respect to the reverse lexicographical total order on $\mathbb{Z}_+^m$. Note that $\deg(0)$ is not defined and whenever we write $\deg(v)$ we mean that $v \neq 0$. Suppose that $v \in W_0 \setminus \mathbb{C}w_{\varphi_m}$, $\deg(v) = \boldsymbol{i}$ and $p = \min\{s : i_s \neq 0\}$. Then it is straightforward to check that

$$
\deg((h_{m-p-\frac{1}{2}} - \varphi_m(h_{m-p-\frac{1}{2}}))v) = \boldsymbol{i} - \epsilon_p,
$$

where ϵ_p denotes the element $(\ldots, 0, 1, 0, \ldots) \in \mathbb{Z}_+^m$ with 1 being in the $(p+1)$-th position from right. Thus W_0 is an irreducible $\mathcal{D}^{(0,0)}$-module. Claim 3 is proved.

Therefore, $W_{\varphi_m} \cong \mathrm{Ind}_{\mathcal{D}^{(0,0)}}^{\mathcal{D}} W_0$ is an irreducible $\mathcal{D}$-module by Lemma 2.5 (with $q = 0$ and $t = m - 1$).

Necessity. Assume that $\varphi_m(h_{m-\frac{1}{2}}) = 0$. Let $w = h_{-\frac{1}{2}} w_{\varphi_m} \in W_{\varphi_m}$. Then it is clear that $\boldsymbol{l}w = 0$, $\boldsymbol{c}w = \varphi_m(\boldsymbol{c})w$ and

$$
d_{m+i}w = \varphi_m(d_{m+i})w \quad \text{and} \quad h_{i+\frac{1}{2}}w = \varphi_m(h_{i+\frac{1}{2}})w, \quad \forall i \in \mathbb{Z}_+,
$$

since $\varphi_m(\boldsymbol{l}) = 0$ and $h_{i-\frac{1}{2}} w_{\varphi_m} = 0$ for $i \geqslant m$. Let $\langle w \rangle$ denote the submodule generated by w of W_{φ_m}. By the PBW theorem, we know that $\langle w \rangle$ has a basis

$$
d_s^{p_s} \cdots d_{m-2}^{p_{m-2}} d_{m-1}^{p_{m-1}} h_{-r-\frac{1}{2}}^{q_r} \cdots h_{-1-\frac{1}{2}}^{q_1} h_{-\frac{1}{2}}^{q_0} w,
$$

where $p_i, q_j, r \in \mathbb{Z}_+, s \in \mathbb{Z}_{<m}$ for $s \leqslant i \leqslant m - 1$ and $0 \leqslant j \leqslant r$. It is easy to see that $w_{\varphi_m} \notin \langle w \rangle$, which implies that $\langle w \rangle$ is a nonzero proper submodule of W_{φ_m}. So W_{φ_m} is not irreducible if $\varphi_m(h_{m-\frac{1}{2}}) = 0$. $\quad\square$

Now we combine Propositions 3.2 and 3.3 into the following main theorem.

Theorem 3.4. *Let* $m \in \mathbb{N}$ *and* $\varphi_m : \mathcal{D}^{(m,0)} \to \mathbb{C}$ *be a Whittaker function.*

(1) *If* $\varphi_m(\boldsymbol{l}) \neq 0$, *then the Whittaker* $\mathcal{D}$-*module* W_{φ_m} *is irreducible if and only if*

$$\varphi_m(d_{2m}) \neq 0 \quad or \quad 2\varphi_m(\boldsymbol{l})\varphi_m(d_{2m-1}) - \varphi_m(h_{m-\frac{1}{2}})^2 \neq 0.$$

(2) *If* $\varphi_m(\boldsymbol{l}) = 0$, *then the Whittaker* $\mathcal{D}$-*module* W_{φ_m} *is irreducible if and only if* $\varphi_m(h_{m-\frac{1}{2}}) \neq 0$.

4 $\mathcal{U}(\mathbb{C}d_0)$-free modules over $\mathcal{D}$

In this section, we determine the $\mathcal{D}$-module structure on $\mathcal{U}(\mathbb{C}d_0)$. Also, we give the necessary and sufficient conditions for these modules to be irreducible.

For any $\lambda \in \mathbb{C}^*$ and $\alpha, \beta \in \mathbb{C}$, denote by $\Omega(\lambda, \alpha, \beta) = \mathbb{C}[t]$ the polynomial algebra over $\mathbb{C}$. It is not hard to see that we can give $\Omega(\lambda, \alpha, \beta)$ a $\mathcal{D}$-module structure via the following actions:

$$d_m(f(t)) = \lambda^m(t + m\alpha)f(t + m),$$
$$h_r(f(t)) = \beta\lambda^r f(t + r),$$
$$\boldsymbol{c}(f(t)) = \boldsymbol{l}(f(t)) = 0$$

for $m \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}$ and $f(t) \in \Omega(\lambda, \alpha, \beta)$.

Remark 4.1. In the above actions, we always fix a $\lambda_0 \in \mathbb{C}^*$ such that $\lambda = \lambda_0^2$. Then we denote $\lambda^r = \lambda_0^{2r}$ for $r \in \mathbb{Z} + \frac{1}{2}$.

Moreover, we have the following lemma.

Lemma 4.2. (1) $\Omega(\lambda, 0, 0)$ *has an irreducible submodule* $t\Omega(\lambda, 0, 0)$.

(2) $\Omega(\lambda, \alpha, \beta)$ *is an irreducible* $\mathcal{D}$-*module if and only if* $\alpha \neq 0$ *or* $\beta \neq 0$.

Proof. It is clear from the irreducibility of $\mathcal{V}$-module $\Omega(\lambda, \alpha)$ and some simple computations. □

Now, we state the main result of this section.

Theorem 4.3. *Let* M *be a* $\mathcal{U}(\mathcal{D})$-*module such that* M, *when considered as a* $\mathcal{U}(\mathbb{C}d_0)$-*module, is free of rank* 1. *Then* $M \cong \Omega(\lambda, \alpha, \beta)$ *for some* $\lambda \in \mathbb{C}^*$ *and* $\alpha, \beta \in \mathbb{C}$. *Moreover,* M *is irreducible if and only if* $M \cong \Omega(\lambda, \alpha, \beta)$ *for some* $\lambda \in \mathbb{C}^*$ *and* $\alpha, \beta \in \mathbb{C}$ *with* $(\alpha, \beta) \neq (0, 0)$.

Proof. It is clear that $M \cong \Omega(\lambda, \alpha) = \mathbb{C}[t]$ as $\mathcal{V}$-modules from Lemma 2.7, where $\lambda \in \mathbb{C}^*, \alpha \in \mathbb{C}$ and $\mathbb{C}[t]$ is the polynomial algebra. So we can assume that

$$\boldsymbol{c}f(t) = 0, \quad d_m f(t) = \lambda^m(t + m\alpha)f(t + m)$$

for $m \in \mathbb{Z}$ and $f(t) \in \mathbb{C}[t]$. Now, we consider the action of h_r for $r \in \mathbb{Z} + \frac{1}{2}$. First, it is not hard to see that

$$h_r(f(t)) = h_r f(d_0)(1) = f(t + r)h_r(1) \tag{4.1}$$

for any $r \in \mathbb{Z} + \frac{1}{2}$ and $f(t) \in \mathbb{C}[t]$. Next, we consider the following two cases.

Case 1. $h_r(1) = 0$ for some $r \in \mathbb{Z} + \frac{1}{2}$.

It is easy to get that $h_r M = 0$ by the equality (4.1). Then we have $h_s M = 0$ for any $s \in \mathbb{Z} + \frac{1}{2}$ by the defining relations of $\mathcal{D}$. Thus $\mathcal{H}M = 0$. It shows that $M \cong \Omega(\lambda, \alpha, 0)$ as $\mathcal{D}$-modules.

Case 2. $h_r(1) \neq 0$ for any $r \in \mathbb{Z} + \frac{1}{2}$.

First, we show the following claim.

Claim 4. $h_r(1) \in \mathbb{C}^*$ for any $r \in \mathbb{Z} + \frac{1}{2}$.

Assume that there exists $r_0 \in \mathbb{Z} + \frac{1}{2}$ such that $h_{r_0}(1) = \sum_{i=0}^{k_{r_0}} a_i t^i$ with $k_{r_0} > 0$ and $a_{k_{r_0}} \neq 0$. Choose $s \in \mathbb{Z} + \frac{1}{2}$ such that $sr_0 < 0$ and $s + r_0 \neq 0$. Define

$$X(t) = h_{r_0}(1) = \sum_{i=0}^{k_{r_0}} a_i t^i \quad \text{and} \quad Y(t) = h_s(1) = \sum_{j=0}^{k_s} b_j t^j,$$

where $b_{k_s} \neq 0$. Now, we have $[h_s, h_{r_0}](1) = 0$. Thus

$$
\begin{aligned}
0 &= h_s h_{r_0}(1) - h_{r_0} h_s(1) = h_s X(t) - h_{r_0} Y(t) \\
&= X(t+s) Y(t) - Y(t+r_0) X(t) \\
&= \left(\sum_{i=0}^{k_{r_0}} a_i (t+s)^i \right) \left(\sum_{j=0}^{k_s} b_j t^j \right) - \left(\sum_{j=0}^{k_s} b_j (t+r_0)^j \right) \left(\sum_{i=0}^{k_{r_0}} a_i t^i \right) \\
&= a_{k_{r_0}} b_{k_s} (s r_0 - r_0 k_s) t^{k_{r_0}+k_s-1} + (\text{lower order terms in } t),
\end{aligned}
$$

which leads to a contradiction. Claim 4 is proved.

Now it is clear that $\boldsymbol{l}M = 0$ from Claim 4. Define $h_r(1) = a_r \in \mathbb{C}^*$ for $r \in \mathbb{Z} + \frac{1}{2}$. Then

$$
\begin{aligned}
-r a_{m+r} = -r h_{m+r}(1) &= [d_m, h_r](1) \\
&= d_m h_r(1) - h_r d_m(1) = a_r d_m(1) - h_r d_m(1) \\
&= a_r (\lambda^m (t + m\alpha)) - (\lambda^m (t + r + m\alpha)) a_r \\
&= -r a_r \lambda^m,
\end{aligned}
$$

which implies that $a_{m+r} = \lambda^m a_r$ for $m \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$. So we have a $\mathcal{D}$-module isomorphism

$$
M \to \Omega(\lambda, \alpha, \lambda^{-\frac{1}{2}} a_{\frac{1}{2}}), \quad t^i \mapsto t^i.
$$

The remaining part follows from Lemma 4.2. In conclusion, we complete the proof. $\qquad \square$

5 Tensor products of Whittaker modules and $\mathcal{U}(\mathbb{C}d_0)$-free $\mathcal{D}$-modules

In this section, we show that if $\Omega(\lambda, \alpha, \beta)$ and W_{φ_m} are irreducible $\mathcal{D}$-modules, then the tensor product module $\Omega(\lambda, \alpha, \beta) \otimes W_{\varphi_m}$ is also irreducible for $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$ and $m \in \mathbb{Z}_+$. Moreover, we prove that $\Omega(\lambda, \alpha, \beta) \otimes W_{\varphi_m} \cong \Omega(\lambda_1, \alpha_1, \beta_1) \otimes W_{\varphi_{m_1}}$ if and only if $\Omega(\lambda, \alpha, \beta) \cong \Omega(\lambda_1, \alpha_1, \beta_1)$ and $W_{\varphi_m} \cong W_{\varphi_{m_1}}$. In fact, we get the more general results.

Proposition 5.1. Let $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$ with $(\alpha, \beta) \neq (0, 0)$, and V be an irreducible restricted module over $\mathcal{D}$. Then $\Omega(\lambda, \alpha, \beta) \otimes V$ is an irreducible $\mathcal{D}$-module.

Proof. It is clear that for any $v \in V$, there exists $N(v) \in \mathbb{Z}_+$ such that $d_m v = h_r v = 0$ for $m, r \geqslant N(v)$ by the definition of restricted module. Now suppose that M is a nonzero submodule of $\Omega(\lambda, \alpha, \beta) \otimes V$. We only need to show $M = \Omega(\lambda, \alpha, \beta) \otimes V$.

Claim 5. There exists a $v \in V \setminus \{0\}$ such that $1 \otimes v \in M$.

Take a nonzero element $w = \sum_{i=0}^s t^i \otimes v_i \in M$ such that $v_i \in V$, $v_s \neq 0$ and s is minimal.

Let $N = \max\{N(v_i) : i = 0, 1, \ldots, s\}$. Then $d_m(v_i) = h_r(v_i) = 0$ for all $m, r \geqslant N$ and $i = 0, 1, \ldots, s$. If $\beta \neq 0$, then we get that

$$
\beta^{-1} \lambda^{-r} h_r w = \sum_{i=0}^s (t+r)^i \otimes v_i \in M
$$

for $r \geqslant N$. We can write the right-hand side in the form

$$
\sum_{i=0}^s r^i w_i \in M, \quad \forall r > N,
$$

where $w_i \in \Omega(\lambda, \alpha, \beta) \otimes V$ are independent of r. In particular, $w_s = 1 \otimes v_s$. Taking $r = \frac{1}{2} + N$, $\frac{1}{2} + (N+1), \ldots, \frac{1}{2} + (N+s)$, we see that the coefficient matrix of w_i is a Vandermonde matrix. Thus each $w_i \in M$. In particular, $w_s = 1 \otimes v_s \in M$.

If $\beta = 0$, then $\alpha \neq 0$. Similarly, using the action of $d_m w$ ($m > N$), we can show that Claim 5 holds as well.

Claim 6. $M = \Omega(\lambda, \alpha, \beta) \otimes V$.

From Claim 5, we have that $1 \otimes v \in M$ for some nonzero $v \in V$. Note that

$$d_m(t^j \otimes v) = \lambda^m(t + m\alpha)(t + m)^j \otimes v = \lambda^m t(t + m)^j \otimes v + \lambda^m m\alpha(t + m)^j \otimes v \qquad (5.1)$$

for any $j \in \mathbb{Z}_+$ and $m \geqslant N(v)$. Thus by induction on j, we deduce that $t^j \otimes v \in M$ for all $j \in \mathbb{Z}_+$, i.e., $\Omega(\lambda, \alpha, \beta) \otimes v \in M$. Define

$$W = \{w \in V : \Omega(\lambda, \alpha, \beta) \otimes w \subseteq M\}.$$

It is clear that $W \neq 0$ since $v \in W$. We need to prove that W is $\mathcal{D}$-submodule of V. For any $w \in W$, $m \in \mathbb{Z}$ and $r \in \frac{1}{2} + \mathbb{Z}$, we compute that

$$d_m(\Omega(\lambda, \alpha, \beta) \otimes w) = d_m\Omega(\lambda, \alpha, \beta) \otimes w + \Omega(\lambda, \alpha, \beta) \otimes d_m w \subseteq M,$$
$$h_r(\Omega(\lambda, \alpha, \beta) \otimes w) = h_r\Omega(\lambda, \alpha, \beta) \otimes w + \Omega(\lambda, \alpha, \beta) \otimes h_r w \subseteq M.$$

Thus, $\Omega(\lambda, \alpha, \beta) \otimes d_m w$ and $\Omega(\lambda, \alpha, \beta) \otimes h_r w \subseteq M$, i.e., $d_m w, h_r w \in W$. These show that W is a nonzero submodule of V. Thus $W = V$ since V is irreducible, which implies $M = \Omega(\lambda, \alpha, \beta) \otimes V$. We complete the proof. $\qquad\square$

Proposition 5.2. *Let $\lambda, \lambda_1 \in \mathbb{C}^*$ and $\alpha, \alpha_1, \beta, \beta_1 \in \mathbb{C}$. Assume that V and V_1 are irreducible restricted modules over $\mathcal{D}$. Then $\Omega(\lambda, \alpha, \beta) \otimes V$ and $\Omega(\lambda_1, \alpha_1, \beta_1) \otimes V_1$ are isomorphic as $\mathcal{D}$-modules if and only if $(\lambda, \alpha, \beta) = (\lambda_1, \alpha_1, \beta_1)$ and $V \cong V_1$ as $\mathcal{D}$-modules.*

Proof. It is clear that the sufficiency is trivial. We only need to show the necessity. Let $\psi : \Omega(\lambda, \alpha, \beta) \otimes V \to \Omega(\lambda_1, \alpha_1, \beta_1) \otimes V_1$ be a module isomorphism. For any nonzero element $1 \otimes v \in \Omega(\lambda, \alpha, \beta) \otimes V$, suppose that $\psi(1 \otimes v) = \sum_{i=0}^{p} t^i \otimes w_i$, where $w_i \in V_1$ and $w_p \neq 0$. Let $N_v = \max\{N(v), N(w_0), N(w_1), \ldots, N(w_p)\}$. Then $d_m v = d_m(w_i) = h_r v = h_r(w_i) = 0$ for $m, r \geqslant N_v$ and $i = 0, 1, \ldots, p$.

Claim 7. $\lambda = \lambda_1$, $\alpha = \alpha_1$ and $\beta = \beta_1$.

Taking $m, m_1 \geqslant N_v$, we have

$$(\lambda^{-m} d_m - \lambda^{-m_1} d_{m_1})(1 \otimes v) = \alpha(m - m_1)(1 \otimes v).$$

By $(\lambda^{-m} d_m - \lambda^{-m_1} d_{m_1})\psi(1 \otimes v) = \alpha(m - m_1)\psi(1 \otimes v)$, we deduce that

$$\alpha(m - m_1) \sum_{i=0}^{p} t^i \otimes w_i = \sum_{i=0}^{p} \left\{ \left(\frac{\lambda_1}{\lambda}\right)^m (t + m\alpha_1)(t + m)^i - \left(\frac{\lambda_1}{\lambda}\right)^{m_1} (t + m_1\alpha_1)(t + m_1)^i \right\} \otimes w_i. \quad (5.2)$$

Thus $((\frac{\lambda_1}{\lambda})^m - (\frac{\lambda_1}{\lambda})^{m_1})t^{p+1} \otimes w_p = 0$ for $m, m_1 \geqslant N_v$. So $\lambda = \lambda_1$. Therefore, the equation (5.2) becomes

$$\alpha(m - m_1) \sum_{i=0}^{p} t^i \otimes w_i = \sum_{i=0}^{p} t\{(t + m)^i - (t + m_1)^i\} \otimes w_i + \sum_{i=0}^{p} \alpha_1\{m(t + m)^i - m_1(t + m_1)^i\} \otimes w_i$$

for $m, m_1 \geqslant N_v$. Then the coefficient of $t^p \otimes w_p$ on the right-hand side is $(\alpha_1 + p)(m - m_1)$, which implies $\alpha = p + \alpha_1$, i.e., $\alpha - \alpha_1 = p \geqslant 0$. Similarly, using ψ^{-1} we can get $\alpha - \alpha_1 \geqslant 0$. So $\alpha = \alpha_1$ and $p = 0$. Thus, $\psi(1 \otimes v) = 1 \otimes w_0$. Note that for any $r \geqslant N_v$, we have

$$\beta\lambda^r(1 \otimes w_0) = \beta\lambda^r\psi(1 \otimes v) = \psi(h_r(1 \otimes v)) = h_r(1 \otimes w_0) = \beta_1\lambda_1^r(1 \otimes w_0).$$

Thus $\beta = \beta_1$. Hence, Claim 7 is proved.

Now we define the linear map $\xi : V \to V_1$ such that $\psi(1 \otimes v) = 1 \otimes \xi(v)$ for $v \in V$. It is clear that ξ is injective. Note that for any $r \in \mathbb{Z} + \frac{1}{2}$,

$$\beta\lambda^r(1 \otimes \xi(v)) + 1 \otimes \xi(h_r(v)) = \psi(h_r(1 \otimes v)) = h_r(\psi(1 \otimes v))$$
$$= h_r(1 \otimes \xi(v)) = \beta\lambda^r(1 \otimes \xi(v)) + 1 \otimes h_r(\xi(v)),$$

which implies

$$\xi(h_r(v)) = h_r(\xi(v)), \quad \forall\, v \in V, \quad r \in \mathbb{Z} + \frac{1}{2}. \tag{5.3}$$

Next, we consider the equation $\psi(d_m(1 \otimes v)) = d_m(\psi(1 \otimes v))$ for $m \geqslant N_v$. We deduce

$$\lambda^m \psi(t \otimes v) + m\alpha\lambda^m(1 \otimes \xi(v)) = \lambda^m t \otimes \xi(v) + m\alpha\lambda^m(1 \otimes \xi(v)).$$

So $\psi(t \otimes v) = t \otimes \xi(v)$. Hence $\psi(d_n(1) \otimes v) = d_n(1) \otimes \xi(v)$ for $n \in \mathbb{Z}$. Using the equation $\psi(d_n(1 \otimes v)) = d_n\psi(1 \otimes v)$ for $n \in \mathbb{Z}$ and $v \in V$, we deduce $1 \otimes \xi(d_n(v)) = 1 \otimes d_n(\xi(v))$. It shows

$$\xi(d_n(v)) = d_n(\xi(v)), \quad \forall\, v \in V, \quad n \in \mathbb{Z}. \tag{5.4}$$

It is clear that $\psi(\boldsymbol{c}(1 \otimes v)) = \boldsymbol{c}\psi(1 \otimes v)$ and $\psi(\boldsymbol{l}(1 \otimes v)) = \boldsymbol{l}\psi(1 \otimes v)$ imply that

$$\xi(\boldsymbol{c}(v)) = \boldsymbol{c}\xi(v) \quad \text{and} \quad \xi(\boldsymbol{l}(v)) = \boldsymbol{l}\xi(v). \tag{5.5}$$

Therefore, ξ is a $\mathcal{D}$-module homomorphism by the equations (5.3)–(5.5). Since $\xi(V) \neq 0$ and V_1 is irreducible, $\xi(V) = V_1$. Thus ξ is a $\mathcal{D}$-module isomorphism from V to V_1. In conclusion, we complete the proof. $\qquad\square$

It is clear that Whittaker $\mathcal{D}$-modules W_{φ_m}'s are restricted modules over $\mathcal{D}$. So we have the following two main theorems from Theorems 3.4 and 4.3 and Propositions 5.1 and 5.2.

Theorem 5.3. *Let $m \in \mathbb{N}$, $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$, and $\varphi_m : \mathcal{D}^{(m,0)} \to \mathbb{C}$ be a Whittaker function.*
(1) *If $\varphi_m(\boldsymbol{l}) \neq 0$, then $\Omega(\lambda, \alpha, \beta) \otimes W_{\varphi_m}$ is an irreducible $\mathcal{D}$-module if and only if*

$$(\varphi_m(d_{2m}), 2\varphi_m(\boldsymbol{l})\varphi_m(d_{2m-1}) - \varphi_m(h_{m-\frac{1}{2}})^2) \neq (0,0) \quad and \quad (\alpha, \beta) \neq (0,0).$$

(2) *If $\varphi_m(\boldsymbol{l}) = 0$, then $\Omega(\lambda, \alpha, \beta) \otimes W_{\varphi_m}$ is an irreducible $\mathcal{D}$-module if and only if*

$$\varphi_m(h_{m-\frac{1}{2}}) \neq 0 \quad and \quad (\alpha, \beta) \neq (0,0).$$

Theorem 5.4. *Let $m, m_1 \in \mathbb{Z}_+$, $\lambda, \lambda_1 \in \mathbb{C}^*$ and $\alpha, \alpha_1, \beta, \beta_1 \in \mathbb{C}$. Suppose that $\varphi_m : \mathcal{D}^{(m,0)} \to \mathbb{C}$ and $\varphi_{m_1} : \mathcal{D}^{(m_1,0)} \to \mathbb{C}$ are the Whittaker functions such that the corresponding Whittaker $\mathcal{D}$-modules W_{φ_m} and $W_{\varphi_{m_1}}$ are all irreducible. The following results hold:*
(1) *$\Omega(\lambda, \alpha, \beta) \cong \Omega(\lambda_1, \alpha_1, \beta_1)$ if and only if $(\lambda, \alpha, \beta) = (\lambda_1, \alpha_1, \beta_1)$.*
(2) *$\Omega(\lambda, \alpha, \beta) \otimes W_{\varphi_m} \cong \Omega(\lambda_1, \alpha_1, \beta_1) \otimes W_{\varphi_{m_1}}$ if and only if $(\lambda, \alpha, \beta) = (\lambda_1, \alpha_1, \beta_1)$ and $W_{\varphi_m} \cong W_{\varphi_{m_1}}$.*

Remark 5.5. Using Theorems 3.4, 4.3 and 5.3, we can obtain many new irreducible non-weight modules over $\mathcal{D}$.

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