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Global well-posedness for the 3-D MHD equations with partial diffusion in the periodic domain

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Abstract In this paper, we prove the global well-posedness of the 3-D magnetohydrodynamics (MHD) equations with partial diffusion in the periodic domain when the initial velocity is small and the initial magnetic field is close to a background magnetic field satisfying the Diophantine condition.

Keywords MHD equations, global well-posedness, Diophantine condition

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1 Introduction

In this paper, we study the 3-D incompressible MHD equations in the periodic domain $\mathbb{T}^3$:

$$\begin{cases} \partial_t u - \mu\Delta u + u\cdot\nabla u + \nabla p = b\cdot\nabla b, \\ \partial_t b + u\cdot\nabla b - \nu\Delta b = b\cdot\nabla u, \\ \nabla\cdot b = \nabla\cdot u = 0, \\ u(0) = u_0(x,y,z), \quad b(0) = b_0(x,y,z). \end{cases} \tag{1.1}$$

Here, (u, b, p) denotes the velocity of the fluid, the magnetic field and the pressure, respectively. The constants $\mu, \nu \geqslant 0$ are the viscosity coefficient and magnetic diffusivity, respectively.

When $\mu > 0$ and $\nu > 0$, the local existence and the uniqueness of the classical solution, and the global existence of the weak solution to the MHD system (1.1) are classical. In the 2-D space, it is well known that the classical solution is global in time, and the weak solution is regular and unique. However, the global existence of the classical solution and the regularity of the weak solution are challenging open problems for the 3-D MHD system. Let us refer to [16] and the references therein for more classical results and the introduction.

When $\mu > 0$ and $\nu = 0$, the global existence of the weak solution remains open even for the 2-D MHD system. When $\mu = 0$ and $\nu > 0$, the global existence of the weak solution to the 2-D MHD system was

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proved by Lei and Zhou [9]. In the latter case, the regularity and the uniqueness of the weak solution are very interesting open problems (see [6] for further results).

In [6], Cao and Wu studied the 2-D anisotropic MHD system as follows:

$$\begin{cases}\partial_t u - \nu_1\partial_x^2 u - \nu_2\partial_y^2 u + u\cdot\nabla u + \nabla p = b\cdot\nabla b,\\ \partial_t b - \eta_1\partial_x^2 b - \eta_2\partial_y^2 b + u\cdot\nabla b = b\cdot\nabla u.\end{cases} \tag{1.2}$$

They proved the global well-posedness of the system (1.2) in the following two cases: $\nu_1>0,\ \eta_2>0$ or $\nu_2>0,\ \eta_1>0$. However, it remains open for the other two cases: $\nu_1>0,\ \eta_1>0$ or $\nu_2>0,\ \eta_2>0$ (see [5] for further results).

Initiated by Lin and Zhang [12], there are many works [11,14,15,20] devoted to the global well-posedness of the MHD system with partial diffusion $\mu>0$ and $\nu=0$ in $\mathbb{R}^2$ under a strong constant magnetic field $B=(0,1)$ (see also [10]). In this case, the linearized MHD system could be reduced to a degenerate damped wave equation

$$\partial_t^2 u - \mu\Delta u_t - \partial_y^2 u = 0.$$

Thus, the background magnetic field provides a weak stabilizing (damping) effect. On the other hand, null structure of nonlinear terms due to the divergence-free condition plays an important role for nonlinear stability. In the case of $\mathbb{R}^3$, the problem becomes more difficult due to the absence of null structure. Abidi and Zhang [1] proved the global well-posedness of the MHD system in $\mathbb{R}^3$ under a magnetic field $B=(0,0,1)$, and Deng and Zhang [7] further proved the explicit decay estimates of the velocity and the magnetic field. One of the key ingredients is to introduce a new coordinate system based on the Frobenius-type theorem, which helps overcome the trouble due to the absence of null structure in three dimensions. Let us mention some global well-posedness results [3,4,8,17,18] under a strong magnetic field when μ and ν are small or $\mu=\nu=0$.

In this paper, we will study the global well-posedness of the MHD system in $\mathbb{T}^3$ with $\mu>0,\ \nu=0$ or $\mu=0,\ \nu>0$ under some background magnetic field B. Before stating our main result, let us first point out the main trouble in this case. For example, when $B=(0,0,1)$ and $\mu>0,\ \nu=0$, we consider the solution to (1.1) of the form

$$u=(u_1(t,x,y),u_2(t,x,y),0),\quad b=(b_1(t,x,y),b_2(t,x,y),1). \tag{1.3}$$

Then $u^h=(u_1(t,x,y),u_2(t,x,y))$ and $b^h=(b_1(t,x,y),b_2(t,x,y))$ satisfy the following MHD system in $\mathbb{T}^2$:

$$\begin{cases}\partial_t u^h - \mu\Delta u^h + u^h\cdot\nabla u^h + \nabla p = b^h\cdot\nabla b^h,\\ \partial_t b^h + u^h\cdot\nabla b^h = b^h\cdot\nabla u^h.\end{cases} \tag{1.4}$$

To prove the global well-posedness of the MHD system in $\mathbb{T}^3$, the first step is that the global well-posedness of the system (1.4) should be established for small data. However, it remains open to our knowledge. Recently, under some symmetry assumptions on the solution which exclude the existence of the nontrivial solution of the form (1.3), Pan et al. [13] proved the global well-posedness of the MHD system in $\mathbb{T}^3$.

To overcome the difficulties mentioned above, we consider the background magnetic field $\omega\in\mathbb{R}^3$ satisfying the so-called Diophantine condition: for any $k\in\mathbb{Z}^3\setminus\{0\}$,

$$|\omega\cdot k| \geqslant \frac{c}{|k|^r} \tag{1.5}$$

for some $c>0$ and $r>2$. The key point is that for ω satisfying the Diophantine condition, it holds that

$$\|f\|_{L^2}\leqslant C\|\omega\cdot\nabla f\|_{H^r},\quad \text{if } \int_{\mathbb{T}^3} f dxdydz = 0.$$

For simplicity, we still use the notation b to denote the perturbation $b-\omega$. Then the perturbation (u,b) satisfies

$$\begin{cases}\partial_t u-\mu\Delta u+u\cdot\nabla u+\nabla p=\omega\cdot\nabla b+b\cdot\nabla b,\\ \partial_t b+u\cdot\nabla b-\nu\Delta b=\omega\cdot\nabla u+b\cdot\nabla u,\\ \nabla\cdot b=\nabla\cdot u=0,\\ u(0)=u_0,\quad b(0)=b_0.\end{cases}\tag{1.6}$$

Our main result is stated as follows.

Theorem 1.1. *Consider the MHD system* (1.6) *with* $\mu=1,\ \nu=0$ *or* $\mu=0,\ \nu=1$. *Assume that* $(u_0,b_0)\in H^N$ *for* $N\geqslant 4r+7$ *satisfies*

$$\begin{aligned}&\|u_0\|_{H^N}+\|b_0\|_{H^N}\leqslant\varepsilon,\\ &\int_{\mathbb{T}^3}u_0(x,y,z)dxdydz=\int_{\mathbb{T}^3}b_0(x,y,z)dxdydz=0.\end{aligned}\tag{1.7}$$

If ε *is small enough, then there exists a unique global solution* $(u,b)\in C([0,+\infty);H^N)$ *to the system* (1.6) *satisfying*

$$\begin{aligned}&\|u(t)\|_{H^{r+4}}+\|b(t)\|_{H^{r+4}}\leqslant C(1+t)^{-\frac{3}{2}},\\ &\|u(t)\|_{H^N}+\|b(t)\|_{H^N}\leqslant C\varepsilon\end{aligned}$$

for any $t\in[0,+\infty)$.

Remark 1.2. Let us give some remarks about our result.

(1) If the initial data satisfies (1.7), the solution to the system (1.6) will preserve this property, i.e.,

$$\int_{\mathbb{T}^3}u(t,x,y,z)dxdydz=\int_{\mathbb{T}^3}b(t,x,y,z)dxdydz=0$$

for any $t\in[0,+\infty)$.

(2) If the initial data is smooth, then the decay rate of (u,b) can be arbitrarily big.

(3) In the next section, we will show that the Diophantine condition is satisfied for almost all the vector fields ω in $\mathbb{R}^3$. However, when the components of ω are rational numbers or when one component of ω is zero, ω does not satisfy the Diophantine condition.

(4) In [19], Wei and Zhang proved the global well-posedness of the 2-D MHD system with $\mu=0$ and $\nu=1$ in the periodic domain when the velocity and the magnetic field are small. Therefore, there is no obstacle mentioned above in this case. Then a natural question is whether the Diophantine condition can be removed in this case.

2 The Diophantine condition and the Poincaré type inequality

Let us first prove that for almost all $\omega\in\mathbb{R}^3$, there exists $c=c(\omega)$ so that the Diophantine condition (1.5) holds. Indeed, for any $\epsilon>0$ and ball B, consider the set

$$E_{k,\epsilon}=\{\omega\in B:|k\cdot\omega|\leqslant\epsilon|k|^{-r}\}.$$

Then we have

$$|E_{k,\epsilon}|\leqslant C\epsilon|k|^{-r-1},$$

which gives

$$\Bigg|\bigcup_{k\in\mathbb{Z}^3,k\neq0}E_{k,\epsilon}\Bigg|\leqslant C\epsilon\sum_{k\in\mathbb{Z}^3,k\neq0}|k|^{-r-1}\leqslant C\epsilon$$

due to $r > 2$. This means that

$$\left|\bigcap_{\epsilon>0}\bigcup_{k\in\mathbb{Z}^3,k\neq 0} E_{k,\epsilon}\right| = 0.$$

Next, we explain why the Diophantine condition is reasonable for our result by comparison with the proof in [7], where Deng and Zhang considered the 3-D MHD system with $\mu = 1,\ \nu = 0$ in $\mathbb{R}^3$. They used the Lagrangian formulation. In Lagrangian coordinates, the perturbation system can be written as the form

$$Y_{tt} - \Delta_y Y_t - \partial_{b_0}^2 Y = F(Y),$$

where $\partial_{b_0} = b_0\cdot\nabla_y$. A key idea is to introduce a new coordinate system so that ∂_{b_0} is transformed into $\omega\cdot\nabla$ if $b_0 - \omega$ is small and compactly supported. However, in the torus, a similar coordinate transformation can be made only when ω satisfies the Diophantine condition (see [2, p. 137]).

In this paper, we will not follow the approach of the coordinate transformation introduced in [7]. Instead, we find that the following Poincaré type inequality with the derivative loss is enough to obtain our result.

Lemma 2.1. *If $\omega \in \mathbb{R}^3$ satisfies the Diophantine condition* (1.5)*, then it holds that for any $s \in \mathbb{R}$,*

$$\|f\|_{H^s} \leqslant C\|\omega\cdot\nabla f\|_{H^{s+r}}$$

if $f \in H^{s+r}(\mathbb{T}^3)$ satisfies $\int_{\mathbb{T}^3} f dxdydz = 0$.

The proof is a direct consequence of the Plancherel formula for the Fourier series.

Let us conclude this section by the following classical lemma.

Lemma 2.2. *Let $k \geqslant 1$. It holds that*

$$\begin{aligned}&\|fg\|_{H^k} \leqslant C(\|f\|_{L^\infty}\|g\|_{H^k} + \|g\|_{L^\infty}\|f\|_{H^k}),\\ &\|\nabla^k(fg) - f\nabla^k g\|_{L^2} \leqslant C(\|\nabla f\|_{L^\infty}\|g\|_{H^{k-1}} + \|g\|_{L^\infty}\|f\|_{H^k}).\end{aligned}$$

3 The MHD system without magnetic diffusion

In this section, we consider the 3-D MHD system without the magnetic diffusion

$$\begin{cases}\partial_t u - \Delta u + u\cdot\nabla u + \nabla p = \omega\cdot\nabla b + b\cdot\nabla b,\\ \partial_t b + u\cdot\nabla b = \omega\cdot\nabla u + b\cdot\nabla u,\\ \nabla\cdot b = \nabla\cdot u = 0.\end{cases} \tag{3.1}$$

3.1 Energy estimates

Let $(u, b) \in C([0, T]; H^N)$ be a solution to the MHD system (3.1).

Lemma 3.1. *For any $l \in [0, N]$ and $t \in [0, T]$, it holds that*

$$\begin{aligned}&\frac{d}{dt}(\|u(t)\|_{H^l}^2 + \|b(t)\|_{H^l}^2) + \|\nabla u(t)\|_{H^l}^2\\ &\quad\leqslant C(\|u(t)\|_{H^3} + \|b(t)\|_{H^3} + \|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2)(\|u(t)\|_{H^l}^2 + \|b(t)\|_{H^l}^2).\end{aligned}$$

Proof. For $l = 0$, the basic energy law gives

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + \|\nabla u(t)\|_{L^2}^2 = 0.$$

Taking ∇^k $(1 \leqslant k \leqslant l)$ to the system (3.1), and then making the L^2 inner product with $(\nabla^k u, \nabla^k b)$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\nabla^k b\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2) + \|\nabla^{k+1} u\|_{L^2}^2$$

$$
\begin{aligned}
&= \int_{\mathbb{T}^3} \nabla^k (b\cdot\nabla u)\cdot\nabla^k b dx + \int_{\mathbb{T}^3} \nabla^k(\omega\cdot\nabla u)\cdot\nabla^k b dx - \int_{\mathbb{T}^3}\nabla^k(u\cdot\nabla b)\cdot\nabla^k b dx \\
&\quad + \int_{\mathbb{T}^3}\nabla^k(b\cdot\nabla b)\cdot\nabla^k u dx + \int_{\mathbb{T}^3}\nabla^k(\omega\cdot\nabla b)\cdot\nabla^k u dx - \int_{\mathbb{T}^3}\nabla^k(u\cdot\nabla u)\cdot\nabla^k u dx \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \qquad (3.2)
\end{aligned}
$$

It is easy to see that

$$I_2 + I_5 = 0. \qquad (3.3)$$

By Lemma 2.2, we have

$$
\begin{aligned}
|I_1| &\leqslant C\|b\|_{L^\infty}\|\nabla u\|_{H^k}\|\nabla^k b\|_{L^2} + C\|\nabla u\|_{L^\infty}\|b\|_{H^k}\|\nabla^k b\|_{L^2} \\
&\leqslant C(\|b\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty})\|\nabla^k b\|_{L^2}^2 + \frac{1}{8}\|\nabla^{k+1}u\|_{L^2}^2. \qquad (3.4)
\end{aligned}
$$

Here, we used the facts that

$$\|b\|_{H^k} \leqslant C\|\nabla^k b\|_{L^2}, \quad \|\nabla u\|_{H^k} \leqslant C\|\nabla^{k+1}u\|_{L^2}.$$

Similarly, we have

$$|I_6| \leqslant C(\|u\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty})\|\nabla^k u\|_{L^2}^2 + \frac{1}{8}\|\nabla^{k+1}u\|_{L^2}^2. \qquad (3.5)$$

For I_4, we get by integration by parts that

$$I_4 = -\int_{\mathbb{T}^3}\nabla^k(b\otimes b)\cdot\nabla^k\nabla u dx,$$

which along with Lemma 2.2 gives

$$|I_4| \leqslant C\|b\|_{L^\infty}^2\|\nabla^k b\|_{L^2}^2 + \frac{1}{8}\|\nabla^{k+1}u\|_{L^2}^2. \qquad (3.6)$$

For I_3, we have

$$I_3 = -\int_{\mathbb{T}^3}(\nabla^k(u\cdot\nabla b) - u\cdot\nabla\nabla^k b)\cdot\nabla^k b dx,$$

which along with Lemma 2.2 gives

$$|I_3| \leqslant C\|\nabla u\|_{L^\infty}\|\nabla^k b\|_{L^2}^2 + C\|\nabla b\|_{L^\infty}\|\nabla^k u\|_{L^2}\|\nabla^k b\|_{L^2}. \qquad (3.7)$$

Summing up (3.2)–(3.7), we conclude that

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla^k b\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2) + \|\nabla^{k+1}u\|_{L^2}^2 \\
&\quad \leqslant C(\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|b\|_{L^\infty}^2)(\|\nabla^k u\|_{L^2}^2 + \|\nabla^k b\|_{L^2}^2) \\
&\quad \leqslant C(\|\nabla u\|_{H^2} + \|\nabla b\|_{H^2} + \|u\|_{H^2}^2 + \|b\|_{H^2}^2)(\|\nabla^k u\|_{L^2}^2 + \|\nabla^k b\|_{L^2}^2).
\end{aligned}
$$

This proves the lemma. □

Lemma 3.2. *Assume that*

$$\sup_{t\in[0,T]}(\|u(t)\|_{H^N} + \|b(t)\|_{H^N}) \leqslant \delta \qquad (3.8)$$

for some $0 < \delta < 1$*. Then it holds that*

$$
\begin{aligned}
&-\sum_{0\leqslant s\leqslant r+3}\frac{d}{dt}\int_{\mathbb{T}^3}\nabla^s u\cdot\nabla^s(\omega\cdot\nabla b)dx + \frac{1}{2}\|\omega\cdot\nabla b\|_{H^{r+3}}^2 \\
&\quad \leqslant (2 + C\delta)\|u\|_{H^{r+5}}^2 + C\delta^2\|b\|_{H^3}^2.
\end{aligned}
$$

Proof. Applying ∇^s $(0 \leqslant s \leqslant r+3)$ to the first equation of (3.1), and multiplying it by $\nabla^s(\omega\cdot\nabla b)$, then integrating over $\mathbb{T}^3$, we obtain

$$\begin{aligned}\|\nabla^s(\omega\cdot\nabla b)\|_{L^2}^2 &= \int_{\mathbb{T}^3}\nabla^s\partial_t u\cdot\nabla^s(\omega\cdot\nabla b)dx + \int_{\mathbb{T}^3}\nabla^s(u\cdot\nabla u)\cdot\nabla^s(\omega\cdot\nabla b)dx\\ &\quad - \int_{\mathbb{T}^3}\nabla^s\Delta u\cdot\nabla^s(\omega\cdot\nabla b)dx - \int_{\mathbb{T}^3}\nabla^s(b\cdot\nabla b)\cdot\nabla^s(\omega\cdot\nabla b)dx\\ &=: J_1+J_2+J_3+J_4. \end{aligned}\tag{3.9}$$

For J_3, we have

$$J_3 \leqslant \frac{1}{4}\|\nabla^s(\omega\cdot\nabla b)\|_{L^2}^2 + \|\nabla^{s+2}u\|_{L^2}^2. \tag{3.10}$$

By Lemma 2.2 and (3.8), we get

$$\begin{aligned}|J_2| &\leqslant \|\nabla^s(u\cdot\nabla u)\|_{L^2}\|\nabla^s(\omega\cdot\nabla b)\|_{L^2}\\ &\leqslant C(\|u\|_{L^\infty}\|\nabla u\|_{H^s} + \|\nabla u\|_{L^\infty}\|u\|_{H^s})\|\nabla^s(\omega\cdot\nabla b)\|_{L^2}\\ &\leqslant \frac{1}{8}\|\nabla^s(\omega\cdot\nabla b)\|_{L^2}^2 + C\delta^2\|\nabla u\|_{H^s}^2. \end{aligned}\tag{3.11}$$

Similarly, we have

$$|J_4| \leqslant \frac{1}{8}\|\nabla^s(\omega\cdot\nabla b)\|_{L^2}^2 + C\delta^2\|b\|_{H^3}^2. \tag{3.12}$$

For J_1, we use (3.1) to obtain that

$$\begin{aligned}J_1 &= \frac{d}{dt}\int_{\mathbb{T}^3}\nabla^s u\cdot\nabla^s(\omega\cdot\nabla b)dx - \int_{\mathbb{T}^3}\nabla^s u\cdot\nabla^s(\omega\cdot\nabla\partial_t b)dx\\ &= \frac{d}{dt}\int_{\mathbb{T}^3}\nabla^s u\cdot\nabla^s(\omega\cdot\nabla b)dx + \int_{\mathbb{T}^3}\nabla^s(\omega\cdot\nabla u)\cdot\nabla^s(b\cdot\nabla u)dx\\ &\quad + \int_{\mathbb{T}^3}\nabla^s(\omega\cdot\nabla u)\cdot\nabla^s(\omega\cdot\nabla u)dx - \int_{\mathbb{T}^3}\nabla^s(\omega\cdot\nabla u)\cdot\nabla^s(u\cdot\nabla b)dx\\ &=: \frac{d}{dt}\int_{\mathbb{T}^3}\nabla^s u\cdot\nabla^s(\omega\cdot\nabla b)dx + J_{11}+J_{12}+J_{13}. \end{aligned}$$

By Lemma 2.2 and (3.8), we get

$$\begin{aligned}&|J_{11}|+|J_{13}| \leqslant C\delta\|u\|_{H^{s+1}}^2,\\ &|J_{12}| \leqslant \|\nabla u\|_{H^s}^2. \end{aligned}$$

This shows that

$$J_1 \leqslant \frac{d}{dt}\int_{\mathbb{T}^3}\nabla^s u\cdot\nabla^s(\omega\cdot\nabla b)dx + (1+C\delta)\|u\|_{H^{s+1}}^2. \tag{3.13}$$

Summing up (3.9)–(3.13), we conclude the lemma. □

3.2 Proof of Theorem 1.1

Given the initial data $(u_0,b_0)\in H^N$, the local well-posedness of the system (3.1) could be easily proved by using the energy method. Thus, we may assume that there exist $T>0$ and a unique solution $(u,b)\in C([0,T];H^N)$ to the system (3.1). Furthermore, we may assume that

$$\sup_{t\in[0,T]}(\|u(t)\|_{H^N}+\|b(t)\|_{H^N}) \leqslant \delta$$

for some $0<\delta<1$ to be determined later.

First of all, it follows from Lemmas 3.1 and 3.2 that

$$\begin{aligned}&\frac{d}{dt}\bigg\{A(\|u(t)\|_{H^{r+4}}^2+\|b(t)\|_{H^{r+4}}^2)-\sum_{0\leqslant s\leqslant r+3}\int_{\mathbb{T}^3}\nabla^s u\cdot\nabla^s(\omega\cdot\nabla b)dx\bigg\}\\&\quad+A\|\nabla u(t)\|_{H^{r+4}}^2+\frac{1}{2}\|\omega\cdot\nabla b\|_{H^{3+r}}^2\\&\quad\leqslant CA\delta\|u\|_{H^{r+4}}^2+CA(\|u(t)\|_{H^3}+\|b(t)\|_{H^3})\|b(t)\|_{H^{r+4}}^2\\&\qquad+(2+C\delta)\|u\|_{H^{r+5}}^2+C\delta^2\|b\|_{H^3}^2,\end{aligned}$$

where $A>1$ is a constant to be determined later.

Thanks to the fact that

$$\int_{\mathbb{T}^3}udx=\int_{\mathbb{T}^3}bdx=0,$$

we deduce from Lemma 2.1 that

$$\|u\|_{H^{r+5}}\leqslant C\|\nabla u\|_{H^{r+4}},\quad \|b\|_{H^3}\leqslant C\|\omega\cdot\nabla b\|_{H^{r+3}}.$$

By the interpolation, we have

$$\|b\|_{H^{r+4}}^2\leqslant\|b\|_{H^3}\|b\|_{H^N}\leqslant C\delta\|\omega\cdot\nabla b\|_{H^{r+3}}$$

as long as $N\geqslant 2r+5$. We define

$$\begin{aligned}E(t)&=A(\|u(t)\|_{H^{r+4}}^2+\|b(t)\|_{H^{r+4}}^2)-\sum_{0\leqslant s\leqslant r+3}\int_{\mathbb{T}^3}\nabla^s u\cdot\nabla^s(\omega\cdot\nabla b)dx,\\D(t)&=A\|\nabla u(t)\|_{H^{r+4}}^2+\frac{1}{2}\|\omega\cdot\nabla b\|_{H^{3+r}}^2.\end{aligned}$$

We take $A>1$ so that

$$E(t)\geqslant(\|u(t)\|_{H^{r+4}}^2+\|b(t)\|_{H^{r+4}}^2).$$

Thus, by taking $\delta>0$ suitably small, we can conclude that

$$\frac{d}{dt}E(t)+\frac{1}{2}D(t)\leqslant 0.$$

Furthermore, if we take $N\geqslant 4r+7$, we get by the interpolation that

$$\|b\|_{H^{r+4}}^2\leqslant\|b\|_{H^3}^{\frac{3}{2}}\|b\|_{H^N}^{\frac{1}{2}}\leqslant C\delta^{\frac{1}{2}}\|\omega\cdot\nabla b\|_{H^{r+3}}^{\frac{3}{2}},$$

which gives

$$E(t)\leqslant C\delta^{\frac{1}{2}}D(t)^{\frac{3}{4}}\leqslant D(t)^{\frac{3}{4}},$$

i.e., $D(t)\geqslant E(t)^{\frac{4}{3}}$. Thus, we obtain

$$\frac{d}{dt}E(t)+cE(t)^{\frac{4}{3}}\leqslant 0,$$

which implies

$$E(t)\leqslant C(1+t)^{-3}. \tag{3.14}$$

Using Lemma 3.1 with $l=N$ again, we obtain

$$\frac{d}{dt}(\|u(t)\|_{H^N}^2+\|b(t)\|_{H^N}^2)$$

$$\leqslant C(\|u(t)\|_{H^3}+\|b(t)\|_{H^3}+\|u(t)\|_{H^2}^2+\|b(t)\|_{H^2}^2)(\|u(t)\|_{H^N}^2+\|b(t)\|_{H^N}^2),$$

from which, together with Gronwall's inequality and (3.14), we infer that

$$\begin{aligned}&(\|u(t)\|_{H^N}^2+\|b(t)\|_{H^N}^2)\\&\quad\leqslant(\|u_0\|_{H^N}^2+\|b_0\|_{H^N}^2)\exp\left(\int_0^t(\|u(\tau)\|_{H^3}+\|b(\tau)\|_{H^3}+\|u(\tau)\|_{H^2}^2+\|b(\tau)\|_{H^2}^2)d\tau\right)\\&\quad\leqslant C(\|u_0\|_{H^N}^2+\|b_0\|_{H^N}^2)\leqslant C\varepsilon^2.\end{aligned}$$

Taking ε small enough so that $C\varepsilon\leqslant\frac{1}{2}\delta$, we deduce from a continuity argument that the local solution can be extended as a global one in time, and it holds that

$$\begin{aligned}&\|u(t)\|_{H^{r+4}}+\|b(t)\|_{H^{r+4}}\leqslant C(1+t)^{-\frac{3}{2}},\\&\|u(t)\|_{H^N}+\|b(t)\|_{H^N}\leqslant C\varepsilon\end{aligned}$$

for any $t\in[0,+\infty)$. This completes the proof of Theorem 1.1.

4 The MHD system without viscous diffusion

In this section, we consider the 3-D MHD system without the viscous diffusion

$$\begin{cases}\partial_t u+u\cdot\nabla u+\nabla p=\omega\cdot\nabla b+b\cdot\nabla b,\\\partial_t b+u\cdot\nabla b-\Delta b=\omega\cdot\nabla u+b\cdot\nabla u,\\\nabla\cdot b=\nabla\cdot u=0.\end{cases}\tag{4.1}$$

We basically follow the proof of Theorem 1.1. Here, we just present a sketch of the proof. Assume that $(u,b)\in C([0,T];H^N)$ is a solution to the MHD system (4.1).

Lemma 4.1. *For any $l\in[0,N]$ and $t\in[0,T]$, it holds that*

$$\begin{aligned}&\frac{d}{dt}(\|u(t)\|_{H^l}^2+\|b(t)\|_{H^l}^2)+\|\nabla b(t)\|_{H^l}^2\\&\quad\leqslant C(\|u(t)\|_{H^3}+\|b(t)\|_{H^3}+\|u(t)\|_{H^2}^2+\|b(t)\|_{H^2}^2)(\|u(t)\|_{H^l}^2+\|b(t)\|_{H^l}^2).\end{aligned}$$

The proof is similar to that of Lemma 3.1. So we omit the details.

Lemma 4.2. *Assume that*

$$\sup_{t\in[0,T]}(\|u(t)\|_{H^N}+\|b(t)\|_{H^N})\leqslant\delta\tag{4.2}$$

for some $0<\delta<1$. Then it holds that

$$\begin{aligned}&-\frac{d}{dt}\sum_{0\leqslant s\leqslant r+3}\int_{\mathbb{T}^3}\nabla^s b\cdot\nabla^s(\omega\cdot\nabla u)dx+\frac{1}{2}\|\omega\cdot\nabla u\|_{H^{3+r}}^2\\&\quad\leqslant\frac{d}{dt}\int_{\mathbb{T}^3}\nabla^s b\cdot\nabla^s(\omega\cdot\nabla u)dx+C\|b\|_{H^{r+5}}^2+C\delta^2\|u\|_{H^3}^2.\end{aligned}$$

Proof. Applying ∇^s $(0\leqslant s\leqslant r+3)$ to the second equation of (4.1), then multiplying it by $\nabla^s(w\cdot\nabla u)$, and integrating over $\mathbb{T}^3$, we obtain

$$\begin{aligned}\|\nabla^s(\omega\cdot\nabla u)\|_{L^2}^2&=\int_{\mathbb{T}^3}\nabla^s b_t\cdot\nabla^s(\omega\cdot\nabla u)dx+\int_{\mathbb{T}^3}\nabla^s(u\cdot\nabla b)\cdot\nabla^s(\omega\cdot\nabla u)dx\\&\quad-\int_{\mathbb{T}^3}\nabla^s\Delta b\cdot\nabla^s(\omega\cdot\nabla u)dx-\int_{\mathbb{T}^3}\nabla^s(b\cdot\nabla u)\cdot\nabla^s(\omega\cdot\nabla u)dx\\&=:H_1+H_2+H_3+H_4.\end{aligned}$$

Taking into account the first equation of (4.1) for H_1, we infer that

$$\begin{aligned} H_1 &= \frac{d}{dt}\int_{\mathbb{T}^3} \nabla^s b \cdot \nabla^s(\omega\cdot\nabla u)dx - \int_{\mathbb{T}^3} \nabla^s b\cdot\nabla^s(\omega\cdot\nabla u_t)dx \\ &= \frac{d}{dt}\int_{\mathbb{T}^3} \nabla^s b\cdot\nabla^s(\omega\cdot\nabla u)dx + \int_{\mathbb{T}^3}\nabla^s(\omega\cdot\nabla b)\cdot\nabla^s(b\cdot\nabla b)dx \\ &\quad + \|\nabla^s(\omega\cdot\nabla b)\|_{L^2}^2 - \int_{\mathbb{T}^3}\nabla^s(\omega\cdot\nabla b)\cdot\nabla^s(u\cdot\nabla u)dx. \end{aligned}$$

By Lemma 2.2 and (4.2), it is easy to prove that

$$\begin{aligned} H_1 &\leqslant \frac{d}{dt}\int_{\mathbb{T}^3}\nabla^s b\cdot\nabla^s(\omega\cdot\nabla u)dx + C\delta\|\nabla b\|_{H^s}^2 + \|\nabla b\|_{H^s}^2 + C\delta\|\nabla b\|_{H^s}\|u\|_{H^3} \\ &\leqslant \frac{d}{dt}\int_{\mathbb{T}^3}\nabla^s b\cdot\nabla^s(\omega\cdot\nabla u)dx + C\|\nabla b\|_{H^s}^2 + C\delta^2\|u\|_{H^3}^2 \end{aligned}$$

and

$$\begin{aligned} &H_2 \leqslant C\delta\|\nabla b\|_{H^s}\|\omega\cdot\nabla u\|_{H^s} \leqslant \frac{1}{8}\|\omega\cdot\nabla u\|_{H^s}^2 + C\delta^2\|\nabla b\|_{H^s}^2, \\ &H_3 \leqslant \|\Delta b\|_{H^s}^2 + \frac{1}{4}\|\omega\cdot\nabla u\|_{H^s}^2, \\ &H_4 \leqslant \frac{1}{8}\|\omega\cdot\nabla u\|_{H^s}^2 + C\delta^2\|b\|_{H^s}^2. \end{aligned}$$

Summing up, we conclude the lemma. □

Now we are in a position to prove Theorem 1.1 in the case where $\mu = 0$ and $\nu = 1$.

Proof of Theorem 1.1 *when* $\mu = 0$ *and* $\nu = 1$. First of all, we introduce

$$\begin{aligned} &E(t) = A(\|u(t)\|_{H^{r+4}}^2 + \|b(t)\|_{H^{r+4}}^2) - \sum_{0\leqslant s\leqslant r+3}\int_{\mathbb{T}^3}\nabla^s b\cdot\nabla^s(\omega\cdot\nabla u)dx, \\ &D(t) = A\|\nabla b(t)\|_{H^{r+4}}^2 + \frac{1}{2}\|\omega\cdot\nabla u\|_{H^{3+r}}^2. \end{aligned}$$

By taking A suitably large and δ small, we can deduce from Lemmas 4.1 and 4.2 that

$$\frac{d}{dt}E(t) + \frac{1}{2}D(t) \leqslant 0,$$

which implies

$$E(t) \leqslant C(1+t)^{-3}.$$

Then we infer from Lemma 4.1 and Gronwall's inequality that

$$(\|u(t)\|_{H^N}^2 + \|b(t)\|_{H^N}^2) \leqslant C(\|u_0\|_{H^N}^2 + \|b_0\|_{H^N}^2) \leqslant C\varepsilon^2.$$

This completes the proof of Theorem 1.1. □

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