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A family of mixed finite elements for the biharmonic equations on triangular and tetrahedral grids

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Abstract This paper introduces a new family of mixed finite elements for solving a mixed formulation of the biharmonic equations in two and three dimensions. The symmetric stress $\boldsymbol{\sigma} = -\nabla^2 u$ is sought in the Sobolev space $H(\mathrm{div}\mathbf{div}, \Omega; \mathbb{S})$ simultaneously with the displacement u in $L^2(\Omega)$. By stemming from the structure of $H(\mathbf{div}, \Omega; \mathbb{S})$ conforming elements for the linear elasticity problems proposed by Hu and Zhang (2014), the $H(\mathrm{div}\mathbf{div}, \Omega; \mathbb{S})$ conforming finite element spaces are constructed by imposing the normal continuity of $\mathbf{div}\boldsymbol{\sigma}$ on the $H(\mathbf{div}, \Omega; \mathbb{S})$ conforming spaces of P_k symmetric tensors. The inheritance makes the basis functions easy to compute. The discrete spaces for u are composed of the piecewise P_{k-2} polynomials without requiring any continuity. Such mixed finite elements are inf-sup stable on both triangular and tetrahedral grids for $k \geqslant 3$, and the optimal order of convergence is achieved. Besides, the superconvergence and the postprocessing results are displayed. Some numerical experiments are provided to demonstrate the theoretical analysis.

Keywords biharmonic equation, symmetric stress tensor, conforming finite element, mixed finite element method

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1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz polyhedral domain with $d = 2$ or $d = 3$. Given a load $f \in L^2(\Omega)$, consider the biharmonic equation

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, Δ^2 is the biharmonic operator, $\boldsymbol{n}$ is the unit outer normal to the boundary $\partial\Omega$, and $u_n := \partial u/\partial \boldsymbol{n}$.

Many attempts have been made to approach the biharmonic problem (1.1), ranging from conforming and classical nonconforming finite element methods, discontinuous Galerkin methods to mixed methods,

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such as [9,12,21,27,30,37–39], to name just a few. On triangular grids, the lowest order of polynomials of the H^2 conforming finite elements is 5, i.e., the Argyris element [2,20], and it can be reduced to the Bell element [20,43] with 18 degrees of the freedom. On tetrahedral grids, a P_9 element constructed in [49] is the lowest order conforming element. In general, due to the high degrees of the freedom with higher order derivatives of the H^2 conforming elements, in addition to the complexity in construction, the computation is relatively costing. Nevertheless, some conforming finite elements are developed [20,22,32,36,42–44,48].

One way to reduce the high degrees of the freedom is to use nonconforming finite elements, such as the Morley element [20, 41, 43], the Adini element [1, 20, 43], the Veubake element [23], a class of Zienkiewicz-type nonconforming elements in any dimensions designed in [45], and other higher order nonconforming methods [14, 26, 29, 33, 37, 46]. The other way is to adopt different variational principles to avoid computational difficulty. A popular choice is mixed finite element methods. For example, the Ciarlet-Raviart method [21] turns (1.1) into a lower order system by introducing an auxiliary variable $\phi = -\Delta u$, and casts the new system in the variational form, and then considers the Ritz-Galerkin method corresponding to this variational formulation. However, such decoupling may not be valid if the polygonal domain is not convex (see [50]). Instead of $\phi = -\Delta u$, the matrix of the second partial derivatives of u, $\boldsymbol{\sigma} = -\nabla^2 u$ is introduced in the Hermann-Miyoshi method [30, 40]. A further mixed method for (1.1) is the Hermann-Johnson element, and the auxiliary variable introduced is the same as the Hermann-Miyoshi method, while the continuity of $\boldsymbol{n}^{\mathrm{T}}\boldsymbol{\sigma}\boldsymbol{n}$ is imposed on $\boldsymbol{\sigma}$.

In this paper, a more intrinsic variational formulation is considered, and it is also known as the Hodge-Laplacian boundary value problem of the divdiv complex. In [7], the well-posedness of the Hodge-Laplacian boundary value problem is discussed. The mixed finite element method seeks the stress $\boldsymbol{\sigma} = -\nabla^2 u$ in the Sobolev space $H(\mathrm{div}\,\mathbf{div}, \Omega; \mathbb{S})$ with

$$H(\mathrm{div}\,\mathbf{div}, \Omega; \mathbb{S}) := \{\boldsymbol{\tau} \in L^2(\Omega; \mathbb{S}) : \mathrm{div}\mathbf{div}\boldsymbol{\tau} \in L^2(\Omega)\}, \tag{1.2}$$

equipped with the squared norm

$$\|\boldsymbol{\tau}\|^2_{H(\mathrm{div}\mathbf{div})} := \|\boldsymbol{\tau}\|_0^2 + \|\mathrm{div}\mathbf{div}\boldsymbol{\tau}\|_0^2. \tag{1.3}$$

Here, $\mathbb{S}$ denotes the set of symmetric $\mathbb{R}^{d\times d}$ matrices. Simultaneously, the mixed method seeks $u \in L^2(\Omega)$ such that

$$\begin{aligned} &(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\mathbf{div}\,\boldsymbol{\tau}, u) = 0 \quad \text{for all } \boldsymbol{\tau} \in H(\mathrm{div}\,\mathbf{div}, \Omega; \mathbb{S}),\\ &(\mathrm{div}\,\mathbf{div}\boldsymbol{\sigma}, v) = -(f, v) \quad \text{for all } v \in L^2(\Omega). \end{aligned} \tag{1.4}$$

It is not easy to construct an $H(\mathrm{div}\,\mathbf{div}, \Omega; \mathbb{S})$ conforming element, and the symmetry of the tensor makes things more complex. A family of $H(\mathbf{div}; \mathbb{S})$ conforming finite elements for elasticity equations is proposed in [31,34,35]. If $\mathbf{div}\boldsymbol{\sigma} \in H(\mathrm{div})$ holds for all $\boldsymbol{\sigma} \in H(\mathbf{div}; \mathbb{S})$, then $\boldsymbol{\sigma} \in H(\mathrm{div}\mathbf{div}; \mathbb{S})$ follows. The relation triggers an idea to obtain the $H(\mathrm{div}\,\mathbf{div}, \Omega; \mathbb{S})$ conforming elements by imposing the continuity of $\boldsymbol{n}^{\mathrm{T}}\mathbf{div}\boldsymbol{\sigma}$ on $H(\mathbf{div}; \mathbb{S})$ conforming spaces. A question arises naturally how to characterize this additional continuity appropriately.

Attempts have been made in [47][1)], where the stress space is composed by the aforementioned $H(\mathbf{div}; \mathbb{S})$ conforming elements [31, 34, 35], and the displacement space chooses the P_k conforming finite element with $k \geqslant 2$. However, the L^2 norms are not optimal. In [24, 25], a depiction of the Sobolev space $H(\mathrm{div}\,\mathbf{div}, \Omega; \mathbb{S})$ is introduced, and the discontinuous Petrov-Galerkin method is considered. Recently, some finite element spaces for $H(\mathrm{div}\,\mathbf{div}, \Omega; \mathbb{S})$ conforming symmetric tensors are constructed on triangles [15] and tetrahedrons [16]. These elements are exploited to solve the mixed problem (1.4) and the optimal order of convergence is achieved. In two dimensions, a simple application of Green's formula shows

$$(\mathrm{div}\mathbf{div}\boldsymbol{\sigma}, v)_K = (\boldsymbol{\sigma}, \nabla^2 v)_K + \sum_{e\in\mathcal{E}(K)} (\boldsymbol{n}^{\mathrm{T}}\mathbf{div}\boldsymbol{\sigma}, v)_e - \sum_{e\in\mathcal{E}(K)} (\boldsymbol{\sigma}\boldsymbol{n}, \nabla v)_e$$

1) Hu J, Ma R, Yang X. A new mixed FEM for fourth order elliptic problems. Unpublished, 2017

with the unit out normal vector $\boldsymbol{n} = (n_1, n_2)^{\mathrm{T}}$ and the unit tangent vector $\boldsymbol{t} = (-n_2, n_1)^{\mathrm{T}}$ below. Expand $(\boldsymbol{\sigma n}, \nabla v)_e = (\boldsymbol{n}^{\mathrm{T}}\boldsymbol{\sigma n}, \partial_n v)_e + (\boldsymbol{t}^{\mathrm{T}}\boldsymbol{\sigma n}, \partial_t v)_e$. A further integration by parts gives rise to

$$(\mathrm{div}\,\mathbf{div}\boldsymbol{\sigma}, v)_K = (\boldsymbol{\sigma}, \nabla^2 v)_K - \sum_{e\in\mathcal{E}(K)}\sum_{a\in\partial e}\mathrm{sign}_{e,a}(\boldsymbol{t}^{\mathrm{T}}\boldsymbol{\sigma n})(a)v(a) - \sum_{e\in\mathcal{E}(K)}[(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{\sigma n}, \partial_n v)_e - (\partial_t(\boldsymbol{t}^{\mathrm{T}}\boldsymbol{\sigma n}) + \boldsymbol{n}^{\mathrm{T}}\mathbf{div}\boldsymbol{\sigma}, v)_e] \tag{1.5}$$

with

$$\mathrm{sign}_{e,a} := \begin{cases} 1, & \text{if } a \text{ is the end point of } e, \\ -1, & \text{if } a \text{ is the start point of } e. \end{cases}$$

Based on (1.5), besides the normal-normal continuity, the stress tensor is continuous at vertices and another trace involving the combination of derivatives of the stress is identified. The basic design of the $H(\mathrm{div}\,\mathbf{div}, \Omega; \mathbb{S})$ conforming finite elements in [15] follows.

However, it is arduous to compute the basis functions for the elements in [15, 16]. Motivated by [31, 34, 35], this paper introduces a more straightforward characterization of the $H(\mathrm{div}\,\mathbf{div}; \mathbb{S}) \cap H(\mathbf{div}; \mathbb{S})$ space. Instead of involving combination of derivatives of stresses, the continuity of $\boldsymbol{\sigma n}$ and $\boldsymbol{n}^{\mathrm{T}}\mathbf{div}\boldsymbol{\sigma}$ is imposed in the design of the new $H(\mathrm{div}\,\mathbf{div}; \mathbb{S})$ conforming elements. Correspondingly, the finite elements obtained in this paper are more regular than those in [15, 16]. Actually, the new elements are subspaces of the elements proposed in [15, 16]. The $H(\mathbf{div}; \mathbb{S})$ bubble functions presented in [31, 34, 35] possess vanishing $\boldsymbol{\sigma n}$ on each face. Therefore, the basis functions corresponding to the degrees of the freedom $\boldsymbol{n}^{\mathrm{T}}\mathbf{div}\boldsymbol{\sigma}$ can be expressed linearly by the basis of these bubbles. The remainder basis functions can be derived by the former $\boldsymbol{n}^{\mathrm{T}}\mathbf{div}\boldsymbol{\sigma}$ basis and the basis functions given by [31, 34, 35]. Besides, the new $H(\mathrm{div}\,\mathbf{div}; \mathbb{S})$ conforming finite elements in two and three dimensions can be constructed in an almost unified way, while the degrees of the freedom in [16] are fairly sophisticated.

In addition, a vectorial H^1 conforming finite element on triangular grids is introduced, and this element plus the $H(\mathrm{div}\,\mathbf{div}; \mathbb{S})$ conforming finite element forms the discrete divdiv complex. In this paper, the exactness of the finite element analogy of divdiv complex is proved on a contractible domain. Actually, by rotation, the two-dimensional divdiv complex is equivalent to the strain complex. Conforming finite elements for $H(\mathrm{rot}\,\mathbf{rot}; \mathbb{S})$ are obtained in [15] in two dimensions. By using piecewise polynomials based on the Clough-Tocher split of the triangle, some lower-order $H(\mathrm{rot}\,\mathbf{rot}; \mathbb{S})$ conforming finite elements are constructed to obtain the discrete strain complex in [18].

Furthermore, the new $H(\mathrm{div}\,\mathbf{div}, \Omega; \mathbb{S})$ conforming finite elements space developed for d being 2 and 3 are capable of discretizing the mixed formulation (1.4) with the optimal order of convergence.

The rest of the paper is organized as follows. In the subsequent section, the construction of $H(\mathrm{div}\,\mathbf{div}; \mathbb{S})$ conforming finite elements in two dimensions as well as in three dimensions is presented. Correspondingly, a vectorial H^1 conforming finite element in two dimensions is introduced to establish the discrete complex, which is proved to be exact on a contractible domain. In Section 3, the new conforming elements are exploited to discrete the mixed problem (1.4). The well-posedness is proved and the error analysis follows. Besides, superconvergence and postprocessing results are displayed. In Section 4, numerical examples are presented to demonstrate the theoretical analysis results. In the end, the appendix provides some ideas to construct the basis functions by a specific example.

Throughout the paper, an inequality $\alpha \lesssim \beta$ replaces $\alpha \leqslant c\beta$ with some multiplicative mesh-size independent constant $c > 0$, which depends on Ω only. While $\alpha \sim \beta$ means that $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$ hold simultaneously. Standard notation on Lebesgue and Sobolev spaces are employed. For a subset $G \subset \Omega$, $(\cdot, \cdot)_G$ denotes the L^2 scalar product over G, and $\|\cdot\|_{0,G}$ denotes the L^2 norm over a set G. $\|\cdot\|_0$ abbreviates $\|\cdot\|_{0,\Omega}$. Other cases are similar. Let $\mathcal{D}(G)$ denote the set of all the infinitely differentiable compactly supported functions on G. Let $P_l(G)$ stand for the set of all the polynomials with the total degree no greater than l over G. Notation $\mathbb{X}$ could be $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{M}$, $\mathbb{T}$, $\mathbb{S}$ and $\mathbb{K}$ in the text. Correspondingly, it denotes the space of scalars, vectors in d dimensions, matrices in $\mathbb{R}^{d\times d}$, traceless matrices in $\mathbb{R}^{d\times d}$, symmetric matrices in $\mathbb{R}^{d\times d}$, and skew-symmetric matrices in $\mathbb{R}^{d\times d}$, respectively. Dimension d is either 2 or 3 in this paper, and it coincides with the shape of G. For example, any variable in $\mathcal{D}(G;\mathbb{S})$ is a symmetric matrix on G and it is infinitely differentiable compactly supported. Similarly, $P_l(G;\mathbb{X})$ can be defined in the same way. Generally, $\mathcal{D}(G;\mathbb{R})$ is simply abbreviated as $\mathcal{D}(G)$, and so does $P_l(G)$ for $P_l(G;\mathbb{R})$. Define the **curl** operators below:

$$\mathbf{curl}\varphi = (-\partial_y\varphi, \partial_x\varphi)^{\mathrm{T}} \quad \text{for all } \varphi \in \mathcal{D}(\Omega;\mathbb{R}), \quad d=2,$$
$$\mathbf{curl}\boldsymbol{\varphi} = (\partial_y\varphi_3 - \partial_z\varphi_2, \partial_z\varphi_1 - \partial_x\varphi_3, \partial_x\varphi_2 - \partial_y\varphi_1)^{\mathrm{T}} \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega;\mathbb{R}^3), \quad d=3.$$

The symmetric gradient operator is denoted by $\boldsymbol{\varepsilon}(u) = 1/2(\nabla u + (\nabla u)^{\mathrm{T}})$. Generally, for a column vector function, differential operators for scalar functions will be applied row-wise to produce a matrix function. Similarly for a matrix function, differential operators for vector functions are applied row-wise. However, $\mathbf{curl}^*$ will be the **curl** operator applied column-rise.

2 The conforming finite element spaces

This section covers some preliminaries and the construction of the new $H(\operatorname{div}\mathbf{div},\Omega;\mathbb{S})$ conforming finite elements in both two and three dimensions. Besides, a vectorial $H^1(\Omega;\mathbb{R}^2)$ conforming finite element space is introduced, and a discrete case of Hilbert complex is obtained.

2.1 Notation

Suppose that $\mathcal{T}_h$ is a shape regular subdivision of Ω consisting of triangles in two dimensions and tetrahedrons in three dimensions. Define h the maximum of the diameters of all the elements $K \in \mathcal{T}_h$. Let $\mathcal{E}_h$, $\mathcal{F}_h$ and $\mathcal{V}_h$ be the set of all the edges, faces, and vertices of Ω regarding to $\mathcal{T}_h$, respectively. Given $K \in \mathcal{T}_h$, let $\mathcal{E}(K)$ denote the set of all the edges of K, and h_e stands for the diameter of edge $e \in \mathcal{E}_h$. Furthermore, when $d=3$, define the set of all the facets of the tetrahedron K as $\mathcal{F}(K)$, and h_F stands for the diameter of the face $F \in \mathcal{F}_h$. Let $\boldsymbol{n}$ and $\boldsymbol{t}$ be the unit outer normal and unit tangential vector of ∂K, respectively. More specifically, when $d=2$, $\boldsymbol{t}_e$ denotes the unit tangential vector along $e \in \mathcal{E}(K)$, and $\boldsymbol{n}_e$ is the normal counterpart. While $d=3$, given $e \in \mathcal{E}(K)$, the unit tangential vector $\boldsymbol{t}_e$, as well as two unit normal vectors, $\boldsymbol{n}_{e,1}$ and $\boldsymbol{n}_{e,2}$ are fixed. For a facet $F \in \mathcal{F}(K)$, the unit outer normal vector $\boldsymbol{n}_F$ as well as two unit tangential vectors $\boldsymbol{t}_{F,1}$ and $\boldsymbol{t}_{F,2}$ are fixed. Within the context, $\boldsymbol{t}_i$ and $\boldsymbol{n}_i$ abbreviate $\boldsymbol{t}_{F,i}$ and $\boldsymbol{n}_{e,i}$, respectively, $i=1,2$. Besides, the union of all the vertices of K is denoted by $\mathcal{V}(K)$. The jump of u across an interior $d-1$ face G shared by neighboring elements K_+ and K_- is defined by

$$[u]_G := u\,|_{K_+} - u\,|_{K_-}.$$

When it comes to any boundary face $G \subset \partial\Omega$, the jump $[\cdot]_G$ reduces to the trace.

For ensuing analysis, let $RM(K)$ denote local rigid motions. When K is a triangle with $\boldsymbol{x} = (x,y)^{\mathrm{T}} \in K$,

$$RM_{\triangle_2}(K) = \left\{ \begin{pmatrix} c_1 + c_3 y \\ c_2 - c_3 x \end{pmatrix} : c_1, c_2, c_3 \in \mathbb{R} \right\}. \tag{2.1}$$

If K is a tetrahedron with $\boldsymbol{x} = (x,y,z)^{\mathrm{T}} \in K$, then

$$RM_{\triangle_3}(K) = \left\{ \begin{pmatrix} c_1 - c_4 y - c_5 z \\ c_2 + c_4 x - c_6 z \\ c_3 + c_5 x + c_6 y \end{pmatrix} : c_1, c_2, c_3, c_4, c_5, c_6 \in \mathbb{R} \right\}. \tag{2.2}$$

Define

$$P_h := \{q \in L^2(\Omega) : q\,|_K \in P_{k-2}(K) \text{ for all } K \in \mathcal{T}_h\}.$$

Set P_h to be $P_{h,\triangle_2}$ and $P_{h,\triangle_3}$ in two and three dimensions, respectively.

Besides, RT_k is the Raviart-Thomas element space [13], i.e.,

$$RT_k(K;\mathbb{R}^d) = P_k(K;\mathbb{R}^d) + \boldsymbol{x}P_k(K).$$

Notice that

$$\dim RT_k(K;\mathbb{R}^3) = \frac{(k+1)(k+2)(k+4)}{2}.$$

Denote by RT the lowest order Raviart-Thomas element space on Ω.

2.2 The construction of the conforming elements on triangular grids

On each triangle K, denote by λ_i, $i = 1, 2, 3$ the barycenter coordinates. The finite element shape functions are simply formed by $P_k(K;\mathbb{S})$, $k \geqslant 3$. Some results are presented in the following two lemmas for later use.

Lemma 2.1 (See [11]). *Given $K \in \mathcal{T}_h$, suppose that $\boldsymbol{\psi} \in P_k(K;\mathbb{R}^2)$ satisfies $\mathrm{div}\boldsymbol{\psi} = 0$ and $\boldsymbol{\psi}\cdot\boldsymbol{n}\,|_{\partial K} = 0$. Then there exists some $q \in \lambda_1\lambda_2\lambda_3 P_{k-2}(K)$ such that*

$$\boldsymbol{\psi} = \mathbf{curl}\, q.$$

Lemma 2.2 (See [8, 17]). *Given $K \in \mathcal{T}_h$, suppose that $\boldsymbol{\tau} \in P_k(K;\mathbb{S})$ satisfies $\mathbf{div}\boldsymbol{\tau} = 0$ and $\boldsymbol{\tau}\boldsymbol{n}\,|_{\partial K} = 0$. Then there exists some $q \in (\lambda_1\lambda_2\lambda_3)^2 P_{k-4}(K)$ such that*

$$\boldsymbol{\tau} = \mathcal{J}q$$

with

$$\mathcal{J}q := \begin{pmatrix} \frac{\partial^2 q}{\partial y^2} & -\frac{\partial^2 q}{\partial x\,\partial y} \\ -\frac{\partial^2 q}{\partial x\,\partial y} & \frac{\partial^2 q}{\partial x^2} \end{pmatrix}. \tag{2.3}$$

The degrees of the freedom are defined as follows:

$$\boldsymbol{\sigma}(a) \quad \text{for all } a \in \mathcal{V}(K), \tag{2.4}$$
$$(\boldsymbol{\sigma}\boldsymbol{n}, \boldsymbol{\phi})_e \quad \text{for all } \boldsymbol{\phi} \in P_{k-2}(e;\mathbb{R}^2), \quad e \in \mathcal{E}(K), \tag{2.5}$$
$$(\mathbf{div}\boldsymbol{\sigma}\cdot\boldsymbol{n}, q)_e \quad \text{for all } q \in P_{k-1}(e), \quad e \in \mathcal{E}(K), \tag{2.6}$$
$$(\boldsymbol{\sigma}, \nabla^2 q)_K \quad \text{for all } q \in P_{k-2}(K), \tag{2.7}$$
$$(\boldsymbol{\sigma}, \nabla\mathbf{curl}\, q)_K \quad \text{for all } q \in \lambda_1\lambda_2\lambda_3 P_{k-3}(K)/P_0(K), \tag{2.8}$$
$$(\boldsymbol{\sigma}, \mathcal{J}q)_K \quad \text{for all } q \in (\lambda_1\lambda_2\lambda_3)^2 P_{k-4}(K). \tag{2.9}$$

Remark 2.3. Any function $\varphi \in \lambda_1\lambda_2\lambda_3 P_{k-3}(K)/P_0(K)$ means $\varphi = \lambda_1\lambda_2\lambda_3 q$ for some $q \in P_{k-3}(K)$ as well as $\int_K \varphi d\boldsymbol{x} = 0$.

Degrees of the freedom (2.4)–(2.6) characterize the continuity of the space $H(\mathrm{div}\mathbf{div};\mathbb{S})$. With the help of Lemmas 2.1 and 2.2, (2.8)–(2.9) can be used to derive the unisolvence. Besides, the degrees of the freedom (2.4)–(2.5) are exactly the characterization of the continuity of $H(\mathbf{div};\mathbb{S})$ in [31, 34], and the continuity of (2.6) across edges leads to $\mathbf{div}\boldsymbol{\sigma} \in H(\mathrm{div};\mathbb{R}^2)$.

The global finite element space is defined by

$$\begin{aligned}\Sigma_{k,\triangle_2} := \{&\boldsymbol{\tau} \in H(\mathrm{div}\mathbf{div},\Omega;\mathbb{S}) : \boldsymbol{\tau}\,|_K \in P_k(K;\mathbb{S}) \text{ for all } K \in \mathcal{T}_h, \\ &\text{all the degrees of the freedom (2.4)–(2.9) are single-valued}\}.\end{aligned} \tag{2.10}$$

Theorem 2.4. *The degrees of the freedom (2.4)–(2.9) uniquely determine a polynomial of $P_k(K;\mathbb{S})$ in the space $\Sigma_{k,\triangle_2}$ defined in (2.10).*

Proof. To start with, it is easy to check that the number of the degrees of the freedom (2.4)–(2.9) is

$$9+6(k-1)+3k+\frac{k(k-1)}{2}-3+\frac{(k-1)(k-2)}{2}-1+\frac{(k-2)(k-3)}{2}$$
$$=\frac{3(k+1)(k+2)}{2}=\dim P_k(K;\mathbb{S}).$$

It suffices to prove if degrees of the freedom (2.4)–(2.9) vanish for $\boldsymbol{\sigma}\in P_k(K;\mathbb{S})$, then $\boldsymbol{\sigma}=0$. Given any $v\in P_{k-2}(K)$, integration by parts and the zero degrees of the freedom (2.5)–(2.7) lead to

$$(\mathrm{div}\,\mathbf{div}\boldsymbol{\sigma},v)_K=(\boldsymbol{\sigma},\nabla^2v)_K-\sum_{e\in\mathcal{E}(K)}(\boldsymbol{\sigma n},\nabla v)_e+\sum_{e\in\mathcal{E}(K)}(\mathrm{div}\boldsymbol{\sigma}\cdot\boldsymbol{n},v)_e=0. \tag{2.11}$$

This implies $\mathrm{div}\,\mathbf{div}\boldsymbol{\sigma}=0$. Together with (2.6), according to Lemma 2.1, there exists some $\varphi\in\lambda_1\lambda_2\lambda_3P_{k-3}(K)$ such that

$$\mathbf{div}\boldsymbol{\sigma}=\mathbf{curl}\varphi. \tag{2.12}$$

For any function $\vartheta\in\lambda_1\lambda_2\lambda_3P_{k-3}(K)/P_0(K)$, integration by parts plus (2.5) and (2.8) shows

$$\begin{aligned}(\mathbf{curl}\varphi,\mathbf{curl}\vartheta)_K&=(\mathbf{div}\boldsymbol{\sigma},\mathbf{curl}\vartheta)_K\\&=-(\boldsymbol{\sigma},\nabla\mathbf{curl}\vartheta)_K+\sum_{e\in\mathcal{E}(K)}(\boldsymbol{\sigma n},\mathbf{curl}\vartheta)_e=0.\end{aligned} \tag{2.13}$$

Besides, (2.4)–(2.5) result in

$$(\mathbf{div}\boldsymbol{\sigma},\boldsymbol{v})_K=0\quad\text{for all } \boldsymbol{v}\in RM_{\triangle_2}(K). \tag{2.14}$$

Taking $\boldsymbol{v}=(-y,x)^{\mathrm{T}}$ in (2.14) and using (2.12) for replacement, we have

$$(\mathbf{div}\boldsymbol{\sigma},\boldsymbol{v})_K=(\mathbf{curl}\varphi,\boldsymbol{v})_K=2\int_K\varphi d\boldsymbol{x}=0. \tag{2.15}$$

Note that (2.13) and (2.15) lead to $\varphi=0$, and thus $\mathbf{div}\boldsymbol{\sigma}=\mathbf{0}$. Furthermore, due to (2.4)–(2.5), according to Lemma 2.2, $\mathbf{div}\boldsymbol{\sigma}=0$ entails the relation $\boldsymbol{\sigma}=\mathcal{J}\zeta$ for some $\zeta\in(\lambda_1\lambda_2\lambda_3)^2P_{k-4}(K)$. This and (2.9) conclude $\boldsymbol{\sigma}=0$ immediately. □

Remark 2.5. The degrees of the freedom $\mathbf{div}\boldsymbol{\sigma}\cdot\boldsymbol{n}$ in (2.6) can be replaced by $\partial_n(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{\sigma n})$ since

$$\mathbf{div}\boldsymbol{\sigma}\cdot\boldsymbol{n}=\mathbf{div}(\boldsymbol{\sigma n})=\partial_t(\boldsymbol{t}^{\mathrm{T}}\boldsymbol{\sigma n})+\partial_n(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{\sigma n}).$$

Let $a_{e,1}$ and $a_{e,2}$ be the start and end point of the edge e, respectively. Integration by parts leads to

$$(\partial_t(\boldsymbol{t}^{\mathrm{T}}\boldsymbol{\sigma n}),v)_e=\boldsymbol{t}^{\mathrm{T}}\boldsymbol{\sigma n}v\,|_{a_{e,1}}^{a_{e,2}}-(\boldsymbol{t}^{\mathrm{T}}\boldsymbol{\sigma n},\partial_tv)_e\quad\text{for all } v\in P_{k-1}(e).$$

The first term can be covered by the degrees of the freedom (2.4), and the second term can be derived by the degrees of the freedom (2.5).

2.3 The construction of the finite element divdiv complex on triangular grids

Define

$$H^1(\mathrm{div},\Omega;\mathbb{R}^2):=\{\boldsymbol{v}\in H^1(\Omega;\mathbb{R}^2):\mathrm{div}\boldsymbol{v}\in H^1(\Omega)\}.$$

The vectorial space $V_h\subset H^1(\mathrm{div},\Omega;\mathbb{R}^2)$ is introduced in this subsection, and the discrete exact complex is established. On a triangle $K\in\mathcal{T}_h$, the shape function space is $P_{k+1}(K;\mathbb{R}^2)$, and the degrees of the freedom are

$$\boldsymbol{v}(a),\ \nabla\boldsymbol{v}(a)\quad\text{for all } a\in\mathcal{V}(K), \tag{2.16}$$

$$(\boldsymbol{v}, \boldsymbol{\phi})_e \quad \text{for all } \boldsymbol{\phi} \in P_{k-3}(e; \mathbb{R}^2), \quad e \in \mathcal{E}(K), \tag{2.17}$$
$$(\mathrm{div}\boldsymbol{v}, q)_e \quad \text{for all } q \in P_{k-2}(e), \quad e \in \mathcal{E}(K), \tag{2.18}$$
$$(\boldsymbol{v}, \nabla q)_K \quad \text{for all } q \in P_{k-3}(K), \tag{2.19}$$
$$(\boldsymbol{v}, \mathbf{curl} q)_K \quad \text{for all } q \in (\lambda_1\lambda_2\lambda_3)^2 P_{k-4}(K). \tag{2.20}$$

Then the space V_h is defined by

$$\begin{aligned} V_h := \{\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^2) : \boldsymbol{v}\,|_K \in P_{k+1}(K; \mathbb{R}^2) \text{ for all } K \in \mathcal{T}_h,\\ \text{all the degrees of the freedom (2.16)–(2.20) are single-valued}\}. \end{aligned} \tag{2.21}$$

Theorem 2.6. *The degrees of the freedom* (2.16)–(2.20) *uniquely determine a polynomial of* $P_{k+1}(K; \mathbb{R}^2)$ *in the space* V_h *defined in* (2.21).

Proof. To start with, it is easy to check that the number of the degrees of the freedom (2.16)–(2.20) equals the dimension of $P_{k+1}(K; \mathbb{R}^2)$. In fact, both of them are

$$(k+3)(k+2).$$

It suffices to prove if degrees of the freedom (2.16)–(2.20) vanish for $\boldsymbol{v} \in P_{k+1}(K; \mathbb{R}^2)$, then $\boldsymbol{v} = 0$. Actually, (2.16)–(2.17) lead to

$$\boldsymbol{v}\,|_e = 0 \quad \text{for all } e \in \mathcal{E}(K). \tag{2.22}$$

The combination of (2.16) and (2.18) results in

$$\mathrm{div}\boldsymbol{v}\,|_e = 0 \quad \text{for all } e \in \mathcal{E}(K). \tag{2.23}$$

This leads to $\mathrm{div}\boldsymbol{v} = \lambda_1\lambda_2\lambda_3 r$ for some $r \in P_{k-3}(K)$. Besides, according to (2.19),

$$(\mathrm{div}\boldsymbol{v}, q)_K = 0 \quad \text{for all } q \in P_{k-3}(K). \tag{2.24}$$

Thus $r = 0$ and $\mathrm{div}\boldsymbol{v} = 0$. This and (2.22)–(2.23) guarantee that there exists some $p \in (\lambda_1\lambda_2\lambda_3)^2 P_{k-4}(K)$ such that

$$\boldsymbol{v} = \mathbf{curl} p.$$

This and (2.20) conclude $\boldsymbol{v} = 0$. □

According to (2.22)–(2.23), the vectorial H^1 conforming finite element space V_h is $H^1(\mathrm{div})$ conforming.

Remark 2.7. For $k = 3$, V_h is a piecewise polynomial of degree 4, which happens to be $\mathcal{P}_4\Lambda^1$ presented in [4, Section 7]. It is an ingredient in the Bernšteĭn-Gelfand-Gelfand (BGG) approach [10] for the Arnold-Winther elasticity element [5].

Remark 2.8. It is straightforward that the space $\Sigma_{k,\triangle_2}$ is a subset of $H(\mathrm{div}\mathbf{div}, \Omega; \mathbb{S}) \cap H(\mathbf{div}, \Omega; \mathbb{S})$. Actually, $\Sigma_{k,\triangle_2}$ is able to preserve the Hilbert complex

$$RT \xrightarrow{\subset} H^1(\mathrm{div}, \Omega; \mathbb{R}^2) \xrightarrow{\mathrm{sym}\mathbf{curl}} H(\mathrm{div}\mathbf{div}, \Omega; \mathbb{S}) \cap H(\mathbf{div}, \Omega; \mathbb{S}) \xrightarrow{\mathrm{div}\mathbf{div}} L^2(\Omega) \longrightarrow 0$$

in the discrete case on a contractible domain Ω. The commuting diagram in [15] can also be constructed here.

Before establishing the exact complex for the finite elements, the exact complex for bubble function spaces is constructed below. Define

$$\mathring{V}_{k+1}(K) := \{\boldsymbol{v} \in P_{k+1}(K; \mathbb{R}^2) : \text{all the degrees of the freedom (2.16)–(2.18) vanish}\}, \tag{2.25}$$
$$\mathring{\Sigma}_k(K) := \{\boldsymbol{\sigma} \in P_k(K; \mathbb{S}) : \text{all the degrees of the freedom (2.4)–(2.6) vanish}\}, \tag{2.26}$$
$$\mathring{P}_{k-2}(K) := P_{k-2}(K)/P_1(K). \tag{2.27}$$

Lemma 2.9. *Given $K \in \mathcal{T}_h$, it holds that*

$$\mathrm{div}\mathbf{div}\mathring{\Sigma}_k(K) = \mathring{P}_{k-2}(K).$$

Proof. It is straightforward from (2.11) that

$$\mathrm{div}\mathbf{div}\mathring{\Sigma}_k(K) \subseteq \mathring{P}_{k-2}(K).$$

It suffices to prove $\mathring{P}_{k-2}(K) \subseteq \mathrm{div}\mathbf{div}\mathring{\Sigma}_k(K)$. Actually, if the inclusion does not hold, then there exists some $q \in \mathring{P}_{k-2}(K)$ and $q \neq 0$ such that

$$(\mathrm{div}\mathbf{div}\boldsymbol{\tau}, q)_K = 0 \quad \text{for all } \boldsymbol{\tau} \in \mathring{\Sigma}_k(K).$$

Integration by parts as in (2.11) leads to

$$(\boldsymbol{\tau}, \nabla^2 q)_K = 0 \quad \text{for all } \boldsymbol{\tau} \in \mathring{\Sigma}_k(K).$$

According to the degrees of the freedom (2.4)–(2.9), there exists $\boldsymbol{\tau} \in \mathring{\Sigma}_k(K)$ such that $(\boldsymbol{\tau}, \nabla^2 q)_K \neq 0$ as long as $\nabla^2 q \neq 0$. Hence,

$$\nabla^2 q = 0.$$

This implies $q \in P_1(K)$. The contradiction occurs. This concludes the proof. □

Lemma 2.10. *For any triangle K, the polynomial complexes*

$$RT \xrightarrow{\subset} P_{k+1}(K;\mathbb{R}^2) \xrightarrow{\mathrm{sym}\mathbf{curl}} P_k(K;\mathbb{S}) \xrightarrow{\mathrm{div}\mathbf{div}} P_{k-2}(K) \longrightarrow 0$$

and

$$0 \xrightarrow{\subset} \mathring{V}_{k+1}(K) \xrightarrow{\mathrm{sym}\mathbf{curl}} \mathring{\Sigma}_k(K) \xrightarrow{\mathrm{div}\mathbf{div}} \mathring{P}_{k-2}(K) \longrightarrow 0$$

are exact.

Proof. The first polynomial complex follows directly from [15, Lemma 3.1]. The exactness also follows from the existence of homotopy operators (see [18,19]). To obtain the second complex, let $\boldsymbol{\sigma} := \mathrm{sym}\mathbf{curl}\boldsymbol{v}$ for any $\boldsymbol{v} \in \mathring{V}_{k+1}(K)$. It suffices to prove $\boldsymbol{\sigma} \in \mathring{\Sigma}_k(K)$. According to (2.16), $\boldsymbol{\sigma}(a) = 0$ for all $a \in \mathcal{V}(K)$. For $\boldsymbol{\phi} \in P_{k-2}(e;\mathbb{R}^2)$,

$$(\boldsymbol{\sigma n}, \boldsymbol{\phi})_e = (\boldsymbol{n}^{\mathrm{T}}\boldsymbol{\sigma n}, \boldsymbol{\phi}\cdot\boldsymbol{n})_e + (\boldsymbol{t}^{\mathrm{T}}\boldsymbol{\sigma n}, \boldsymbol{\phi}\cdot\boldsymbol{t})_e =: I + II.$$

The calculations in [15, Lemma 2.2] lead to some identities

$$\boldsymbol{n}^{\mathrm{T}}\boldsymbol{\sigma n} = \boldsymbol{n}^{\mathrm{T}}\partial_t\boldsymbol{v}, \tag{2.28}$$

$$\boldsymbol{t}^{\mathrm{T}}\boldsymbol{\sigma n} = \boldsymbol{t}^{\mathrm{T}}\partial_t\boldsymbol{v} - \frac{1}{2}\mathbf{div}\boldsymbol{v}, \tag{2.29}$$

$$\mathbf{div}\boldsymbol{\sigma}\cdot\boldsymbol{n} = \frac{1}{2}\partial_t\mathbf{div}\boldsymbol{v}. \tag{2.30}$$

Combined with (2.16)–(2.17), (2.28) leads to $I = 0$. Combined with (2.16)–(2.18), (2.29) leads to $II = 0$. The identity (2.30) plus (2.16) and (2.18) result in

$$\mathbf{div}\boldsymbol{\sigma}\cdot\boldsymbol{n} = 0 \quad \text{on each } e \in \mathcal{E}(K).$$

The previous arguments lead to $\boldsymbol{\sigma} \in \mathring{\Sigma}_k(K)$ and $\mathrm{sym}\mathbf{curl}\mathring{V}_{k+1}(K) \subset \mathring{\Sigma}_k(K)$.

On the other hand, a direct calculation leads to

$$\begin{aligned}
&\dim(\overset{\circ}{\Sigma}_k(K)) = (k-1)^2 + \frac{(k-2)(k-3)}{2} - 4,\\
&\dim(\mathrm{div}\mathbf{div}\overset{\circ}{\Sigma}_k(K)) = \frac{1}{2}k(k-1) - 3 = \dim(\overset{\circ}{P}_{k-2}(K)),\\
&\dim(\overset{\circ}{V}_{k+1}(K)) = (k-2)^2 - 1.
\end{aligned}$$

These result in

$$\dim(\mathrm{div}\mathbf{div}\overset{\circ}{\Sigma}_k(K)) = \dim(\overset{\circ}{\Sigma}_k(K)) - \dim(\overset{\circ}{V}_{k+1}(K)).$$

Together with Lemma 2.9, the exactness of the complex follows. □

Lemma 2.11 (See [7])**.** *The divdiv Hilbert complex*

$$RT \xrightarrow{\subset} H^3(\Omega;\mathbb{R}^2) \xrightarrow{\mathrm{sym}\mathbf{curl}} H^2(\Omega;\mathbb{S}) \xrightarrow{\mathrm{div}\mathbf{div}} L^2(\Omega) \longrightarrow 0$$

is exact on a contractible domain Ω*.*

Similar to [15, Subsection 3.3], the interpolations with commuting properties can be constructed as follows. Denote by $\Pi_{K,\triangle_2} : H^2(K;\mathbb{S}) \to P_k(K;\mathbb{S})$ the local nodal interpolation operator based on the degrees of the freedom (2.4)–(2.9). For any $\boldsymbol{\tau} \in P_k(K;\mathbb{S})$, $\Pi_{K,\triangle_2}\boldsymbol{\tau} = \boldsymbol{\tau}$ is easy to verify. For the shape regular mesh $\mathcal{T}_h$,

$$\|\boldsymbol{\tau} - \Pi_{K,\triangle_2}\boldsymbol{\tau}\|_{0,K} + h_K|\boldsymbol{\tau} - \Pi_{K,\triangle_2}\boldsymbol{\tau}|_{1,K} + h_K^2|\boldsymbol{\tau} - \Pi_{K,\triangle_2}\boldsymbol{\tau}|_{2,K} \lesssim h_K^s|\boldsymbol{\tau}|_{s,K} \tag{2.31}$$

holds for $\boldsymbol{\tau} \in H^s(K;\mathbb{S})$ with $2 \leqslant s \leqslant k+1$. Integration by parts leads to

$$\mathrm{div}\mathbf{div}(\Pi_{K,\triangle_2}\boldsymbol{\tau}) = \mathcal{Q}_{k-2}^K\mathrm{div}\mathbf{div}\boldsymbol{\tau} \quad \text{for all } \boldsymbol{\tau} \in H^2(K;\mathbb{S}). \tag{2.32}$$

Here, $\mathcal{Q}_{k-2}^K : L^2(K) \to P_{k-2}(K)$ is the L^2 projection operator. It may be later denoted by $\mathcal{Q}_{k-2,\triangle_d}^K$, $d = 2,3$, to distinguish the dimension of K.

Denote by $\tilde{I}_K : H^3(K;\mathbb{R}^2) \to P_{k+1}(K;\mathbb{R}^2)$ the local nodal interpolation operator based on the degrees of the freedom (2.16)–(2.20). For any $\boldsymbol{v} \in P_{k+1}(K;\mathbb{R}^2)$, $\tilde{I}_K\boldsymbol{v} = \boldsymbol{v}$ is easy to verify. For the shape regular mesh $\mathcal{T}_h$,

$$\|\boldsymbol{v} - \tilde{I}_K\boldsymbol{v}\|_{0,K} + h_K|\boldsymbol{v} - \tilde{I}_K\boldsymbol{v}|_{1,K} \lesssim h_K^s|\boldsymbol{v}|_{s,K} \tag{2.33}$$

holds for $\boldsymbol{v} \in H^s(K;\mathbb{R}^2)$ with $3 \leqslant s \leqslant k+2$. The proof of Lemma 2.10 shows

$$\Pi_{K,\triangle_2}(\mathrm{sym}\mathbf{curl}\boldsymbol{v}) - \mathrm{sym}\mathbf{curl}(\tilde{I}_K\boldsymbol{v}) \in \overset{\circ}{\Sigma}_k(K).$$

Hence, according to Lemma 2.10, there exists $\tilde{\boldsymbol{v}} \in \overset{\circ}{V}_{k+1}(K)$ such that

$$\mathrm{sym}\mathbf{curl}\tilde{\boldsymbol{v}} = \Pi_{K,\triangle_2}(\mathrm{sym}\mathbf{curl}\boldsymbol{v}) - \mathrm{sym}\mathbf{curl}(\tilde{I}_K\boldsymbol{v}), \tag{2.34}$$

$$\|\tilde{\boldsymbol{v}}\|_{0,K} \lesssim h_K\|\Pi_{K,\triangle_2}(\mathrm{sym}\mathbf{curl}\boldsymbol{v}) - \mathrm{sym}\mathbf{curl}(\tilde{I}_K\boldsymbol{v})\|_{0,K}. \tag{2.35}$$

Let $I_K\boldsymbol{v} := \tilde{I}_K\boldsymbol{v} + \tilde{\boldsymbol{v}}$. It is also easy to verify $I_K\boldsymbol{v} = \boldsymbol{v}$ for any $\boldsymbol{v} \in P_{k+1}(K;\mathbb{R}^2)$ and

$$\mathrm{sym}\mathbf{curl}(I_K\boldsymbol{v}) = \Pi_{K,\triangle_2}(\mathrm{sym}\mathbf{curl}\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in H^3(K;\mathbb{R}^2). \tag{2.36}$$

It follows from (2.31) and (2.33) that

$$\|\boldsymbol{v} - I_K\boldsymbol{v}\|_{0,K} + h_K|\boldsymbol{v} - I_K\boldsymbol{v}|_{1,K} \lesssim h_K^s|\boldsymbol{v}|_{s,K} \tag{2.37}$$

with $3 \leqslant s \leqslant k+2$.

For each $K \in \mathcal{T}_h$, let $I_h : H^3(\Omega;\mathbb{R}^2) \to V_h$ be defined by $(I_h \boldsymbol{v})|_K := I_K(\boldsymbol{v}|_K)$. $\Pi_{h,\triangle_2} : H^2(\Omega;\mathbb{S}) \to \Sigma_{k,\triangle_2}$ is defined by $(\Pi_{h,\triangle_2}\boldsymbol{\tau})|_K := \Pi_{K,\triangle_2}(\boldsymbol{\tau}|_K)$ as well as $\mathcal{Q}_{h,\triangle_2} : L^2(\Omega) \to P_{h,\triangle_2}$ is defined by $(\mathcal{Q}_{h,\triangle_2} q)|_K := \mathcal{Q}^K_{k-2,\triangle_2}(q|_K)$.

It follows immediately

$$\mathrm{div}\mathbf{div}(\Pi_{h,\triangle_2}\boldsymbol{\tau}) = \mathcal{Q}_{h,\triangle_2}\mathrm{div}\mathbf{div}\boldsymbol{\tau} \quad \text{for all } \boldsymbol{\tau} \in H^2(\Omega;\mathbb{S}), \tag{2.38}$$

$$\mathrm{sym}\mathbf{curl}(I_h\boldsymbol{v}) = \Pi_{h,\triangle_2}(\mathrm{sym}\mathbf{curl}\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in H^3(\Omega;\mathbb{R}^2). \tag{2.39}$$

Lemma 2.12. *The sequence*

$$RT \xrightarrow{\subset} V_h \xrightarrow{\mathrm{sym}\mathbf{curl}} \Sigma_{k,\triangle_2} \xrightarrow{\mathrm{div}\mathbf{div}} P_{h,\triangle_2} \longrightarrow 0$$

is a complex, which is exact on contractible domains.

Proof. It is straightforward that

$$\mathrm{div}\mathbf{div}\Sigma_{k,\triangle_2} \subseteq P_{h,\triangle_2}.$$

To obtain $P_{h,\triangle_2} = \mathrm{div}\mathbf{div}\Sigma_{k,\triangle_2}$, it suffices to prove $P_{h,\triangle_2} \subseteq \mathrm{div}\mathbf{div}\Sigma_{k,\triangle_2}$. If the inclusion does not hold, then there exists some $q \in P_{h,\triangle_2}$ and $q \neq 0$ such that

$$(\mathrm{div}\mathbf{div}\boldsymbol{\tau}, q)_\Omega = 0 \quad \text{for all } \boldsymbol{\tau} \in \Sigma_{k,\triangle_2}.$$

Lemma 2.9 shows $q|_K \in P_1(K)$ for all $K \in \mathcal{T}_h$. Let the only nonzero degrees of the freedom of $\boldsymbol{\tau}$ be

$$(\mathbf{div}\boldsymbol{\tau}\cdot\boldsymbol{n}_e, [q]_e)_e = ([q]_e, [q]_e)_e \quad \text{on some } e \in \mathcal{E}_h.$$

Integration by parts leads to

$$0 = (\mathrm{div}\mathbf{div}\boldsymbol{\tau}, q)_\Omega = \sum_{K\in\mathcal{T}_h} (\mathrm{div}\boldsymbol{\tau}\cdot\boldsymbol{n}, q)_{\partial K} = \|[q]_e\|^2_{0,e}.$$

This shows $[q]_e = 0$. The arbitrariness of the choice of $e \in \mathcal{E}_h$ leads to $q = 0$. The contradiction occurs.

In addition, (2.39) implies

$$\mathrm{sym}\mathbf{curl}V_h \subseteq \Sigma_{h,\triangle_2}.$$

By counting the dimensions,

$$\begin{aligned}
&\mathrm{dim}\Sigma_{h,\triangle_2} = 3\#\mathcal{V}_h + (3k-2)\#\mathcal{E}_h + \frac{3}{2}k(k-3)\#\mathcal{T}_h,\\
&\mathrm{dimsym}\mathbf{curl}V_h = 6\#\mathcal{V}_h + (3k-5)\#\mathcal{E}_h + (k-1)(k-3)\#\mathcal{T}_h - 3,\\
&\mathrm{dimdiv}\mathbf{div}\Sigma_{h,\triangle_2} = \frac{1}{2}k(k-1)\#\mathcal{T}_h.
\end{aligned}$$

Here, $\#\mathcal{S}$ is the number of the elements in the finite set S. According to the Euler's formula $\#\mathcal{E}_h + 1 = \#\mathcal{V}_h + \#\mathcal{T}_h$,

$$\mathrm{dim}\Sigma_{h,\triangle_2} = \mathrm{dimsym}\mathbf{curl}V_h + \mathrm{dimdiv}\mathbf{div}\Sigma_{h,\triangle_2}. \tag{2.40}$$

This concludes that the complex is exact. □

2.4 The construction of the conforming elements on tetrahedral grids

In this subsection, Ω is a bounded polyhedron in $\mathbb{R}^3$. Given a tetrahedron $K \in \mathcal{T}_h$, the finite element shape functions are formed by $P_k(K;\mathbb{S})$, $k \geqslant 3$. Some results and notation are introduced here for ensuing use.

Lemma 2.13 (See [11]). *Suppose that $K \in \mathcal{T}_h$, $\boldsymbol{\psi} \in P_k(K;\mathbf{R}^3)$ satisfies $\mathbf{div}\boldsymbol{\psi} = 0$, and $\boldsymbol{\psi} \cdot \boldsymbol{n}\,|_{\partial K} = 0$. Then there exists some $\boldsymbol{\vartheta} \in W_{k+1}(K;\mathbb{R}^3)$ such that*

$$\boldsymbol{\psi} = \mathbf{curl}\boldsymbol{\vartheta},$$

where $W_{k+1}(K;\mathbb{R}^3)$ is defined by

$$W_{k+1}(K;\mathbb{R}^3) := \{\boldsymbol{\phi} \in P_{k+1}(K;\mathbb{R}^3) : \boldsymbol{\phi} \times \boldsymbol{n}\,|_F = 0 \text{ for all } F \in \mathcal{F}(K)\}.$$

Define

$$M_{k+2}(K;\mathbb{S}) := \{\boldsymbol{\tau} \in P_{k+2}(K;\mathbb{S}) : (I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})\boldsymbol{\tau}(I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})\,|_F = \Lambda_F(\tau) = 0 \text{ for all } F \in \mathcal{F}(K)\}$$

with $\Lambda_F(\boldsymbol{\tau}) : F \to (I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})\mathbb{S}(I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})$ being defined by

$$\Lambda_F(\boldsymbol{\tau}) = (I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})(2\varepsilon(\boldsymbol{\tau}\boldsymbol{n}) - \partial_n\boldsymbol{\tau})(I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}).$$

Here, F is a plane with unit normal $\boldsymbol{n}$.

Lemma 2.14 (See [3]). *Suppose that $K \in \mathcal{T}_h$ is a tetrahedron, $\boldsymbol{\tau} \in P_k(K;\mathbb{S})$ satisfies $\mathbf{div}\boldsymbol{\tau} = 0$ and $\boldsymbol{\tau}\boldsymbol{n}\,|_{\partial K} = 0$. Then there exists some $\boldsymbol{\zeta} \in M_{k+2}(K;\mathbb{S})$ such that*

$$\boldsymbol{\tau} = \mathbf{curl}\mathbf{curl}^*\boldsymbol{\zeta}.$$

In addition, define

$$\mathcal{W}_{k-1}(K;\mathbb{R}^3) := \mathbf{curl}W_k(K;\mathbb{R}^3)/RM_{\triangle_3}(K) \tag{2.41}$$

and

$$\mathcal{M}_k(K;\mathbb{S}) := \mathbf{curl}\mathbf{curl}^*\, M_{k+2}(K;\mathbb{S}). \tag{2.42}$$

The degrees of the freedom are

$$\boldsymbol{\sigma}(a) \quad \text{for all } a \in \mathcal{V}(K), \tag{2.43}$$

$$(\boldsymbol{t}_e^{\mathrm{T}}\boldsymbol{\sigma}\boldsymbol{n}_j, q)_e, \quad (\boldsymbol{n}_i^{\mathrm{T}}\boldsymbol{\sigma}\boldsymbol{n}_j, q)_e, \quad 1 \leqslant i,j \leqslant 2 \quad \text{for all } q \in P_{k-2}(e), \quad e \in \mathcal{E}(K), \tag{2.44}$$

$$(\boldsymbol{\sigma}\boldsymbol{n}, \boldsymbol{\phi})_F \quad \text{for all } \boldsymbol{\phi} \in P_{k-3}(F;\mathbb{R}^3), \quad F \in \mathcal{F}(K), \tag{2.45}$$

$$(\mathbf{div}\boldsymbol{\sigma} \cdot \boldsymbol{n}, q)_F \quad \text{for all } q \in P_{k-1}(F), \quad F \in \mathcal{F}(K), \tag{2.46}$$

$$(\boldsymbol{\sigma}, \nabla^2 q)_K \quad \text{for all } q \in P_{k-2}(K), \tag{2.47}$$

$$(\boldsymbol{\sigma}, \nabla\boldsymbol{\phi})_K \quad \text{for all } \boldsymbol{\phi} \in \mathcal{W}_{k-1}(K;\mathbb{R}^3), \tag{2.48}$$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_K \quad \text{for all } \boldsymbol{\tau} \in \mathcal{M}_k(K;\mathbb{S}). \tag{2.49}$$

The degrees of the freedom (2.43)–(2.45) are exactly the characterization of the continuity of $H(\mathbf{div};\mathbb{S})$ in [31, 34], and the continuity of (2.46) across each interior face leads to $\mathbf{div}\boldsymbol{\sigma} \in H(\mathrm{div};\mathbb{R}^3)$.

The global conforming finite element space is defined by

$$\begin{aligned}\Sigma_{k,\triangle_3} := \{&\boldsymbol{\tau} \in H(\mathrm{div}\mathbf{div}, \Omega;\mathbb{S}) : \boldsymbol{\tau}\,|_K \in P_k(K;\mathbb{S}) \text{ for all } K \in \mathcal{T}_h,\\ &\text{all the degrees of the freedom (2.43)–(2.49) are single-valued}\}.\end{aligned} \tag{2.50}$$

Theorem 2.15. *The degrees of the freedom* (2.43)–(2.49) *uniquely determine a polynomial of $P_k(K;\mathbb{S})$ defined in* (2.50).

Proof. Note that given any $\boldsymbol{\psi} \in W_k(K;\mathbb{R}^3)$, $\boldsymbol{\psi} \times \boldsymbol{n}|_{\partial K} = 0$ and an integration by parts lead to $(\mathbf{curl}\boldsymbol{\psi}, \boldsymbol{u})_K = (\boldsymbol{\psi}, \mathbf{curl}\boldsymbol{u})_K$ for any $\boldsymbol{u} \in RM_{\triangle_3}(K)$. This, (2.41) and $\mathbf{curl}RM_{\triangle_3}(K) = P_0(K;\mathbb{R}^3)$ imply

$$\begin{aligned}\dim\mathcal{W}_{k-1}(K;\mathbb{R}^3) &= \dim\mathbf{curl}W_k(K;\mathbb{R}^3) - 3\\ &= \dim W_k(K;\mathbb{R}^3) - \dim\nabla\overset{\circ}{P}_{k+1}(K) - 3\\ &= \dim RT_{k-3}(K;\mathbb{R}^3) - \dim P_{k-3}(K) - 3\\ &= \frac{2k^3 - 3k^2 - 5k - 12}{6}.\end{aligned}$$

The number of degrees of the freedom (2.49) reads (see [3, Theorem 7.2])

$$\dim\mathcal{M}_k(K;\mathbb{S}) = \frac{k^3-3k^2-4k+12}{2}.$$

The remainder degrees of the freedom can be counted easily. Thus the number of all the degrees of the freedom (2.43)–(2.49) is

$$(k+1)(k+2)(k+3),$$

which equals $\dim P_k(K;\mathbb{S})$.

Suppose that $\boldsymbol{\sigma}\in P_k(K;\mathbb{S})$ and all the degrees of the freedom (2.43)–(2.49) are zero. Then the unisolvence for $P_k(K;\mathbb{S})$ follows from $\boldsymbol{\sigma}=0$. For $v\in P_{k-2}(K)$, integration by parts and the zero degrees of the freedom (2.45)–(2.47) lead to

$$(\operatorname{div}\mathbf{div}\boldsymbol{\sigma},v)_K = (\boldsymbol{\sigma},\nabla^2 v)_K - \sum_{F\in\mathcal{F}(K)}(\boldsymbol{\sigma}\boldsymbol{n},\nabla v)_F + \sum_{F\in\mathcal{F}(K)}(\operatorname{div}\boldsymbol{\sigma}\cdot\boldsymbol{n},v)_F = 0.$$

Therefore,

$$\operatorname{div}\mathbf{div}\boldsymbol{\sigma}=0. \tag{2.51}$$

This, Lemma 2.13, (2.46) and (2.51) ensure that there exists a function $\boldsymbol{\psi}\in W_k(K;\mathbb{R}^3)$ such that

$$\mathbf{div}\boldsymbol{\sigma}=\mathbf{curl}\boldsymbol{\psi}.$$

Furthermore, for all $\boldsymbol{\vartheta}\in W_k(K;\mathbb{R}^3)$ with $\mathbf{curl}\boldsymbol{\vartheta}\perp RM_{\triangle_3}(K)$, (2.48) and (2.45) result in

$$\begin{aligned}(\mathbf{curl}\boldsymbol{\psi},\mathbf{curl}\boldsymbol{\vartheta})_K &= (\mathbf{div}\boldsymbol{\sigma},\mathbf{curl}\boldsymbol{\vartheta})_K\\ &= -(\boldsymbol{\sigma},\nabla\mathbf{curl}\boldsymbol{\vartheta})_K + \sum_{F\in\mathcal{F}(K)}(\boldsymbol{\sigma}\boldsymbol{n},\mathbf{curl}\boldsymbol{\vartheta})_F = 0.\end{aligned} \tag{2.52}$$

On the other hand, (2.43)–(2.45) lead to the following orthogonality:

$$(\mathbf{div}\boldsymbol{\sigma},\boldsymbol{v})_K=0 \quad \text{for all } \boldsymbol{v}\in RM_{\triangle_3}(K). \tag{2.53}$$

This and (2.52) prove $\boldsymbol{\psi}=\mathbf{0}$. Hence $\mathbf{div}\boldsymbol{\sigma}=\mathbf{0}$. Furthermore, (2.43)–(2.45) lead to $\boldsymbol{\sigma}\boldsymbol{n}=0$ on ∂K. According to Lemma 2.14, $\mathbf{div}\boldsymbol{\sigma}=\mathbf{0}$ entails the relation $\boldsymbol{\sigma}=\mathbf{curl}\mathbf{curl}^*\boldsymbol{\varphi}$ for some $\boldsymbol{\varphi}\in M_{k+2}(K;\mathbb{S})$. Consequently, (2.49) concludes $\boldsymbol{\sigma}=\mathbf{0}$. □

Remark 2.16. Alternatively, $\mathcal{M}_k(K;\mathbb{S})$ from (2.42) can be defined by

$$\mathcal{M}_k(K;\mathbb{S}) := \{\boldsymbol{\sigma}\in P_k(K;\mathbb{S}) : \mathbf{div}\boldsymbol{\sigma}=0,\, \boldsymbol{\sigma}\boldsymbol{n}=0\}.$$

The number of the basis of the $H(\mathbf{div};\mathbb{S})-P_k$ bubble function space $\Sigma_{K,b}:=\{\boldsymbol{\sigma}\in P_k(K;\mathbb{S}):\boldsymbol{\sigma}\boldsymbol{n}=0\}$ introduced in [35, (2.9)] is $\dim P_{k-2}(K;\mathbb{S})$, and the range of $\mathbf{div}\Sigma_{K,b}$ is P_{k-1}/RM. Furthermore, restricted to the bubble functions, the adjoint of $\mathbf{div}$ operator is $-\boldsymbol{\varepsilon}$. The dimension of $\mathcal{M}_k(K;\mathbb{S})$ can also be derived by the subtraction of the dimension of the range of $\boldsymbol{\varepsilon}(P_{k-1}/RM)$ from $\dim P_{k-2}(K;\mathbb{S})$, which reads

$$\dim\mathcal{M}_k(K;\mathbb{S}) = 6\dim P_{k-2}(K) - 3\dim P_{k-1}(K) + 6 = \frac{k^3-3k^2-4k+12}{2}. \tag{2.54}$$

In addition, the basis functions of the space $\mathcal{M}_k(K;\mathbb{S})$ can be constructed by those bubbles in [35].

Remark 2.17. The continuity of $\mathbf{div}\boldsymbol{\sigma}\cdot\boldsymbol{n}$ can be replaced by $\partial_n(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{\sigma}\boldsymbol{n})$. However, different from two dimensions, the replacement cannot be done for the interpolation of the degrees of the freedom (2.46). In fact, for any $v\in P_{k-1}(F)$,

$$\begin{aligned}(\mathbf{div}\boldsymbol{\sigma}\cdot\boldsymbol{n},v)_F &= (\mathbf{div}_F(\boldsymbol{\sigma}\boldsymbol{n}),v)_F + (\partial_{\boldsymbol{n}}(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{\sigma}\boldsymbol{n}),v)_F\\ &= -(\boldsymbol{\sigma}\boldsymbol{n},\nabla_F v)_F + (\boldsymbol{n}_{\partial F}^{\mathrm{T}}\boldsymbol{\sigma}\boldsymbol{n},v)_{\partial F} + (\partial_{\boldsymbol{n}}(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{\sigma}\boldsymbol{n}),v)_F.\end{aligned} \tag{2.55}$$

Here, $\mathbf{div}_F(\boldsymbol{\sigma}\boldsymbol{n}) := (\boldsymbol{n}\times\nabla)\cdot(\boldsymbol{n}\times(\boldsymbol{\sigma}\boldsymbol{n}))$ and $\nabla_F v := (\boldsymbol{n}\times\nabla v)\times\boldsymbol{n}$. The first two terms of (2.55) are not any of the degrees of the freedom defined in (2.43)–(2.49).

3 Mixed finite element methods

Recall that the dimension d in this paper is either 2 or 3. This section exploits the space $P_{h,\triangle_d}$ and the anterior $H(\mathrm{div}\mathbf{div},\Omega;\mathbb{S})$ conforming finite element spaces $\Sigma_{k,\triangle_d}$, $d=2,3$, to discretize the biharmonic equation. The mixed finite element approximation for (1.4) is to find $\boldsymbol{\sigma}_h\in\Sigma_{k,\triangle_d}$, and $u_h\in P_{h,\triangle_d}$ such that

$$\begin{aligned}&(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)+(\mathrm{div}\mathbf{div}\boldsymbol{\tau}_h,u_h)=0\quad\text{for all }\boldsymbol{\tau}_h\in\Sigma_{k,\triangle_d},\\&(\mathrm{div}\mathbf{div}\boldsymbol{\sigma}_h,v_h)=-(f,v_h)\quad\text{for all }v_h\in P_{h,\triangle_d}.\end{aligned}\tag{3.1}$$

3.1 The BB condition

In this subsection, the discrete inf-sup condition is proved to obtain the well-posedness of the mixed finite problem (3.1). Define $T(\mathbb{X}):=\{K\in\mathcal{T}_h:K\cap\mathbb{X}\neq\emptyset\}$ and $N(T(\mathbb{X})):=\#T(\mathbb{X})$ with $\mathbb{X}$ being a vertex $a\in\mathcal{V}_h$ or an edge $e\in\mathcal{E}_h$. The proof of the Babuška-Brezzi (BB) condition is based on a quasi-interpolation $\widetilde{\Pi}_{h,\triangle_d}$ with $d=2,3$.

Recall the L^2 projection $\mathcal{Q}^K_{k,\triangle_d}$ onto $P_{k,\triangle_d}(K)$. When $d=2$, define

$$\widetilde{\Pi}_{h,\triangle_2}:H^1(\Omega;\mathbb{S})\cap\{\boldsymbol{\tau}\in L^2(\Omega;\mathbb{S}):\mathbf{div}\boldsymbol{\tau}\in H^1(\Omega;\mathbb{R}^2)\}\to\Sigma_{k,\triangle_2}$$

as follows: for any $\boldsymbol{\tau}\in H^1(\Omega;\mathbb{S})\cap\{\boldsymbol{\tau}\in L^2(\Omega;\mathbb{S}):\mathbf{div}\boldsymbol{\tau}\in H^1(\Omega;\mathbb{R}^2)\}$,

$$\begin{aligned}&\widetilde{\Pi}_{h,\triangle_2}\boldsymbol{\tau}(a)=\frac{1}{N(T(a))}\sum_{K'\in T(a)}(\mathcal{Q}^{K'}_{k,\triangle_2}\boldsymbol{\tau})(a),\\&((\widetilde{\Pi}_{h,\triangle_2}\boldsymbol{\tau})\boldsymbol{n},\boldsymbol{\phi})_e=(\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{\phi})_e\quad\text{for all }\boldsymbol{\phi}\in P_{k-2}(e;\mathbb{R}^2),\\&(\mathbf{div}(\widetilde{\Pi}_{h,\triangle_2}\boldsymbol{\tau})\cdot\boldsymbol{n},q)_e=(\mathbf{div}\boldsymbol{\tau}\cdot\boldsymbol{n},q)_e\quad\text{for all }q\in P_{k-1}(e),\\&(\widetilde{\Pi}_{h,\triangle_2}\boldsymbol{\tau},\nabla^2q)_K=(\boldsymbol{\tau},\nabla^2q)_K\quad\text{for all }q\in P_{k-2}(K),\\&(\widetilde{\Pi}_{h,\triangle_2}\boldsymbol{\tau},\nabla\mathbf{curl}q)_K=(\boldsymbol{\tau},\nabla\mathbf{curl}q)_K\quad\text{for all }q\in\lambda_1\lambda_2\lambda_3P_{k-3}(K)/P_0(K),\\&(\widetilde{\Pi}_{h,\triangle_2}\boldsymbol{\tau},\mathcal{J}q)_K=(\boldsymbol{\tau},\mathcal{J}q)_K\quad\text{for all }q\in(\lambda_1\lambda_2\lambda_3)^2P_{k-4}(K)\end{aligned}$$

for each $a\in\mathcal{V}_h$, $e\in\mathcal{E}_h$ and $K\in\mathcal{T}_h$.

When $d=3$, define $\widetilde{\Pi}_{h,\triangle_3}:H^1(\Omega;\mathbb{S})\cap\{\boldsymbol{\tau}\in L^2(\Omega;\mathbb{S}):\mathbf{div}\boldsymbol{\tau}\in H^1(\Omega;\mathbb{R}^3)\}\to\Sigma_{k,\triangle_3}$ as follows: for any $\boldsymbol{\tau}\in H^1(\Omega;\mathbb{S})\cap\{\boldsymbol{\tau}\in L^2(\Omega;\mathbb{S}):\mathbf{div}\boldsymbol{\tau}\in H^1(\Omega;\mathbb{R}^3)\}$,

$$\begin{aligned}&\widetilde{\Pi}_{h,\triangle_3}\boldsymbol{\tau}(a)=\frac{1}{N(T(a))}\sum_{K'\in T(a)}(\mathcal{Q}^{K'}_{k,\triangle_3}\boldsymbol{\tau})(a),\\&(\boldsymbol{t}_e^{\mathrm{T}}\widetilde{\Pi}_{h,\triangle_3}\boldsymbol{\tau}\boldsymbol{n}_j,q)_e=\frac{1}{N(T(e))}\sum_{K'\in T(e)}(\boldsymbol{t}_e^{\mathrm{T}}(\mathcal{Q}^{K'}_{k,\triangle_3}\boldsymbol{\tau})\boldsymbol{n}_j,q)_e,\quad 1\leqslant j\leqslant 2\quad\text{for all }q\in P_{k-2}(e),\\&(\boldsymbol{n}_i^{\mathrm{T}}\widetilde{\Pi}_{h,\triangle_3}\boldsymbol{\tau}\boldsymbol{n}_j,q)_e=\frac{1}{N(T(e))}\sum_{K'\in T(e)}(\boldsymbol{n}_i^{\mathrm{T}}(\mathcal{Q}^{K'}_{k,\triangle_3}\boldsymbol{\tau})\boldsymbol{n}_j,q)_e,\quad 1\leqslant i,j\leqslant 2\quad\text{for all }q\in P_{k-2}(e),\\&(\widetilde{\Pi}_{h,\triangle_3}\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{\phi})_F=(\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{\phi})_F\quad\text{for all }\boldsymbol{\phi}\in P_{k-3}(F;\mathbb{R}^3),\\&(\mathbf{div}\widetilde{\Pi}_{h,\triangle_3}\boldsymbol{\tau}\cdot\boldsymbol{n},q)_F=(\mathbf{div}\boldsymbol{\tau}\cdot\boldsymbol{n},q)_F\quad\text{for all }q\in P_{k-1}(F),\\&(\widetilde{\Pi}_{h,\triangle_3}\boldsymbol{\tau},\nabla^2q)_K=(\boldsymbol{\tau},\nabla^2q)_K\quad\text{for all }q\in P_{k-2}(K),\\&(\widetilde{\Pi}_{h,\triangle_3}\boldsymbol{\tau},\nabla\boldsymbol{\phi})_K=(\boldsymbol{\tau},\nabla\boldsymbol{\phi})_K\quad\text{for all }\boldsymbol{\phi}\in\mathcal{W}_{k-1}(K;\mathbb{R}^3),\\&(\widetilde{\Pi}_{h,\triangle_3}\boldsymbol{\tau},\boldsymbol{\psi})_K=(\boldsymbol{\tau},\boldsymbol{\psi})_K\quad\text{for all }\boldsymbol{\psi}\in\mathcal{M}_k(K;\mathbb{S})\end{aligned}$$

for each $a\in\mathcal{V}_h$, $e\in\mathcal{E}_h$, $F\in\mathcal{F}_h$ as well as $K\in\mathcal{T}_h$.

Theorem 3.1. *Assume the triangulation $\mathcal{T}_h$ is shape regular. There exists a constant β independent of h such that the following BB condition holds*:

$$\inf_{v_h\in P_{h,\triangle_d}}\sup_{\boldsymbol{\tau}_h\in\Sigma_{k,\triangle_d}}\frac{(\mathrm{div}\mathbf{div}\boldsymbol{\tau}_h,v_h)}{\|\boldsymbol{\tau}_h\|_{H(\mathrm{div}\mathbf{div})}\|v_h\|_0}\geqslant\beta>0.\tag{3.2}$$

Furthermore, the stability for (3.1) *is obtained.*

Proof. For any $v_h \in P_{h,\triangle_d}$, according to [28], there exists some $\boldsymbol{\phi} \in H^1(\Omega;\mathbb{R}^d)$ such that $\mathrm{div}\boldsymbol{\phi} = v_h$ and $\|\boldsymbol{\phi}\|_1 \lesssim \|v_h\|_0$. There exists some $\boldsymbol{\tau}_0 \in H^1(\Omega;\mathbb{S})$ such that $\mathbf{div}\boldsymbol{\tau}_0 = \boldsymbol{\phi}$ and $\|\boldsymbol{\tau}_0\|_1 \lesssim \|\boldsymbol{\phi}\|_0$.

For any $q \in P_{h,\triangle_d}$, integration by parts leads to

$$\begin{aligned}(\mathrm{div}\mathbf{div}\widetilde{\Pi}_{h,\triangle_d}\boldsymbol{\tau}, q) &= (\widetilde{\Pi}_{h,\triangle_d}\boldsymbol{\tau}, \nabla^2 q) - \sum_{K\in\mathcal{T}_h}(\widetilde{\Pi}_{h,\triangle_d}\boldsymbol{\tau}\cdot\boldsymbol{n}, \nabla q)_{\partial K} + \sum_{K\in\mathcal{T}_h}(\mathbf{div}(\widetilde{\Pi}_{h,\triangle_d}\boldsymbol{\tau})\cdot\boldsymbol{n}, q)_{\partial K}\\ &= (\boldsymbol{\tau}, \nabla^2 q) - \sum_{K\in\mathcal{T}_h}(\boldsymbol{\tau}\cdot\boldsymbol{n}, \nabla q)_{\partial K} + \sum_{K\in\mathcal{T}_h}(\mathbf{div}\boldsymbol{\tau}\cdot\boldsymbol{n}, q)_{\partial K}\\ &= (\mathrm{div}\mathbf{div}\boldsymbol{\tau}, q).\end{aligned}$$

This implies

$$\mathrm{div}\mathbf{div}\widetilde{\Pi}_{h,\triangle_d}\boldsymbol{\tau} = \mathcal{Q}_{h,\triangle_d}(\mathrm{div}\mathbf{div}\boldsymbol{\tau}). \tag{3.3}$$

The estimates

$$\|\boldsymbol{\tau} - \Pi_{h,\triangle_d}\boldsymbol{\tau}\|_i \lesssim h^{s-i}|\boldsymbol{\tau}|_s + h^{s+1-i}\|\mathbf{div}\boldsymbol{\tau}\|_s, \quad s \geqslant 1, \quad i = 0, 1 \tag{3.4}$$

follow by standard techniques.

Due to (3.3)–(3.4), it holds that

$$\|\mathrm{div}\mathbf{div}\widetilde{\Pi}_{h,\triangle_d}\boldsymbol{\tau}_0\|_0 = \|\mathcal{Q}_{h,\triangle_d}\mathrm{div}\mathbf{div}\boldsymbol{\tau}_0\|_0 = \|\mathcal{Q}_{h,\triangle_d}\mathrm{div}\boldsymbol{\phi}\|_0 \lesssim \|\boldsymbol{\phi}\|_1 \lesssim \|v_h\|_0.$$

Thus $\|\widetilde{\Pi}_{h,\triangle_d}\boldsymbol{\tau}_0\|_{H(\mathrm{div}\mathbf{div})} \lesssim \|v_h\|_0$. The replacement $\boldsymbol{\tau}_h = \widetilde{\Pi}_{h,\triangle_d}\boldsymbol{\tau}_0$ proves the BB condition (3.2).

Additionally, by the Babuška-Brezzi theory [11,13], for any $\widetilde{\boldsymbol{\tau}}_h \in \Sigma_{k,\triangle_d}$ and $\widetilde{v}_h \in P_{h,\triangle_d}$,

$$\|\widetilde{\boldsymbol{\tau}}_h\|_{H(\mathrm{div}\mathbf{div})} + \|\widetilde{v}_h\|_0 \lesssim \sup_{\substack{\boldsymbol{\tau}_h\in\Sigma_{k,\triangle_d},\\ v_h\in P_{h,\triangle_d}}} \frac{(\widetilde{\boldsymbol{\tau}}_h, \boldsymbol{\tau}_h) + (\mathrm{div}\mathbf{div}\boldsymbol{\tau}_h, \widetilde{v}_h) + (\mathrm{div}\mathbf{div}\widetilde{\boldsymbol{\tau}}_h, v_h)}{\|\boldsymbol{\tau}_h\|_{H(\mathrm{div}\mathbf{div})} + \|v_h\|_0}. \tag{3.5}$$

This ensures that the problem (3.1) is well posed. □

Remark 3.2. The $H^2(\Omega;\mathbb{S})$ regularity for $\boldsymbol{\tau}$ is required if one employs the interpolation operator $\Pi_{h,\triangle_2}$ in (3.2). The proof of Theorem 3.1 somehow reduces the regularity requirement of the interpolation. Nevertheless, the exactness of the divdiv Hilbert complex in Lemma 2.11 ensures the existence of $\boldsymbol{\tau} \in H^2(\Omega;\mathbb{S})$ for any $v_h \in P_{h,\triangle_2}$.

3.2 Error analysis

The stability of (3.1) allows the following error estimates.

Theorem 3.3. *Let* $(\boldsymbol{\sigma}, u) \in H(\mathrm{div}\mathbf{div}, \Omega;\mathbb{S}) \times L^2(\Omega)$ *be the solution of* (1.4) *and* $(\boldsymbol{\sigma}_h, u_h) \in \Sigma_{k,\triangle_d} \times P_{h,\triangle_d}$ *be the solution of* (3.1). *Assume* $\boldsymbol{\sigma} \in H^{k+1}(\Omega;\mathbb{S})$, $u \in H^{k-1}(\Omega)$ *and* $f \in H^{k-1}(\Omega)$, $k \geqslant 3$. *Then*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \lesssim h^{k+1}|\boldsymbol{\sigma}|_{k+1}, \tag{3.6}$$

$$\|u - u_h\|_0 \lesssim h^{k+1}|\boldsymbol{\sigma}|_{k+1} + h^{k-1}|u|_{k-1}, \tag{3.7}$$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\mathrm{div}\mathbf{div})} \lesssim h^{k+1}|\boldsymbol{\sigma}|_{k+1} + h^{k-1}|f|_{k-1}. \tag{3.8}$$

Proof. Theorem 3.1 leads to

$$\begin{aligned}&\|\Pi_{h,\triangle_d}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\mathrm{div}\mathbf{div})} + \|\mathcal{Q}_{h,\triangle_d}u - u_h\|_0\\ &\quad\lesssim \sup_{\substack{\boldsymbol{\tau}_h\in\Sigma_{k,\triangle_d},\\ v_h\in P_{h,\triangle_d}}} \frac{(\Pi_{h,\triangle_d}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\mathrm{div}\mathbf{div}\boldsymbol{\tau}_h, \mathcal{Q}_{h,\triangle_d}u - u_h) + (\mathrm{div}\mathbf{div}(\Pi_{h,\triangle_d}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), v_h)}{\|\boldsymbol{\tau}_h\|_{H(\mathrm{div}\mathbf{div})} + \|v_h\|_0}.\end{aligned}$$

According to (1.4) and (3.1),

$$
\begin{aligned}
&(\Pi_{h,\triangle_d}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)+(\operatorname{div}\mathbf{div}\boldsymbol{\tau}_h,\mathcal{Q}_{h,\triangle_d}u-u_h)+(\operatorname{div}\mathbf{div}(\Pi_{h,\triangle_d}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),v_h)\\
&\quad=(\Pi_{h,\triangle_d}\boldsymbol{\sigma}-\boldsymbol{\sigma},\boldsymbol{\tau}_h).
\end{aligned}
$$

This shows

$$
\|\Pi_{h,\triangle_d}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(\operatorname{div}\mathbf{div})}+\|\mathcal{Q}_{h,\triangle_d}u-u_h\|_0\lesssim\|\Pi_{h,\triangle_d}\boldsymbol{\sigma}-\boldsymbol{\sigma}\|_0. \tag{3.9}
$$

Additionally, together with (2.31) the standard interpolation error estimates for $d=2$, as well as the same results hold for $d=3$, the convergence results (3.6)–(3.8) follow from (3.9). □

3.3 Superconvergence

Introduce the space

$$
H^2(\mathcal{T}_h):=\{v\in L^2(\Omega):v\,|_K\in H^2(K)\text{ for all }K\in\mathcal{T}_h\}.
$$

Define the corresponding mesh-dependent norm in two dimensions,

$$
|v|^2_{2,h,\triangle_2}:=\sum_{K\in\mathcal{T}_h}|v|^2_{2,K}+\sum_{e\in\mathcal{E}_h}(h_e^{-3}\|[v]\|^2_{0,e}+h_e^{-1}\|[\nabla v]_e\|^2_{0,e}),
$$

as well as in three dimensions,

$$
|v|^2_{2,h,\triangle_3}:=\sum_{K\in\mathcal{T}_h}|v|^2_{2,K}+\sum_{F\in\mathcal{F}_h}(h_F^{-3}\|[v]\|^2_{0,F}+h_F^{-1}\|[\nabla v]_F\|^2_{0,F}).
$$

Lemma 3.4. *For d being either 2 or 3, there exists some constant $\beta>0$ such that the following BB condition regarding to the mesh-dependent norm holds*:

$$
\sup_{\boldsymbol{\tau}_h\in\Sigma_{k,\triangle_d}}\frac{(\operatorname{div}\mathbf{div}\boldsymbol{\tau}_h,v_h)}{\|\boldsymbol{\tau}_h\|_0}\geqslant\beta|v_h|_{2,h,\triangle_d}\quad\text{for all }\ v_h\in P_{h,\triangle_d}. \tag{3.10}
$$

Proof. Let $v_h\in P_{h,\triangle_d}$. For $d=2$, let the degrees of the freedom of $\boldsymbol{\tau}_h\in\Sigma_{k,\triangle_2}$ for each $K\in\mathcal{T}_h$ be

$$
\begin{aligned}
&\boldsymbol{\tau}_h(a)=0\quad\text{for all }a\in\mathcal{V}(K),\\
&(\boldsymbol{\tau}_h\boldsymbol{n},\boldsymbol{\phi})_e=(h_e^{-1}[\nabla v_h],\boldsymbol{\phi})_e\quad\text{for all }\boldsymbol{\phi}\in P_{k-2}(e;\mathbb{R}^2),\quad e\in\mathcal{E}(K),\\
&(\mathbf{div}\boldsymbol{\tau}_h\cdot\boldsymbol{n},q)_e=-(h_e^{-3}[v_h],\boldsymbol{\phi})_e\quad\text{for all }q\in P_{k-1}(e),\quad e\in\mathcal{E}(K),\\
&(\boldsymbol{\tau}_h,\nabla^2 q)_K=(\nabla^2v_h,\nabla^2q)_K\quad\text{for all }q\in P_{k-2}(K),\\
&(\boldsymbol{\tau}_h,\nabla\mathbf{curl}\,q)_K=0\quad\text{for all }q\in\lambda_1\lambda_2\lambda_3P_{k-3}(K)/P_0(K),\\
&(\boldsymbol{\tau}_h,\mathcal{J}q)_K=0\quad\text{for all }q\in\mathring{\mathcal{B}}_{Arg,k+2}(K).
\end{aligned}
$$

Consider

$$
\begin{aligned}
(\operatorname{div}\mathbf{div}\boldsymbol{\tau}_h,v_h)&=\sum_{K\in\mathcal{T}_h}(\operatorname{div}\mathbf{div}\boldsymbol{\tau}_h,v_h)_K\\
&=\sum_{K\in\mathcal{T}_h}(\boldsymbol{\tau}_h,\nabla^2v_h)_K-\sum_{e\in\mathcal{E}_h}(\mathbf{div}\boldsymbol{\tau}_h\cdot\boldsymbol{n},[v_h])_e+\sum_{e\in\mathcal{E}_h}(\boldsymbol{\tau}_h\boldsymbol{n},[\nabla v_h])_e\\
&=\sum_{K\in\mathcal{T}_h}\|\nabla^2v_h\|_0^2+\sum_{e\in\mathcal{E}_h}(h_e^{-3}\|[v_h]\|^2_{0,e}+h_e^{-1}\|[\nabla v_h]\|^2_{0,e})\\
&=|v_h|^2_{2,h,\triangle_2}.
\end{aligned}
$$

The scaling argument leads to

$$
\|\boldsymbol{\tau}_h\|_0\lesssim|v_h|_{2,h,\triangle_2}.
$$

Therefore,

$$\frac{(\operatorname{div}\mathbf{div}\boldsymbol{\tau}_h, v_h)}{\|\boldsymbol{\tau}_h\|_0} \gtrsim |v_h|_{2,h,\triangle_2}.$$

This proves (3.10) in two dimensions.

When it comes to $d=3$, the same techniques are applied. Let $\boldsymbol{\tau}_h \in \Sigma_{k,\triangle_3}$ on each $K \in \mathcal{T}_h$ with

$$\begin{aligned}
&\boldsymbol{\tau}_h(a) = 0 \quad \text{for all } a \in \mathcal{V}(K),\\
&(\boldsymbol{t}_e^{\mathrm{T}}\boldsymbol{\sigma}\boldsymbol{n}_j, q)_e = 0, \quad (\boldsymbol{n}_i^{\mathrm{T}}\boldsymbol{\sigma}\boldsymbol{n}_j, q)_e = 0, \quad 1 \leqslant i,j \leqslant 2 \quad \text{for all } q \in P_{k-2}(e), \quad e \in \mathcal{E}(K),\\
&(\boldsymbol{\tau}_h\boldsymbol{n}, \boldsymbol{\phi})_F = (h_F^{-1}[\nabla v_h], \boldsymbol{\phi})_F \quad \text{for all } \boldsymbol{\phi} \in P_{k-3}(F;\mathbb{R}^3), \quad F \in \mathcal{F}(K),\\
&(\mathbf{div}\boldsymbol{\tau}_h \cdot \boldsymbol{n}, q)_F = -(h_F^{-3}[v_h], q)_F \quad \text{for all } q \in P_{k-1}(F), \quad F \in \mathcal{F}(K),\\
&(\boldsymbol{\tau}_h, \nabla^2 q)_K = 0 \quad \text{for all } q \in P_{k-2}(K),\\
&(\boldsymbol{\tau}_h, \nabla\boldsymbol{\phi})_K = 0 \quad \text{for all } \boldsymbol{\phi} \in \mathcal{W}_{k-1}(K;\mathbb{R}^3),\\
&(\boldsymbol{\tau}_h, \boldsymbol{\tau})_K = 0 \quad \text{for all } \boldsymbol{\tau} \in \mathcal{M}_k(K;\mathbb{S}).
\end{aligned}$$

This leads to

$$(\operatorname{div}\mathbf{div}\boldsymbol{\tau}_h, v_h) = |v_h|^2_{2,h,\triangle_3}.$$

The scaling argument in this scenario results in

$$\|\boldsymbol{\tau}_h\|_0 \lesssim |v_h|_{2,h,\triangle_3}.$$

This proves (3.10) in three dimensions. □

Babuška-Brezzi theory [11, 13] and the BB condition (3.10) lead to the following stability results: for any $\widetilde{\boldsymbol{\tau}}_h \in \Sigma_{k,\triangle_d}$ and $\widetilde{v}_h \in P_{h,\triangle_d}$,

$$\|\widetilde{\boldsymbol{\tau}}_h\|_0 + |\widetilde{v}_h|_{2,h,\triangle_d} \lesssim \sup_{\substack{\boldsymbol{\tau}_h \in \Sigma_{k,\triangle_d},\\ v_h \in P_{h,\triangle_d}}} \frac{(\widetilde{\boldsymbol{\tau}}_h, \boldsymbol{\tau}_h) + (\operatorname{div}\mathbf{div}\boldsymbol{\tau}_h, \widetilde{v}_h) + (\operatorname{div}\mathbf{div}\widetilde{\boldsymbol{\tau}}_h, v_h)}{\|\boldsymbol{\tau}_h\|_0 + |v_h|_{2,h,\triangle_d}}. \tag{3.11}$$

The stability result (3.11) gives rise to the following superconvergence results.

Theorem 3.5. *Suppose that* $(\boldsymbol{\sigma}_h, u_h) \in \Sigma_{k,\triangle_d} \times P_{h,\triangle_d}$ *is the solution of the mixed finite element method* (3.1)*. Assume* $\boldsymbol{\sigma} \in H^{k+1}(\Omega;\mathbb{S})$*. Then*

$$|\mathcal{Q}_h u - u_h|_{2,h,\triangle_d} \lesssim h^{k+1}|\boldsymbol{\sigma}|_{k+1}.$$

3.4 Postprocessing

The superconvergence of $|Q_h u - u_h|_{2,h,\triangle_d}$ is used to get a high order approximation of displacement in this subsection. Define $u_h^* \in P_{k+2}(\mathcal{T}_h)$ as follows: for each $K \in \mathcal{T}_h$,

$$(\nabla^2 u_h^*, \nabla^2 q)_K = -(\boldsymbol{\sigma}_h, \nabla^2 q)_K \quad \text{for all } q \in P_{k+2}(\mathcal{T}_h), \tag{3.12}$$

$$(u_h^*, q)_K = (u_h, q)_K \quad \text{for all } q \in P_1(\mathcal{T}_h). \tag{3.13}$$

Theorem 3.6. *Suppose that* $(\boldsymbol{\sigma}_h, u_h) \in \Sigma_{h,\triangle_d} \times P_{h,\triangle_d}$ *is the solution of the mixed finite element method* (3.1)*. Assume* $u \in H^{k+3}(\Omega;\mathbb{S})$*. Then*

$$|u - u_h^*|_{2,h,\triangle_d} \lesssim h^{k+1}|u|_{k+3}.$$

Proof. The proof of Theorem 3.6 is similar to [15, Theorem 4.4], and the details are omitted here. □

4 Numerical results

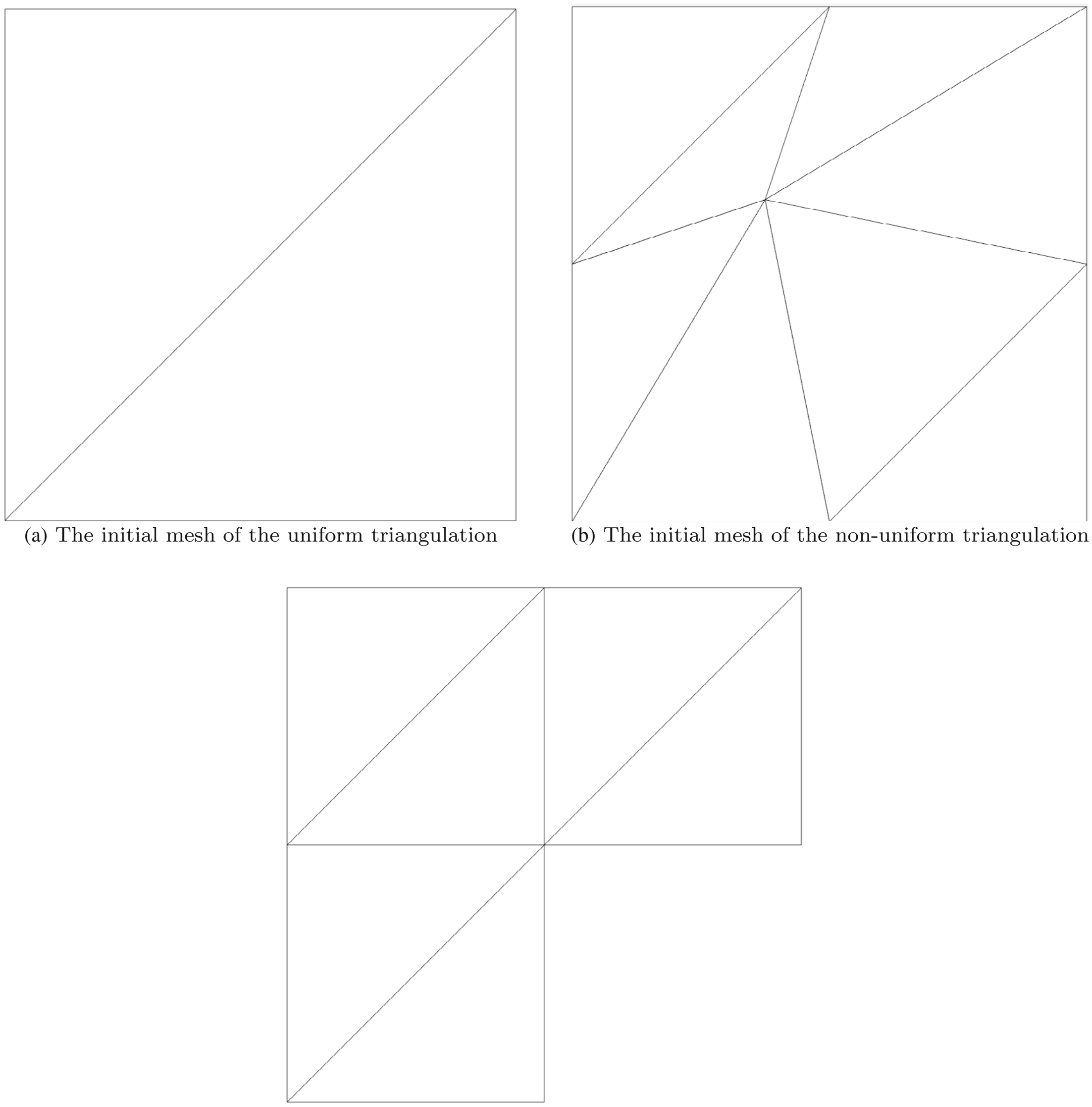
Some numerical results are presented in this section to verify the error analysis and convergence results in the previous sections.

4.1 Example 1

The computational domain is $\Omega = (0,1) \times (0,1)$ with the homogeneous boundary condition. The load function $f = \Delta^2 u$ in (1.1) is derived by the exact solution

$$u(x,y) = x^2 y^2 (y-1)^2 (1-x)^2.$$

Use the $H(\mathrm{div}\mathbf{div};\mathbb{S})$ conforming finite element $\Sigma_{3,\triangle_2}$ for $\boldsymbol{\sigma}_h$ in the problem (3.1) and the piecewise linear space $P_{h,\triangle_2}$ for u_h. $\mathcal{T}_h$ is uniform in this example. The initial mesh is shown in Figure 1. The errors are reported in Table 1. As shown in Theorem 3.3, the optimal order of convergence for both $\boldsymbol{\sigma}$ and u is achieved in the computation. Besides, the superconvergence can be observed. The errors $\|Q_h u - u_h\|_0$ and $|Q_h u - u_h|_{2,h,\triangle_2}$ are fourth order of convergence, and $|Q_h u - u_h|_{2,h,\triangle_2}$ are fourth order higher than the optimal one. In addition, fourth order of convergence is achieved for $|u - u_h^*|_{2,h,\triangle_2}$ with the postprocessing solution u_h^*.



(a) The initial mesh of the uniform triangulation

(b) The initial mesh of the non-uniform triangulation

(c) The initial mesh of the triangulation for the L-shape domain

Figure 1 The initial mesh

Table 1 The error and the order of convergence on uniform meshes

	$\|\sigma-\sigma_h\|_0$	h^n	$\|\mathrm{div}\mathbf{div}(\sigma-\sigma_h)\|_0$	h^n	$\|u-u_h\|_0$	h^n
1	1.3900E−02	–	1.5552E+00	–	9.2528E−04	–
2	5.1722E−03	1.43	1.0180E+00	0.61	2.8527E−04	1.70
3	4.2279E−04	3.61	3.0838E−01	1.72	1.0686E−04	1.42
4	2.9243E−05	3.85	8.0510E−02	1.94	3.0080E−05	1.83
5	1.9079E−06	3.94	2.0341E−02	1.98	7.7431E−06	1.96
	$\|Q_hu-u_h\|_0$	h^n	$\|Q_hu-u_h\|_{2,h,\triangle_2}$	h^n	$\|u-u_h^*\|_{2,h,\triangle_2}$	h^n
1	3.2820E−04	–	2.8601E−03	–	2.4999E−02	–
2	9.7954E−05	1.74	1.8199E−03	0.65	5.9936E−03	2.06
3	7.3858E−06	3.73	1.7841E−04	3.35	5.0637E−04	3.57
4	4.7511E−07	3.96	1.3361E−05	3.74	3.4275E−05	3.88
5	2.9848E−08	3.99	9.1053E−07	3.88	2.1900E−06	3.97

Table 2 The error and the order of convergence on non-uniform meshes

	$\|\sigma-\sigma_h\|_0$	h^n	$\|\mathrm{div}\mathbf{div}(\sigma-\sigma_h)\|_0$	h^n	$\|u-u_h\|_0$	h^n
1	5.7827E−03	–	1.0671E+00	–	3.4324E−04	–
2	4.4357E−04	3.70	3.1255E−01	1.77	1.1681E−04	1.56
3	3.1802E−05	3.80	8.1188E−02	1.94	3.3490E−05	1.80
4	2.1183E−06	3.91	2.0490E−02	1.99	8.6457E−06	1.95
5	1.3611E−07	3.96	5.1346E−03	2.00	2.1787E−06	1.99
	$\|Q_hu-u_h\|_0$	h^n	$\|Q_hu-u_h\|_{2,h,\triangle_2}$	h^n	$\|u-u_h^*\|_{2,h,\triangle_2}$	h^n
1	1.1435E−04	–	1.9830E−03	–	6.7217E−03	–
2	7.7707E−06	3.88	1.8557E−04	3.42	5.3202E−04	3.66
3	5.1056E−07	3.93	1.4758E−05	3.65	3.8076E−05	3.80
4	3.2260E−08	3.98	1.0220E−06	3.85	2.5243E−06	3.91
5	2.0211E−09	4.00	6.6842E−08	3.93	1.6174E−07	3.96

4.2 Example 2

Compute Example 1 on non-uniform triangulations. The initial mesh is shown in Figure 1. The errors and convergence rates are displayed in Table 2. The computation shows that the nonuniformity of the mesh does not downgrade approximability.

4.3 Example 3

The L-shape domain $\Omega=(-1,1)\times(-1,1)\backslash([0,1]\times[-1,0])$. Figure 1 shows its initial mesh. Let $\omega:=3\pi/2$, and $\alpha=0.544483736782464$ is a non-characteristic root of $\sin^2(\alpha\omega)=\alpha^2\sin^2(\omega)$ with

$$\begin{aligned}g_{\alpha,\omega}(\theta)&=g_1(\cos((\alpha-1)\theta)-\cos((\alpha+1)\theta))\\&\quad-g_2\left(\frac{1}{\alpha-1}\sin((\alpha-1)\theta)-\frac{1}{\alpha+1}\sin((\alpha+1)\theta)\right)\end{aligned}$$

and

$$\begin{aligned}g_1&=\frac{1}{\alpha-1}\sin((\alpha-1)\omega)-\frac{1}{\alpha+1}\sin((\alpha+1)\omega),\\g_2&=\cos((\alpha-1)\omega)-\cos((\alpha+1)\omega).\end{aligned}$$

The load function $f=\Delta^2u$ in (1.1) is derived by the exact solution

$$u(x,y)=(1-x^2)^2(1-y^2)^2(\sqrt{x^2+y^2})^{1+\alpha}g_{\alpha,\omega}(\theta).$$

Table 3 The error and the order of convergence for the L-shape domain

	$\Vert\sigma-\sigma_h\Vert_0$	h^n	$\Vert\mathrm{div}\mathbf{div}(\sigma-\sigma_h)\Vert_0$	h^n	$\Vert u-u_h\Vert_0$	h^n
1	3.0154E+00	–	1.1771E+02	–	2.7847E−01	–
2	1.6652E+00	0.86	4.9184E+01	1.26	4.8223E−02	2.53
3	1.1244E+00	0.57	2.1869E+01	1.17	2.1671E−02	1.15
4	7.7274E−01	0.54	1.3617E+01	0.68	7.2229E−03	1.59
5	5.3096E−01	0.54	9.2420E+00	0.56	2.5820E−03	1.48
	$\Vert Q_hu-u_h\Vert_0$	h^n	$\vert Q_hu-u_h\vert_{2,h,\triangle_2}$	h^n	$\vert u-u_h^*\vert_{2,h,\triangle_2}$	h^n
1	5.1725E−02	–	2.9152E−01	–	3.8066E+00	–
2	2.1221E−02	1.29	2.2187E−01	0.39	2.0648E+00	0.88
3	9.5846E−03	1.15	1.6895E−01	0.39	1.3956E+00	0.57
4	4.4346E−03	1.11	1.1977E−01	0.50	9.5956E−01	0.54
5	2.0731E−03	1.10	8.2704E−02	0.53	6.5952E−01	0.54

Use the $H(\mathrm{div}\mathbf{div};\mathbb{S})$ conforming finite element $\Sigma_{3,\triangle_2}$ for $\boldsymbol{\sigma}_h$ in the problem (3.1) and the piecewise linear space $P_{h,\triangle_2}$ for u_h. The triangulation $\mathcal{T}_h$ is uniform. The numerical results are presented in Table 3. The convergence can still be observed in the L-shape domain. The convergence rate is degenerate because the solution possesses singularities at the origin. Nevertheless, it is noted that the convergence rate of $\Vert u-u_h\Vert_0$ is higher than the other errors.

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Appendix A

This appendix provides some ideas to construct the basis for $\Sigma_{k,\triangle_d}$. It is discussed for $k=3$ and $d=2$ while the ideas apply for $k\geqslant 3$ and $d=3$.

For the case where $d=2$ and $k=3$, let $\boldsymbol{x}_1,\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ be the vertices of a element $K\in\mathcal{T}_h$. The affine mapping $F:\widehat{K}\to K$ reads

$$\boldsymbol{x}=F(\widehat{\boldsymbol{x}})=B\widehat{\boldsymbol{x}}+\boldsymbol{x}_1$$

with

$$B=(\boldsymbol{x}_2-\boldsymbol{x}_1,\,\boldsymbol{x}_3-\boldsymbol{x}_1).$$

Suppose that the triangle $\widehat{K}$ is spanned by $(0,0)$, $(1,0)$ and $(0,1)$, and use $\widehat{\boldsymbol{x}}=(\widehat{x},\widehat{y})^{\mathrm{T}}$ for the vector in that coordinate. Thus

$$\widehat{\boldsymbol{x}}_1=\begin{pmatrix}0\\0\end{pmatrix},\quad \widehat{\boldsymbol{x}}_2=\begin{pmatrix}1\\0\end{pmatrix},\quad \widehat{\boldsymbol{x}}_3=\begin{pmatrix}0\\1\end{pmatrix}. \tag{A.1}$$

For each edge $e_i\in\mathcal{E}(K)$, the corresponding tangent vector is $\boldsymbol{t}_i=\boldsymbol{x}_{i-1}-\boldsymbol{x}_{i+1}$, $i=1,2,3$, where the indices are calculated mod 3. The unit outward normal vector of e_i is denoted by $\boldsymbol{n}_i$. By the affine mapping,

$$\boldsymbol{n}_i=\frac{B^{-\mathrm{T}}\widehat{\boldsymbol{n}}_i}{|B^{-\mathrm{T}}\widehat{\boldsymbol{n}}_i|},\quad \boldsymbol{t}_i=B\,\widehat{\boldsymbol{t}}_i. \tag{A.2}$$

The barycenter coordinates read

$$\lambda_2=\boldsymbol{n}_2\cdot(\boldsymbol{x}_1-\boldsymbol{x}), \tag{A.3}$$

$$\lambda_3=\boldsymbol{n}_3\cdot(\boldsymbol{x}_1-\boldsymbol{x}), \tag{A.4}$$

$$\lambda_1=1-\lambda_2-\lambda_3. \tag{A.5}$$

Define $\mathrm{J}:=\det(B)$. Note that J does not vanish at any point. Define for $\boldsymbol{\tau}\in H(\mathrm{div}\,\mathbf{div},K;\mathbb{S})$, by the Piola transform [6],

$$\boldsymbol{\tau}(\boldsymbol{x}):=\frac{1}{\mathrm{J}}\,B\,\widehat{\boldsymbol{\tau}}(\widehat{\boldsymbol{x}})\,B^{\mathrm{T}}. \tag{A.6}$$

Some fundamental properties of the Piola transform (A.6) are presented in the subsequent lemmas.

Lemma A.1. *If* $\widehat{\boldsymbol{\tau}}\in H(\mathbf{div},\widehat{K};\mathbb{S})$ *satisfies* $\widehat{\boldsymbol{\tau}}\widehat{\boldsymbol{n}}\,|_{\partial\widehat{K}}=0$, *then* $\boldsymbol{\tau}\in H(\mathbf{div},K;\mathbb{S})$ *defined in* (A.6) *satisfies* $\boldsymbol{\tau}\boldsymbol{n}\,|_{\partial K}=0$.

Proof. The combination of (A.2) and (A.6) shows that on each edge $e\in\mathcal{E}(K)$,

$$\boldsymbol{\tau}\boldsymbol{n}=\frac{1}{\mathrm{J}}B\widehat{\boldsymbol{\tau}}B^{\mathrm{T}}\boldsymbol{n}=\frac{B\widehat{\boldsymbol{\tau}}\widehat{\boldsymbol{n}}}{\mathrm{J}|B^{-\mathrm{T}}\widehat{\boldsymbol{n}}|}. \tag{A.7}$$

Thus $\widehat{\boldsymbol{\tau}}\widehat{\boldsymbol{n}}\,|_{\widehat{e}}=0$ implies $\boldsymbol{\tau}\boldsymbol{n}\,|_e=0$. □

Lemma A.2. *Suppose* $\boldsymbol{\tau}\in H(\mathrm{div}\,\mathbf{div},K;\mathbb{S})$, $q\in P_{k-1}(e)$ *and* $e\in\mathcal{E}(K)$. *If* $\mathrm{J}>0$, *then*

$$(\mathbf{div}\boldsymbol{\tau}\cdot\boldsymbol{n},q)_e=(\widehat{\mathbf{div}}\widehat{\boldsymbol{\tau}}\cdot\widehat{\boldsymbol{n}},\widehat{q})_{\widehat{e}}.$$

If $\mathrm{J}<0$, *then*

$$(\mathbf{div}\boldsymbol{\tau}\cdot\boldsymbol{n},q)_e=-(\widehat{\mathbf{div}}\widehat{\boldsymbol{\tau}}\cdot\widehat{\boldsymbol{n}},\widehat{q})_{\widehat{e}}.$$

Proof. From [6], we have

$$\mathbf{div}\boldsymbol{\tau}=\frac{1}{\mathrm{J}}B\,\widehat{\mathbf{div}}\,\widehat{\boldsymbol{\tau}}.$$

Let $q(x)=\widehat{q}(\widehat{x})$. It holds that

$$(\mathbf{div}\boldsymbol{\tau}\cdot\boldsymbol{n},q)_e=\left(\frac{B\widehat{\mathbf{div}}\widehat{\boldsymbol{\tau}}B^{-\mathrm{T}}\widehat{\boldsymbol{n}}|e|}{\mathrm{J}|B^{-\mathrm{T}}\widehat{\boldsymbol{n}}||\widehat{e}|},\widehat{q}\right)_{\widehat{e}}=\left(\frac{\widehat{\boldsymbol{n}}^{\mathrm{T}}\widehat{\mathbf{div}}\widehat{\boldsymbol{\tau}}|e|}{\mathrm{J}|B^{-\mathrm{T}}\widehat{\boldsymbol{n}}||\widehat{e}|},\widehat{q}\right)_{\widehat{e}}=\frac{|\mathrm{J}|}{\mathrm{J}}(\widehat{\mathbf{div}}\widehat{\boldsymbol{\tau}}\cdot\widehat{\boldsymbol{n}},\widehat{q})_{\widehat{e}}.$$

This completes the proof. □

The basis for $\Sigma_{k,\triangle_2}$ are formed as follows. For $k=3$, only the degrees of the freedom (2.4)–(2.6) are adopted. The first step is to construct basis functions for the degrees of the freedom (2.6), which are denoted by $\boldsymbol{\tau}_{h,i}$, $i=1,2,\ldots,9$.

Recall the $H(\mathbf{div},K;\mathbb{S})$ bubble functions $\boldsymbol{\vartheta}_h$ introduced in [34],

$$\boldsymbol{\vartheta}_h \in \Sigma_{K,b} := \sum_{1\leqslant i\leqslant 3} \lambda_{i-1}\lambda_{i+1}P_1(K)\boldsymbol{t}_i\boldsymbol{t}_i^{\mathrm{T}}$$

with

$$\boldsymbol{\vartheta}_h\boldsymbol{n}_j\,|_{e_j} = \mathbf{0}, \quad j=1,2,3.$$

Lemma A.1 ensures that $\boldsymbol{\tau}_{h,i}$ can be obtained from the basis functions $\widehat{\boldsymbol{\tau}}_{h,i}$ defined on the reference element $\widehat{K}$. Let the nine basis functions of $\widehat{\Sigma}_{\widehat{K},b}$ be $\widehat{\boldsymbol{\vartheta}}_{h,i}$, $i=1,2,\ldots,9$. To be precise,

$$\begin{aligned}
\widehat{\boldsymbol{\vartheta}}_{h,1} &= \frac{9}{2}\widehat{\lambda}_2\widehat{\lambda}_3(3\widehat{\lambda}_2-1)\widehat{\boldsymbol{t}}_1\widehat{\boldsymbol{t}}_1^{\mathrm{T}},\\
\widehat{\boldsymbol{\vartheta}}_{h,2} &= \frac{9}{2}\widehat{\lambda}_3\widehat{\lambda}_2(3\widehat{\lambda}_3-1)\widehat{\boldsymbol{t}}_1\widehat{\boldsymbol{t}}_1^{\mathrm{T}},\\
\widehat{\boldsymbol{\vartheta}}_{h,3} &= \frac{9}{2}\widehat{\lambda}_3\widehat{\lambda}_1(3\widehat{\lambda}_3-1)\widehat{\boldsymbol{t}}_2\widehat{\boldsymbol{t}}_2^{\mathrm{T}},\\
\widehat{\boldsymbol{\vartheta}}_{h,4} &= \frac{9}{2}\widehat{\lambda}_1\widehat{\lambda}_3(3\widehat{\lambda}_1-1)\widehat{\boldsymbol{t}}_2\widehat{\boldsymbol{t}}_2^{\mathrm{T}},\\
\widehat{\boldsymbol{\vartheta}}_{h,5} &= \frac{9}{2}\widehat{\lambda}_1\widehat{\lambda}_2(3\widehat{\lambda}_1-1)\widehat{\boldsymbol{t}}_3\widehat{\boldsymbol{t}}_3^{\mathrm{T}},\\
\widehat{\boldsymbol{\vartheta}}_{h,6} &= \frac{9}{2}\widehat{\lambda}_2\widehat{\lambda}_1(3\widehat{\lambda}_2-1)\widehat{\boldsymbol{t}}_3\widehat{\boldsymbol{t}}_3^{\mathrm{T}},\\
\widehat{\boldsymbol{\vartheta}}_{h,7} &= 27\widehat{\lambda}_1\widehat{\lambda}_2\widehat{\lambda}_3\widehat{\boldsymbol{t}}_1\widehat{\boldsymbol{t}}_1^{\mathrm{T}},\\
\widehat{\boldsymbol{\vartheta}}_{h,8} &= 27\widehat{\lambda}_1\widehat{\lambda}_2\widehat{\lambda}_3\widehat{\boldsymbol{t}}_2\widehat{\boldsymbol{t}}_2^{\mathrm{T}},\\
\widehat{\boldsymbol{\vartheta}}_{h,9} &= 27\widehat{\lambda}_1\widehat{\lambda}_2\widehat{\lambda}_3\widehat{\boldsymbol{t}}_3\widehat{\boldsymbol{t}}_3^{\mathrm{T}}.
\end{aligned}$$

Assume

$$\widehat{\boldsymbol{\tau}}_{h,i} = \sum_{i=1}^{9}\alpha_j^{(i)}\widehat{\boldsymbol{\vartheta}}_{h,j}, \quad \alpha_j^i\in\mathbb{R}.$$

The corresponding basis functions for degrees of the freedom (2.6) on $\widehat{K}$ can be calculated immediately. Suppose that C denotes the 9×9 coefficients matrix consisting of $\alpha_j^{(i)}$, and let $C(i,j)=\alpha_j^{(i)}$. Then

$$C = \begin{pmatrix}
0 & 0 & 4/9 & 2/9 & 2/3 & 4/3 & 1/3 & -1/3 & 1/9\\
0 & 0 & 4/3 & 2/3 & 2/9 & 4/9 & 1/9 & -1/3 & 1/3\\
0 & 0 & -40/9 & -20/9 & -20/9 & -40/9 & -10/9 & 20/9 & -10/9\\
-4/3 & -8/3 & 0 & 0 & -4/9 & -2/9 & 2/9 & 0 & -1/3\\
-4/9 & -8/9 & 0 & 0 & -4/3 & -2/3 & 2/9 & -2/9 & -1/9\\
40/9 & 80/9 & 0 & 0 & 40/9 & 20/9 & -20/9 & 10/9 & 10/9\\
-8/9 & -4/9 & -2/3 & -4/3 & 0 & 0 & -1/9 & -2/9 & 2/9\\
-8/3 & -4/3 & -2/9 & -4/9 & 0 & 0 & -1/3 & 0 & 2/9\\
80/9 & 40/9 & 20/9 & 40/9 & 0 & 0 & 10/9 & 10/9 & -20/9
\end{pmatrix}.$$

This leads to

$$\widehat{\boldsymbol{\tau}}_{h,1} = \begin{pmatrix} 9\widehat{\lambda}_1\widehat{\lambda}_2^2 & -9\widehat{\lambda}_1\widehat{\lambda}_2\widehat{\lambda}_3\\ -9\widehat{\lambda}_1\widehat{\lambda}_2\widehat{\lambda}_3 & 3\widehat{\lambda}_1\widehat{\lambda}_3^2\end{pmatrix}, \quad \widehat{\boldsymbol{\tau}}_{h,2} = \begin{pmatrix} 3\widehat{\lambda}_1\widehat{\lambda}_2^2 & -9\widehat{\lambda}_1\widehat{\lambda}_2\widehat{\lambda}_3\\ -9\widehat{\lambda}_1\widehat{\lambda}_2\widehat{\lambda}_3 & 9\widehat{\lambda}_1\widehat{\lambda}_3^2\end{pmatrix},$$

$$\widehat{\boldsymbol{\tau}}_{h,3} = \begin{pmatrix} -30\widehat{\lambda}_1\widehat{\lambda}_2^2 & 60\widehat{\lambda}_1\widehat{\lambda}_2\widehat{\lambda}_3\\ 60\widehat{\lambda}_1\widehat{\lambda}_2\widehat{\lambda}_3 & -30\widehat{\lambda}_1\widehat{\lambda}_3^2\end{pmatrix},$$

$$\widehat{\boldsymbol{\tau}}_{h,4} = \begin{pmatrix} -3\widehat{\lambda}_2(\widehat{\lambda}_2^2+8\widehat{\lambda}_2\widehat{\lambda}_3-\widehat{\lambda}_2+10\widehat{\lambda}_3^2-7\widehat{\lambda}_3+\widehat{\lambda}_1) & 9\widehat{\lambda}_2\widehat{\lambda}_3(\widehat{\lambda}_3-\widehat{\lambda}_1) \\ 9\widehat{\lambda}_2\widehat{\lambda}_3(\widehat{\lambda}_3-\widehat{\lambda}_1) & -9\widehat{\lambda}_2\widehat{\lambda}_3^2 \end{pmatrix},$$

$$\widehat{\boldsymbol{\tau}}_{h,5} = \begin{pmatrix} -3\widehat{\lambda}_2(3\widehat{\lambda}_2^2+12\widehat{\lambda}_2\widehat{\lambda}_3-3\widehat{\lambda}_2+10\widehat{\lambda}_3^2-9\widehat{\lambda}_3+3\widehat{\lambda}_1) & 3\widehat{\lambda}_2\widehat{\lambda}_3(\widehat{\lambda}_3-3\widehat{\lambda}_1) \\ 3\widehat{\lambda}_2\widehat{\lambda}_3(\widehat{\lambda}_3-3\widehat{\lambda}_1) & -3\widehat{\lambda}_2\widehat{\lambda}_3^2 \end{pmatrix},$$

$$\widehat{\boldsymbol{\tau}}_{h,6} = \begin{pmatrix} 30\widehat{\lambda}_2(\widehat{\lambda}_2^2+6\widehat{\lambda}_2\widehat{\lambda}_3-\widehat{\lambda}_2+6\widehat{\lambda}_3^2-5\widehat{\lambda}_3+\widehat{\lambda}_1) & 30\widehat{\lambda}_2\widehat{\lambda}_3(2\widehat{\lambda}_1-\widehat{\lambda}_3) \\ 30\widehat{\lambda}_2\widehat{\lambda}_3(2\widehat{\lambda}_1-\widehat{\lambda}_3) & 30\widehat{\lambda}_1\widehat{\lambda}_2^2 \end{pmatrix},$$

$$\widehat{\boldsymbol{\tau}}_{h,7} = \begin{pmatrix} -3\widehat{\lambda}_2^2\widehat{\lambda}_3 & 3\widehat{\lambda}_2\widehat{\lambda}_3(2\widehat{\lambda}_2-\widehat{\lambda}_1) \\ 3\widehat{\lambda}_2\widehat{\lambda}_3(2\widehat{\lambda}_2-\widehat{\lambda}_1) & -3\widehat{\lambda}_3(10\widehat{\lambda}_2^2+12\widehat{\lambda}_2\widehat{\lambda}_3-9\widehat{\lambda}_2+3\widehat{\lambda}_3^2-3\widehat{\lambda}_3+3\widehat{\lambda}_1) \end{pmatrix},$$

$$\widehat{\boldsymbol{\tau}}_{h,8} = \begin{pmatrix} -9\widehat{\lambda}_2^2\widehat{\lambda}_3 & 9\widehat{\lambda}_2\widehat{\lambda}_3(2\widehat{\lambda}_2-\widehat{\lambda}_1) \\ 9\widehat{\lambda}_2\widehat{\lambda}_3(2\widehat{\lambda}_2-\widehat{\lambda}_1) & -3\widehat{\lambda}_3(10\widehat{\lambda}_2^2+8\widehat{\lambda}_2\widehat{\lambda}_3-7\widehat{\lambda}_2+\widehat{\lambda}_3^2-\widehat{\lambda}_3+\widehat{\lambda}_1) \end{pmatrix},$$

$$\widehat{\boldsymbol{\tau}}_{h,9} = \begin{pmatrix} 30\widehat{\lambda}_2^2\widehat{\lambda}_3 & 30\widehat{\lambda}_2\widehat{\lambda}_3(2\widehat{\lambda}_1-\widehat{\lambda}_2) \\ 30\widehat{\lambda}_2\widehat{\lambda}_3(2\widehat{\lambda}_1-\widehat{\lambda}_2) & 30\widehat{\lambda}_3(6\widehat{\lambda}_2^2+6\widehat{\lambda}_2\widehat{\lambda}_3-5\widehat{\lambda}_2+\widehat{\lambda}_3^2-\widehat{\lambda}_3+\widehat{\lambda}_1) \end{pmatrix}.$$

Hence $\boldsymbol{\tau}_{h,i}$, $i=1,2,\ldots,9$ follow by the Piola transform (A.6). These basis $\boldsymbol{\tau}_{h,i}$, $i=1,2,\ldots,9$ satisfy

$$\begin{aligned} &\boldsymbol{\tau}_{h,i}\boldsymbol{n}_j\,|_{e_j}=0, \quad 1\leqslant i\leqslant 9, \quad j=1,2,3,\\ &d_{i,e}(\boldsymbol{\tau}_{h,j})=\delta_{ij}, \quad 1\leqslant i,j\leqslant 9. \end{aligned}$$

Here, $d_{i,e}(\cdot)$, $i=1,2,\ldots,9$ are defined by

$$\begin{aligned} d_{1,e}(\cdot)&=\int_{e_1}\boldsymbol{n}_1^{\mathrm{T}}\mathbf{div}(\cdot)\lambda_2 ds, & d_{2,e}(\cdot)&=\int_{e_1}\boldsymbol{n}_1^{\mathrm{T}}\mathbf{div}(\cdot)\lambda_3 ds, & d_{3,e}(\cdot)&=\int_{e_1}\boldsymbol{n}_1^{\mathrm{T}}\mathbf{div}(\cdot)\lambda_2\lambda_3 ds,\\ d_{4,e}(\cdot)&=\int_{e_2}\boldsymbol{n}_2^{\mathrm{T}}\mathbf{div}(\cdot)\lambda_3 ds, & d_{5,e}(\cdot)&=\int_{e_2}\boldsymbol{n}_2^{\mathrm{T}}\mathbf{div}(\cdot)\lambda_1 ds, & d_{6,e}(\cdot)&=\int_{e_2}\boldsymbol{n}_2^{\mathrm{T}}\mathbf{div}(\cdot)\lambda_3\lambda_1 ds,\\ d_{7,e}(\cdot)&=\int_{e_3}\boldsymbol{n}_3^{\mathrm{T}}\mathbf{div}(\cdot)\lambda_1 ds, & d_{8,e}(\cdot)&=\int_{e_3}\boldsymbol{n}_3^{\mathrm{T}}\mathbf{div}(\cdot)\lambda_2 ds, & d_{9,e}(\cdot)&=\int_{e_3}\boldsymbol{n}_3^{\mathrm{T}}\mathbf{div}(\cdot)\lambda_1\lambda_2 ds. \end{aligned}$$

The second step is to construct the remainder 21 basis functions $\boldsymbol{\tau}_{h,i}$, $i=10,11,\ldots,30$ for $\Sigma_{3,\triangle_2}$. These basis satisfy

$$\int_e \boldsymbol{n}_e^{\mathrm{T}}\mathbf{div}\boldsymbol{\tau}_{h,i}p_2 ds=0 \quad \text{for all } p_2\in P_2(e), \quad e\in\mathcal{E}(K).$$

Similarly, recall the rest two types of basis functions in [34], which are vertex-based basis functions and edge-based basis functions with nonzero fluxes. On the element $K\in\mathcal{T}_h$, the remainder 21 basis functions of $\Sigma_{3,\triangle_2}$ can be derived from the following two classes of basis functions in [34]:

(1) (Vertex-based basis functions) The 9 basis functions in [34] are defined by

$$\begin{aligned} &\boldsymbol{\varphi}_{h,i}=\phi_1\mathbb{T}_i, \quad i=1,2,3,\\ &\boldsymbol{\varphi}_{h,i+3}=\phi_2\mathbb{T}_i, \quad i=1,2,3,\\ &\boldsymbol{\varphi}_{h,i+6}=\phi_3\mathbb{T}_i, \quad i=1,2,3 \end{aligned}$$

with the Lagrange nodal basis functions in $P_3(K)$:

$$\phi_i=\frac{1}{2}\lambda_i(3\lambda_i-1)(3\lambda_i-2), \quad i=1,2,3$$

and

$$\mathbb{T}_1:=\begin{pmatrix}1&0\\0&0\end{pmatrix}, \quad \mathbb{T}_2:=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \quad \mathbb{T}_3:=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

(2) (Edge-based basis functions with nonzero fluxes) For $e_i\in\mathcal{E}(K)$, $i=1,2,3$, define

$$\boldsymbol{x}_{e_i,1}=\frac{1}{3}(2\boldsymbol{x}_{i+1}+\boldsymbol{x}_{i-1}), \quad \boldsymbol{x}_{e_i,2}=\frac{1}{3}(2\boldsymbol{x}_{i-1}+\boldsymbol{x}_{i+1}).$$

The associated Lagrange nodal basis functions are

$$\phi_{e_i,1} = \frac{9}{2}\lambda_{i+1}\lambda_{i-1}(3\lambda_{i+1} - 1), \quad \phi_{e_i,2} = \frac{9}{2}\lambda_{i-1}\lambda_{i+1}(3\lambda_{i-1} - 1).$$

The 12 edge-based basis functions $\boldsymbol{\varphi}_{h,10}, \ldots, \boldsymbol{\varphi}_{h,21}$ (with nonzero fluxes) in [34] are

$$\phi_{e_i,j}\boldsymbol{n}_i\boldsymbol{n}_i^{\mathrm{T}}, \quad \frac{1}{2}\phi_{e_i,j}(\boldsymbol{t}_i\boldsymbol{n}_i^{\mathrm{T}} + \boldsymbol{n}_i\boldsymbol{t}_i^{\mathrm{T}}), \quad i = 1, 2, 3, \quad j = 1, 2,$$

respectively.

The basis functions of $\Sigma_{h,\triangle_2}$ have the forms

$$\boldsymbol{\tau}_{h,i+9} = \boldsymbol{\varphi}_{h,i} - \sum_{j=1}^{9} \beta_j^{(i)} \boldsymbol{\tau}_{h,j}, \quad i = 1, 2, \ldots, 21.$$

The coefficients $\beta_1^{(i)}, \ldots, \beta_9^{(i)}$ are constants, given by

$$(\boldsymbol{n}_l^{\mathrm{T}}\mathbf{div}\boldsymbol{\varphi}_{h,i}, \lambda_{l+1})_{e_l}, \quad (\boldsymbol{n}_l^{\mathrm{T}}\mathbf{div}\boldsymbol{\varphi}_{h,i}, \lambda_{l-1})_{e_l}, \quad (\boldsymbol{n}_l^{\mathrm{T}}\mathbf{div}\boldsymbol{\varphi}_{h,i}, \lambda_{l-1}\lambda_{l+1})_{e_l}, \quad l = 1, 2, 3,$$

respectively.

Remark A.3. The implementation of $\boldsymbol{\tau}_{h,i}$, $i = 10, 11, \ldots, 30$ can also rely on the reference element $\widehat{K}$. For example, according to Lemma A.2,

$$(\boldsymbol{n}_1^{\mathrm{T}}\mathbf{div}\boldsymbol{\varphi}_{h,1}, \lambda_2)_{e_l} = (\widehat{\boldsymbol{n}}_1^{\mathrm{T}}\widehat{\mathbf{div}}\widehat{\boldsymbol{\varphi}}_{h,1}, \widehat{\lambda}_2)_{\widehat{e}_1}, \tag{A.8}$$

where

$$\widehat{\boldsymbol{\varphi}}_{h,1} = \mathrm{J}\, B^{-1}\boldsymbol{\varphi}_{h,1}B^{-\mathrm{T}} = \mathrm{J}\, B^{-1}\phi_1\mathbb{T}_1B^{-\mathrm{T}} =: \phi_1 M_1.$$

Note that the matrix $M_1 = \mathrm{J}B^{-1}\mathbb{T}_1B^{-\mathrm{T}}$. This shows

$$\begin{aligned}(\boldsymbol{n}_1^{\mathrm{T}}\mathbf{div}\boldsymbol{\varphi}_{h,1}, \lambda_2)_{e_1} &= (\widehat{\boldsymbol{n}}_1^{\mathrm{T}}\widehat{\mathbf{div}}(\phi_1 M_1), \widehat{\lambda}_2)_{\widehat{e}_1}\\ &= (\widehat{\boldsymbol{n}}_1^{\mathrm{T}} M_1 \widehat{\nabla}\phi_1, \widehat{\lambda}_2)_{\widehat{e}_1}\\ &= \widehat{\boldsymbol{n}}_1^{\mathrm{T}} M_1 (\widehat{\nabla}\phi_1, \widehat{\lambda}_2)_{\widehat{e}_1}.\end{aligned}$$

The term $(\widehat{\nabla}\phi_1, \widehat{\lambda}_2)_{\widehat{e}_1}$ can be calculated exactly on $\widehat{K}$. Then some transformations lead to

$$(\boldsymbol{n}_1^{\mathrm{T}}\mathbf{div}\boldsymbol{\varphi}_{h,1}, \lambda_2)_{e_1},$$

which is β_1^1.