SCIENCE CHINA
Mathematics

Special Issue on Differential Geometry
Progress of Projects Supported by NSFC
• ARTICLES •

July 2021 Vol. 64 No. 7: 1493–1504
https://doi.org/10.1007/s11425-020-1841-y

On the generalized Chern conjecture for hypersurfaces with constant mean curvature in a sphere

In Memory of Professor Zhengguo Bai (1916–2015)

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Received September 30, 2020; accepted January 28, 2021; published online April 9, 2021

Abstract Let M be a compact hypersurface with constant mean curvature in $\mathbb{S}^{n+1}$. Denote by H and S the mean curvature and the squared norm of the second fundamental form of M, respectively. We verify that there exists a positive constant $\gamma(n)$ depending only on n such that if $|H| \leqslant \gamma(n)$ and $\beta(n,H) \leqslant S \leqslant \beta(n,H) + \frac{n}{18}$, then $S \equiv \beta(n,H)$ and M is a Clifford torus. Here, $\beta(n,H) = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}$.

Keywords generalized Chern conjecture, hypersurfaces with constant mean curvature, rigidity theorem, scalar curvature, the second fundamental form

MSC(2020) 53C24, 53C40

Citation: Lei L, Xu H W, Xu Z Y. On the generalized Chern conjecture for hypersurfaces with constant mean curvature in a sphere. Sci China Math, 2021, 64: 1493–1504, https://doi.org/10.1007/s11425-020-1841-y

1 Introduction

One of the most important results on rigidity of minimal submanifolds is the following theorem due to Simons [21], Lawson [14], Chern et al. [8] and Li and Li [16].

Theorem A. *Let M be an n-dimensional oriented compact minimal submanifold in an $(n+p)$-dimensional unit sphere $\mathbb{S}^{n+p}$. If the squared length of the second fundamental form of M satisfies $S \leqslant \max\{\frac{n}{2-1/p}, \frac{2}{3}n\}$, then M must be one of the following*:

(i) *the great sphere $\mathbb{S}^n$ with $S \equiv 0$;*

(ii) *the Clifford torus $\mathbb{S}^k(\sqrt{\frac{k}{n}}) \times \mathbb{S}^{n-k}(\sqrt{\frac{n-k}{n}})$ with $S \equiv n$ for $1 \leqslant k \leqslant n-1$;*

(iii) *the Veronese surface in $\mathbb{S}^4$ with $S \equiv \frac{4}{3}$.*

This derives an optimal pinching theorem for minimal hypersurfaces in a sphere.

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Theorem B. *If M is a compact minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$, and if the squared length of the second fundamental form of M satisfies $S \leqslant n$, then $S \equiv 0$ and M is the great sphere $\mathbb{S}^n$, or $S \equiv n$ and M is one of the Clifford tori $\mathbb{S}^k(\sqrt{\frac{k}{n}}) \times \mathbb{S}^{n-k}(\sqrt{\frac{n-k}{n}})$, $1 \leqslant k \leqslant n-1$.*

According to this result, Chern et al. [7,8] proposed the famous Chern conjecture for minimal hypersurfaces in a sphere (the standard version of the Chern conjecture) in the late 1960s. This conjecture was listed in the well-known problem section by Yau [35] in 1982. In [17], Münzner proved that if M is a compact isoparametric minimal hypersurface in $\mathbb{S}^{n+1}$, then $g \in \{1, 2, 3, 4, 6\}$ and $S = (g-1)n$, where g is the number of distinct principal curvatures of M. In 1986, Verstraelen, Montiel, Ros and Urbano (see [23]) gave the refined version of the Chern conjecture. Afterwards, Xu and Xu [30,31] formulated the stronger version of the Chern conjecture.

The Chern conjecture for minimal hypersurfaces in spheres can be summarized as follows.

Chern conjecture. Let M be a compact minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$.

(A) (Standard version) If M has constant scalar curvature, then the possible values of the scalar curvature of M form a discrete set.

(B) (Refined version) If M has constant scalar curvature, then M is isoparametric.

(C) (Stronger version) Denote by S the squared length of the second fundamental form of M. Set $a_k = (k - \mathrm{sgn}(5-k))n$ for $k \in \{m \in \mathbb{Z}^+; 1 \leqslant m \leqslant 5\}$. Then we have

(i) for any fixed $k \in \{m \in \mathbb{Z}^+; 1 \leqslant m \leqslant 4\}$, if $a_k \leqslant S \leqslant a_{k+1}$, then M is isoparametric, and $S \equiv a_k$ or $S \equiv a_{k+1}$;

(ii) if $S \geqslant a_5$, then M is isoparametric, and $S \equiv a_5$.

It is seen from the above that the Chern conjecture consists of several pinching problems. Notice that the first pinching problem has been solved due to Theorem B. The second pinching problem is an important part of the Chern conjecture, which has been open for almost fifty years.

The second pinching problem. Let M be a compact minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$.

(i) If S is constant, and if $n \leqslant S \leqslant 2n$, then $S = n$ or $S = 2n$.

(ii) If $n \leqslant S \leqslant 2n$, then $S \equiv n$ or $S \equiv 2n$.

In 1983, Peng and Terng [19,20] initiated the study of the second pinching problem for minimal hypersurfaces in the unit sphere, and made the following breakthrough on the Chern conjecture.

Theorem C. *Let M be a compact minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$.*

(i) *If S is constant, and if $n \leqslant S \leqslant n + \frac{1}{12n}$, then $S = n$.*

(ii) *If $n \leqslant 5$, and if $n \leqslant S \leqslant n + \delta_1(n)$, where $\delta_1(n)$ is a positive constant depending only on n, then $S \equiv n$.*

During the past three decades, there has been some important progress on the Chern conjecture (see [6,12,13,23] for more details). On the aspect of the standard version of the Chern conjecture, Yang and Cheng [32–34] improved the pinching constant $\frac{1}{12n}$ in Theorem C(i) to $\frac{n}{3}$. Later, Suh and Yang [22] improved this pinching constant to $\frac{3}{7}n$. A general version of Suh and Yang's result for hypersurfaces with constant mean curvature and constant scalar curvature can be found in [28].

In 1993, Chang [3] proved the Chern conjecture (B) in dimension three. More generally, Chang [4] (see also [9]) proved that all the compact hypersurfaces with constant scalar curvature and constant mean curvature in the unit sphere $\mathbb{S}^4$ are isoparametric. In 2017, Deng et al. [10] showed that any closed Willmore minimal hypersurface with constant scalar curvature in $\mathbb{S}^5$ must be isoparametric.

On the stronger version of the Chern conjecture, Wei and Xu [24] investigated the Chern conjecture (C) and proved that if M is a compact minimal hypersurface in $\mathbb{S}^{n+1}$, $n = 6$ and $n = 7$, and if $n \leqslant S \leqslant n + \delta_2(n)$, where $\delta_2(n)$ is a positive constant depending only on n, then $S \equiv n$. Later, Zhang [36] extended the second pinching theorem due to Peng and Terng [20] and Wei and Xu [24] to the case of $n = 8$. In 2011, Ding and Xin [11] obtained the following important rigidity result.

Theorem D. *Let M be an n-dimensional compact minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$. If $n \geqslant 6$, the squared norm of the second fundamental form satisfies $0 \leqslant S - n \leqslant \frac{n}{23}$, then $S \equiv n$, i.e., M is a Clifford torus.*

In 2016, Xu and Xu [31] gave a refined version of Ding and Xin's rigidity theorem.

Theorem E. *Let M be an n-dimensional compact minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$. If the squared length of the second fundamental form satisfies $0 \leqslant S - n \leqslant \frac{n}{22}$, then $S \equiv n$ and M is a Clifford torus.*

In 2017, Lei et al. [15] verified the following rigidity theorem on the stronger version of the Chern conjecture for minimal hypersurfaces in a sphere.

Theorem F. *Let M be an n-dimensional compact minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$. If the squared length of the second fundamental form satisfies $0 \leqslant S - n \leqslant \frac{n}{18}$, then $S \equiv n$ and M is a Clifford torus.*

It is well known that the possible values of the squared length of the second fundamental forms of all the closed isoparametric hypersurfaces with constant mean curvature H in the unit sphere form a discrete set I ($\subset \mathbb{R}$). More generally, the following conjecture can be viewed as a general version of the Chern conjecture.

Generalized Chern conjecture. Let M be an n-dimensional closed hypersurface with constant mean curvature H in the unit sphere $\mathbb{S}^{n+1}$.

(i) Assume that $a < b$ and $[a, b] \cap I = \{a, b\}$. If $a \leqslant S \leqslant b$, then $S \equiv a$ or $S \equiv b$, and M is an isoparametric hypersurface in $\mathbb{S}^{n+1}$.

(ii) Set $c = \sup_{t \in I} t$. If $S \geqslant c$, then $S \equiv c$, and M is an isoparametric hypersurface in $\mathbb{S}^{n+1}$.

In 1990, Cheng and Nakagawa [5] and Xu [26, 27] provided an affirmative answer to the generalized Chern conjecture for the case of $a = nH^2$ and $b = \alpha(n, H)$, independently. Precisely, they proved the generalized Simons-Lawson-Chern-do Carmo-Kobayashi theorem for compact hypersurfaces with constant mean curvature in spheres.

Theorem G. *Let M be an n-dimensional compact hypersurface with constant mean curvature in $\mathbb{S}^{n+1}$. If $S \leqslant \alpha(n, H)$, then either $S \equiv nH^2$ and M is a totally umbilic sphere, or $S \equiv \alpha(n, H)$ and M is a Clifford torus. Here, $\alpha(n, H) = n + \frac{n^3}{2(n-1)} H^2 - \frac{n(n-2)}{2(n-1)} \sqrt{n^2 H^4 + 4(n-1) H^2}$, and H is the mean curvature of M.*

In the case of low dimensions, the second pinching problem for compact hypersurfaces with constant mean curvature in spheres was studied by many authors. In the case of arbitrary dimensions, Xu and Xu [30] proved the following second pinching theorem.

Theorem H. *Let M be an n-dimensional compact hypersurface with constant mean curvature in the unit sphere $\mathbb{S}^{n+1}$. There exists a positive constant $\gamma_0(n)$ depending only on n, such that if $|H| \leqslant \gamma_0(n)$, and $\beta(n, H) \leqslant S \leqslant \beta(n, H) + \frac{n}{23}$, then $S \equiv \beta(n, H)$ and M is one of the following cases*: (i) $\mathbb{S}^k(\sqrt{\frac{k}{n}}) \times \mathbb{S}^{n-k}(\sqrt{\frac{n-k}{n}})$ *and* $1 \leqslant k \leqslant n-1$; (ii) $\mathbb{S}^1(\frac{1}{\sqrt{1+\mu^2}}) \times \mathbb{S}^{n-1}(\frac{\mu}{\sqrt{1+\mu^2}})$. *Here,*

$$\beta(n, H) = n + \frac{n^3}{2(n-1)} H^2 + \frac{n(n-2)}{2(n-1)} \sqrt{n^2 H^4 + 4(n-1) H^2} \quad \text{and} \quad \mu = \frac{n|H| + \sqrt{n^2 H^2 + 4(n-1)}}{2}.$$

In this paper, we improve Theorem H and prove the following second pinching theorem for hypersurfaces with small constant mean curvature in a sphere.

Main theorem. *Let M be an n-dimensional compact hypersurface with constant mean curvature in the unit sphere $\mathbb{S}^{n+1}$. There exists a positive constant $\gamma(n)$ depending only on n, such that if $|H| \leqslant \gamma(n)$, and $\beta(n, H) \leqslant S \leqslant \beta(n, H) + \frac{n}{18}$, then $S \equiv \beta(n, H)$ and M is one of the following cases*: (i) $\mathbb{S}^k(\sqrt{\frac{k}{n}}) \times \mathbb{S}^{n-k}(\sqrt{\frac{n-k}{n}})$ *and* $1 \leqslant k \leqslant n-1$; (ii) $\mathbb{S}^1(\frac{1}{\sqrt{1+\mu^2}}) \times \mathbb{S}^{n-1}(\frac{\mu}{\sqrt{1+\mu^2}})$.

Remark 1.1. When $n \in \{6, 12, 24\}$, we construct a family of compact isoparametric hypersurfaces $\{M_t\}_{t \in (t_2, t_1)}$ in $\mathbb{S}^{n+1}$, whose squared norm of the second fundamental form S_t satisfies the pinching condition $\beta(n, H) \leqslant S_t \leqslant \beta(n, H) + \frac{n}{18}$. But the conclusion of the main theorem does not hold for $\{M_t\}_{t \in (t_2, t_1)}$. Therefore, the assumption $|H| \leqslant \gamma(n)$ in the main theorem cannot be removed.

2 Hypersurfaces with constant mean curvature in a sphere

Let M be a hypersurface in the unit sphere $\mathbb{S}^{n+1}$. Denote by $\overline{\nabla}$ and ∇ the Levi-Civita connection on $\mathbb{S}^{n+1}$ and the induced connection on M, respectively. Let h be the second fundamental form of M. For the tangent vector fields X and Y over M, we have the Gauss formula $\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$ (see [1,25]). We shall make use of the following convention on the range of indices: $1 \leqslant i,j,k,\ldots \leqslant n$. Choose a local orthonormal frame $\{e_i\}$ for the tangent bundle over M. Let ν be a local unit normal vector field of M. Set $h(e_i,e_j) = h_{ij}\nu$. Denote by S the squared length of the second fundamental form of M, i.e., $S = \sum_{i,j} h_{ij}^2$. We denote by h_{ijk} the covariant derivative of h_{ij}. It follows from the Codazzi equation that h_{ijk} is symmetric in i,j and k.

From now on, we assume that M is a compact hypersurface with constant mean curvature in $\mathbb{S}^{n+1}$. Then $\sum_i h_{ii} = nH$. The following Simons type formula can be found in [29,30]:

$$\frac{1}{2}\Delta S = S(n-S) - n^2H^2 + nHf_3 + |\nabla h|^2, \tag{2.1}$$

$$\frac{1}{2}\Delta|\nabla h|^2 = (2n+3-S)|\nabla h|^2 - \frac{3}{2}|\nabla S|^2 + |\nabla^2 h|^2 - 3(A-2B) + 3nHC, \tag{2.2}$$

where

$$A = \sum_{i,j,k,l,m} h_{ijk}h_{ijl}h_{km}h_{ml}, \quad B = \sum_{i,j,k,l,m} h_{ijk}h_{klm}h_{im}h_{jl},$$
$$C = \sum_{i,j,k,l} h_{ijk}h_{ijl}h_{kl}, \quad f_k = \operatorname{Trace} h^k.$$

$A-2B$ is called the Peng-Terng invariant of M (see also [15]). Notice that

$$|HC| \leqslant |H|\sqrt{S}|\nabla h|^2. \tag{2.3}$$

Integrating (2.1) and (2.2), we get

$$\int_M |\nabla h|^2 dM = \int_M [S(S-n) + n^2H^2 - nHf_3] dM \tag{2.4}$$

and

$$\int_M |\nabla^2 h|^2 dM = \int_M \left[(S-2n-3)|\nabla h|^2 + 3(A-2B) + \frac{3}{2}|\nabla S|^2 - 3nHC\right] dM. \tag{2.5}$$

Choose a local orthonormal frame such that $h_{ij} = \lambda_i\delta_{ij}$. Then $\sum_i \lambda_i = nH$, $\sum_i \lambda_i^2 = S$, $f_k = \sum_i \lambda_i^k$ and

$$A - 2B = \sum_{i,j,k} h_{ijk}^2(\lambda_i^2 - 2\lambda_i\lambda_j).$$

We have the following integral formula (see [29]):

$$\int_M (A-2B) dM = \int_M \left(nHf_3 - S^2 - f_3^2 + Sf_4 - \frac{|\nabla S|^2}{4}\right) dM. \tag{2.6}$$

Define

$$G = \sum_{i,j} (\lambda_i - \lambda_j)^2(1+\lambda_i\lambda_j)^2 = 2[Sf_4 - f_3^2 - S^2 - S(S-n) + 2nHf_3 - n^2H^2].$$

Then we get

$$\int_M (A-2B) dM = \int_M \left(\frac{1}{2}G + |\nabla h|^2 - \frac{1}{4}|\nabla S|^2\right) dM. \tag{2.7}$$

Following [29], we have

$$3(A-2B) \leqslant \frac{\sqrt{17}+1}{2} S|\nabla h|^2. \tag{2.8}$$

Denote by $\mathring{h}$ the traceless second fundamental form of M. By diagonalizing h_{ij}, we have $\mathring{h}_{ij} = \mathring{\lambda}_i\delta_{ij}$, where $\mathring{\lambda}_i = \lambda_i - H$. Taking $\mathring{S} = |\mathring{h}|^2$ and $\mathring{f}_k = \sum_i(\mathring{\lambda}_i)^k$, we have $\mathring{S} = S - nH^2$ and $f_3 = \mathring{f}_3 + 3H\mathring{S} + nH^3$. So $\nabla\mathring{h} = \nabla h$ and $\nabla\mathring{S} = \nabla S$. From (2.1), we obtain

$$\frac{1}{2}\Delta\mathring{S} = -F + |\nabla\mathring{h}|^2, \tag{2.9}$$

where $F = \mathring{S}^2 - n\mathring{S} - nH^2\mathring{S} - nH\mathring{f}_3$. So

$$|\nabla\mathring{S}|^2 = \frac{1}{2}\Delta\mathring{S}^2 - \mathring{S}\Delta\mathring{S} = \frac{1}{2}\Delta\mathring{S}^2 + 2\mathring{S}F - 2\mathring{S}|\nabla\mathring{h}|^2.$$

Thus, we have

$$\int_M |\nabla\mathring{S}|^2 dM = \int_M (2\mathring{S}F - 2\mathring{S}|\nabla\mathring{h}|^2) dM, \tag{2.10}$$

$$\int_M F dM = \int_M |\nabla\mathring{h}|^2 dM. \tag{2.11}$$

Put

$$\beta(n,H) = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}, \quad \mathring{\beta}(n,H) = \beta(n,H) - nH^2.$$

As we did in [29], when $\mathring{S} \geqslant \mathring{\beta}(n,H)$, we have

$$F \geqslant 0. \tag{2.12}$$

Moreover, if $F = 0$, then $\mathring{S} = \mathring{\beta}(n,H)$. Following [30], we have the following estimate:

$$|\nabla^2 h|^2 \geqslant \frac{3}{4}G + \frac{3F^2}{2(n+4)\mathring{S}}. \tag{2.13}$$

3 An estimate for $A - 2B$

We need the following two inequalities from [15].

Lemma 3.1. *For all $x, y, z \in \mathbb{R}$, we have*

$$-2(xy + yz + zx + 2)^3 < (x-y)^2(xy+1)^2 + (x-z)^2(xz+1)^2 + (z-y)^2(yz+1)^2.$$

Lemma 3.2. *Let s be a positive number satisfying $s \geqslant 6$, and let $D_s = \{(x,y) \mid x^2 + y^2 \leqslant s\}$. For any point $(x,y) \in D_s$, we have*

$$-(x^2 + 4xy + 4)^3 < \frac{16}{5}\left(3 - \frac{10}{s}\right)(x-y)^2(1+xy)^2.$$

With the aid of the above lemmas, we now prove the following estimate.

Theorem 3.1. *Let M be an n $(\geqslant 6)$-dimensional hypersurface in $\mathbb{S}^{n+1}$. Let η be a positive number. If $\beta(n,H) \leqslant S \leqslant \beta(n,H) + \frac{n}{\eta}$, then $3(A - 2B) \leqslant (S + 4 + \sqrt[3]{\psi G})|\nabla h|^2$. Here,*

$$\psi = \frac{24}{5} - \frac{16}{(1+\eta^{-1})n} + q_5, \quad q_5 = \frac{16(\beta(n,H) - n)}{(1+\eta^{-1})n(\beta(n,H) + \frac{n}{\eta})}.$$

Proof. From the definitions of A and B, we have

$$3(A - 2B) = \sum_{i,j,k \text{ distinct}} h_{ijk}^2(\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - 2\lambda_i\lambda_j - 2\lambda_j\lambda_k - 2\lambda_i\lambda_k)$$

$$+3\sum_{i,j\text{ distinct}} h_{iij}^2(\lambda_j^2-4\lambda_i\lambda_j)-3\sum_i h_{iii}^2\lambda_i^2.$$

By the definition of G, for distinct i,j,k, we get

$$G\geqslant 2(\lambda_i-\lambda_j)^2(1+\lambda_i\lambda_j)^2+2(\lambda_j-\lambda_k)^2(1+\lambda_j\lambda_k)^2+2(\lambda_i-\lambda_k)^2(1+\lambda_i\lambda_k)^2.$$

Applying Lemma 3.1, we obtain

$$\begin{aligned}&-2\lambda_i\lambda_j-2\lambda_j\lambda_k-2\lambda_i\lambda_k-4\\&\quad<[4(\lambda_i-\lambda_j)^2(1+\lambda_i\lambda_j)^2+4(\lambda_j-\lambda_k)^2(1+\lambda_j\lambda_k)^2+4(\lambda_i-\lambda_k)^2(1+\lambda_i\lambda_k)^2]^{\frac{1}{3}}\\&\quad\leqslant\sqrt[3]{2G}.\end{aligned}$$

Since $\beta(n,H)\geqslant n$, we have $\psi>\frac{24}{5}-\frac{16}{6}>2$. Thus we get

$$\lambda_i^2+\lambda_j^2+\lambda_k^2-2\lambda_i\lambda_j-2\lambda_j\lambda_k-2\lambda_i\lambda_k<S+4+\sqrt[3]{\psi G}.$$

Note that $S\geqslant\beta(n,H)\geqslant 6$. By Lemma 3.2, we have

$$\begin{aligned}-(4\lambda_i\lambda_j+\lambda_i^2+4)&<\left[\frac{16}{5}\left(3-\frac{10}{S}\right)(\lambda_i-\lambda_j)^2(1+\lambda_i\lambda_j)^2\right]^{\frac{1}{3}}\\&\leqslant\left[\left(\frac{24}{5}-\frac{16}{S}\right)G\right]^{\frac{1}{3}}\\&\leqslant\left[\left(\frac{24}{5}-\frac{16}{\beta(n,H)+\frac{n}{\eta}}\right)G\right]^{\frac{1}{3}}\\&=\sqrt[3]{\psi G}.\end{aligned}$$

Thus we get

$$\lambda_j^2-4\lambda_i\lambda_j=\lambda_i^2+\lambda_j^2+4-(4\lambda_i\lambda_j+\lambda_i^2+4)<S+4+\sqrt[3]{\psi G}.$$

Therefore, we obtain

$$\begin{aligned}3(A-2B)&\leqslant(S+4+\sqrt[3]{\psi G})\left(\sum_{i,j,k\text{ distinct}}h_{ijk}^2+3\sum_{i,j\text{ distinct}}h_{iij}^2\right)\\&\leqslant(S+4+\sqrt[3]{\psi G})|\nabla h|^2.\end{aligned}$$

This completes the proof. □

4 Proof of the main theorem

Now we are in a position to prove our rigidity theorem on the generalized Chern conjecture for hypersurfaces with constant mean curvature in a sphere.

Proof of Main theorem. Without loss of generality, we assume that $H\geqslant 0$. Notice that we have the pinching condition

$$\beta(n,H)\leqslant S\leqslant\beta(n,H)+\frac{n}{18}.$$

Combining (2.3), (2.5), (2.7), (2.10) and (2.13), we have

$$\begin{aligned}0&\leqslant\int_M\left[(S-2n-3)|\nabla h|^2+3(A-2B)+\frac{3}{2}|\nabla S|^2-3nHC-\frac{3}{4}G\right]dM\\&\leqslant\int_M\left[\left(S-2n-\frac{3}{2}\right)|\nabla h|^2+\frac{3}{2}(A-2B)+\frac{9}{4}(\mathring{S}F-\mathring{S}|\nabla\mathring{h}|^2)+3nH\sqrt{S}|\nabla h|^2\right]dM\end{aligned}$$

$$\leqslant \int_M \bigg\{ \bigg(-\frac{5}{4}S - 2n - \frac{3}{2} \bigg) |\nabla h|^2 + \frac{3}{2}(A - 2B) + 3nH\sqrt{S}|\nabla h|^2 + \frac{9}{4}\bigg[\bigg(n + D(n,H) + \frac{n}{18} - nH^2 \bigg) F + nH^2 |\nabla h|^2 \bigg] \bigg\} dM$$
$$\leqslant \int_M \bigg[\bigg(\frac{3n}{8} - \frac{5}{4}S - \frac{3}{2} + q_0 \bigg) |\nabla h|^2 + \frac{3}{2}(A - 2B) \bigg] dM, \tag{4.1}$$

where $q_0 = q_0(n,H) = \frac{9}{4}D(n,H) + 3nH\sqrt{\beta(n,H) + \frac{n}{18}}$ and $D(n,H) = \beta(n,H) - n$.

When $n = 3$, from (2.8) and (4.1), we have

$$0 \leqslant \int_M \bigg[\frac{3n}{8} + \frac{\sqrt{17} - 4}{4}\bigg(D(n,H) + \frac{19n}{18} \bigg) - \frac{3}{2} + q_0 \bigg] |\nabla h|^2 dM$$
$$\leqslant \int_M \bigg[\frac{9}{8} + \frac{\sqrt{17} - 4}{4} \times \frac{19}{6} - \frac{3}{2} + q_1 \bigg] |\nabla h|^2 dM, \tag{4.2}$$

where $q_1 = \frac{\sqrt{17}-4}{4}D(n,H) + q_0$. There exists an explicit positive constant $\gamma(n)$, such that if $|H| \leqslant \gamma(n)$, then $q_1 \leqslant 0.2$. Then the coefficient of the integral in (4.2) is negative for $|H| \leqslant \gamma(n)$.

When $4 \leqslant n \leqslant 5$, with the aid of [30, Lemma 3.2], we have

$$0 \leqslant \int_M \bigg[\frac{3n}{8} - \bigg(\frac{5}{4} - \frac{2 + \zeta(n)}{2} \bigg) \beta(n,H) - \frac{3}{2} + q_0 \bigg] |\nabla h|^2 dM$$
$$= \int_M \bigg[\frac{n}{8} + \frac{\zeta(n)}{2}n - \frac{3}{2} + q_2 \bigg] |\nabla h|^2 dM$$
$$\leqslant \int_M \bigg[\frac{5}{8} + \frac{0.23 \times 5}{2} - \frac{3}{2} + q_2 \bigg] |\nabla h|^2 dM, \tag{4.3}$$

where $q_2 = q_0 - (\frac{5}{4} - \frac{2+\zeta(n)}{2})D(n,H)$, $\zeta(4) = 0.16$ and $\zeta(5) = 0.23$. There exists an explicit positive constant $\gamma(n)$, such that if $|H| \leqslant \gamma(n)$, then $q_2 \leqslant 0.2$. Then the coefficient of the integral in (4.3) is negative for $|H| \leqslant \gamma(n)$.

When $n \geqslant 6$, we assume that M is a compact hypersurface with constant mean curvature in $\mathbb{S}^{n+1}$, which satisfies $|H| \leqslant \gamma(n)$ and $\beta(n,H) \leqslant S \leqslant \beta(n,H) + \frac{n}{\eta}$, where η is a positive parameter.

Combining (2.5), (2.7) and (2.13), we have

$$\int_M (A - 2B) dM = \int_M \bigg[\frac{1}{2}G + |\nabla h|^2 - \frac{1}{4}|\nabla S|^2 \bigg] dM$$
$$\leqslant \int_M \bigg[\frac{2}{3}|\nabla^2 h|^2 - \frac{F^2}{(n+4)\mathring{S}} + |\nabla h|^2 - \frac{1}{4}|\nabla S|^2 \bigg] dM$$
$$= \int_M \bigg[\bigg(\frac{2}{3}S - \frac{4}{3}n - 1 \bigg) |\nabla h|^2 + 2(A - 2B) - \frac{F^2}{(n+4)\mathring{S}} + \frac{3}{4}|\nabla S|^2 - 2nHC \bigg] dM. \tag{4.4}$$

We estimate $\frac{F}{\mathring{S}}$ as follows:

$$\frac{F}{\mathring{S}} = \mathring{S} - n - nH^2 - \frac{nH\mathring{f}_3}{\mathring{S}}$$
$$\geqslant \mathring{S} - n - nH^2 - \frac{nH\mathring{S}^{3/2}}{\mathring{S}}$$
$$\geqslant \mathring{S} - \mathring{\beta} - nH^2 - nH\sqrt{\mathring{\beta}(n,H) + \frac{n}{\eta}}$$
$$= \mathring{S} - \mathring{\beta} - q_3 \tag{4.5}$$

and

$$\begin{aligned}\frac{F}{\mathring{S}} &\leqslant \mathring{S} - n - nH^2 + \frac{nH\mathring{S}^{3/2}}{\mathring{S}}\\ &\leqslant \mathring{S} - n - nH^2 + nH\sqrt{\mathring{\beta}(n,H) + \frac{n}{\eta}}\\ &= S - n + q_4,\end{aligned} \tag{4.6}$$

where $q_3 = nH\sqrt{\mathring{\beta} + \frac{n}{\eta}} + nH^2$ and $q_4 = nH\sqrt{\mathring{\beta} + \frac{n}{\eta}} - 2nH^2$. This together with (2.10) implies

$$\begin{aligned}&\int_M \left[\frac{3}{4}|\nabla S|^2 - \frac{F^2}{(n+4)\mathring{S}}\right]dM\\ &= \int_M \left\{\left[\frac{3}{2}(\mathring{S} - \mathring{\beta}) - \frac{F}{(n+4)\mathring{S}}\right]F + \frac{3}{2}(\mathring{\beta} - \mathring{S})|\nabla h|^2\right\}dM\\ &\leqslant \int_M \left\{\left[b(\mathring{S} - \mathring{\beta}) + \frac{q_3}{n+4}\right]F + \frac{3}{2}(\mathring{\beta} - \mathring{S})|\nabla h|^2\right\}dM\\ &\leqslant \int_M \left[\left(\frac{bn}{\eta} + \frac{q_3}{n+4}\right)|\nabla h|^2 + \frac{3}{2}(\beta - S)|\nabla h|^2\right]dM,\end{aligned} \tag{4.7}$$

where $b = \frac{3}{2} - \frac{1}{n+4}$. From (4.4) and (4.7), we obtain

$$\int_M (A - 2B)dM \geqslant \int_M \left[\left(1 - \frac{n}{6} + \frac{5}{6}S - \frac{bn}{\eta} - \frac{3(\beta - n)}{2} - \frac{q_3}{n+4}\right)|\nabla h|^2 + 2nHC\right]dM. \tag{4.8}$$

Let σ be a positive parameter. Using Theorem 3.1 and Young's inequality, we get

$$3(A - 2B) \leqslant (S + 4 + \sqrt[3]{\psi G})|\nabla h|^2 \leqslant (S+4)|\nabla h|^2 + \frac{1}{3}\psi\sigma^2 G + \frac{2}{3\sigma}|\nabla h|^3. \tag{4.9}$$

Let ε and κ be positive parameters. From (2.9), we have the following estimate:

$$\begin{aligned}\int_M |\nabla h|^3 dM &= \int_M \left[F|\nabla h| + \frac{1}{2}|\nabla h|\Delta S\right]dM\\ &= \int_M \left[F|\nabla h| - \frac{1}{2}\langle\nabla|\nabla h|, \nabla S\rangle\right]dM\\ &\leqslant \int_M \left[F|\nabla h| + \varepsilon|\nabla^2 h|^2 + \frac{1}{16\varepsilon}|\nabla S|^2\right]dM.\end{aligned} \tag{4.10}$$

Together with (4.6) and the pinching condition, we obtain

$$\begin{aligned}\int_M F|\nabla h| dM &\leqslant \int_M \left[2\kappa\mathring{S}F + \frac{1}{8\kappa\mathring{S}}F|\nabla h|^2\right]dM\\ &\leqslant \int_M \left[2\left(\mathring{\beta} + \frac{n}{\eta}\right)\kappa F + \frac{S - n + q_4}{8\kappa}|\nabla h|^2\right]dM\\ &= \int_M \left[2\left(\mathring{\beta} + \frac{n}{\eta}\right)\kappa + \frac{1}{8\kappa}(S - n + q_4)\right]|\nabla h|^2 dM.\end{aligned} \tag{4.11}$$

Combining (4.9)–(4.11), we get

$$\begin{aligned}3\int_M (A - 2B)dM \leqslant \int_M \bigg\{&(S+4)|\nabla h|^2 + \frac{1}{3}\psi\sigma^2 G + \frac{2}{3\sigma}\left[\left(2\left(\mathring{\beta} + \frac{n}{\eta}\right)\kappa + \frac{1}{8\kappa}(S - n + q_4)\right)|\nabla h|^2\right.\\ &\left.+ \varepsilon|\nabla^2 h|^2 + \frac{1}{16\varepsilon}|\nabla S|^2\right]\bigg\}dM.\end{aligned}$$

This together with (2.5) and (2.7) implies

$$3\int_M (A-2B)dM \leqslant \int_M \bigg\{(S+4)|\nabla h|^2 + \frac{1}{3}\psi\sigma^2\bigg(2(A-2B) - 2|\nabla h|^2 + \frac{1}{2}|\nabla S|^2\bigg) + \frac{2}{3\sigma}\bigg[\bigg(2\bigg(\mathring{\beta}+\frac{n}{\eta}\bigg)\kappa + \frac{1}{8\kappa}(S-n+q_4)\bigg)|\nabla h|^2 + \frac{1}{16\varepsilon}|\nabla S|^2 + \varepsilon\bigg(-(2n+3-S)|\nabla h|^2 + 3(A-2B) + \frac{3}{2}|\nabla S|^2 - 3nHC\bigg)\bigg]\bigg\}dM.$$

This implies

$$\int_M \bigg[\theta(A-2B) - \tau|\nabla S|^2 + 2n\frac{\varepsilon}{\sigma}HC\bigg]dM \leqslant \int_M \bigg\{S+4-\frac{2}{3}\psi\sigma^2 + \frac{2}{3\sigma}\times\bigg[2\bigg(\mathring{\beta}+\frac{n}{\eta}\bigg)\kappa + \frac{1}{8\kappa}(S-n+q_4) - \varepsilon(2n+3-S)\bigg]\bigg\}|\nabla h|^2dM, \tag{4.12}$$

where

$$\theta = 3 - \frac{2}{3}\psi\sigma^2 - 2\sigma^{-1}\varepsilon, \quad \tau = \frac{1}{6}\psi\sigma^2 + \frac{2}{3\sigma}\bigg(\frac{1}{16\varepsilon} + \frac{3\varepsilon}{2}\bigg).$$

We restrict σ, ε and H such that $\theta \geqslant 0$.

By (2.10), we get

$$\begin{aligned}\int_M |\nabla S|^2 dM &= 2\int_M [(S-\beta)F + (\beta - S)|\nabla h|^2]dM \\ &\leqslant 2\int_M \bigg[\frac{n}{\eta}F + (\beta-S)|\nabla h|^2\bigg]dM \\ &= 2\int_M \bigg(\beta - S + \frac{n}{\eta}\bigg)|\nabla h|^2 dM.\end{aligned} \tag{4.13}$$

Combining (4.8), (4.12) and (4.13), we obtain

$$\begin{aligned}0 \leqslant \int_M \bigg\{&S+4-\frac{2}{3}\psi\sigma^2 + \frac{2}{3\sigma}\times\bigg[2\bigg(\mathring{\beta}+\frac{n}{\eta}\bigg)\kappa + \frac{1}{8\kappa}(S-n+q_4) - \varepsilon(2n+3-S)\bigg] \\ &+ 2\tau\bigg(\beta - S + \frac{n}{\eta}\bigg) - \theta\bigg(1 - \frac{n}{6} + \frac{5}{6}S - \frac{bn}{\eta} - \frac{3(\beta-n)}{2} - \frac{q_3}{n+4}\bigg)\bigg\}|\nabla h|^2 dM \\ &- \int_M 2n\bigg(\theta + \frac{\varepsilon}{\sigma}\bigg)HCdM.\end{aligned} \tag{4.14}$$

Let $\mathcal{O}(x)$ denote any continuous function satisfying $\mathcal{O}(0) = 0$. Then q_3, q_4 and q_5 all belong to the class of $\mathcal{O}(H)$. We also have $\beta = n + \mathcal{O}(H)$. Take

$$\varepsilon = \frac{1}{18}, \quad \sigma = \frac{7}{18}, \quad \kappa = \frac{1}{24}, \quad \eta = 18.$$

Thus we have

$$\psi = -\frac{288}{19n} + \frac{24}{5} + \mathcal{O}(H), \quad \tau = \frac{12431}{5670} - \frac{196}{513n} + \mathcal{O}(H), \quad \theta = \frac{784}{513n} + \frac{6323}{2835} + \mathcal{O}(H).$$

It follows from (2.3) that

$$-2n\bigg(\theta + \frac{\varepsilon}{\sigma}\bigg)HC \leqslant 2n\bigg(\theta + \frac{\varepsilon}{\sigma}\bigg)H\sqrt{\beta + \frac{n}{18}}|\nabla h|^2 = \mathcal{O}(H)|\nabla h|^2.$$

Therefore, from (4.14) we obtain

$$0 \leqslant \int_M \bigg[-\bigg(\frac{784}{1539n} + \frac{13}{2430}\bigg)S + \frac{6745n^2 + 902646n + 4299120}{1939140(n+4)} + \mathcal{O}(H)\bigg]|\nabla h|^2 dM$$

$$
\begin{aligned}
&\leqslant \int_M \left[-\left(\frac{784}{1539n} + \frac{13}{2430}\right)n + \frac{6745n^2 + 902646n + 4299120}{1939140(n+4)} + \mathcal{O}(H)\right] |\nabla h|^2 dM \\
&= \int_M \left[-\frac{3629n^2 + 126690n - 347760}{1939140(n+4)} + \mathcal{O}(H)\right] |\nabla h|^2 dM. \qquad (4.15)
\end{aligned}
$$

When $n \geqslant 6$, there exists a positive constant $\gamma(n)$, such that for $|H| \leqslant \gamma(n)$, we have $\theta > 0$, and the expression in the square bracket of the right-hand side of (4.15) is negative. This implies $|\nabla h| \equiv 0$. From (2.11), we have $\mathring{S} = \mathring{\beta}(n, H)$.

When $H = 0$, $\mathring{S} = \mathring{\beta}(n, H)$ becomes $S = n$. Then M is one of the Clifford tori

$$
\mathbb{S}^k\left(\sqrt{\frac{k}{n}}\right) \times \mathbb{S}^{n-k}\left(\sqrt{\frac{n-k}{n}}\right), \quad 1 \leqslant k \leqslant n-1.
$$

When $H \neq 0$, the principal curvature of M satisfies

$$
\mu_1 = \cdots = \mu_{n-1} = H - \sqrt{\frac{\mathring{\beta}(n, H)}{n(n-1)}}, \quad \mu_n = H + \sqrt{\frac{(n-1)\mathring{\beta}(n, H)}{n}}.
$$

Therefore, M is the Clifford torus

$$
\mathbb{S}^1\left(\frac{1}{\sqrt{1+\mu^2}}\right) \times \mathbb{S}^{n-1}\left(\frac{\mu}{\sqrt{1+\mu^2}}\right) \quad \text{in } S^{n+1},
$$

where $\mu = \frac{nH + \sqrt{n^2H^2 + 4(n-1)}}{2}$. This completes the proof of the main theorem. □

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 11531012), China Postdoctoral Science Foundation (Grant No. BX20180274) and Natural Science Foundation of Zhejiang Province (Grant No. LY20A010024).

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Appendix A

When $n=3k$, where $k\in\{2,4,8\}$, we construct the following example, which implies that the assumption $|H|\leqslant\gamma(n)$ in the main theorem cannot be removed.

We take

$$F(u)=u_5^3-3u_5u_4^2+\frac{3u_5}{2}(U_1\bar{U_1}+U_2\bar{U_2}-2U_3\bar{U_3})+\frac{3\sqrt{3}u_4}{2}(U_1\bar{U_1}-U_2\bar{U_2})+\frac{3\sqrt{3}}{2}(U_1U_2U_3+\bar{U_3}\bar{U_2}\bar{U_1}),$$

where $U_1,U_2,U_3\in\mathbb{F}$, $u_4,u_5\in\mathbb{R}$, $u=(U_1,U_2,U_3,u_4,u_5)\in\mathbb{R}^{3k+2}$, $k\in\{2,4,8\}$ and

$$\mathbb{F}=\begin{cases}\mathbb{C} & \text{for } k=2,\\ \mathbb{H}\ (\text{quaternions}) & \text{for } k=4,\\ \mathbb{O}\ (\text{octonions}) & \text{for } k=8.\end{cases}$$

Cartan [2] showed that $M_t:=\{u\in\mathbb{S}^{3k+1}:F(u)=\cos 3t\}$ forms a compact isoparametric family of hypersurfaces of dimension $3k$ in $\mathbb{S}^{3k+1}$. In fact, for each t, M_t is a tube with a constant radius over a standard Veronese embedding of $\mathbb{FP}^2$ into $\mathbb{S}^{3k+1}$.

Precisely, M_t is an isoparametric hypersurface with 3 distinct principal curvatures $\frac{\cos 3t-\cos t-\sqrt{3}\sin t}{\sin 3t}$, $\frac{\cos 3t-\cos t+\sqrt{3}\sin t}{\sin 3t}$ and $\frac{2\cos t+\cos 3t}{\sin 3t}$ with the same multiplicity k. Hence the mean curvature of M_t is $\cot 3t$. By a direct computation, we have

$$\begin{aligned}
\frac{\cos 3t-\cos t-\sqrt{3}\sin t}{\sin 3t} &= \frac{\cot t+\sqrt{3}}{1-\sqrt{3}\cot t} = \cot\left(t-\frac{\pi}{3}\right),\\
\frac{\cos 3t-\cos t+\sqrt{3}\sin t}{\sin 3t} &= \frac{\cot t-\sqrt{3}}{1+\sqrt{3}\cot t} = \cot\left(t+\frac{\pi}{3}\right),\\
\frac{2\cos t+\cos 3t}{\sin 3t} &= \cot t.
\end{aligned}$$

Using the formula

$$\sum_{i=1}^{N-1}\cot^2\left(y+\frac{i\pi}{N}\right) = N^2\cot^2(Ny)+N(N-1)$$

due to Muto [18] (see also [17]), we obtain that the squared norm of the second fundamental form of M_t equals $6k+9kH^2$. Since

$$\begin{aligned}
\alpha(3k,H) &= 3k+\frac{27k^3}{2(3k-1)}H^2-\frac{3k(3k-2)}{2(3k-1)}\sqrt{9k^2H^4+4(3k-1)H^2},\\
\beta(3k,H) &= 3k+\frac{27k^3}{2(3k-1)}H^2+\frac{3k(3k-2)}{2(3k-1)}\sqrt{9k^2H^4+4(3k-1)H^2},
\end{aligned}$$

$f(H^2) := 6k+9kH^2-\beta(3k,H)$ is strictly decreasing in H^2 with $f(0)=3k$, $\lim_{H\to\infty} f(H^2)=-\infty$, there exist two positive constants H_1 and H_2 with $H_1<H_2$, such that $f(H_1^2)=\frac{3k}{18}$ and $f(H_2^2)=0$. Thus, we have t_1 and t_2 satisfying $H_1=\cot 3t_1$ and $H_2=\cot 3t_2$ for $0<t_2<t_1<\frac{\pi}{6}$. For any $t_0\in(t_2,t_1)$, M_{t_0} is an isoparametric hypersurface with mean curvature $H_0=\cot 3t_0\in(H_1,H_2)$, whose squared norm of the second fundamental form S_0 satisfies $\beta(3k,H_0)<S_0<\beta(3k,H_0)+\frac{3k}{18}$, but the conclusion of our main theorem does not hold for M_{t_0}.

Therefore, the condition $|H|\leqslant\gamma(n)$ cannot be removed for $n\in\{6,12,24\}$.