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Large m asymptotics for minimal partitions of the Dirichlet eigenvalue

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Abstract In this paper, we study large m asymptotics of the l^1 minimal m-partition problem for the Dirichlet eigenvalue. For any smooth domain $\Omega \subset \mathbb{R}^n$ such that $|\Omega| = 1$, we prove that the limit $\lim_{m\to\infty} l^1_m(\Omega) = c_0$ exists, and the constant c_0 is independent of the shape of Ω. Here, $l^1_m(\Omega)$ denotes the minimal value of the normalized sum of the first Laplacian eigenvalues for any m-partition of Ω.

Keywords Dirichlet eigenvalue, l^1 minimal partition problem, large m asymptotics

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1 Introduction

Let Ω be a bounded, smooth domain in $\mathbb{R}^n$, and $m > 1$ be a positive integer. We consider the following so-called l^1 minimal partition problem.

Problem 1.1. *Find a partition of Ω into m, mutually disjoint subsets Ω_j, $j = 1, 2, \ldots, m$, such that $\Omega = \bigcup_{j=1}^m \Omega_j$, and it minimizes the l^1 energy functional $\sum_{j=1}^m \lambda_1(\Omega_j)$ among all admissible partitions. Here, $\lambda_1(A)$ denotes the first eigenvalue of Laplacian Δ on A with the zero Dirichlet boundary condition on ∂A.*

The existence of the minimal partition and regularity of free interfaces have been studied by many authors (see [4, 5, 7–10, 13, 15] and the survey articles [2, 11, 12]). In [9], Caffarelli and Lin proved the equivalence between Problem 1.1 and the following problem.

Problem 1.2. *It holds that*

$$\Sigma^m = \left\{ y \in \mathbb{R}^m, \sum_{k\neq l} y_k^2 y_l^2 = 0 \right\}.$$

Find $u \in H_0^1(\Omega, \Sigma^m)$ such that

$$\int_\Omega u_j^2\, dx = 1 \quad \textit{for any } j = 1, \ldots, m$$

and that u minimizes $\int_\Omega |\nabla u|^2\, dx$ among all such maps in $H_0^1(\Omega, \Sigma^m)$.

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Problem 1.2 obviously admits a minimizer $u = (u_1, u_2, \dots, u_m)$. It is proved in [9] that u is locally Lipschitz continuous in Ω (and Lipschitz continuous up to the boundary when $\partial\Omega$ is smooth), and $\Omega_j = \{x \in \Omega : u_j(x) > 0\}$ $(j = 1, \dots, m)$ are open subsets of Ω whose boundaries $\partial\Omega_j$ are smooth away from a relatively closed subset $S \subset \Omega$ of Hausdorff dimension at most $n-2$. Moreover, $\{\Omega_j\}_{j=1}^m$ gives a partition of Ω that minimizes $\sum_{j=1}^m \lambda_1(\Omega_j)$. It is shown later by Alper [1] that the set S is rectifiable and of bounded $(n-2)$-dimensional Hausdorff measure.

In this paper, we are interested in the asymptotic behavior of the minimal partition as $m \to \infty$. Our main theorem is as follows.

Theorem 1.3. *Let Ω be a bounded, smooth domain in $\mathbb{R}^n$ with $|\Omega| = 1$. Then*

$$\lim_{m\to\infty} l_m^1(\Omega) = c_0 \quad \text{for some positive constant } c_0 \text{ independent of } \Omega. \tag{1.1}$$

Here,

$$l_m^1(\Omega) = \frac{\sum_{j=1}^m \lambda_1(\Omega_j)}{m^{1+\frac{2}{n}}},$$
$$\Omega = \bigcup_{j=1}^m \Omega_j \quad \text{is an } l^1\text{-minimal } m\text{-partition.}$$

Remark 1.4. For $\Omega \subset \mathbb{R}^2$, by the hexagonal tiling construction and the Faber-Krahn inequality, one can easily get the following lower bound and upper bound for the constant c_0:

$$\lambda_1(D) \leqslant c_0 \leqslant \lambda_1(H), \tag{1.2}$$

where D is the 2-D unit-area disk and H is the unit-area regular hexagon.

It should be noted, in the above theorem, the smoothness of Ω does not play any role here, and the smoothness assumption is just for convenience. The problem of large m asymptotics was considered first in [9] and they proved that

$$\sum_{j=1}^m \lambda_1(\Omega_j) \simeq m\lambda_m(\Omega),$$

where $\lambda_m(\Omega)$ is the m-th Dirichlet eigenvalue of Ω. They also made a conjecture that the limit $\lim_{m\to\infty} l_m^1(\Omega)$ exists and for the case $\Omega \subset \mathbb{R}^2$, the minimal partitions for large m will be close to a regular hexagon packing pattern and the constant c_0 equals $\lambda_1(H)$. Theorem 1.3 here verifies the first part of the conjecture, while the second part (regular hexagon pattern) remains open though one can very well expect it in a stochastic sense. In recent years some attempts have been made to a related issue. For examples, Bourgain [3] and Steinerberger [14] have improved the lower bound in (1.2) by showing that $l_m^1(\Omega) > \lambda_1(D) + \varepsilon_0$ for some sufficiently small constant ε_0. Their tools are a quantitative Faber-Krahn inequality and some packing properties of disks in $\mathbb{R}^2$. In [6], Bucur et al. studied this so-called "honeycomb conjecture", and they gave a proof under the assumption that every Ω_j $(j = 1, \dots, m)$ is convex and the regular hexagon minimizes λ_1 among all convex hexagons with the same area, which is itself an interesting open problem.

In Section 2, we prove Theorem 1.3. The proof will be concentrated on the case $n = 2$. For $n \geqslant 3$ one can apply the same arguments with only some obvious modifications. We first prove the limit exists for the unit cube, and then we prove the statement for general domain Ω by approximating it using smaller dyadic cubes of the same size. Here for the upper limit, one uses a simple comparison construction. For the lower bound, we use the setting employed in [9], i.e., by considering harmonic maps into singular spaces. A surgery on minimizing sequences of harmonic maps yields a desired lower limit bound.

2 Proof of Theorem 1.3

We first prove the following lemma.

Lemma 2.1. *Let Q be a unit cube in $\mathbb{R}^2$. For any $m > 0$ and $k \geqslant 1$, it holds that*

$$l^1_m(Q) \geqslant l^1_{mk^2}(Q). \tag{2.1}$$

Proof. Let $s = l^1_m(Q)$. By the existence of the l^1-minimal m-partition, there is an m-partition $\{\Omega_j\}_{j=1}^m$ of Q such that $\sum_{j=1}^m \lambda_1(\Omega_j) = m^2 s$. Now we divide Q into k^2 identical cubes $\{Q^i\}_{i=1}^{k^2}$ with edge length $\frac{1}{k}$. In each Q^i, we put a translated and scaled copy of the same m-partition as $\{\Omega_j\}_{j=1}^m$, which is denoted by $\{\Omega_j^i\}_{j=1}^m$. As a result we get an mk^2-partition of Q, and we have

$$l^1_{mk^2}(Q) \leqslant \left(\sum_{1\leqslant i\leqslant k^2} \left(\sum_{1\leqslant j\leqslant m} \lambda_1(\Omega_j^i) \right) \right) \Big/ (mk^2)^2 = s.$$

Here, we have used the degree -2-homogeneity of λ_1 with respect to scalings. $\square$

Proof of Theorem 1.3. **Step 1.** Consider the unit cube Q. We show that there exists c_0 such that

$$\lim_{m\to\infty} l^1_m(Q) = c_0.$$

Define

$$a(Q) = \liminf_{m\to\infty} l^1_m(Q).$$

For any $\varepsilon > 0$, there exists an integer m_ε such that $l^1_{m_\varepsilon}(Q) \leqslant a(Q) + \frac{\varepsilon}{2}$. For any $m \geqslant m_\varepsilon$, there exists $k \in \mathbb{N}$ such that $k^2 m_\varepsilon \leqslant m \leqslant (k+1)^2 m_\varepsilon$. By Lemma 2.1, we have

$$l^1_{(k+1)^2 m_\varepsilon}(Q) \leqslant l^1_{m_\varepsilon}(Q).$$

Let $\{\Omega_j\}_{j=1}^{(k+1)^2 m_\varepsilon}$ be the minimal $(k+1)^2 m_\varepsilon$-partition of Q. By grouping together some of the subdomains Ω_j, we can obtain a new m-partition of Q, denoted by $\{\Omega'_j\}_{j=1}^m$, and then we deduce that

$$l^1_m(Q) \leqslant \frac{\sum_{j=1}^m \lambda_1(\Omega'_j)}{m^2} \leqslant \frac{((k+1)^2 m_\varepsilon)^2}{m^2} l^1_{m_\varepsilon}(Q) \leqslant \left(\frac{k+1}{k}\right)^4 \left(a(Q) + \frac{\varepsilon}{2}\right).$$

Let k_ε be sufficiently large such that

$$\left(\frac{k_\varepsilon+1}{k_\varepsilon}\right)^4 \left(a(Q) + \frac{\varepsilon}{2}\right) \leqslant a(Q) + \varepsilon.$$

Then for any $m \geqslant k_\varepsilon^2 m_\varepsilon$, we have

$$l^1_m(Q) \leqslant a(Q) + \varepsilon,$$

which implies that

$$\limsup_{m\to\infty} l^1_m(Q) \leqslant a(Q).$$

One can deduce from the above proof that

$$\lim_{m\to\infty} l^1_m(Q) = \lim_{m\to\infty} l^1_{(m+o(m))}(Q).$$

Step 2. For any bounded, smooth domain $\Omega \subset \mathbb{R}^2$ such that $|\Omega| = 1$, we prove

$$\limsup_{m\to\infty} l^1_m(\Omega) \leqslant \lim_{m\to\infty} l^1_m(Q) = a(Q).$$

For any $\varepsilon > 0$, there is $k \in \mathbb{N}$, such that

$$\bigcup_{j=1}^k Q_j \subset \Omega \subset \left(\bigcup_{j=1}^k Q_j\right) \cup \left(\bigcup_{i=1}^l Q_{k+i}\right). \tag{2.2}$$

Here, $\{Q_j\}_{j=1}^k$ and $\{Q_{k+i}\}_{i=1}^l$ are smaller dyadic cubes of the same size and $\{Q_{k+i}\}_{i=1}^l$ satisfies

$$\sum_{i=1}^{l}|Q_{k+i}| \leqslant \frac{\varepsilon}{4}.$$

If $k>1$, we let $m=(k-1)n+t$, where $m,n,t\in\mathbb{N}$ and $t<(k-1)$. Then we have

$$\begin{aligned} l_m^1(\Omega) &\leqslant l_m^1\bigg(\bigcup_{j=1}^k Q_j\bigg)\\ &\leqslant \bigg(\frac{(k-1)n^2 l_n^1(Q)}{|Q_j|}+\frac{t^2 l_t^1(Q)}{|Q_j|}\bigg)\Big/ m^2. \end{aligned}$$

Here, the second inequality comes from the construction of the partition that divides each of Q_j ($j=1,\ldots,(k-1)$) into n sub-domains and divides the last cube Q_k into t sub-domains. Let n be sufficiently large or equivalently m sufficiently large. We can guarantee that the value of the last line is less than $a(Q)(1+\varepsilon)$, which leads to that $\limsup_{m\to\infty} l_m^1(\Omega)\leqslant a(Q)$.

Step 3. We are left to prove $\liminf_{m\to\infty} l_m^1(\Omega)\geqslant a(Q)$. Given $\varepsilon>0$, by (2.2), Ω can be approximated by smaller dyadic cubes. Then we have

$$l_m^1(\Omega)\geqslant l_m^1\bigg(\bigg(\bigcup_{j=1}^k Q_j\bigg)\cup\bigg(\bigcup_{i=1}^l Q_{k+i}\bigg)\bigg).$$

It suffices to show that given m large enough,

$$l_m^1\bigg(\bigg(\bigcup_{j=1}^k Q_j\bigg)\cup\bigg(\bigcup_{i=1}^l Q_{k+i}\bigg)\bigg)\geqslant (1-\varepsilon)a(Q). \tag{2.3}$$

Actually, (2.3) is implied by the following Lemma 2.2.

Lemma 2.2. *Let Ω be a domain in $\mathbb{R}^2$ with $|\Omega|=1$. Γ is a straight line that separates Ω into two sub-domains D_1 and D_2 with the areas α and $1-\alpha$, respectively. Assume there exists a constant c such that*

$$\lim_{m\to\infty} l_m^1\bigg(\frac{1}{\sqrt{\alpha}}D_1\bigg)=\lim_{m\to\infty} l_m^1\bigg(\frac{1}{\sqrt{1-\alpha}}D_2\bigg)=c.$$

Then

$$\lim_{m\to\infty} l_m^1(\Omega)=c.$$

Let us assume this lemma and proceed with our proof. Note that $(\bigcup_{j=1}^k Q_j)\cup(\bigcup_{i=1}^l Q_{k+i})$ is the union of $k+l$ small cubes, whose areas added up to $(1+\delta)$ for some $\delta<\frac{\varepsilon}{4}$. By proper scalings and by repetitive applications of Lemma 2.2, we can then get that

$$\lim_{m\to\infty} l_m^1\bigg(\bigg(\bigcup_{j=1}^k Q_j\bigg)\cup\bigg(\bigcup_{i=1}^l Q_{k+i}\bigg)\bigg)=\frac{1}{1+\delta}\lim_{m\to\infty} l_m^1(Q)\geqslant(1-\varepsilon)a(Q),$$

which yields the conclusion (2.3). The proof of the theorem is then completed. □

Proof of Lemma 2.2. Without loss of generality, one assumes $\Gamma=\{x=0\}$ and

$$D_1=\{z=(x,y)\in\Omega: x<0\},\quad D_2=\{z=(x,y)\in\Omega: x>0\}.$$

Note that by the same arguments as in Step 2, we have

$$\limsup_{m\to\infty} l_m^1(\Omega)\leqslant c.$$

It suffices to prove for any $\varepsilon > 0$, there exists m_ε such that if $m \geqslant m_\varepsilon$, then

$$l_m^1(\Omega) \geqslant c(1-\varepsilon). \tag{2.4}$$

In the rest of proof we always fix $\varepsilon > 0$ and we always assume m is large enough (depending on ε that will be specified later). We need to study Problem 1.2, which is the equivalent formulation of the minimal partition Problem 1.1. Let $u = (u_1, \dots, u_m) \in H_0^1(\Omega, \Sigma^m)$ be a minimizer of Problem 1.2. Then $\{\mathrm{supp}(u_j)\}_{j=1}^m$ gives a minimal m-partition of Ω. Denote

$$\Omega_j = \mathrm{supp}(u_j).$$

Take a fixed small number δ (also depending on ε only, which will be determined later). We define the following regions:

$$\begin{aligned}
&S_\delta = \left\{ z = (x,y) \in \Omega : \mathrm{dist}(z, \Gamma) < \frac{\delta}{2} \right\} = \left\{ z = (x,y) \in \Omega,\ |x| < \frac{\delta}{2} \right\},\\
&D_1' = D_1 \backslash S_\delta, \quad D_2' = D_2 \backslash S_\delta.
\end{aligned}$$

Then we classify the sub-domains in the partition $\{\Omega_j\}_{j=1}^m$ according to their intersections with S_δ, D_1' and D_2', i.e.,

$$\begin{aligned}
&A_\delta = \{\Omega_k : \Omega_k \cap D_2' = \emptyset\},\\
&B_\delta = \{\Omega_k : \Omega_k \cap D_1' = \emptyset\},\\
&C_\delta = \{\Omega_k : \Omega_k \cap D_1' \neq \emptyset,\ \Omega_k \cap D_2' \neq \emptyset\}.
\end{aligned}$$

We are mostly interested in sub-domains in C_δ. Take $\Omega_j \in C_\delta$. Define the sub-region of S_δ:

$$S_{(r, r+\frac{\delta}{2})} = \left\{ z \in S_\delta : r < x < r + \frac{\delta}{2} \right\}, \quad r \in \left[-\frac{\delta}{2}, 0 \right].$$

Note that for each r, $S_{(r,r+\frac{\delta}{2})}$ is a region with half width of S_δ. Obviously, there exists $r_j \in [-\frac{\delta}{2}, 0]$ such that

$$\int_{\Omega_j \cap S_{(r_j, r_j+\frac{\delta}{2})}} |u_j^2| \leqslant \frac{1}{2} \int_{\Omega_j} u_j^2. \tag{2.5}$$

Let ξ_j be a smooth cut-off function such that

$$\xi_j(z) \equiv 1 \quad \text{if } z \notin S_{(r_j, r_j+\frac{\delta}{2})}; \quad \xi_j(z) \equiv 0 \quad \text{on } x = r_j + \frac{\delta}{4}; \quad |\nabla \xi_j| \leqslant \frac{8}{\delta}.$$

Claim. If $\frac{\int_{\Omega_j} |\nabla(\xi_j u_j)|^2}{\int_{\Omega_j} |\xi_j u_j|^2} \geqslant (1+\frac{\varepsilon}{5})\lambda_1(\Omega_j)$, then there exists a constant C_1 which depends on ε such that $\lambda_1(\Omega_j) \leqslant C_1(\varepsilon)$.

Proof. We calculate directly

$$\frac{\int_{\Omega_j} |\nabla(\xi_j u_j)|^2}{\int_{\Omega_j} |\xi_j u_j|^2} \geqslant \left(1 + \frac{\varepsilon}{5}\right) \lambda_1(\Omega_j) \tag{2.6a}$$

$$\Rightarrow \int_{\Omega_j} [|\nabla \xi_j|^2 u_j^2 + 2\xi_j u_j \nabla \xi_j \cdot \nabla u_j + \xi_j^2 |\nabla u_j|^2] \geqslant \left(1 + \frac{\varepsilon}{5}\right) \lambda_1(\Omega_j) \int_{\Omega_j} (u_j \xi_j)^2. \tag{2.6b}$$

By integration by parts, we have

$$\int_{\Omega_j} |\nabla u_j|^2 \xi_j^2 = \int_{\Omega_j} \lambda_1(\Omega_j)(u_j \xi_j)^2 - \int_{\Omega_j} 2\xi_j u_j \nabla \xi \cdot \nabla u_j.$$

Thus (2.6b) implies

$$\int_{\Omega_j} |\nabla \xi_j|^2 u_j^2 \geqslant \frac{\varepsilon}{5} \lambda_1(\Omega_j) \int_{\Omega_j} (u_j \xi_j)^2.$$

By the assumption on ξ_j, we conclude that

$$\lambda_1(\Omega_j) \leqslant \frac{640}{\varepsilon\delta^2} =: C_1(\varepsilon). \qquad \square$$

Let D_δ be the subset of C_δ that consists of all these sub-domains that satisfy (2.6a). According to the above claim, for any $\Omega_j \in D_\delta$, $\lambda_1(\Omega_j) \leqslant C_1(\varepsilon)$, and by the well-known Faber-Krahn inequality, there exists a constant $C_2(\varepsilon)$ such that $|\Omega_j| \geqslant C_2(\varepsilon)$. Then we can control the number of sub-domains in D_δ by a constant only depending on ε, but independent of m, i.e., $\#D_\delta \leqslant C_3(\varepsilon)$.

Based on $u, A_\delta, B_\delta, C_\delta$ and D_δ, we can then define modified vector-valued functions v and w such that $\mathrm{supp}(v) \subset D_1' \cup S_\delta$ and $\mathrm{supp}(w) \subset D_2' \cup S_\delta$. We follow the following schemes:

(i) If $\Omega_j \in A_\delta$, then $v_j = u_j$.

(ii) If $\Omega_j \in B_\delta$, then $w_j = u_j$.

(iii) If $\Omega_j \in C_\delta \backslash D_\delta$, then we have by definition

$$\frac{\int_{\Omega_j} |\nabla(\xi_j u_j)|^2}{\int_{\Omega_j} |\xi_j u_j|^2} \leqslant \left(1 + \frac{\varepsilon}{5}\right)\lambda(\Omega_j). \tag{2.7}$$

By noting that $\xi = 0$ on $\{x = r_j + \frac{\delta}{4}\}$, the line $\{x = r_j + \frac{\delta}{4}\}$ divides Ω_j into two sub-domains Ω_j^1 and Ω_j^2, where

$$\Omega_j^1 \subset D_1' \cup S_\delta, \quad \Omega_j^2 \subset D_2' \cup S_\delta.$$

Moreover, we have $u_j\xi_j \,|_{\Omega_j^1} \in H_0^1(\Omega_j^1)$ and $u_j\xi_j \,|_{\Omega_j^2} \in H_0^1(\Omega_j^2)$. We denote

$$\tau_1 := \frac{\int_{\Omega_j^1} |\nabla(\xi_j u_j)|^2}{\int_{\Omega_j^1} |\xi_j u_j|^2}, \quad \tau_2 := \frac{\int_{\Omega_j^2} |\nabla(\xi_j u_j)|^2}{\int_{\Omega_j^2} |\xi_j u_j|^2}.$$

Clearly, (2.7) implies that

$$\min\{\tau_1, \tau_2\} \leqslant \left(1 + \frac{\delta}{5}\right)\lambda(\Omega_j).$$

If $\tau_1 \leqslant \tau_2$, then we let

$$v_j = \frac{\xi_j u_j}{\sqrt{\int_{\Omega_j^1} |\xi_j u_j|^2}} \quad \text{on } \Omega_j^1, \quad v_j = 0 \text{ elsewhere.}$$

Otherwise, let

$$w_j = \frac{\xi_j u_j}{\sqrt{\int_{\Omega_j^2} |\xi_j u_j|^2}} \quad \text{on } \Omega_j^2, \quad w_j = 0 \text{ elsewhere.}$$

We also denote

$$E_\delta = \{\Omega_j^1 : \Omega_j \in C_\delta \backslash D_\delta,\, \tau_1 \leqslant \tau_2\},$$
$$F_\delta = \{\Omega_j^2 : \Omega_j \in C_\delta \backslash D_\delta,\, \tau_1 > \tau_2\}.$$

(iv) Finally, we rearrange the vector of functions v and w such that

$$\mathrm{supp}\, v_j \neq \emptyset, \quad \mathrm{supp}\, v_j \in A_\delta \cup E_\delta \quad \text{for all } j = 1, \ldots, m_1.$$
$$\mathrm{supp}\, w_j \neq \emptyset, \quad \mathrm{supp}\, w_j \in B_\delta \cup F_\delta \quad \text{for all } j = 1, \ldots, m_2.$$

Here, $m_1 = \#A_\delta + \#E_\delta$ and $m_2 = \#B_\delta + \#F_\delta$.

Now we are ready to prove (2.4). One calculates

$$\begin{aligned}&\frac{\sum_{j=1}^m \int_{\Omega_j} |\nabla u_j|^2}{m^2}\\ &\geqslant \frac{\sum_{\Omega_j \in A_\delta} \int_{\Omega_j} |\nabla u_j|^2 + \sum_{\Omega_j \in B_\delta} \int_{\Omega_j} |\nabla u_j|^2 + \sum_{\Omega_j \in C_\delta \backslash D_\delta} \int_{\Omega_j} |\nabla u_j|^2}{m^2}\end{aligned}$$

$$
\begin{aligned}
&\geqslant \frac{1}{m^2}\left(\sum_{\Omega_j\in A_\delta}\int_{\Omega_j}|\nabla u_j|^2+\sum_{\Omega_j\in B_\delta}\int_{\Omega_j}|\nabla u_j|^2+\frac{\sum_{\Omega_j\in E_\delta}\int_{\Omega_j}|\nabla v_j|^2}{1+\varepsilon/5}+\frac{\sum_{\Omega_j\in F_\delta}\int_{\Omega_j}|\nabla w_j|^2}{1+\varepsilon/5}\right)\\
&\geqslant \frac{(\sum_{\Omega_j\in A_\delta}\int|\nabla v_j|^2+\sum_{\Omega_j\in E_\delta}\int|\nabla v_j|^2+\sum_{\Omega_j\in B_\delta}\int|\nabla w_j|^2+\sum_{\Omega_j\in F_\delta}\int|\nabla w_j|^2)}{m^2(1+\varepsilon/5)}.
\end{aligned}
\tag{2.8}
$$

Define

$$
\tilde{D}_1=\bigcup_{\Omega_j\in A_\delta\cup E_\delta}\Omega_j,\quad \tilde{D}_2=\bigcup_{\Omega_j\in B_\delta\cup F_\delta}\Omega_j.
$$

By the construction above we have

$$
\tilde{D}_i\subset D_i'\cup S_\delta\subset\left(1+\frac{\varepsilon}{10}\right)D_i\quad\text{for }\ i=1,2,
$$

where $\delta<\delta(\varepsilon)$ is small enough. Hence we obtain that

$$
\lim_{m\to\infty}l_m^1(\tilde{D}_1)\geqslant\left(1-\frac{\varepsilon}{5}\right)\lim_{m\to\infty}l_m^1(D_1)=\frac{1-\varepsilon/5}{\alpha}c, \tag{2.9}
$$

$$
\lim_{m\to\infty}l_m^1(\tilde{D}_2)\geqslant\left(1-\frac{\varepsilon}{5}\right)\lim_{m\to\infty}l_m^1(D_2)=\frac{1-\varepsilon/5}{1-\alpha}c. \tag{2.10}
$$

Note that by our construction, $v\in H_0^1(\tilde{D}_1,\Sigma^{m_1})$ and $w\in H_0^1(\tilde{D}_2,\Sigma^{m_2})$, $m_1+m_2=m-C_3(\varepsilon)$. We take m sufficiently large such that

$$
\left(\frac{m-C_3(\varepsilon)}{m}\right)^2\geqslant 1-\frac{\varepsilon}{5},\quad l_{m_1}^1(\tilde{D}_1)\geqslant\frac{1-\varepsilon/4}{\alpha}c,\quad l_{m_2}^1(\tilde{D}_2)\geqslant\frac{1-\varepsilon/4}{1-\alpha}c. \tag{2.11}
$$

Here, we have assumed that m_1 and m_2 also go to infinity when m goes to infinity. If the latter is not true, then it is even easier to conclude (2.4), and we shall omit the details to the readers. By combining (2.8)–(2.11), we can deduce that

$$
\begin{aligned}
\frac{1}{m^2}\left(\sum_{j=1}^m\int_{\Omega_j}|\nabla u_j|^2\right)&\geqslant\frac{1}{m^2(1+\varepsilon/5)}(m_1^2l_{m_1}^1(\tilde{D}_1)+m_2^2l_{m_2}^1(\tilde{D}_2))\\
&\geqslant\frac{c(1-\varepsilon/4)}{(1+\varepsilon/5)m^2}\left(\frac{m_1^2}{\alpha}+\frac{m_2^2}{1-\alpha}\right)\\
&\geqslant\frac{1-\varepsilon/4}{1+\varepsilon/5}\left(\frac{m-C_3(\varepsilon)}{m}\right)^2c\\
&\geqslant\frac{(1-\varepsilon/4)(1-\varepsilon/5)}{1+\varepsilon/5}c\geqslant(1-\varepsilon)c.
\end{aligned}
$$

This completes the proof. □

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