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Hankel operators on exponential Bergman spaces

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Abstract We completely describe the boundedness and compactness of Hankel operators with general symbols acting on Bergman spaces with exponential type weights.

Keywords weighted Bergman space, Hankel operator, $\overline{\partial}$-equation

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1 Introduction

Hankel operator acting on Bergman spaces is an important area of research in the theory of operators acting on spaces of analytic functions. Most of the theory of Hankel operators on standard Bergman spaces is well understood, but not so much is known for large Bergman spaces. The function and operator theory acting on large Bergman spaces on the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$ is just developing, and it is our purpose to study big Hankel operators acting on such spaces. For a strictly subharmonic function φ on $\mathbb{D}$ and $0 < p \leqslant \infty$, let L^p_φ consist of those Lebesgue measurable functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{L^p_\varphi} = \left\{ \int_{\mathbb{D}} |f(z)\mathrm{e}^{-\varphi(z)}|^p dA(z) \right\}^{\frac{1}{p}} < \infty, \quad 0 < p < \infty,$$
$$\|f\|_{L^\infty_\varphi} = \sup_{z\in\mathbb{D}} |f(z)|\mathrm{e}^{-\varphi(z)} < \infty, \quad p = \infty,$$

and consider the weighted Bergman space $A^p_\varphi = L^p_\varphi \cap H(\mathbb{D})$. Here, $H(\mathbb{D})$ denotes the set of all holomorphic functions in $\mathbb{D}$ and dA is the Lebesgue area measure on $\mathbb{C}$. We also use L^p to stand for the usual Lebesgue space $L^p(\mathbb{D}, dA)$.

In this paper we are interested in A^p_φ with the weight function $\varphi \in \mathcal{W}_0$ which was first introduced in [10]. To describe $\mathcal{W}_0$ precisely, let C_0 be the family of all continuous functions ρ on $\mathbb{D}$ satisfying $\lim_{|z|\to 1} \rho(z) = 0$. Set

$$\mathcal{L} = \left\{ \rho : \mathbb{D} \to \mathbb{R} : \rho \in C_0, \ \|\rho\|_L = \sup_{z,w\in\mathbb{D}, z\neq w} \frac{|\rho(z) - \rho(w)|}{|z - w|} < \infty \right\},$$

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and let $\mathcal{L}_0$ consist of those $\rho \in \mathcal{L}$ with the property that for each $\varepsilon > 0$ there is a compact subset $E \subset \mathbb{D}$ with

$$|\rho(z) - \rho(w)| \leqslant \varepsilon |z - w|$$

whenever $z, w \in \mathbb{D} \setminus E$. The class $\mathcal{W}_0$ is defined as

$$\mathcal{W}_0 = \left\{ \varphi \in C^2(\mathbb{D}) : \Delta\varphi > 0, \text{ and } \exists\, \rho \in \mathcal{L}_0 \text{ such that } \frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \right\}.$$

Here and afterward, the expression $A \simeq B$ means that there exist two positive constants c_1 and c_2 independent of the functions being considered such that $c_1 A \leqslant B \leqslant c_2 A$.

It is easy to verify that A^p_φ is a Banach space when $1 \leqslant p \leqslant \infty$, and A^2_φ is a Hilbert space. These spaces are also called large Bergman spaces because they usually contain all the standard Bergman spaces. Examples of weighted Bergman spaces with $\varphi \in \mathcal{W}_0$ include exponential Bergman spaces, double exponential weighted Bergman spaces, and also some non-radial weighted Bergman spaces (see [10, 14]). With the Bergman reproducing kernel $K(\cdot, \cdot)$ on A^2_φ one can define the Bergman projection P as

$$P(g)(z) = \int_{\mathbb{D}} g(\xi) K(z, \xi) \mathrm{e}^{-2\varphi(\xi)} dA(\xi).$$

For $1 \leqslant p \leqslant \infty$, P is bounded from L^p_φ to A^p_φ, and $P|_{A^p_\varphi}$, the restriction on A^p_φ, is just the identity operator Id (see [10] for details).

Given some symbol function f, one defines the so-called Hankel operator H_f as

$$H_f(g) = (\mathrm{Id} - P)(fg). \tag{1.1}$$

From [10] we know that

$$\Gamma = \left\{ \sum_{j=1}^{N} a_j K(\cdot, z_j) : N \in \mathbb{N},\, a_j \in \mathbb{C},\, z_j \in \mathbb{D} \text{ for } 1 \leqslant j \leqslant N \right\}$$

is dense in A^p_φ. Therefore, to let H_f make sense on Γ we naturally consider those f in the symbol class $\mathcal{S}$ defined as

$$\mathcal{S} = \{ f \text{ measurable on } \mathbb{D} : fg \in L^1_\varphi \text{ for } g \in \Gamma \}$$

(from [10, Theorem 3.3], $\|K(\cdot, z)\|_{L^\infty_\varphi} < \infty$ so that $P(fg)(z)$ is well defined for $f \in \mathcal{S}$, $g \in \Gamma$ and $z \in \mathbb{D}$). The purpose of this work is, for $1 \leqslant p, q < \infty$, to characterize those $f \in \mathcal{S}$ such that H_f is bounded (or compact) as an operator acting from A^p_φ to L^q_φ. The descriptions obtained are presented in Section 4.

As in [10], we write $\mathcal{BDK}$ (Borichev-Dhuez-Kellay) to be the weight class introduced by Borichev et al. [3]. We know $\mathcal{BDK} \subset \mathcal{W}_0$ and $\mathcal{W}_0 \setminus \mathcal{BDK} \neq \emptyset$. The Bergman space A^p_φ with $\varphi \in \mathcal{BDK}$ has been studied in [2, 3, 6, 7, 9, 14, 15].

Given Banach spaces X and Y, and some linear operator from X to Y, we use $\|\cdot\|_X$ and $\|T\|_{X\to Y}$ respectively to stand for the norm on X, and the operator norm of T. Throughout this paper, we use C to denote positive constants whose values may change from line to line, but do not depend on functions being considered.

2 Some preliminaries

We are going to present some basic conclusions that will be used in the following sections. Let $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$. We define a distance $d_\rho(z, w)$ on $\mathbb{D}$ as

$$d_\rho(z, w) = \inf_\gamma \int_0^1 |\gamma'(t)| \frac{dt}{\rho(\gamma(t))},$$

where the infimum is taken over all piecewise C^1 curves $\gamma : [0,1] \to \mathbb{D}$ with $\gamma(0) = z$ and $\gamma(1) = w$. It is mentioned in [5] that $d_\rho(\cdot,\cdot)$ is equivalent to the Bergman distance $\beta_\varphi(\cdot,\cdot)$ induced by the Bergman metric

$$\frac{1}{2}\frac{\partial^2 \log K(z,z)}{\partial z \partial \overline{z}} dz \otimes d\overline{z}.$$

The estimates on the Bergman kernel play an important role in our analysis. The following lemma comes from [10].

Lemma 2.1. *Let $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$. There are positive constants C_1, C_2, σ and d such that*

$$|K(z,w)| \leqslant C_1 \frac{\mathrm{e}^{\varphi(z)+\varphi(w)}}{\rho(z)\rho(w)} \mathrm{e}^{-\sigma d_\rho(z,w)} \quad \textit{for } z,w \in \mathbb{D}$$

and

$$|K(z,w)| \geqslant C_2 \frac{\mathrm{e}^{\varphi(z)}\mathrm{e}^{\varphi(w)}}{\rho(z)\rho(w)} \quad \textit{for } d_\rho(z,w) \leqslant d.$$

For $K_z(\cdot) = K(\cdot, z) \in H(\mathbb{D})$ and $0 < p \leqslant \infty$, with Lemma 2.1 and an elementary calculation as that of [10, Corollary 3.2] we obtain

$$\|K_z\|_{L^p_\varphi} \simeq \mathrm{e}^{\varphi(z)} \rho(z)^{\frac{2}{p}-2}. \tag{2.1}$$

Write $k_{z,p} = \frac{K_z}{\|K_z\|_{L^p_\varphi}}$ to denote the normalized reproducing kernels in A^p_φ.

For $z \in \mathbb{D}$ and $r > 0$, set

$$D(z,r) = \{w : |w - z| < r\}$$

to be the Euclidean disc with the center z and the radius r. Write

$$B_\rho(z,r) = \{w \in \mathbb{D} : d_\rho(w,z) < r\} \quad \text{and} \quad D^r(z) = D(z, r\rho(z)).$$

The following lemma is from [10].

Lemma 2.2. *Let $\rho \in \mathcal{L}$ be positive. Then there exists $\alpha > 0$ with the following properties*:

(i) *There exist constants C_1 and C_2 such that*

$$C_1\rho(w) \leqslant \rho(z) \leqslant C_2\rho(w) \tag{2.2}$$

for $z \in \mathbb{D}$ and $w \in D^\alpha(z)$.

(ii) *There exists a constant $B > 0$ such that*

$$D^r(z) \subseteq D^{Br}(w), \quad D^r(w) \subseteq D^{Br}(z) \tag{2.3}$$

for $w \in D^r(z)$ and $0 < r \leqslant \alpha$.

(iii) *There exist positive constants c_1 and c_2 such that*

$$B_\rho(z, c_1 r) \subseteq D^r(z) \subseteq B_\rho(z, c_2 r) \tag{2.4}$$

for $z \in \mathbb{D}$ and $0 < r \leqslant \alpha$.

Moreover, if α is small enough, we can take $C_1 = 1/2$, $C_2 = 2$ in (i) *and $B = 4$ in* (ii).

For our analysis we need a covering lemma which is almost identical to [8, Lemma 3.1].

Lemma 2.3. *Let $\rho \in \mathcal{L}$ be positive. There are positive constants α and s, depending only on $\|\rho\|_L$, such that for $0 < r \leqslant \alpha$ there exists a sequence $\{z_j\}_{j=1}^\infty \subset \mathbb{D}$ satisfying*

(i) $\mathbb{D} = \bigcup_{j\geqslant 1} D^r(z_j)$;

(ii) $D^{sr}(z_j) \cap D^{sr}(z_m) = \emptyset$ *for $m \neq j$;*

(iii) $\{D^{2\alpha}(z_j)\}_{j=1}^\infty$ *is a covering of $\mathbb{D}$ of finite multiplicity.*

A sequence $\{z_j\}_{j=1}^{\infty}$ satisfying Lemmas 2.3(i)–2.3(iii) will be called a (ρ, r)-lattice. Given some (ρ, r)-lattice $\{z_j\}_{j=1}^{\infty}$, by Lemma 2.3(iii) we have some integer N so that

$$1 \leqslant \sum_{j=1}^{\infty} \chi_{D^{Br}(z_j)}(z) \leqslant N \quad \text{for } z \in \mathbb{D}. \tag{2.5}$$

Here and afterward, χ_E is the characteristic function of a subset E of $\mathbb{D}$. In what follows we always take $\alpha > 0$ as that in Lemmas 2.2 and 2.3. The next lemma has already been obtained for $\varphi \in \mathcal{BDK}$ in Arroussi's dissertation [1].

Lemma 2.4. *Let $\varphi \in \mathcal{W}_0$, $0 < p \leqslant \infty$, and let $\{z_j\}_{j=1}^{\infty}$ be some (ρ, r)-lattice with $0 < r \leqslant \alpha$. Then for $\lambda = \{\lambda_j\}_{j=1}^{\infty} \in \ell^p$, we have $\sum_{j=1}^{\infty} \lambda_j k_{z_j,p} \in A^p_\varphi$ with the norm estimate*

$$\Big\| \sum_{j=1}^{\infty} \lambda_j k_{z_j,p} \Big\|_{L^p_\varphi} \leqslant C \|\lambda\|_{\ell^p}. \tag{2.6}$$

Proof. We treat the case where $1 \leqslant p \leqslant \infty$ first. Let q be the conjugate exponent of p. For $f \in H(\mathbb{D})$, by [10, Lemma 3.3] we have

$$|f(z)e^{-\varphi(z)}|^p \leqslant \frac{C}{\rho(z)^2} \int_{D^r(z)} |f(w)e^{-\varphi(w)}|^p dA(w), \quad z \in \mathbb{D}. \tag{2.7}$$

Hence,

$$\begin{aligned}\sum_{j=1}^{\infty} |K_z(z_j)e^{-\varphi(z_j)}|^q \rho(z_j)^2 &\leqslant C \sum_{j=1}^{\infty} \int_{D^r(z_j)} |K_z(\xi)e^{-\varphi(\xi)}|^q dA(\xi) \\ &\leqslant C\|K_z\|^q_{L^q_\varphi}.\end{aligned}$$

Then, for each N, Hölder's inequality implies

$$\begin{aligned}\sum_{j=1}^{N} |\lambda_j k_{z_j,p}(z)| &\leqslant \bigg(\sum_{j=1}^{N} |\lambda_j|^p\bigg)^{\frac{1}{p}} \bigg(\sum_{j=1}^{N} |k_{z_j,p}(z)|^q\bigg)^{\frac{1}{q}} \\ &\leqslant C\|\lambda\|_{\ell^p} \bigg(\sum_{j=1}^{\infty} |K_z(z_j)e^{-\varphi(z_j)}|^q \rho(z_j)^2\bigg)^{\frac{1}{q}} \\ &\leqslant C\|\lambda\|_{\ell^p} \|K_z\|_{L^q_\varphi} < \infty.\end{aligned}$$

This implies that $\sum_{j=1}^{\infty} \lambda_j k_{z_j,p}$ converges uniformly on compact subsets of $\mathbb{D}$. Furthermore, for any $g \in A^q_\varphi$,

$$\begin{aligned}\sum_{j=1}^{\infty} |\langle \lambda_j k_{z_j,p}, g\rangle_{L^2_\varphi}| &= \sum_{j=1} \frac{|\lambda_j \overline{g(z_j)}|}{\|K_{z_j}\|_{L^p_\varphi}} \\ &\leqslant C \sum_{j=1}^{\infty} |\lambda_j| |g(z_j)e^{-\varphi(z_j)}| \rho(z_j)^{2-\frac{2}{p}} \\ &\leqslant C\|\lambda\|_{\ell^p} \bigg(\sum_{j=1}^{\infty} \int_{D^r(z_j)} |g(\xi)e^{-\varphi(\xi)}|^q dA(\xi)\bigg)^{\frac{1}{q}} \\ &\leqslant C\|\lambda\|_{\ell^p} \|g\|_{L^q_\varphi}.\end{aligned}$$

Therefore,

$$\bigg| \bigg\langle \sum_{j=1}^{\infty} \lambda_j k_{z_j,p}, g \bigg\rangle_{L^2_\varphi} \bigg| \leqslant \sum_{j=1}^{\infty} |\langle \lambda_j k_{z_j,p}, g\rangle_{L^2_\varphi}| \leqslant C\|\lambda\|_{\ell^p} \|g\|_{L^q_\varphi}.$$

[10, Theorem 4.3] tells us that the dual of A^p_φ is A^q_φ for $1 \leqslant p < \infty$ and the predual of A^∞_φ is A^1_φ. From these we obtain (2.6) for $1 \leqslant p \leqslant \infty$.

For $0 < p \leqslant 1$, by $(a+b)^p \leqslant a^p + b^p$ for $a, b > 0$ we have

$$\left\| \sum_{j=1}^N \lambda_j k_{z_j,p} \right\|^p_{L^p_\varphi} \leqslant \sum_{j=1}^\infty |\lambda_j|^p \|k_{z_j,p}\|^p_{L^p_\varphi} = \|\lambda\|^p_{\ell^p}.$$

This completes the proof. □

In our analysis, we need to use the notion of Carleson measures. Here is the definition.

Definition 2.5. Suppose μ is a positive Borel measure on $\mathbb{D}$ and $0 < p, q < \infty$. If the embedding $\mathrm{Id} : A^p_\varphi \to L^q(\mathbb{D}, \mathrm{e}^{-q\varphi} d\mu)$ is continuous (or compact) then μ is said to be a q-Carleson measure (or a vanishing q-Carleson measure) for A^p_φ.

As on the classical Bergman spaces we are going to use $\widehat{\mu}_r$ to characterize Carleson measures shown in the following proposition. For $\varphi \in \mathcal{BDK}$, the weight class introduced in [3], all conclusions in Proposition 2.6 except the estimate (2.9) were represented as [14, Theorem 1] (although it is given there in a different form). Fortunately, the proof of that in [14] works well in the present setting with only one adjustment that the test function $F_{a,n,p}(z)$ there should be replaced by

$$F_a(z) = k_{a,\infty}(z) \simeq \rho(a)^2 K_a(z) \mathrm{e}^{-\varphi(a)},$$

because $F_{a,n,p}(z)$ is available only when $\varphi \in \mathcal{BDK}$ (particulary, φ must be radial) (see [3, 14]).

Given μ as above and $0 < r \leqslant \alpha$, set

$$\widehat{\mu}_r(z) = \frac{\mu(D^r(z))}{|D^r(z)|},$$

where $|D^r(z)|$ denotes the area measure of $D^r(z)$. Notice that $|D^r(z)| \simeq \rho(z)^2$.

Proposition 2.6. *Let μ be a positive Borel measure on $\mathbb{D}$.*

(i) *For $0 < p \leqslant q < \infty$, μ is a q-Carleson measure for A^p_φ if and only if*

$$\sup_{z \in \mathbb{D}} \widehat{\mu}_r(z) \rho(z)^{2(1-\frac{q}{p})} < \infty$$

for some (or any) $r \in (0, \alpha]$. In addition, μ is a vanishing q-Carleson measure for A^p_φ if and only if

$$\lim_{|z| \to 1} \widehat{\mu}_r(z) \rho(z)^{2(1-\frac{q}{p})} = 0$$

for some (or any) $r \in (0, \alpha]$.

(ii) *For $0 < q < p < \infty$, μ is a q-Carleson measure for A^p_φ if and only if μ is a vanishing q-Carleson measure for A^p_φ if and only if*

$$\widehat{\mu}_r \in L^{\frac{p}{p-q}}$$

for some (or any) $r \in (0, \alpha]$.

When μ is a q-Carleson measure for A^p_φ, it holds that

$$\|\mathrm{Id}\|_{A^p_\varphi \to L^q(\mathbb{D}, \mathrm{e}^{-q\varphi} d\mu)} \simeq \|(\widehat{\mu}_r)^{\frac{1}{q}} \rho(z)^{2(\frac{1}{q}-\frac{1}{p})}\|_{L^\infty} \quad \text{if } 0 < p \leqslant q < \infty \tag{2.8}$$

and

$$\|\mathrm{Id}\|_{A^p_\varphi \to L^q(\mathbb{D}, \mathrm{e}^{-q\varphi} d\mu)} \simeq \|(\widehat{\mu}_r)^{\frac{1}{q}}\|_{L^{\frac{pq}{p-q}}} \quad \text{if } 0 < q < p < \infty. \tag{2.9}$$

Proof. We only present the proof of the estimate (2.9). For this purpose we first prove

$$\|(\widehat{\mu}_r)^{\frac{1}{q}}\|_{L^{\frac{pq}{p-q}}} \leqslant C \|\mathrm{Id}\|_{A^p_\varphi \to L^q(\mathbb{D}, \mathrm{e}^{-q\varphi} d\mu)}. \tag{2.10}$$

As in [14] we use an argument of Luecking [13]. Let $\{z_j\}_{j=1}^{\infty}$ be some (ρ,r)-lattice, and take $\{\phi_j\}_{j=1}^{\infty}$ to be a sequence of Rademacher functions on $[0,1]$. For $\lambda=\{\lambda_j\}_{j=1}^{\infty}\in\ell^p$ consider the function G_t defined as

$$G_t(z)=\sum_{j=1}^{\infty}\lambda_j\phi_j(t)k_{z_j,p}(z).$$

From Lemma 2.4 we know $\|G_t\|_{A^p_\varphi}\leqslant C\|\lambda\|_{\ell^p}$. If μ is a q-Carleson measure for A^p_φ, then

$$\int_{\mathbb{D}}|G_t(z)|^q\mathrm{e}^{-q\varphi(z)}d\mu(z)\leqslant\|\mathrm{Id}\|^q_{A^p_\varphi\to L^q(\mathbb{D},\mathrm{e}^{-q\varphi}d\mu)}\|\lambda\|^q_{\ell^p}.$$

Integrating with respect to t from 0 to 1, applying Fubini's theorem, and invoking Khintchine's inequality we obtain

$$\int_{\mathbb{D}}\left(\sum_{j=1}^{\infty}|\lambda_j|^2|k_{z_j,p}(z)|^2\right)^{\frac{q}{2}}\mathrm{e}^{-q\varphi(z)}d\mu(z)\leqslant C\|\mathrm{Id}\|^q_{A^p_\varphi\to L^q(\mathbb{D},\mathrm{e}^{-q\varphi}d\mu)}\|\lambda\|^q_{\ell^p}.$$

On the other hand, by Lemmas 2.1–2.3 and (2.7), one gets

$$\begin{aligned}&\int_{\mathbb{D}}\left(\sum_{j=1}^{\infty}|\lambda_j|^2|k_{z_j,p}(z)|^2\right)^{\frac{q}{2}}\mathrm{e}^{-q\varphi(z)}d\mu(z)\\&\quad\geqslant C\sum_{k=1}^{\infty}\int_{D^r(z_k)}\left(\sum_{j=1}^{\infty}|\lambda_j|^2|k_{z_j,p}(z)|^2\right)^{\frac{q}{2}}\mathrm{e}^{-q\varphi(z)}d\mu(z)\\&\quad\geqslant C\sum_{k=1}^{\infty}\int_{D^r(z_k)}|\lambda_k|^q|k_{z_k,p}(z)|^q\mathrm{e}^{-q\varphi(z)}d\mu(z)\\&\quad\geqslant C\sum_{k=1}^{\infty}|\lambda_k|^q\rho(z_k)^{2-\frac{2q}{p}}\widehat{\mu}_r(z_k).\end{aligned}$$

Therefore,

$$\sum_{k=1}^{\infty}|\lambda_j|^q(\rho(z_k)^{2-\frac{2q}{p}}\widehat{\mu}_r(z_k))\leqslant C\|\mathrm{Id}\|^q_{A^p_\varphi\to L^q(\mathbb{D},\mathrm{e}^{-q\varphi}d\mu)}\|\lambda\|^q_{\ell^p}.$$

By the duality between $\ell^{p/q}$ and $\ell^{p/(p-q)}$ we have

$$\left(\sum_{k=1}^{\infty}\rho(z_k)^2\widehat{\mu}_r(z_k)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\leqslant C\|\mathrm{Id}\|^q_{A^p_\varphi\to L^q(\mathbb{D},\mathrm{e}^{-q\varphi}d\mu)}.\tag{2.11}$$

Meanwhile, it is easy to verify that for $z\in D^r(z_k)$,

$$\rho(z_k)^2\widehat{\mu}_r(z)^{\frac{p}{p-q}}\leqslant C\sum_{j:D^r(z_j)\cap D^r(z_k)\neq\emptyset}\rho(z_j)^2\widehat{\mu}_r(z_j)^{\frac{p}{p-q}}.$$

Therefore,

$$\|\widehat{\mu}_r\|^{\frac{p}{p-q}}_{L^{\frac{p}{p-q}}}\leqslant\sum_{k=1}^{\infty}\int_{D^r(z_k)}\widehat{\mu}_r(z)^{\frac{p}{p-q}}dA(z)\leqslant C\sum_{j=1}^{\infty}\rho(z_j)^2\widehat{\mu}_r(z_j)^{\frac{p}{p-q}}.$$

This and (2.11) imply (2.10).

To prove the other direction, for $f \in H(\mathbb{D})$, applying (2.7) and Hölder's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|^q \mathrm{e}^{-q\varphi(z)} d\mu(z) &\leqslant \sum_{j=1}^{\infty} \int_{D^r(z_j)} |f(z)|^q \mathrm{e}^{-q\varphi(z)} d\mu(z)\\ &\leqslant C \sum_{j=1}^{\infty} \widehat{\mu}_r(z_j)\rho(z_j)^2 \sup_{\xi \in D^r(z_j)} |f(\xi)|^q \mathrm{e}^{-q\varphi(\xi)}\\ &\leqslant C\bigg(\sum_{j=1}^{\infty} \widehat{\mu}_r(z_j)^{\frac{p}{p-q}} \rho(z_j)^2\bigg)^{\frac{p-q}{p}} \bigg(\sum_{j=1}^{\infty} \rho(z_j)^2 \sup_{\xi \in D^r(z_j)} |f(\xi)|^p \mathrm{e}^{-p\varphi(\xi)}\bigg)^{\frac{q}{p}}\\ &\leqslant C\bigg(\sum_{j=1}^{\infty} \widehat{\mu}_r(z_j)^{\frac{p}{p-q}} \rho(z_j)^2\bigg)^{\frac{p-q}{p}} \bigg(\sum_{j=1}^{\infty} \int_{D^{2r}(z_j)} |f(\zeta)|^p \mathrm{e}^{-p\varphi(\zeta)} dA(\zeta)\bigg)^{\frac{q}{p}}\\ &\leqslant C\|\widehat{\mu}_r\|_{L^{\frac{p}{p-q}}} \|f\|^q_{L^p_\varphi}.\end{aligned}$$

This means

$$\|\mathrm{Id}\|_{A^p_\varphi \to L^q(\mathbb{D},\, \mathrm{e}^{-q\varphi} d\mu)} \leqslant C\|\widehat{\mu}_r\|^{\frac{1}{q}}_{L^{\frac{p}{p-q}}} = C\|(\widehat{\mu}_r)^{\frac{1}{q}}\|_{L^{\frac{pq}{p-q}}}.$$

From this and (2.10) we obtain (2.9). □

3 Some $\overline{\partial}$-estimates

By Lemmas 2.1 and 2.2(iii), we have some $\alpha > 0$ such that $K_z(\xi) = K(\xi, z)$ does not vanish for $\xi \in D^\alpha(z)$. Given any $r \in (0, \alpha/3]$ and a (ρ, r)-lattice $\{z_j\}_{j=1}^\infty$, let $\{\psi_j\}_{j=1}^\infty$ be some partition of the unity subordinate to the covering $\{D^r(z_j)\}_{j=1}^\infty$. Precisely,

$$\psi_j \in C^\infty(\mathbb{D}), \quad \mathrm{Supp}\,\psi_j \subset D^r(z_j) \quad \text{and} \quad \psi_j \geqslant 0, \quad \sum_{j=1}^{\infty} \psi_j = 1.$$

Set

$$G(z, \xi) = \frac{1}{(\xi - z)\rho(\xi)} \sum_{j=1}^{\infty} \frac{K_{z_j}(z)\psi_j(\xi)}{K_{z_j}(\xi)}.$$

Define an integral operator T as

$$T(f)(z) = \int_{\mathbb{D}} G(z, \xi) f(\xi) dA(\xi).$$

Lemma 3.1. *Let $\varphi \in \mathcal{W}_0$ and $1 \leqslant p \leqslant \infty$. Then T is a bounded linear operator on L^p_φ.*

Proof. We use interpolation to prove this lemma. By (2.7) and Lemmas 2.1 and 2.2 we have

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{|K_{z_j}(z)|\psi_j(\xi)}{|K_{z_j}(\xi)|} &\simeq \rho(\xi)^2 \mathrm{e}^{-\varphi(z_j)-\varphi(\xi)} \sum_{j \in \{k: \xi \in D^r(z_k)\}} |K_z(z_j)|\psi_j(\xi)\\ &\leqslant C\mathrm{e}^{-\varphi(\xi)} \int_{D^{2r}(\xi)} |K_z(\zeta)| \mathrm{e}^{-\varphi(\zeta)} dA(\zeta).\end{aligned}$$

Write

$$Q(z, \xi) = \frac{\mathrm{e}^{-\varphi(\xi)}}{|\xi - z|\rho(\xi)} \int_{D^{2r}(\xi)} |K_\zeta(z)| \mathrm{e}^{-\varphi(\zeta)} dA(\zeta). \tag{3.1}$$

We have

$$|G(z, \xi)| \leqslant C\, Q(z, \xi). \tag{3.2}$$

For f measurable on $\mathbb{D}$, set

$$T_1(f)(z) = \int_{D^r(z)} Q(z,\xi) f(\xi) dA(\xi)$$

and

$$T_2(f)(z) = \int_{\mathbb{D}\setminus D^r(z)} Q(z,\xi) f(\xi) dA(\xi).$$

To prove the conclusion of the lemma, from (3.2) we need only to prove that both T_1 and T_2 are bounded on L^p_φ. For T_1, by Lemma 2.2, we have

$$\begin{aligned}\|T_1(f)\|_{L^1_\varphi} &\leqslant \int_{\mathbb{D}} \bigg(\int_{\mathbb{D}} \chi_{D^r(z)}(\xi) Q(z,\xi) |f(\xi)| dA(\xi) \bigg) \mathrm{e}^{-\varphi(z)} dA(z) \\ &= \int_{\mathbb{D}} |f(\xi)| \bigg(\int_{\mathbb{D}} \chi_{D^r(z)}(\xi) Q(z,\xi) \mathrm{e}^{-\varphi(z)} dA(z) \bigg) dA(\xi) \\ &\leqslant \int_{\mathbb{D}} |f(\xi)| \bigg(\int_{D^{2r}(\xi)} Q(z,\xi) \mathrm{e}^{-\varphi(z)} dA(z) \bigg) dA(\xi).\end{aligned}$$

Putting the expression of $Q(z,\xi)$ inside and using (2.1), we obtain

$$\begin{aligned}\|T_1(f)\|_{L^1_\varphi} &\leqslant \int_{\mathbb{D}} |f(\xi)| \mathrm{e}^{-\varphi(\xi)} \bigg(\int_{D^{2r}(\xi)} \frac{\mathrm{e}^{-\varphi(z)}}{|\xi - z|\rho(\xi)} \int_{D^{2r}(\xi)} |K_\zeta(z)| \mathrm{e}^{-\varphi(\zeta)} dA(\zeta) dA(z) \bigg) dA(\xi) \\ &\leqslant \int_{\mathbb{D}} |f(\xi)| \mathrm{e}^{-\varphi(\xi)} \bigg(\int_{D^{2r}(\xi)} \frac{\mathrm{e}^{-\varphi(z)}}{|\xi - z|\rho(\xi)} \|K_z\|_{L^1_\varphi} \, dA(z) \bigg) dA(\xi) \\ &\leqslant C \int_{\mathbb{D}} |f(\xi)| \mathrm{e}^{-\varphi(\xi)} dA(\xi) \int_{D^{2r}(\xi)} \frac{1}{|\xi - z|\rho(\xi)} dA(z).\end{aligned}$$

By using polar coordinates, it is easy to see that

$$\int_{D^{2r}(\xi)} \frac{1}{|\xi - z|} dA(z) \leqslant C\rho(\xi),$$

so that we finally obtain

$$\|T_1(f)\|_{L^1_\varphi} \leqslant C \int_{\mathbb{D}} |f(\xi)| \mathrm{e}^{-\varphi(\xi)} dA(\xi) = C\|f\|_{L^1_\varphi}$$

proving that T_1 is bounded on L^1_φ. Similarly,

$$\begin{aligned}\|T_1(f)\|_{L^\infty_\varphi} &= \sup_{z\in\mathbb{D}} \mathrm{e}^{-\varphi(z)} \int_{D^r(z)} Q(z,\xi) |f(\xi)| dA(\xi) \\ &\leqslant C\|f\|_{L^\infty_\varphi} \sup_{z\in\mathbb{D}} \int_{D^r(z)} \frac{1}{|\xi - z|\rho(\xi)} \int_{D^{2r}(\xi)} |K_\zeta(z)| \mathrm{e}^{-\varphi(z)-\varphi(\zeta)} dA(\zeta) \, dA(\xi) \\ &\leqslant C\|f\|_{L^\infty_\varphi} \sup_{z\in\mathbb{D}} \int_{D^r(z)} \frac{1}{|\xi - z|\rho(\xi)} dA(\xi) \\ &\leqslant C\|f\|_{L^\infty_\varphi}.\end{aligned}$$

Set M_{e^φ} to be the multiplier that $M_{\mathrm{e}^\varphi}(f) = f\mathrm{e}^\varphi$. It is easy to see that M_{e^φ} is an isometry from L^p to L^p_φ with the inverse $M_{\mathrm{e}^{-\varphi}}$. Therefore, $M_{\mathrm{e}^{-\varphi}} T_1 M_{\mathrm{e}^\varphi}$ is bounded both on L^1 and L^∞. By interpolation, $M_{\mathrm{e}^{-\varphi}} T_1 M_{\mathrm{e}^\varphi}$ is bounded on L^p which implies T_1 is bounded on L^p_φ.

For T_2, applying Lemma 2.1, we have

$$\begin{aligned}|T_2 f(z)| &\leqslant \int_{\mathbb{D}\setminus D^r(z)} \frac{|f(\xi)| \mathrm{e}^{-\varphi(\xi)}}{|\xi - z|\rho(\xi)} \bigg(\int_{D^{2r}(\xi)} |K(\zeta, z)| \mathrm{e}^{-\varphi(\zeta)} dA(\zeta) \bigg) dA(\xi) \\ &\leqslant C \, \frac{\mathrm{e}^{\varphi(z)}}{\rho(z)} \int_{\mathbb{D}\setminus D^r(z)} \frac{|f(\xi)| \mathrm{e}^{-\varphi(\xi)}}{|\xi - z|\rho(\xi)} \bigg(\int_{D^{2r}(\xi)} \frac{\mathrm{e}^{-\sigma d_\rho(\zeta,z)} dA(\zeta)}{\rho(\zeta)} \bigg) dA(\xi)\end{aligned}$$

$$\leqslant C\frac{e^{\varphi(z)}}{\rho(z)^2}\int_{\mathbb{D}\setminus D^r(z)}\frac{|f(\xi)|e^{-\varphi(\xi)}}{\rho(\xi)^2}\bigg(\int_{D^{2r}(\xi)}e^{-\sigma d_\rho(\zeta,z)}dA(\zeta)\bigg)dA(\xi).$$

On the other hand, $d_\rho(\cdot,\cdot)$ is a distance on $\mathbb{D}$. From Lemma 2.2(iii), there is some constant C such that for $\zeta\in D^{2r}(\xi)$,

$$d_\rho(\xi,z)\leqslant d_\rho(\xi,\zeta)+d_\rho(\zeta,z)\leqslant C+d_\rho(\zeta,z).$$

Thus, for $\zeta\in D^{2r}(\xi)$, we have

$$e^{-\sigma d_\rho(\zeta,z)}\leqslant C\,e^{-\sigma d_\rho(\xi,z)}.$$

It follows that

$$|T_2f(z)|\leqslant C\,\frac{e^{\varphi(z)}}{\rho(z)^2}\int_{\mathbb{D}\setminus D^r(z)}|f(\xi)|e^{-\varphi(\xi)}e^{-\sigma d_\rho(\xi,z)}dA(\xi).$$

With this estimate and [10, Corollary 3.1] we obtain

$$\begin{aligned}\|T_2f\|_{L^1_\varphi}&\leqslant C\int_{\mathbb{D}}\bigg(\int_{\mathbb{D}}|f(\xi)|e^{-\varphi(\xi)}e^{-\sigma d_\rho(\xi,z)}dA(\xi)\bigg)\frac{dA(z)}{\rho(z)^2}\\&=C\int_{\mathbb{D}}|f(\xi)|e^{-\varphi(\xi)}\bigg(\int_{\mathbb{D}}\frac{e^{-\sigma d_\rho(\xi,z)}}{\rho(z)^2}dA(z)\bigg)dA(\xi)\\&\leqslant C\|f\|_{L^1_\varphi}.\end{aligned}$$

Similarly, for $p=\infty$ we have

$$\begin{aligned}\|T_2f\|_{L^\infty_\varphi}&\leqslant\sup_{z\in\mathbb{D}}\frac{1}{\rho(z)^2}\int_{\mathbb{D}\setminus D^r(z)}|f(\xi)|e^{-\varphi(\xi)}e^{-\sigma d_\rho(\xi,z)}dA(\xi).\\&\leqslant C\|f\|_{L^\infty_\varphi}\sup_{z\in\mathbb{D}}\int_{\mathbb{D}}\frac{e^{-\sigma d_\rho(\xi,z)}}{\rho(z)^2}dA(\xi)\\&\leqslant C\|f\|_{L^\infty_\varphi}.\end{aligned}$$

With the same approach for T_1, by interpolation we know that T_2 is bounded on L^p_φ as well. □

Set C_c^∞ to be the family of all C^∞ functions with compact support in $\mathbb{D}$. Given f Lebesgue measurable on $\mathbb{D}$, for $z=x+iy$ one can define the weak derivative $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ (see [4]). Set

$$\frac{\partial f}{\partial z}=\frac{1}{2}\bigg\{\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\bigg\}$$

and

$$\frac{\partial f}{\partial \overline{z}}=\frac{1}{2}\bigg\{\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\bigg\}.$$

Since we deal with functions of one complex variable, we can use $\overline{\partial}f$ to stand for $\frac{\partial f}{\partial\overline{z}}$ for short.

Theorem 3.2. *Let $\varphi\in\mathcal{W}_0$. Given f measurable on $\mathbb{D}$ such that $\rho f\in L^1_\varphi$, set*

$$u(z)=\sum_{j=1}^\infty K_{z_j}(z)\int_{\mathbb{D}}\frac{\psi_j(\xi)}{(\xi-z)K_{z_j}(\xi)}f(\xi)dA(\xi).\tag{3.3}$$

Then u solves the equation $\overline{\partial}u=f$ weakly in $\mathbb{D}$. Furthermore, for $1\leqslant p<\infty$ there is some constant $C>0$ such that

$$\|u\|_{L^p_\varphi}\leqslant C\|\rho f\|_{L^p_\varphi}.\tag{3.4}$$

Proof. For a function f with $\rho f \in L^p_\varphi$, one has $u(z) = T(f\rho)(z)$. Then Lemma 3.1 implies $\|u\|_{L^p_\varphi} \leqslant C\|f\rho\|_{L^p_\varphi}$, which gives (3.4).

For $f \in C^1(\overline{\mathbb{D}})$, the Cauchy-Pompeiu formula tells us that (see [4, Theorem 2.1.2])

$$\frac{\partial}{\partial \overline{z}} \int_{\mathbb{D}} \frac{f(\xi)}{\xi - z} dA(\xi) = f(z) \quad \text{for } z \in \mathbb{D}. \tag{3.5}$$

Then for $\phi \in C_c^\infty(\mathbb{D})$ and $f \in L^1_{\text{loc}}$, (3.5) and the fact that $K_{z_j} \in H(\mathbb{D})$ imply

$$\left\langle K_{z_j}(\cdot) \int_{\mathbb{D}} \frac{\psi_j(\xi)}{(\xi - \cdot)K_{z_j}(\xi)} f(\xi) dA(\xi), \frac{\partial \phi}{\partial z} \right\rangle_{L^2} = -\langle f\psi_j, \phi\rangle_{L^2}.$$

Set

$$U(z) = \sum_{j=1}^\infty |K_{z_j}(z)| \int_{\mathbb{D}} \frac{\psi_j(\xi)}{|(\xi - z)K_{z_j}(\xi)|} |f(\xi)| dA(\xi).$$

We have

$$|u(z)| \leqslant U(z).$$

By the fact that $\operatorname{Supp} \psi_j \subset D^r(z_j)$, applying Lemma 2.1 and [10, Corollary 3.1], we get

$$\begin{aligned} U(z) &\leqslant C \sum_{j=1}^\infty \frac{\mathrm{e}^{\varphi(z_j)+\varphi(z)}}{\rho(z_j)\rho(z)} \int_{D^r(z_j)} \frac{\psi_j(\xi)}{|\xi - z|} |f(\xi)| \frac{\rho(z_j)\rho(\xi)}{\mathrm{e}^{\varphi(z_j)+\varphi(\xi)}} dA(\xi) \\ &\leqslant C \sum_{j=1}^\infty \frac{\mathrm{e}^{\varphi(z)}}{\rho(z)} \int_{D^r(z_j)} \frac{\psi_j(\xi)}{|\xi - z|} |\rho(\xi) f(\xi) \mathrm{e}^{-\varphi(\xi)}| dA(\xi) \\ &= C \sum_{j=1}^\infty \frac{\mathrm{e}^{\varphi(z)}}{\rho(z)} \int_{\mathbb{D}} \frac{\psi_j(\xi)}{|\xi - z|} |\rho(\xi) f(\xi) \mathrm{e}^{-\varphi(\xi)}| dA(\xi). \end{aligned}$$

Write $\Omega = \operatorname{Supp} \phi$ which is compact. Then,

$$\begin{aligned} \int_{\mathbb{D}} U(z) \left|\frac{\partial \phi}{\partial z}(z)\right| dA(z) &\leqslant C \int_\Omega \left|\frac{\partial \phi}{\partial z}(z)\right| dA(z) \sum_{j=1}^\infty \frac{\mathrm{e}^{\varphi(z)}}{\rho(z)} \int_{\mathbb{D}} \frac{\psi_j(\xi)}{|\xi - z|} |\rho(\xi) f(\xi) \mathrm{e}^{-\varphi(\xi)}| dA(\xi) \\ &\leqslant C \left\| \frac{\mathrm{e}^{\varphi(z)} \overline{\partial} \phi}{\rho(z)} \right\|_{L^\infty(\Omega)} \int_{\mathbb{D}} |\rho(\xi) f(\xi) \mathrm{e}^{-\varphi(\xi)}| dA(\xi) \sum_{j=1}^\infty \psi_j(\xi) \int_\Omega \frac{1}{|\xi - z|} dA(z) \\ &\leqslant C \left\| \frac{\mathrm{e}^{\varphi(z)} \overline{\partial} \phi}{\rho(z)} \right\|_{L^\infty(\Omega)} \int_{\mathbb{D}} |\rho(\xi) f(\xi) \mathrm{e}^{-\varphi(\xi)}| \sum_{j=1}^\infty \psi_j(\xi) dA(\xi) \\ &\leqslant C \left\| \frac{\mathrm{e}^{\varphi(z)} \overline{\partial} \phi}{\rho(z)} \right\|_{L^\infty(\Omega)} \int_{\mathbb{D}} |\rho(\xi) f(\xi) \mathrm{e}^{-\varphi(\xi)}| dA(\xi) \\ &< \infty. \end{aligned}$$

Hence, we can apply Fubini's theorem to obtain

$$\begin{aligned} \int_{\mathbb{D}} u(z) \overline{\frac{\partial \phi}{\partial z}}(z) dA(z) &= \int_{\mathbb{D}} \left(\sum_{j=1}^\infty K_{z_j}(z) \int_{\mathbb{D}} \frac{\psi_j(\xi)}{(\xi - z)K_{z_j}(\xi)} f(\xi) dA(\xi) \right) \overline{\frac{\partial \phi}{\partial z}}(z) dA(z) \\ &= \sum_{j=1}^\infty \int_{\mathbb{D}} \left(K_{z_j}(z) \int_{\mathbb{D}} \frac{\psi_j(\xi)}{(\xi - z)K_{z_j}(\xi)} f(\xi) dA(\xi) \right) \overline{\frac{\partial \phi}{\partial z}}(z) dA(z). \end{aligned}$$

Therefore,

$$\left\langle u, \frac{\partial \phi}{\partial z} \right\rangle_{L^2} = \sum_{j=1}^\infty \left\langle K_{z_j}(\cdot) \int_{\mathbb{D}} \frac{\psi_j(\xi)}{(\xi - \cdot)K_{z_j}(\xi)} f(\xi) dA(\xi), \frac{\partial \phi}{\partial z} \right\rangle_{L^2}$$

$$= -\sum_{j=1}^{\infty} \langle f\psi_j, \phi\rangle_{L^2}$$
$$= -\langle f, \phi\rangle_{L^2}.$$

With this we know $\frac{\partial u}{\partial \bar z} = f$ weakly. □

4 Hankel operators from A^p_φ to L^q_φ

Recall that

$$\Gamma = \left\{ \sum_{j=1}^{N} a_j K_{z_j} : N \in \mathbb{N},\, a_j \in \mathbb{C},\, z_j \in \mathbb{D} \text{ for } 1 \leqslant j \leqslant N \right\}$$

and

$$\mathcal{S} = \{f \text{ measurable on } \mathbb{D} : fg \in L^1_\varphi \text{ for } g \in \Gamma\}.$$

[10, Corollary 4.2] tells us that Γ is dense in A^p_φ for all $0 < p < \infty$. From [10, Theorem 3.3], $\|K(\cdot, z)\|_{L^\infty_\varphi} < \infty$. This implies that $P(fg)(z)$ is well defined for $f \in \mathcal{S}$, $g \in \Gamma$ and $z \in \mathbb{D}$. Hence, for $f \in \mathcal{S}$ the Hankel operator H_f is densely defined on A^p_φ. Therefore, a function $f \in \mathcal{S}$ can be used as the symbol to define a Hankel operator on A^p_φ.

The following lemma sets up a bridge between Hankel operators and the solution to the $\overline{\partial}$-equation in Theorem 3.2.

Lemma 4.1. *Let $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$, and suppose that $f \in \mathcal{S}$ with $\rho\overline{\partial} f \in \mathcal{S}$, where the derivative is in the weak sense. Then for $g \in \Gamma$, it holds that*

$$H_f(g) = u - P(u), \tag{4.1}$$

where

$$u(z) = \sum_{j=1}^{\infty} K_{z_j}(z) \int_{\mathbb{D}} \frac{\psi_j(\xi)}{(\xi - z) K_{z_j}(\xi)} g(\xi)\, \overline{\partial} f(\xi)\, dA(\xi). \tag{4.2}$$

Proof. Since $\rho\overline{\partial} f \in \mathcal{S}$, for $g \in \Gamma$ we have $g\rho\overline{\partial} f \in L^1_\varphi$. For u defined as in (4.2), Theorem 3.2 implies $u \in L^p_\varphi$ with

$$\|u\|_{L^p_\varphi} \leqslant C \|g(\rho\overline{\partial} f)\|_{L^p_\varphi}. \tag{4.3}$$

Meanwhile, $fg \in L^1_\varphi$ for $g \in \Gamma$. Then, $fg - u \in L^1_\varphi$, and Theorem 3.2 tells us that

$$\overline{\partial}(fg - u) = g\overline{\partial} f - \overline{\partial} u = 0$$

showing that $fg - u \in A^1_\varphi$. Since $P\,|_{A^1_\varphi} = \mathrm{Id}$, we have

$$P(fg - u) = fg - u.$$

Therefore,

$$H_f(g) - (u - P(u)) = fg - P(fg) - (u - P(u)) = (fg - u) - P(fg - u) = 0,$$

from which (4.1) follows. □

To characterize the boundedness (or compactness) of Hankel operators H_f, we need an auxiliary function $G_{q,r}(f)$ which is an analogue of the one first introduced in [12], when Luecking studied Hankel operators on the standard Bergman space A^p. Let $q \geqslant 1$ and $0 < r \leqslant \alpha$. For $f \in L^q_{\mathrm{loc}}$ we define $G_{q,r}(f)$ to be

$$G_{q,r}(f)(z) = \inf \left\{ \left(\frac{1}{|D^r(z)|} \int_{D^r(z)} |f - h|^q dA \right)^{\frac{1}{q}} : h \in H(D^r(z)) \right\}, \quad z \in \mathbb{D}.$$

For $f \in L^1_{\rm loc}(\mathbb{D})$, $1 \leqslant q < \infty$ and $0 < r \leqslant \alpha$, write

$$M_{q,r}(f)(z) = \left\{ \frac{1}{|D^r(z)|} \int_{D^r(z)} |f|^q dA \right\}^{\frac{1}{q}}$$

to be the q-th mean of $|f|$ over $D^r(z)$.

Our analysis on the Hankel operator going from A^p_φ to L^q_φ will be carried out in two cases where $1 \leqslant p \leqslant q < \infty$ and $1 \leqslant q < p < \infty$, respectively.

Theorem 4.2. *Let $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$, and let $1 \leqslant p \leqslant q < \infty$. Set $s = \frac{1}{q} - \frac{1}{p}$. Then for $f \in \mathcal{S}$, the following statements are equivalent*:

(i) *$H_f : A^p_\varphi \to L^q_\varphi$ is bounded*;

(ii) *for some* (*or any*) *$0 < r \leqslant \alpha$, $\rho^{2s} G_{q,r}(f) \in L^\infty$*;

(iii) *f admits a decomposition $f = f_1 + f_2$, where $f_1 \in C^1(\mathbb{D})$ satisfying*

$$\rho^{2s+1} |\overline{\partial} f_1| \in L^\infty, \tag{4.4}$$

and f_2 has the property that for some (*or any*) *$0 < r \leqslant \alpha$,*

$$\rho^{2s} M_{q,r}(f_2) \in L^\infty. \tag{4.5}$$

Furthermore, for $0 < r \leqslant \alpha$,

$$\|H_f\|_{A^p_\varphi \to L^q_\varphi} \simeq \|\rho^{2s} G_{q,r}(f)\|_{L^\infty}. \tag{4.6}$$

Proof. (i) $\Rightarrow$ (ii). For α as in Lemma 2.2, Lemma 2.1 tells us that there is some constant $C > 0$ such that

$$\inf_{\xi \in D^\alpha(z)} |k_{z,p}(\xi)| \geqslant C\rho(z)^{-\frac{2}{p}} \mathrm{e}^{\varphi(\xi)} > 0 \quad \text{for } z \in \mathbb{D}.$$

Then,

$$\frac{1}{k_{z,p}} P(f k_{z,p}) \in H(D^r(z))$$

and

$$\begin{aligned} \|H_f(k_{z,p})\|^q_{L^q_\varphi} &= \int_{\mathbb{D}} |f k_{z,p}(\xi) - P(f k_{z,p})(\xi)|^q \mathrm{e}^{-q\varphi(\xi)} dA(\xi) \\ &\geqslant \int_{D^r(z)} |k_{z,p}(\xi)|^q \left| f(\xi) - \frac{1}{k_{z,p}(\xi)} P(f k_{z,p})(\xi) \right|^q \mathrm{e}^{-q\varphi(\xi)} dA(\xi) \\ &\geqslant C\rho(z)^{-\frac{2q}{p}} \int_{D^r(z)} \left| f(\xi) - \frac{1}{k_{z,p}(\xi)} P(f k_{z,p})(\xi) \right|^q dA(\xi) \\ &\geqslant C\{\rho(z)^{2s} G_{q,r}(f)(z)\}^q. \end{aligned} \tag{4.7}$$

On the other hand,

$$\|H_f(k_{z,p})\|^q_{L^q_\varphi} \leqslant \|H_f\|^q_{A^p_\varphi \to L^q_\varphi} \|k_{z,p}\|^q_{L^p_\varphi} = \|H_f\|^q_{A^p_\varphi \to L^q_\varphi}.$$

Therefore, we have

$$\rho(z)^{2s} G_{q,r}(f)(z) \leqslant C\|H_f\|_{A^p_\varphi \to L^q_\varphi} \quad \text{for all } z \in \mathbb{D}. \tag{4.8}$$

From this, the statement (ii) follows.

(ii) $\Rightarrow$ (iii). Suppose $\|\rho^{2s} G_{q,r}(f)\|_{L^\infty} < \infty$ for some $r \in (0, \alpha]$. Fix a $(\rho, \frac{r}{2})$-lattice $\{z_j\}_{j=1}^\infty$, and take $\{\psi_j\}_{j=1}^\infty$ to be a partition of the unity subordinate to $\{D^{\frac{r}{2}}(z_j)\}_{j=1}^\infty$ satisfying $\rho(z_j)|\overline{\partial}\psi_j| \leqslant C$ for $j = 1, 2, \ldots$ With a normal family argument we may find some function $h_j \in H(D^r(z_j))$ such that

$$\frac{1}{|D^r(z_j)|} \int_{D^r(z_j)} |f - h_j|^q dA = G^q_{q,r}(f)(z_j), \quad j = 1, 2, \ldots \tag{4.9}$$

Set

$$f_1(z)=\sum_{j=1}^{\infty}h_j(z)\psi_j(z)\in C^{\infty}(\mathbb{D})$$

and $f_2=f-f_1$. Define $J_z=\{j:z\in D^r(z_j)\}$. Then, $\rho(z_j)\simeq\rho(z)$ for $j\in J_z$, and

$$|J_z|:=\sum_{j=1}^{\infty}\chi_{D^r(z_j)}(z)\leqslant C. \tag{4.10}$$

As that on [13, pp. 254–255], for $z\in\mathbb{D}$ it holds that

$$\rho(z)|\overline{\partial}f_1(z)|\leqslant C\sum_{j\in J_z}G_{q,r}(f)(z_j). \tag{4.11}$$

This implies

$$\rho(z)^{2s+1}|\overline{\partial}f_1(z)|\leqslant C\|\rho^{2s}G_{q,r}(f)\|_{L^\infty}\quad\text{for } z\in\mathbb{D}. \tag{4.12}$$

On the other hand,

$$f_2(z)=\sum_{j=1}^{\infty}(f(z)-h_j(z))\psi_j(z),$$

and by (2.5) only at most N terms are not zero in this summation. Hölder's inequality implies

$$|f_2(z)|^q\leqslant C\sum_{j=1}^{\infty}|f(z)-h_j(z)|^q\psi_j(z).$$

Then, by (4.9),

$$\begin{aligned}M_{q,r}(f_2)(z)&\leqslant C\sum_{j=1}^{\infty}\left(\frac{1}{|D^r(z)|}\int_{D^r(z)}|(f-h_j)|^q\psi_j dA\right)^{\frac{1}{q}}\\&\leqslant C\sum_{j=1}^{\infty}\left(\frac{1}{|D^r(z)|}\int_{D^r(z)\cap D^{r/2}(z_j)}|f-h_j|^q dA\right)^{\frac{1}{q}}\\&\leqslant C\sum_{j\in J_z}G_{q,r}(f)(z_j).\end{aligned} \tag{4.13}$$

Hence,

$$\rho(z)^{2s}\,M_{q,r}(f_2)(z)\leqslant C\|\rho^{2s}G_{q,r}(f)\|_{L^\infty}\quad\text{for } z\in\mathbb{D}. \tag{4.14}$$

Notice that the condition (4.5) is independent of $r\in(0,\alpha]$. We reach the condition (iii) from (4.12) and (4.14).

(iii) $\Rightarrow$ (i). If we set $d\mu=|f_2|^q dA$, then

$$\widehat{\mu}_r(z)^{\frac{1}{q}}=M_{q,r}(f_2)(z). \tag{4.15}$$

The assumption (4.5) and Proposition 2.6 imply that μ is a q-Carleson measure for A^p_φ with

$$\|\mathrm{Id}\|_{A^p_\varphi\to L^q(\mathbb{D},\mathrm{e}^{-q\varphi}d\mu)}\simeq\|\rho^{2s}M_{q,r}(f_2)\|_{L^\infty}.$$

By the boundedness of the Bergman projection on L^q_φ,

$$\begin{aligned}\|H_{f_2}g\|_{L^q_\varphi}&\leqslant C\|f_2 g\|_{L^q_\varphi}\\&=C\bigg(\int_{\mathbb{D}}|g|^q\,\mathrm{e}^{-q\varphi}d\mu\bigg)^{\frac{1}{q}}\end{aligned}$$

$$\leqslant C\|\mathrm{Id}\|_{A^p_\varphi \to L^q(\mathbb{D}, \mathrm{e}^{-q\varphi}d\mu)}\|g\|_{L^p_\varphi}$$
$$\leqslant C\|\rho^{2s}M_{q,r}(f_2)\|_{L^\infty}\|g\|_{L^p_\varphi}. \tag{4.16}$$

Next, we suppose that f_1 satisfies (4.4). With the fact that $s \leqslant 0$ and $\rho^{2s+1}\overline{\partial} f \in L^\infty$, we know $\rho|\overline{\partial} f_1| \in L^\infty$. Now, for $g \in \Gamma$, take u as in (4.1) so that

$$u(z) = \sum_{j=1}^{\infty} K_{z_j}(z)\int_{\mathbb{D}} \frac{\psi_j(\xi)}{(\xi - z)K_{z_j}(\xi)} g(\xi)\overline{\partial} f_1(\xi)dA(\xi).$$

Theorem 3.2 and Lemma 4.1 tell us

$$H_{f_1}(g) = u - P(u) \quad \text{and} \quad \|u\|_{L^q_\varphi} \leqslant C\|g(\rho\overline{\partial} f_1)\|_{L^q_\varphi}.$$

From the boundedness of P on L^q_φ we obtain

$$\|H_{f_1}g\|_{L^q_\varphi} \leqslant (1 + \|P\|_{L^q_\varphi \to L^q_\varphi})\|u\|_{L^q_\varphi} \leqslant C\left\|g(\rho\overline{\partial} f_1)\right\|_{L^q_\varphi}. \tag{4.17}$$

Meanwhile, if we consider the measure $d\nu = [\rho|\overline{\partial} f_1|]^q dA$, it is easy to see that

$$\widehat{\nu}_r(z)^{\frac{1}{q}} \leqslant C \sup_{\xi \in D^r(z)} \rho(\xi)|\overline{\partial} f_1(\xi)|. \tag{4.18}$$

Hence,

$$\rho(z)^{2s}\widehat{\nu}_r(z)^{\frac{1}{q}} \leqslant C\|\rho^{2s+1}|\overline{\partial} f_1|\|_{L^\infty}.$$

It follows from (4.4) and Proposition 2.6 that ν is a q-Carleson measure for A^p_φ with

$$\|\mathrm{Id}\|_{A^p_\varphi \to L^q(\mathbb{D}, \mathrm{e}^{-q\varphi}d\nu)} \simeq \|\rho^{2s+1}|\overline{\partial} f_1|\|_{L^\infty}.$$

Then

$$\|g(\rho\overline{\partial} f_1)\|_{L^q_\varphi} \leqslant C\|\rho^{2s+1}|\overline{\partial} f_1|\|_{L^\infty} \cdot \|g\|_{L^p_\varphi}.$$

Hence,

$$\|H_{f_1}g\|_{L^q_\varphi} \leqslant C\|\rho^{2s+1}|\overline{\partial} f_1|\|_{L^\infty} \cdot \|g\|_{L^p_\varphi}.$$

With this and (4.16), we obtain

$$\|H_f\|_{A^p_\varphi \to L^q_\varphi} \leqslant C\{\|\rho^{2s+1}|\overline{\partial} f_1|\|_{L^\infty} + \|\rho^{2s}M_{q,r}(f_2)\|_{L^\infty}\}. \tag{4.19}$$

This gives the implication (iii) $\Rightarrow$ (i), finishing the proof of the equivalence among (i)–(iii). The norm estimates (4.6) come from (4.8), (4.12), (4.14) and (4.19). □

The next result describes the compactness of H_f when $p \leqslant q$. For $q \geqslant 1$, we understand that $H_f : A^p_\varphi \to L^q_\varphi$ is compact if and only if whenever $\{g_m\}_{m=1}^\infty$ is a bounded sequence in A^p_φ converging to zero on compact subsets of $\mathbb{D}$, it follows that $\|H_f g_m\|_{L^q_\varphi}$ tends to zero.

Theorem 4.3. *Let $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$, and let $1 \leqslant p \leqslant q < \infty$. Set $s = \frac{1}{q} - \frac{1}{p}$. Then for $f \in \mathcal{S}$, the following statements are equivalent*:

(i) *$H_f : A^p_\varphi \to L^q_\varphi$ is compact*;

(ii) *for some (or any) $0 < r \leqslant \alpha$, $\lim_{|z|\to 1} \rho^{2s}G_{q,r}(f)(z) = 0$*;

(iii) *f admits a decomposition $f = f_1 + f_2$, where $f_1 \in C^1(\mathbb{D})$ satisfying*

$$\lim_{|z|\to 1} \rho(z)^{2s+1}|\overline{\partial} f_1(z)| = 0 \tag{4.20}$$

and

$$\lim_{|z|\to 1} \rho(z)^{2s}M_{q,r}(f_2)(z) = 0 \tag{4.21}$$

for some (or any) $0 < r \leqslant \alpha$.

Proof. Let H_f be compact from A^p_φ to L^q_φ. It is easy to see that $\{k_{z,p} : z \in \mathbb{D}\}$ tends to 0 weakly in A^p_φ as $|z| \to 1$. Then, for $0 < r \leqslant \alpha$ fixed, from (4.7) we have

$$\rho(z)^{2s} G_{q,r}(f)(z) \leqslant C \|H_f(k_{z,p})\|_{L^q_\varphi} \to 0$$

as $|z| \to 1$. So, (i) implies (ii).

Suppose now that (ii) holds for some $r \in (0, \alpha]$. From (4.11) and (4.13) we know

$$\rho(z)^{2s+1} |\overline{\partial} f_1(z)| \leqslant C \sum_{j \in J_z} \rho(z_j)^{2s} G_{q,r}(f)(z_j)$$

and

$$\rho(z)^{2s} M_{q,r}(f_2)(z) \leqslant C \sum_{j \in J_z} \rho(z_j)^{2s} G_{q,r}(f)(z_j).$$

From these estimates, the statement (iii) follows easily.

Finally, we prove the implication (iii) $\Rightarrow$ (i). As in the proof of Theorem 4.2, we know that both $d\mu = |f_2|^q dA$ and $d\nu = [\rho|\overline{\partial} f_1|]^q dA$ are vanishing q-Carleson measures for A^p_φ. With (2.7) we know that the unit ball of A^p_φ is a normal family. Then, for any bounded sequence $\{g_m\}$ in A^p_φ converging to zero uniformly on compact subsets of $\mathbb{D}$, we have

$$\|H_{f_2}(g_m)\|_{L^q_\varphi} \leqslant C \bigg(\int_{\mathbb{D}} |f_2|^q |g_m|^q \mathrm{e}^{-q\varphi} dA \bigg)^{\frac{1}{q}} \to 0,$$

and by (4.17),

$$\|H_{f_1}(g_m)\|_{L^q_\varphi} \leqslant C \|(\rho|\overline{\partial} f_1|) g_m\|_{L^q_\varphi} \to 0.$$

Then, $\lim_{m\to\infty} \|H_f(g_m)\|_{L^q_\varphi} = 0$, and this tells us that H_f is compact from A^p_φ to L^q_φ. □

Next, we proceed to characterize the boundedness and compactness in the case where $1 \leqslant q < p < \infty$.

Theorem 4.4. *Let $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$, and let $1 \leqslant q < p < \infty$. Set $s = \frac{1}{q} - \frac{1}{p}$. Then for $f \in \mathcal{S}$, the following statements are equivalent*:

(i) *$H_f : A^p_\varphi \to L^q_\varphi$ is bounded*;

(ii) *$H_f : A^p_\varphi \to L^q_\varphi$ is compact*;

(iii) *for some (or any) $0 < r \leqslant \frac{\alpha}{2}$, $G_{q,r}(f)(z) \in L^{\frac{1}{s}}$*;

(iv) *f admits a decomposition $f = f_1 + f_2$, where*

$$f_1 \in C^1(\mathbb{D}), \quad \rho|\overline{\partial} f_1| \in L^{\frac{1}{s}} \quad \text{and} \quad M_{q,r}(f_2) \in L^{\frac{1}{s}} \tag{4.22}$$

for some (or any) $0 < r \leqslant \alpha$.

Furthermore, for $0 < r \leqslant \frac{\alpha}{2}$ fixed,

$$\|H_f\|_{A^p_\varphi \to L^q_\varphi} \simeq \|G_{q,r}(f)\|_{L^{\frac{1}{s}}}. \tag{4.23}$$

Proof. (ii) $\Rightarrow$ (i) is trivial. We need only to prove the implications (i) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (iii). For $r \in (0, \alpha]$ fixed, take $\{z_j\}_{j=1}^\infty$ to be some $(r/4, \rho)$-lattice. By Lemma 2.4, for $\lambda = \{\lambda_j\} \in \ell^p$, we have

$$\bigg\| \sum_{j=1}^\infty \lambda_j k_{z_j,p} \bigg\|_{L^p_\varphi} \leqslant C \|\lambda\|_{\ell^p}.$$

As in [13] again, take $\{\phi_j\}_{j=1}^\infty$ to be a sequence of Rademacher functions in $[0, 1]$. From the boundedness of H_f, we have

$$\bigg\| H_f \bigg(\sum_{j=1}^\infty \lambda_j \phi_j(t) k_{z_j,p} \bigg) \bigg\|_{L^q_\varphi} \leqslant \|H_f\|_{A^p_\varphi \to L^q_\varphi} \cdot \bigg\| \sum_{j=1}^\infty \lambda_j \phi_j(t) k_{z_j,p} \bigg\|_{L^p_\varphi} \leqslant C \|H_f\|_{A^p_\varphi \to L^q_\varphi} \|\lambda\|_{\ell^p}.$$

Meanwhile, by Khintchine's inequality,

$$\begin{aligned}&\int_0^1 \left\|H_f\left(\sum_{j=1}^{\infty}\lambda_j\phi_j(t)k_{z_j,p}\right)\right\|_{L^q_\varphi}^q dt\\&\quad=\int_{\mathbb{D}}\mathrm{e}^{-q\varphi(z)}dA(z)\int_0^1\left|\sum_{j=1}^{\infty}\lambda_j\phi_j(t)H_f(k_{z_j,p})(z)\right|^q dt\\&\quad\simeq\int_{\mathbb{D}}\left(\sum_{j=1}^{\infty}|\lambda_j|^2|H_f(k_{z_j,p})(z)|^2\right)^{\frac{q}{2}}\mathrm{e}^{-q\varphi(z)}dA(z).\end{aligned}$$

This, together with the previous estimate, gives

$$\int_{\mathbb{D}}\left(\sum_{j=1}^{\infty}|\lambda_j|^2|H_f(k_{z_j,p})(z)|^2\right)^{\frac{q}{2}}\mathrm{e}^{-q\varphi(z)}dA(z)\leqslant C\|H_f\|^q_{A^p_\varphi\to L^q_\varphi}\|\lambda\|^q_{\ell^p}.$$

On the other hand,

$$\begin{aligned}&\int_{\mathbb{D}}\left(\sum_{j=1}^{\infty}|\lambda_j|^2|H_f(k_{z_j,p})(z)|^2\right)^{\frac{q}{2}}\mathrm{e}^{-q\varphi(z)}dA(z)\\&\quad\geqslant C\sum_{k=1}^{\infty}\int_{D^r(z_k)}(|\lambda_k||H_f(k_{z_k,p})(z)|)^q\mathrm{e}^{-q\varphi(z)}dA(z)\\&\quad=C\sum_{k=1}^{\infty}|\lambda_k|^q\int_{D^r(z_k)}|f(z)k_{z_k,p}(z)-P(fk_{z_k,p})(z)|^q\mathrm{e}^{-q\varphi(z)}dA(z).\end{aligned}$$

As in (4.7),

$$\int_{D^r(z_k)}|f(z)k_{z_k,p}(z)-P(fk_{z_k,p})(z)|^q\mathrm{e}^{-q\varphi(z)}dA(z)\geqslant C\{\rho(z_k)^{2s}G_{q,r}(f)(z_k)\}^q.$$

Therefore, joining the previous estimates, we obtain

$$\sum_{k=1}^{\infty}|\lambda_k|^q\{\rho(z_k)^{2s}G_{q,r}(f)(z_k)\}^q\leqslant C\|H_f\|^q_{A^p_\varphi\to L^q_\varphi}\|\{|\lambda_j|^q\}\|_{\ell^{p/q}}.$$

By the duality between $\ell^{p/q}$ and $\ell^{p/(p-q)}$, we have

$$\sum_{k=1}^{\infty}[G_{q,r}(f)(z_k)]^{\frac{pq}{p-q}}\rho(z_k)^2=\sum_{k=1}^{\infty}[\rho(z_k)^{2s}G_{q,r}(f)(z_k)]^{\frac{pq}{p-q}}\leqslant C\|H_f\|^{\frac{pq}{p-q}}_{A^p_\varphi\to L^q_\varphi}.$$

This can be viewed as the discrete version of the statement (iii). Since

$$G_{q,\frac{r}{2}}(f)(w)\leqslant CG_{q,r}(f)(z)\quad\text{for } w\in D^{\frac{r}{2}}(z), \tag{4.24}$$

we have

$$\begin{aligned}\int_{\mathbb{D}}G_{q,\frac{r}{2}}(f)^{\frac{pq}{p-q}}dA&\leqslant\sum_{k=1}^{\infty}\int_{D^{\frac{r}{2}}(z_k)}G_{q,\frac{r}{2}}(f)^{\frac{pq}{p-q}}(u)dA(u)\\&\leqslant C\sum_{k=1}^{\infty}|D^{\frac{r}{2}}(z_k)|G_{q,r}(f)^{\frac{pq}{p-q}}(z_k)\\&\leqslant C\|H_f\|^{\frac{pq}{p-q}}_{A^p_\varphi\to L^q_\varphi}.\end{aligned}\tag{4.25}$$

This gives the statement (iii).

(iii) ⇒ (iv). As in the proof of Theorem 4.2, set $f_1 = \sum_{j=1}^{\infty} h_j\, \psi_j \in C^\infty(\mathbb{D})$ and $f_2 = f - f_1$. By (4.24),

$$G_{q,\frac{r}{2}}(f)^{\frac{pq}{p-q}}(z_j) \leqslant C \frac{1}{|D^{\frac{r}{2}}(z_j)|} \int_{D^{\frac{r}{2}}(z_j)} G_{q,r}(f)^{\frac{pq}{p-q}}(u)dA(u).$$

From (4.11) we have

$$\begin{aligned}[\rho(z)|\overline{\partial} f_1(z)|]^{\frac{pq}{p-q}} &\leqslant C \sum_{j\in J_z} G_{q,\frac{r}{2}}(f)^{\frac{pq}{p-q}}(z_j)\\ &\leqslant \frac{C}{|D^r(z)|} \sum_{j\in J_z} \int_{D^r(z_j)} G_{q,r}(f)^{\frac{pq}{p-q}}(u)dA(u)\\ &\leqslant \frac{C}{|D^r(z)|} \int_{D^{2r}(z)} G_{q,r}(f)^{\frac{pq}{p-q}}(u)dA(u).\end{aligned}$$

Integrating both sides on $\mathbb{D}$ against the measure dA, and applying Fubini's theorem, one gets

$$\begin{aligned}&\int_{\mathbb{D}} [\rho(z)|\overline{\partial} f_1(z)|]^{\frac{pq}{p-q}} dA(z)\\ &\quad\leqslant C \int_{\mathbb{D}} \frac{1}{|D^r(z)|} dA(z) \int_{\mathbb{D}} \chi_{D^{2r}(z)}(u) G_{q,r}(f)^{\frac{pq}{p-q}}(u)dA(u)\\ &\quad\leqslant C \int_{\mathbb{D}} G_{q,r}(f)^{\frac{pq}{p-q}}(u)dA(u).\end{aligned} \tag{4.26}$$

Notice that $\frac{1}{s} > 1$. By (4.13) and (4.24) we obtain

$$\begin{aligned}M_{q,r}(f_2)(z) &\leqslant C \sum_{j=1}^{\infty} \bigg(\frac{1}{|D^r(z)|} \int_{D^r(z)\cap D^{r/2}(z_j)} |(f-h_j)|^q dA\bigg)^{\frac{1}{q}}\\ &\leqslant C \frac{1}{|D^r(z)|} \int_{D^{2r}(z)} G_{q,2r}(f)(\xi)dA(\xi)\\ &\leqslant C \bigg\{\frac{1}{|D^r(z)|} \int_{D^{2r}(z)} G_{q,2r}^{\frac{1}{s}}(f)(\xi)dA(\xi)\bigg\}^s.\end{aligned}$$

This and Fubini's theorem turn out

$$\|M_{q,r}(f_2)\|_{L^{\frac{1}{s}}} \leqslant C \|G_{q,2r}(f)\|_{L^{\frac{1}{s}}}. \tag{4.27}$$

In addition, it is trivial that the condition $M_{q,r}(f_2) \in L^{\frac{1}{s}}$ is independent of r. We see that (4.26) and (4.27) give the statement (iv).

Now we prove (iv) ⇒ (ii). First, we claim that both f_1 and $\rho|\overline{\partial} f_1|$ are in $\mathcal{S}$. In fact, apply [10, Lemma 3.3] to get

$$\begin{aligned}&\int_{\mathbb{D}} |f_2(\xi)K_z(\xi)|\mathrm{e}^{-\varphi(\xi)}dA(\xi)\\ &\quad\leqslant C \int_{\mathbb{D}} |f_2(\xi)| \bigg(\frac{1}{\rho(\xi)^2} \int_{D^{r/2}(\xi)} |K_z(\zeta)|\mathrm{e}^{-\varphi(\zeta)}dA(\zeta)\bigg) dA(\xi)\\ &\quad= C \int_{\mathbb{D}} |K_z(\zeta)|\mathrm{e}^{-\varphi(\zeta)} \int_{\mathbb{D}} \chi_{D^{r/2}(\xi)}(\zeta)|f_2(\xi)| \frac{1}{\rho(\xi)^2}\, dA(\xi)\, dA(\zeta)\\ &\quad\leqslant C \int_{\mathbb{D}} M_{1,r}(|f_2|)(\zeta)|K_z(\zeta)|\mathrm{e}^{-\varphi(\zeta)}dA(\zeta)\\ &\quad\leqslant C \int_{\mathbb{D}} M_{q,r}(|f_2|)(\zeta)|K_z(\zeta)|\mathrm{e}^{-\varphi(\zeta)}dA(\zeta).\end{aligned}$$

By Hölder's inequality with the exponent $\frac{1}{s} = \frac{pq}{p-q}$ and its conjugate exponent denoted by t, notice also that $\|K_z\|_{L^t_\varphi} < \infty$ and

$$\int_{\mathbb{D}} |f_2(\xi)K_z(\xi)|\mathrm{e}^{-\varphi(\xi)}dA(\xi) \leqslant C\|M_{q,r}(f_2)\|_{L^{\frac{1}{s}}} \cdot \|K_z\|_{L^t_\varphi} < \infty.$$

This implies $f_2 \in \mathcal{S}$, and $f_1 = f - f_2 \in \mathcal{S}$. For $\rho|\overline{\partial} f_1|$, notice that $\rho|\overline{\partial} f_1| \in L^{\frac{1}{s}}$ with $\frac{1}{s} = \frac{pq}{p-q} > 1$. Then

$$\int_{\mathbb{D}} \rho(\xi)|\overline{\partial} f_1(\xi) K_z(\xi)|\mathrm{e}^{-\varphi(\xi)} dA(\xi) \leqslant \bigg\{ \int_{\mathbb{D}} |\rho(\xi)\overline{\partial} f_1(\xi)|^{\frac{1}{s}} dA(\xi) \bigg\}^s \|K_z\|_{L^t_\varphi} < \infty.$$

It follows that $\rho|\overline{\partial} f_1| \in \mathcal{S}$.

As before, write $d\nu = [\rho|\overline{\partial} f_1|]^q dA$. Applying Hölder's inequality with the exponent $\frac{p}{p-q}$ and its conjugate p/q, we get

$$\begin{aligned}
\|\widehat{\nu}_r\|^{\frac{p}{p-q}}_{L^{\frac{p}{p-q}}} &= \int_{\mathbb{D}} \bigg\{ \frac{\int_{D^r(\xi)} [\rho|\overline{\partial} f_1(\zeta)|]^q dA(\zeta)}{|D^r(\xi)|} \bigg\}^{\frac{p}{p-q}} dA(\xi) \\
&\leqslant C \int_{\mathbb{D}} \bigg\{ \int_{D^r(\xi)} [\rho(\zeta)|\overline{\partial} f_1(\zeta)|]^{\frac{pq}{p-q}} dA(\zeta) \bigg\} \frac{1}{\rho(\xi)^2} dA(\xi) \\
&\simeq C \int_{\mathbb{D}} [\rho(\zeta)|\overline{\partial} f_1(\zeta)|]^{\frac{1}{s}} dA(\zeta) < \infty.
\end{aligned}$$

Lemma 2.6 tells us that ν is a q-Carleson measure for A^p_φ. Equivalently, the embedding

$$\mathrm{Id} : A^p_\varphi \hookrightarrow L^q(\mathbb{D}, \mathrm{e}^{-q\varphi} d\nu)$$

is compact with

$$\|\mathrm{Id}\|^q_{A^p_\varphi \hookrightarrow L^q(\mathbb{D}, \mathrm{e}^{-q\varphi} d\nu)} \leqslant C \|\widehat{\nu}_r\|_{L^{\frac{p}{p-q}}} \leqslant C \|\rho|\overline{\partial} f_1|\|^q_{L^{\frac{1}{s}}} < \infty.$$

Meanwhile, since both f_1 and $\rho|\overline{\partial} f_1|$ are in $\mathcal{S}$, for $g \in \Gamma$, as in (4.17), we have

$$\|H_{f_1} g\|_{L^q_\varphi} \leqslant C \|g(\rho\overline{\partial} f_1)\|_{L^q_\varphi} = C \|\mathrm{Id}(g)\|_{L^q(\mathbb{D}, \mathrm{e}^{-q\varphi} d\nu)}.$$

Hence H_{f_1} is bounded from A^p_φ to L^q_φ with the norm estimate

$$\|H_{f_1}\|_{A^p_\varphi \to L^q_\varphi} \leqslant C \|\rho|\overline{\partial} f_1|\|_{L^{\frac{1}{s}}}. \tag{4.28}$$

We claim that H_{f_1} is compact as well. To see this, let $\{g_m\}_{m=1}^\infty$ be any bounded sequence in A^p_φ with the property that $\lim_{m\to\infty} \sup_{z\in K} |g_m(z)| = 0$ on any compact subset $K \subset \mathbb{D}$. We are going to prove $H_{f_1}(g_m) \to 0$ in L^q_φ as $m \to \infty$. For this purpose, for each m pick some $h_m \in \Gamma$ so that

$$\|g_m - h_m\|_{L^p_\varphi} < \frac{1}{m}.$$

Set

$$u_m(z) = \sum_{j=1}^\infty K_{z_j}(z) \int_{\mathbb{D}} \frac{\phi_j(\xi)}{(\xi - z) K_{z_j}(\xi)} h_m(\xi) \overline{\partial} f_1(\xi) dA(\xi).$$

Then, $\overline{\partial} u_m = h_m \overline{\partial} f_1$ and

$$\|u_m\|_{L^q_\varphi} \leqslant C \|h_m(\rho\overline{\partial} f_1)\|_{L^q_\varphi} = C \|h_m\|_{L^q(\mathbb{D},\, \mathrm{e}^{-q\varphi} d\nu)}.$$

Notice that $\mathrm{Id} : A^p_\varphi \hookrightarrow L^q(\mathbb{D},\, \mathrm{e}^{-q\varphi} d\nu)$ is compact, so $\lim_{m\to\infty} \|h_m\|_{L^q(\mathbb{D},\, \mathrm{e}^{-q\varphi} d\nu)} = 0$, showing that

$$\lim_{m\to\infty} \|u_m\|_{L^q_\varphi} = 0.$$

Then, as $H_{f_1}(h_m) = u_m - P(u_m)$, we get

$$\lim_{m\to\infty} \|H_{f_1}(h_m)\|_{L^q_\varphi} \leqslant (1 + \|P\|_{L^q_\varphi \to L^q_\varphi}) \lim_{m\to\infty} \|u_m\|_{L^q_\varphi} = 0. \tag{4.29}$$

On the other hand, by (4.28),

$$\lim_{m\to\infty} \|H_{f_1}(g_m - h_m)\|_{L^q_\varphi} \leqslant \|H_{f_1}\|_{A^p_\varphi \to L^q_\varphi} \lim_{m\to\infty} \|g_m - h_m\|_{L^p_\varphi} = 0.$$

This, together with (4.29), implies

$$\lim_{m\to\infty}\|H_{f_1}(g_m)\|_{L^q_\varphi}\leqslant\lim_{m\to\infty}\{\|H_{f_1}(g_m-h_m)\|_{L^q_\varphi}+\|H_{f_1}(h_m)\|_{L^q_\varphi}\}=0,$$

which gives the compactness of H_{f_1} from A^p_φ to L^q_φ.

Finally, we consider the compactness of H_{f_2}. Similarly, $d\mu=|f_2|^q dA$ is a vanishing q-Carleson measure for A^p_φ. Equivalently, $\mathrm{Id}: A^p_\varphi\to L^q(\mathbb{D},\mathrm{e}^{-q\varphi}d\mu)$ is compact. By

$$\|H_{f_2}(g)\|_{L^q_\varphi}\leqslant C\|f_2 g\|_{L^q_\varphi}=C\|\mathrm{Id}(g)\|_{L^q(\mathbb{D},\,\mathrm{e}^{-q\varphi}d\mu)} \tag{4.30}$$

with the similar approach for H_{f_1} above we know H_{f_2} is compact from A^p_φ to L^q_φ as well. This finishes the proof of the implication (iv) $\Rightarrow$ (ii).

Furthermore, from (4.28), (4.30) and (4.26), (4.27), we have

$$\|H_f\|_{A^p_\varphi\to L^q_\varphi}\leqslant C\inf\{\|H_{f_1}\|_{A^p_\varphi\to L^q_\varphi}+\|H_{f2}\|_{A^p_\varphi\to L^q_\varphi}\}\leqslant C\|G_{q,r}(f)\|_{L^{\frac{1}{s}}},$$

where the "inf" is taken over all the decomposition $f=f_1+f_2$ as (4.22). This and (4.25) imply (4.23). The proof is completed. □

5 Simultaneous boundedness of H_f and $H_{\overline{f}}$

For $f\in L^q_{\mathrm{loc}}(\mathbb{D})$ with $1\leqslant q<\infty$ and $0<r<\alpha$, set

$$f_{D^r(z)}=\frac{1}{|D^r(z)|}\int_{D^r(z)}f dA,$$
$$MO_{q,r}(f)(z)=\left\{\frac{1}{|D^r(z)|}\int_{D^r(z)}|f-f_{D^r(z)}|^q dA\right\}^{\frac{1}{q}}$$

and

$$\mathrm{Osc}_r(f)(z)=\sup_{\xi\in B(z,r)}|f(\xi)-f(z)|.$$

Lemma 5.1. *Let $1\leqslant q<\infty$, $0<s\leqslant\infty$, $-\infty<\gamma<\infty$, and let $f\in L^q_{\mathrm{loc}}(\mathbb{D})$. Then the following statements are equivalent*:

(i) *for some* (*or any*) $0<r\leqslant\alpha$, *both* $\rho^\gamma G_{q,r}(f)$ *and* $\rho^\gamma G_{q,r}(\overline{f})$ *are in* L^s;

(ii) *for some* (*or any*) $0<r\leqslant\alpha$, *one has* $\rho^\gamma MO_{q,r}(f)\in L^s$;

(iii) $f=f_1+f_2$ *with* $f_1\in C(\mathbb{D})$, *and for some* (*or any*) $0<r\leqslant\alpha$,

$$\rho^\gamma\mathrm{Osc}_r(f_1)\in L^s\quad\text{and}\quad\rho^\gamma M_{q,r}(f_2)\in L^s. \tag{5.1}$$

Furthermore,

$$\|\rho^\gamma G_{q,r}(f)\|_{L^s}+\|\rho^\gamma G_{q,r}(\overline{f})\|_{L^s}\simeq\|\rho^\gamma MO_{q,r}(f)\|_{L^s}. \tag{5.2}$$

Proof. By definition, we know

$$G_{q,r}(f)(z)\leqslant MO_{q,r}(f)(z)\quad\text{and}\quad G_{q,r}(\overline{f})(z)\leqslant MO_{q,r}(f)(z), \tag{5.3}$$

which give the implication (ii) $\Rightarrow$ (i).

Similar to [11, the estimate (2.7)], for fixed $r>0$, we have some constant C independent of z such that

$$\|u-u(z)\|_{L^q(D^r(z),dA)}\leqslant C\|v\|_{L^q(D^r(z),dA)}$$

for all real-valued functions u and v so that $u+\mathrm{i}v\in H(D^r(z))$. From this, as done in [11, Proposition 2.5], we know

$$MO_{q,r}(f)(z)\leqslant C\{G_{q,r}(f)(z)+G_{q,r}(\overline{f})(z)\}. \tag{5.4}$$

This means that (i) implies (ii).

Suppose $f = f_1 + f_2$ is as in the statement (iii). From

$$MO_{q,r}(f_1)(z) = \left\{\frac{1}{|D^r(z)|}\int_{D^r(z)}\left|\frac{1}{|D^r(z)|}\int_{D^r(z)}(f_1(\xi)-f_1(\zeta))dA(\zeta)\right|^q dA(\xi)\right\}^{\frac{1}{q}}$$
$$\leqslant 2\mathrm{Osc}_r(f_1)(z)$$

and $MO_{q,r}(f_2)(z) \leqslant 2M_{q,r}(f_2)(z)$, we know that f satisfies (ii).

To prove the implication (ii) $\Rightarrow$ (iii) we set $f_1(z) = f_{D^r(z)}$ and $f_2 = f - f_1$. As in the proof of [16, Lemma 8.3] we have

$$\mathrm{Osc}_{r/2}(f_1)(z) \leqslant CMO_{q,r}(f)(z) \quad \text{and} \quad M_{q,r/2}(f_2)(z) \leqslant CMO_{q,r}(f)(z).$$

In addition, it is easy to see that the condition (5.1) is independent of $r \in (0, \alpha]$. Then (iii) follows from (ii). The equivalence (5.2) comes from (5.3) and (5.4). $\square$

Lemma 5.2. *Let $1 \leqslant q < \infty$, $0 < s \leqslant \infty$, $-\infty < \gamma < \infty$, and let $f \in L^q_{\mathrm{loc}}(\mathbb{D})$. Then the following statements are equivalent:*

(i) *for some (or any) $0 < r \leqslant \alpha$, $\lim_{|z|\to 1}\{\rho(z)^\gamma G_{q,r}(f)(z) + \rho(z)^\gamma G_{q,r}(\overline{f})(z)\} = 0$;*

(ii) *for some (or any) $0 < r \leqslant \alpha$, $\lim_{|z|\to 1}\rho(z)^\gamma MO_{q,r}(f)(z) = 0$;*

(iii) *$f = f_1 + f_2$ with $f_1 \in C(\mathbb{D})$, and for some (or any) $0 < r \leqslant \alpha$,*

$$\lim_{|z|\to 1}\{\rho(z)^\gamma \mathrm{Osc}_r(f_1)(z) + \rho(z)^\gamma M_{q,r}(f_2)(z)\} = 0.$$

The proof of this lemma can be carried out with the same approach as that of Lemma 5.1 and will be omitted here.

Here are three theorems for simultaneous boundedness (or compactness) of Hankel operators H_f and $H_{\overline{f}}$ from A^p_φ to L^q_φ.

Theorem 5.3. *Let $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$, and let $1 \leqslant p \leqslant q < \infty$. Set $s = \frac{1}{q} - \frac{1}{p}$. Then for $f \in \mathcal{S}$, the following statements are equivalent:*

(i) *$H_f, H_{\overline{f}} : A^p_\varphi \to L^q_\varphi$ are simultaneously bounded;*

(ii) *for some (or any) $0 < r \leqslant \alpha$, $\rho^{2s} MO_{q,r}(f) \in L^\infty$;*

(iii) *f admits a decomposition $f = f_1 + f_2$, where $f_1 \in C^1(\mathbb{D})$ satisfying*

$$\rho^{2s}\mathrm{Osc}_r(f_1) \in L^\infty \quad \text{and} \quad \rho^{2s} M_{q,r}(f_2) \in L^\infty$$

for some (or any) $0 < r \leqslant \alpha$.

Furthermore,

$$\|H_f\|_{A^p_\varphi \to L^q_\varphi} + \|H_{\overline{f}}\|_{A^p_\varphi \to L^q_\varphi} \simeq \|\rho^{2s} MO_{q,r}(f)\|_{L^\infty}. \tag{5.5}$$

Theorem 5.4. *Let $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$, and let $1 \leqslant p \leqslant q < \infty$. Set $s = \frac{1}{q} - \frac{1}{p}$. Then for $f \in \mathcal{S}$, the following statements are equivalent:*

(i) *$H_f, H_{\overline{f}} : A^p_\varphi \to L^q_\varphi$ are simultaneously compact;*

(ii) *for some (or any) $0 < r \leqslant \alpha$, $\lim_{|z|\to 1}\rho^{2s}(z) MO_{q,r}(f)(z) = 0$;*

(iii) *f admits a decomposition $f = f_1 + f_2$, where $f_1 \in C^1(\mathbb{D})$ satisfying*

$$\lim_{|z|\to 1}\rho^{2s}(z)\mathrm{Osc}_r(f_1)(z) = 0 \quad \text{and} \quad \lim_{|z|\to 1}\rho^{2s}(z) M_{q,r}(f_2)(z) = 0$$

for some (or any) $0 < r \leqslant \alpha$.

Theorem 5.5. *Let $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$, and let $1 \leqslant q < p < \infty$. Set $s = \frac{1}{q} - \frac{1}{p}$. Then for $f \in \mathcal{S}$, the following statements are equivalent:*

(i) *$H_f, H_{\overline{f}} : A^p_\varphi \to L^q_\varphi$ are bounded;*

(ii) $H_f, H_{\overline{f}} : A^p_\varphi \to L^q_\varphi$ *are compact*;

(iii) *for some* (*or any*) $0 < r \leqslant \alpha$, $MO_{q,r}(f) \in L^{\frac{1}{s}}$;

(iv) $f = f_1 + f_2$ *with* $f_1 \in C(\mathbb{D})$,

$$\mathrm{Osc}_r(f_1) \in L^{\frac{1}{s}} \quad \textit{and} \quad M_{q,r}(f_2) \in L^{\frac{1}{s}}$$

for some (*or any*) $0 < r < \alpha$.

Furthermore, $\|H_f\|_{A^p_\varphi \to L^q_\varphi} \simeq \|MO_{q,r}(f)\|_{L^{\frac{1}{s}}}$.

Proof. The proof of Theorem 5.3–5.5 are in the same approach, so we only write out the one for Theorem 5.3 here.

Theorem 4.2 tells us that the statement (i) is equivalent to

$$\rho^{2s} G_{q,r}(f) + \rho^{2s} G_{q,r}(\overline{f}) \in L^\infty.$$

This by Lemma 5.2 is equivalent to the statement (ii). The equivalence between (ii) and (iii) comes from Lemma 5.2 as well. □

When f is holomorphic, it is trivial that $H_f = 0$. Furthermore, for fixed $0 < r \leqslant \alpha$ there are two positive constants C_1 and C_2 such that

$$C_1 \rho(z)|f'(z)| \leqslant MO_{q,r}(f)(z) \leqslant C_2 \sup_{\xi \in D^r(z)} \rho(\xi)|f'(\xi)|.$$

Therefore we have the following theorem on Hankel operators with conjugate holomorphic symbols. The case where $\varphi \in \mathcal{BDK}$ and $p = q = 2$, was previously obtained in [9].

Theorem 5.6. *Let* $\varphi \in \mathcal{W}_0$ *with* $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$, *and set* $s = \frac{1}{q} - \frac{1}{p}$ *for* $1 \leqslant p, q < \infty$. *Then for* $f \in \mathcal{S} \cap H(\mathbb{D})$, *the following statements are true*:

(i) *For* $p \leqslant q$, $H_{\overline{f}}$ *is bounded from* A^p_φ *to* L^q_φ *if and only if* $\rho^{2s+1} f' \in L^\infty$; $H_{\overline{f}}$ *is compact from* A^p_φ *to* L^q_φ *if and only if* $\lim_{|z|\to 1} \rho^{2s+1} f'(z) = 0$.

(ii) *For* $p > q$, $H_{\overline{f}}$ *is bounded from* A^p_φ *to* L^q_φ *if and only if* $H_{\overline{f}}$ *is compact from* A^p_φ *to* L^q_φ *if and only if* $\rho f' \in L^{\frac{1}{s}}$.

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