SCIENCE CHINA
Mathematics

• ARTICLES •

November 2021 Vol. 64 No. 11: 2479–2494
https://doi.org/10.1007/s11425-020-1689-1

Comparison theorems for GJMS operators

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Received January 19, 2020; accepted May 8, 2020; published online August 4, 2020

Abstract In this paper, we compare the first order fractional GJMS (see Graham et al. (1992)) operator P_1 with the conformal Laplacian P_2 on the conformal infinity of a Poincaré-Einstein manifold. We derive some inequalities between the Yamabe constants and the first eigenvalues associated with P_1 and P_2, and prove some rigidity theorems by characterizing the equalities. Similarly, some comparison theorems between P_2 and the Paneitz operator P_4 or the 6th order GJMS operator P_6 are also given.

Keywords GJMS operator, Poincaré-Einstein manifold, comparison theorem

MSC(2020) 53C21

Citation: Wang F, Zhou H H. Comparison theorems for GJMS operators. Sci China Math, 2021, 64: 2479–2494, https://doi.org/10.1007/s11425-020-1689-1

1 Introduction

In this paper, we consider the fractional powers of Laplacian on the conformal infinity of a Poincaré-Einstein manifold introduced by Graham and Zworski [13]. Suppose that (X^{n+1}, g_+) is a $C^{m,\alpha}$ conformally compact Poincaré-Einstein manifold with the conformal infinity $(\partial X, [\hat{g}])$, i.e., X is identified with the interior of a compact manifold with boundary $\overline{X}$, and $\bar{g} = \rho^2 g_+$ can be $C^{m,\alpha}$ extended to $\overline{X}$ and satisfies

$$\begin{cases} \mathrm{Ric}_{g_+} = -n g_+, \\ \bar{g}\,|_{\partial X} \in [\hat{g}], \end{cases}$$

where ρ is a smooth boundary defining function. Here, we require that $n \geqslant 3$ and $m \geqslant 3$. If one fixes a smooth representative $\hat{g}$ on ∂X, and an identification of $\partial X \times [0, \epsilon)$ with the collar neighborhood of ∂X, then by the boundary regularity theorem in [7], there exists a geodesic normal defining function x such that

$$g_+ = x^{-2}(dx^2 + G(x)),$$

where $G(x)$ is a family of metrics on ∂X with the expansion

$$G(x) = \begin{cases} g_0 + x^2 g_2 + \cdots + x^{n-1} g_{n-1} + x^n g_n + O(x^{n+1}), & n \text{ is odd}, \\ g_0 + x^2 g_2 + \cdots + x^{n-2} g_{n-2} + (x^n \log x) h + x^n g_n + O(x^{n+1} \log x), & n \text{ is even}. \end{cases}$$

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Here, $g_0 = \hat{g}$, g_i $(1 \leqslant i \leqslant n-1)$ and h are symmetric 2-tensors determined by $\hat{g}$, and g_n is the global term which cannot be locally determined.

Consider a fractional power $0 < \gamma < \frac{n}{2}$ and define $s = \frac{n}{2} + \gamma$. Assume $\gamma \notin \mathbb{N}$ and $\frac{n^2}{4} - \gamma^2 \notin \mathrm{Spec}(-\Delta_+)$. Then given any $f \in C^\infty(\partial X)$, there is a unique solution satisfying the following equation:

$$-\Delta_+ u - s(n-s)u = 0, \quad x^{s-n} u\,|_{\partial X} = f.$$

Moreover, u takes the form

$$u = x^{n-s}F + x^s G, \quad F\,|_M = f, \quad F, G \in C^{m,\alpha}(\overline{X}).$$

We can define the scattering operator $S(s)$ by

$$S(s)f = G\,|_M.$$

In addition, the fractional GJMS operator is defined by the renormalized scattering operator

$$P_{2\gamma}^{\hat{g}} = 2^{2\gamma}\frac{\Gamma(\gamma)}{\Gamma(-\gamma)} S\left(\frac{n}{2} + \gamma\right), \quad Q_{2\gamma}^{\hat{g}} = \frac{2}{n-2\gamma} P_{2\gamma} 1.$$

Here, $Q_{2\gamma}^{\hat{g}}$ is called the fractional Q-curvature.

This family of operators are intensively studied in [3, 13, 19, 24]. The most important property includes that $P_{2\gamma}^{\hat{g}}$ is a self-adjoint and conformal covariant pseudo-differential elliptic operator with the principal symbol $|\xi|_{\hat{g}}^{2\gamma}$. The meromorphic extension of $S(s)$ given in [19] implies that $P_{2\gamma}^{\hat{g}}$ is a continuous family of operators in the real parameter γ as long as

$$\frac{n^2}{4} - \gamma^2 \notin \mathrm{Spec}(-\Delta_+).$$

While $\gamma = k \leqslant \frac{n}{2}$ is a positive integer, it coincides with the classical GJMS operator of order $2k$, which is only determined by the boundary metric. For example, if $\gamma = 1$, then $P_2^{\hat{g}}$ is the conformal Laplacian

$$P_2^{\hat{g}} = -\Delta_{\hat{g}} + \frac{n-2}{2} J_{\hat{g}}, \quad J_{\hat{g}} = \frac{1}{2(n-1)} R_{\hat{g}}.$$

If $\gamma = 2$, then $P_4^{\hat{g}}$ is the Paneitz operator

$$P_4^{\hat{g}} = (-\Delta_{\hat{g}})^2 + \delta_{\hat{g}}((n-2)J_{\hat{g}} - 4A_{\hat{g}})d + \frac{n-4}{2} Q_4^{\hat{g}}.$$

Here, $\delta_{\hat{g}}$ is the divergence operator with respect to $\hat{g}$, $A_{\hat{g}}$ is the Schouten tensor with respect to $\hat{g}$ and

$$Q_4^{\hat{g}} = -\Delta_{\hat{g}} J_{\hat{g}} + \frac{n}{2} J_{\hat{g}}^2 - 2|A_{\hat{g}}|^2$$

(see Section 3 for more details). In this paper, we mainly study the fractional Yamabe constants $Y_{2\gamma}(\partial X, [\hat{g}])$ and the first eigenvalues associated with $P_{2\gamma}^{\hat{g}}$, which are defined by the following:

$$Y_{2\gamma}(\partial X, [\hat{g}]) = \inf_{f \in C^\infty(\partial X), f > 0} \frac{\oint_{\partial X} f P_{2\gamma}^{\hat{g}} f \, dS_{\hat{g}}}{(\oint_{\partial X} f^{\frac{2n}{n-2\gamma}} dS_{\hat{g}})^{\frac{n-2\gamma}{n}}},$$

$$\lambda_1(P_{2\gamma}^{\hat{g}}) = \inf_{f \in C^\infty(\partial X)} \frac{\oint_{\partial X} f P_{2\gamma}^{\hat{g}} f \, dS_{\hat{g}}}{\oint_{\partial X} f^2 dS_{\hat{g}}}.$$

A standard example is the ball model of the hyperbolic space $\mathbb{H}^{n+1}$, which has the conformal infinity $(\mathbb{S}^n, [g_{\mathbb{S}}])$, where $g_{\mathbb{S}}$ is the round metric on $\mathbb{S}^n$. In this case,

$$\mathrm{Spec}(-\Delta_+) = \left[\frac{n^2}{4}, +\infty\right).$$

Hence for all $\gamma \in (0, \frac{n}{2})$, $P_{2\gamma}^{g_{\mathbb{S}}}$ can be represented by

$$P_{2\gamma}^{g_{\mathbb{S}}} = \frac{\Gamma(B + \frac{1}{2} + \gamma)}{\Gamma(B + \frac{1}{2} - \gamma)}, \quad \text{where } B = \sqrt{-\Delta_{g_{\mathbb{S}}} + \left(\frac{n-1}{2}\right)^2} \tag{1.1}$$

(see [2]). Then the fractional Q-curvature is

$$Q_{2\gamma}^{g_{\mathbb{S}}} = \frac{2}{n-2\gamma} P_{2\gamma}^{g_{\mathbb{S}}}(1) = \frac{2}{n-2\gamma} \frac{\Gamma(\frac{n}{2} + \gamma)}{\Gamma(\frac{n}{2} - \gamma)},$$

the fractional Yamabe constant is

$$Y_{2\gamma}(\mathbb{S}^n, [g_{\mathbb{S}}]) = 2^{\frac{2\gamma}{n}} \pi^{\frac{\gamma(n+1)}{n}} \frac{\Gamma(\frac{n}{2} + \gamma)}{\Gamma(\frac{n}{2} - \gamma)} \left[\Gamma\left(\frac{n+1}{2}\right)\right]^{-\frac{2\gamma}{n}} = \frac{\Gamma(\frac{n}{2} + \gamma)}{\Gamma(\frac{n}{2} - \gamma)} |\mathbb{S}^n|^{\frac{2\gamma}{n}},$$

and the first eigenvalue is

$$\lambda_1(P_{2\gamma}^{g_{\mathbb{S}}}) = P_{2\gamma}^{g_{\mathbb{S}}}(1) = \frac{\Gamma(\frac{n}{2} + \gamma)}{\Gamma(\frac{n}{2} - \gamma)}.$$

We are aiming to understand the family property of those operators by comparing them with the standard model. We first give the comparison theorems between $P_1^{\hat{g}}$ and $P_2^{\hat{g}}$.

Theorem 1.1. *Suppose that (X^{n+1}, g_+) $(n \geqslant 3)$ is a Poincaré-Einstein manifold, which is $C^{3,\alpha}$ conformally compact with the conformal infinity $(\partial X, [\hat{g}])$. Assume that $\frac{n^2-1}{4} \notin \operatorname{Spec}(-\Delta_+)$ and $Y_1(\partial X, [\hat{g}])$ can be achieved by some smooth representative $\hat{g}$. Then*

$$\frac{Y_2(\partial X, [\hat{g}])}{Y_2(\mathbb{S}^n, [g_{\mathbb{S}}])} \leqslant \left(\frac{Y_1(\partial X, [\hat{g}])}{Y_1(\mathbb{S}^n, [g_{\mathbb{S}}])}\right)^2, \tag{1.2}$$

and the equality holds if and only if (X, g_+) is isometric to the hyperbolic space $\mathbb{H}^{n+1}$.

Here, we need a condition that Y_1 can be achieved. This relies on the solvability of the fractional Yamabe problem for $\gamma = \frac{1}{2}$, or equivalently the second type Escobar-Yamabe problem on the manifold with the boundary [9]. Since a $C^{3,\alpha}$ conformal compactification of the Poincaré-Einstein manifold always has umbilical boundary, the condition is satisfied in the following cases:

(a) the dimension $2 \leqslant n \leqslant 6$;

(b) the dimension $n \geqslant 7$ and X is spin;

(c) the dimension $n \geqslant 7$ and X is locally conformally flat.

Please refer to [1, 5, 8–10, 22] for more details. By a certain type compactification, we relate Q_1 to the boundary mean curvature of the compact metric and hence (1.2) can be transformed to the inequality between the boundary Yamabe constant and the second type Yamabe constant for the manifolds with the boundary given in [6] (see Section 2 for more details).

Theorem 1.2. *Suppose that (X^{n+1}, g_+) $(n \geqslant 3)$ is a Poincaé-Einstein manifold with the conformal infinity $(\partial X, [\hat{g}])$. Assume that $\frac{n^2-1}{4} \notin \operatorname{Spec}(\Delta_+)$ and $\bar{g}$ is a $C^{3,\alpha}$ conformal compactification such that the interior scalar curvature $R_{\bar{g}}$ vanishes and the boundary mean curvature $H_{\bar{g}}$ is a constant. Define $\hat{g} = \bar{g}\,|_{\partial X}$. Then*

$$\frac{\lambda_1(P_2^{\hat{g}})}{\lambda_1(P_2^{g_{\mathbb{S}}})} \leqslant \left(\frac{\lambda_1(P_1^{\hat{g}})}{\lambda_1(P_1^{g_{\mathbb{S}}})}\right)^2, \tag{1.3}$$

and the equality holds if and only if (X, g_+) is isometric to the hyperbolic space $\mathbb{H}^{n+1}$ and $(\overline{X}, \bar{g})$ is the flat ball.

By [1, 5, 9, 10, 22, 23], every $(n+1)$-dimensional compact Riemannian manifold with the boundary $(n \geqslant 2)$ carries a conformal scalar flat metric with respect to which its boundary has constant mean curvature. (This metric might not be the minimizer such that Y_1 is achieved in Theorem 1.1.) While taking this metric to be the conformal compactification of the Poincaré-Einstein manifold, we can find a particular compactification, as well as a boundary representative, such that its Q_1 is constant. By relating Q_1 to the boundary mean curvature, (1.3) can be transformed to the inequality given by [25, Theorem 3.1]. We provide a slightly different proof here. We want to point out that for $P_1^{\hat{g}}$ being well defined, the condition

$$\frac{n^2-1}{4} \notin \mathrm{Spec}(\Delta_+),$$

or an equivalent condition for the compactification, is necessary.

We also give some similar comparison theorems between the classical GJMS operators P_2, P_4 and P_6. However, in this case, we only need to work on a general closed Riemannian manifold (M^n, h) with no requirement of Poincaré-Einstein filling-in. Under certain geometric conditions, we derive some inequalities between the Yamabe constants and between the first eigenvalues associated with them. Some characterisation of the equalities and applications in conformal geometry is also given.

Theorem 1.3. *Suppose that (M^n, h) $(n \geqslant 5)$ is a smooth closed compact Riemannian manifold. Then*

$$\frac{Y_4(M,[h])}{Y_4(\mathbb{S}^n,[g_{\mathbb{S}}])} \leqslant \left(\frac{Y_2(M,[h])}{Y_2(\mathbb{S}^n,[g_{\mathbb{S}}])}\right)^2. \tag{1.4}$$

Moreover, if the equality holds, then there exists an Einstein metric $g \in [h]$ such that both $Y_2(M,[h])$ and $Y_4(M,[h])$ can be achieved by g.

A question left is that when h is a positive Einstein whether the equality in (1.4) holds or not. This leads to the validity of Obata's theorem for the 4th order Yamabe problem, which is not completely clear currently. Theorem 1.3 directly implies the following conclusion for the 4th order Yamabe problem.

Corollary 1.4. *Suppose that (M^n, h) $(n \geqslant 5)$ is a smooth closed compact Riemannian manifold such that the Yamabe constant satisfies*

$$-Y_2(\mathbb{S}^n,[g_{\mathbb{S}}]) < Y_2(M,[h]) \leqslant Y_2(\mathbb{S}^n,[g_{\mathbb{S}}]).$$

Then

$$Y_4(M,[h]) \leqslant Y_4(\mathbb{S}^n,[g_{\mathbb{S}}]) \tag{1.5}$$

and the equality holds if and only if (M,h) is conformally equivalent to $(\mathbb{S}^n, g_{\mathbb{S}})$.

See [16–18] for more results on the 4th order Yamabe problem. Similarly, we also have a comparison theorem for the first eigenvalues of P_2^h and P_4^h.

Theorem 1.5. *Suppose that (M^n, h) $(n \geqslant 5)$ is a smooth closed compact Riemannian manifold such that the scalar curvature is constant. Then*

$$\frac{\lambda_1(P_4^h)}{\lambda_1(P_4^{g_{\mathbb{S}}})} \leqslant \left(\frac{\lambda_1(P_2^h)}{\lambda_1(P_2^{g_{\mathbb{S}}})}\right)^2. \tag{1.6}$$

Moreover,

(i) *if (M,h) is a nonnegative Einstein manifold, then the equality holds*;

(ii) *if the equality holds, then (M,h) must be Einstein.*

For P_2^h and P_6^h, we have the following comparison theorem. Let W_h be the Weyl tensor and E_h be the trace free part of the Ricci tensor with respect to the metric h.

Theorem 1.6. *Suppose that (M^n, h) $(n \geqslant 7)$ is a smooth closed compact Riemannian manifold of positive Yamabe type. If h is a Yamabe metric and satisfies*

$$|W_h| + \frac{2|E_h|}{(n-2)^2\sqrt{n(n-1)}} < C(n)J, \quad \text{where } C(n) = \frac{n^3-4n^2-4n+48}{8n(n-2)^2}, \tag{1.7}$$

then

$$\frac{Y_6(M,[h])}{Y_6(\mathbb{S}^n,[g_{\mathbb{S}}])} \leqslant \left(\frac{Y_2(M,[h])}{Y_2(\mathbb{S}^n,[g_{\mathbb{S}}])}\right)^3 \quad \text{and} \quad \frac{\lambda_1(P_6^h)}{\lambda_1(P_6^{g_{\mathbb{S}}})} \leqslant \left(\frac{\lambda_1(P_2^h)}{\lambda_1(P_2^{g_{\mathbb{S}}})}\right)^3.$$

Moreover, if one of the equalities holds, then (M,h) is Einstein.

Notice that the condition (1.7) is fulfilled if a Yamabe metric h is locally conformally flat and satisfies either $Q_4^h \geqslant 0$ or its Schouten tensor A_h is semi-positive (see Section 4 for more details). In addition, a direct conclusion for the 6th Yamabe problem is the following corollary.

Corollary 1.7. *Suppose that* (M^n, h) $(n \geqslant 7)$ *is a closed smooth compact manifold and satisfies the assumption of Theorem* 1.6. *Then*

$$Y_6(M, [h]) \leqslant Y_6(\mathbb{S}^n, [g_{\mathbb{S}}])$$

and the equality holds if and only if (M, h) *is conformally equivalent to* $(\mathbb{S}^n, g_{\mathbb{S}})$.

The rest of this paper is outlined as follows. In Section 2, we mainly study the relationship between $P_1^{\hat{g}}$ and $P_2^{\hat{g}}$ under the Poincaré-Einstein setting $(X, g_+; \partial X, [\hat{g}])$, and prove Theorems 1.1 and 1.2. In Section 3, we study the relationship between P_2^h and P_4^h on a general closed manifold (M^n, h) and prove Theorem 1.3, Corollary 1.4 and Theorem 1.5. In Section 4, we study the relationship between P_2^h and P_6^h and prove Theorem 1.6 and Corollary 1.7.

2 $P_1^{\hat{g}}$ vs. $P_2^{\hat{g}}$

Suppose that (X^{n+1}, g_+) is a Poincaré-Einstein manifold and $\bar{g} = \rho^2 g_+$ is a $C^{3,\alpha}$ conformal compactification. Define

$$\hat{g} = \bar{g}\,|_{\partial X}.$$

Let $R_{\bar{g}}$ and $R_{\hat{g}}$ be the scalar curvature of $\bar{g}$ and $\hat{g}$, and $E_{\bar{g}}$ be the trace free part of the interior Ricci curvature $\mathrm{Ric}_{\bar{g}}$. Let $H_{\bar{g}}$ be the boundary mean curvature. Notice that as a $C^{3,\alpha}$ compactification of the Poincaré-Einstein manifold, $(\overline{X}, \bar{g})$ always has umbilical boundary. Hence the second fundamental form of the boundary is

$$\Pi_{\bar{g}} = \frac{1}{n} H_{\bar{g}} \hat{g}.$$

We first derive some curvature identities for this system. In particular, the following integral formula was first introduced in [15] under a slightly different assumption.

Lemma 2.1. *If* $R_{\bar{g}} = 0$, *then* $H_{\bar{g}} = n Q_1^{\hat{g}}$ *and*

$$\frac{2}{(n-1)^2} \int_X \rho |E_{\bar{g}}|^2 dV_{\bar{g}} = \oint_{\partial X} \left(\frac{1}{n} H_{\bar{g}}^2 - \frac{1}{n-1} R_{\hat{g}} \right) dS_{\hat{g}}. \tag{2.1}$$

Proof. Take x to be the geodesic normal defining function with respect to $(g_+, \hat{g})$. Then by [7],

$$g_+ = x^{-2}(dx^2 + G(x)), \quad G(x) = \hat{g} + x^2 g_2 + \cdots$$

in a collar neighborhood $[0, \epsilon)_x \times \partial X$. In particular, $g_2 = -A_{\hat{g}}$ is the Schouten tensor of $\hat{g}$. For $\bar{g} = \rho^2 g_+$ satisfying $R_{\bar{g}} = 0$, $v = \rho^{\frac{n-1}{2}}$ satisfies

$$\Delta_+ v - \frac{n^2 - 1}{4} v = 0, \quad x^{\frac{1-n}{2}} v\,|_{\partial X} = 1.$$

It is equivalent to saying ρ is the adapted defining function in the sense of [3]. Therefore, v has asymptotical expansion

$$\begin{aligned} &v = x^{\frac{n-1}{2}}(1 + x v_1 + x^2 v_2 + \cdots), \\ &v_1 = -P_1^{\hat{g}}(1) = -\frac{n-1}{2} Q_1^{\hat{g}}, \quad v_2 = \frac{1}{8} R_{\hat{g}}. \end{aligned}$$

This implies that ρ has asymptotical expansion

$$\begin{aligned} &\rho = x(1 + x\rho_1 + x^2 \rho_2 + \cdots), \\ &\rho_1 = -Q_1^{\hat{g}}, \quad \rho_2 = \frac{1}{4(n-1)} R_{\hat{g}} - \frac{n-3}{4} (Q_1^{\hat{g}})^2. \end{aligned}$$

Then

$$\bar{g} = \rho^2 g_+ = [1 + x\rho_1 + x^2\rho_2 + O(x^{2+\alpha})]^2(dx^2 + \hat{g} + x^2 g_2 + \cdots).$$

This implies that $\Pi_{\bar{g}} = -\rho_1 \hat{g}$ with respect to the outward unit normal on the boundary and hence

$$H = -n\rho_1 = nQ_1^{\hat{g}}.$$

Next, the conformal transformation of the Ricci curvature under $g_+ = \rho^{-2}\bar{g}$ gives

$$E_{\bar{g}} = -(n-1)\rho^{-1}\left(\bar{\nabla}^2\rho - \frac{1}{n+1}(\Delta_{\bar{g}}\rho)\bar{g}\right),$$

where $\bar{\nabla}$ and $\Delta_{\bar{g}}$ are the covariant derivative and Beltrami-Laplacian with respect to $\bar{g}$. Therefore,

$$\begin{aligned}
\frac{2}{(n-1)^2}\int_X \rho|E_{\bar{g}}|^2 dV_{\bar{g}} &= -\frac{2}{n-1}\int_X E_{ij}\rho^{ij} dV_{\bar{g}} \\
&= -\frac{2}{n-1}\int_X E_{ij:}{}^{j}\rho^i dV_{\bar{g}} + \frac{2}{n-1}\oint_{\partial X} E_{\bar{g}}(\nu,\nu) dS_{\hat{g}} \\
&= \frac{2}{n-1}\oint_{\partial X} E_{\bar{g}}(\nu,\nu) dS_{\hat{g}}.
\end{aligned}$$

Here, $\nu = -\partial_x$ is the outward unit normal vector field on the boundary. By the Gauss-Codazzi equation and $R_{\bar{g}} = 0$, we have

$$2E_{\bar{g}}(\nu,\nu) = \frac{n-1}{n}H_{\bar{g}}^2 - R_{\hat{g}}.$$

Then the integral formula (2.1) follows. We finish the proof. □

We also recall a rigidity theorem from [4], which is first proved in [6].

Lemma 2.2. *Suppose that $(\overline{X}, \bar{g})$ is a $C^{3,\alpha}$ conformal compactification of the Poincaré-Einstein manifold (X, g_+). If the interior Ricci curvature $\mathrm{Ric}_{\bar{g}}$ vanishes and the boundary mean curvature $H_{\bar{g}}$ is a constant, then $(\overline{X}, \bar{g})$ is isometric to the flat ball $(\mathbb{B}^n, g_{\mathbb{R}})$ and (X, g_+) is isometric to the hyperbolic space $\mathbb{H}^n$.*

Proof. Notice here $R_{\hat{g}} = \frac{n-1}{n}H_{\bar{g}}^2$ by the Gauss-Codazzi equation and hence $R_{\hat{g}}$ is a constant. Consider the transformation of scalar curvature and Ricci curvature under conformal change $\bar{g} = \rho^2 g_+$, which gives

$$\Delta_{\bar{g}}\rho = \frac{n+1}{2}\rho^{-1}(|\bar{\nabla}\rho|_{\bar{g}}^2 - 1), \tag{2.2}$$

$$\bar{\nabla}^2\rho - \frac{1}{n+1}(\Delta_{\bar{g}}\rho)\bar{g} = -\frac{1}{n-1}\rho E_{\bar{g}} = 0. \tag{2.3}$$

By identifying a collar neighborhood of ∂X with $[0,\epsilon)\times\partial X$, $\bar{g}$ takes the normal form

$$\bar{g} = dr^2 + g(r), \tag{2.4}$$

where $g(r)$ is a family of metrics on ∂X with $g(0) = \hat{g}$. Then according to [11], ρ has asymptotical expansion

$$\rho = r + c_2 r^2 + c_3 r^3 + O(r^{3+\alpha}),$$

where

$$c_2 = -\frac{1}{2n}H_{\bar{g}}, \quad c_3 = \frac{1}{6(n-1)}R_{\hat{g}} - \frac{1}{6n}H_{\bar{g}}^2 = 0.$$

Direct computation shows that

$$\Delta_{\bar{g}}\rho\,|_{\partial X} = -\frac{n+1}{n}H_{\bar{g}}$$

and

$$\begin{aligned}
\bar{\nabla}_i(\Delta_{\bar{g}}\rho) &= (n+1)\rho^{-1}\rho_{ij}\rho^j - \frac{n+1}{2}\rho^{-2}(|\bar{\nabla}\rho|_{\bar{g}}^2 - 1)\rho_i \\
&= \rho^{-1}\left[\Delta_{\bar{g}} - \frac{n+1}{2}\rho^{-1}(|\bar{\nabla}\rho|_{\bar{g}}^2 - 1)\right]\rho_{:i} = 0.
\end{aligned}$$

Hence all over $\overline{X}$,

$$\Delta_{\bar{g}}\rho \equiv -\frac{n+1}{n}H_{\bar{g}}.$$

Since $\rho > 0$ in the interior, we have that $H_{\bar{g}}$ must be a positive constant. Then up to a scaling, we set $H_{\bar{g}} = n$ and hence $\Delta_{\bar{g}}\rho = -(n+1)$. Thus the equations (2.2) and (2.3) become

$$|\bar{\nabla}\rho|^2_{\bar{g}} - 1 + 2\rho = 0, \tag{2.5}$$

$$\bar{\nabla}^2\rho + \bar{g} = 0. \tag{2.6}$$

Moreover, $R_{\hat{g}} = \frac{n-1}{n}H^2_{\bar{g}} = n(n-1)$ implies that the boundary $(\partial X, \hat{g})$ has a positive Yamabe constant. By [26], ∂X is connected.

Take any normal geodesic $\gamma(t)$ such that $\gamma(0) = p \in \partial X$. Then $\gamma(t) = (t, p)$. By the equation (2.6), the function $f(t) = \rho(\gamma(t))$ satisfies

$$f''(t) + 1 = 0, \quad f(0) = 0, \quad f'(0) = \partial_r\rho\,|_{\partial X} = 1.$$

Hence in the small collar neighborhood,

$$f(t) = t - \frac{t^2}{2} \Rightarrow \rho = r - \frac{r^2}{2}. \tag{2.7}$$

On each hypersurface $\partial X_r = \{r = \text{constant}\}$ for r small, $\rho\,|_{\partial X_r}$ is a constant. Moreover, by (2.6), $\rho\,|_{\partial X_r}$ satisfies

$$\hat{\nabla}^2\rho - (\partial_r\rho)\Pi(r) + g(r) = 0,$$

where $\Pi(r)$ is the second fundamental form with respect to the outward unit normal $-\partial_r$. Here, $\hat{\nabla}$ is the connection on ∂X_r with respect to the metric $g(r)$. However, we know $\Pi(r) = -\frac{1}{2}g'(r)$ while using the normal form (2.4). This implies that

$$(1-r)g'(r) + 2g(r) = 0 \Rightarrow g(r) = (1-r)^2\hat{g}. \tag{2.8}$$

Those formulae (2.7) and (2.8) hold in the collar neighborhood such that the normal form (2.4) holds. At any point $0 < r_0 < 1$, if (2.8) holds, then (2.4) extends in a neighborhood $[r_0, r_0 + \epsilon)$ and hence (2.7) and (2.8) can also be extended. The extension will not stop until arriving at $r = 1$. Therefore,

$$\bar{g} = dr^2 + (1-r)^2\hat{g}, \quad 0 \leqslant r < 1.$$

When $r \to 1$, ∂X shrinks to one point since it is connected, which corresponds to the unique maximum point of ρ. The maximum point is non-degenerate and smooth. Hence $\hat{g}$ must be the standard sphere metric on $\mathbb{S}^n$. Therefore, by taking $t = 1 - r$,

$$(\overline{X}, \bar{g}) = ([0,1]_t \times \mathbb{S}^n, \bar{g} = dt^2 + t^2 g_{\mathbb{S}}),$$

which is the flat ball of radius one in $\mathbb{R}^{n+1}$. In addition, $g_+ = \rho^{-2}\bar{g}$ with $\rho = (1-t^2)/2$ shows that (X, g_+) is the standard hyperbolic space $\mathbb{H}^{n+1}$. □

Next, we generalize some positivity results for $P_{2\gamma}$ with $\gamma \in (0,1)$ by Guillarmou and Qing [14, Theorem 1.2] to the nonnegative case.

Lemma 2.3. *Suppose $Y_2(\partial X, [\hat{g}]) \geqslant 0$. Then for all $\gamma \in (0,1)$,*

(1) *if $\hat{g}$ is a representative such that $\hat{R} \geqslant 0$, then $Q^{\hat{g}}_{2\gamma} \geqslant 0$;*

(2) *the first eigenvalue of $P^{\hat{g}}_{2\gamma}$ is nonnegative.*

Proof. Fix $\hat{g}$ to be a boundary representative such that $R_{\hat{g}}$ is nonnegative and let x be the geodesic normal defining function with respect to $(g_+, \hat{g})$. We first construct a test function ϕ. Consider the following equation:

$$\Delta_+ v + (n+1)v = 0, \quad xv\,|_{\partial X} = 1.$$

Then the solution v is unique and positive, and has asymptotical expansion

$$v = x^{-1}(1 + x^2 v_2 + \cdots), \quad v_2 = \frac{1}{4n(n-1)} R_{\hat{g}}.$$

Take $\phi = v^{-(n-s)}$. By the computation of [21], for all $s \in (0, n)$,

$$\frac{\Delta_+ \phi}{\phi} - s(n-s) = (n-s)(n-s+1)\left(1 - \frac{|dv|^2_{g_+}}{v^2}\right) > 0 \quad \text{in } X.$$

This shows that $\operatorname{spec}(\Delta_+) = [\frac{n^2}{4}, +\infty)$. Here, $s = \frac{n}{2} + \gamma$ with $\gamma \in (0,1)$. So ϕ has asymptotical expansion

$$\phi = r^{\frac{n}{2}-\gamma}(1 + r^2 \phi_2 + \cdots), \quad \phi_2 = -\frac{(n-2\gamma)}{8n(n-1)} R_{\hat{g}} \leqslant 0.$$

To prove (1), let u be the unique solution of the following equation:

$$\Delta_+ u - \frac{n^2 - \gamma^2}{4} u = 0, \quad x^{-\frac{n-\gamma}{2}} u\,|_{\partial X} = 1.$$

Then u has asymptotical expansion

$$u = x^{\frac{n-\gamma}{2}}\left(1 + x^{2\gamma} S\left(\frac{n}{2} + \gamma\right)1 + x^2 u_2 + \cdots\right), \quad u_2 = \frac{n-2\gamma}{16(1-\gamma)(n-1)} R_{\hat{g}} \geqslant 0.$$

Moreover, u/ϕ satisfies

$$\Delta_+\left(\frac{u}{\phi}\right) = \left(s(n-s) - \frac{\Delta_+\phi}{\phi}\right)\left(\frac{u}{\phi}\right) + 2\nabla\left(\frac{u}{\phi}\right)\nabla(\ln\phi), \quad \frac{u}{\phi}\bigg|_{\partial X} = 1.$$

Applying the maximum principle to the above equation, we conclude that $u \leqslant \phi$ in X. Hence

$$S\left(\frac{n}{2} + \gamma\right)1 \leqslant 0 \Rightarrow Q^{\hat{g}}_{2\gamma} \geqslant 0.$$

To prove (2), let λ_1 be the first eigenvalue of $P^{\hat{g}}_{2\gamma}$ and f be the first eigenfunction, i.e., $P^{\hat{g}}_{2\gamma} f = \lambda_1 f$. Without loss of generality, we can assume

$$\max_{\partial X} f = f(p) > 0.$$

Let w be the unique solution of the following equation:

$$\Delta_+ w - \frac{n^2 - \gamma^2}{4} w = 0, \quad r^{-\frac{n-\gamma}{2}} w\,|_{\partial X} = f.$$

Then w has the asymptotical expansion

$$w = x^{\frac{n-\gamma}{2}}\left(f + x^{2\gamma} S\left(\frac{n}{2} + \gamma\right) f + x^2 w_2 + \cdots\right), \quad w_2 = \frac{1}{4(1-\gamma)}\left(\Delta_{\hat{g}} f + \frac{n-2\gamma}{4(n-1)} R_{\hat{g}} f\right).$$

Obviously, $w_2(p) \geqslant 0$. Similarly, w/ϕ satisfies

$$\Delta_+\left(\frac{w}{\phi}\right) = \left(s(n-s) - \frac{\Delta_+\phi}{\phi}\right)\left(\frac{w}{\phi}\right) + 2\nabla\left(\frac{w}{\phi}\right)\nabla(\ln\phi), \quad \frac{w}{\phi}\bigg|_{\partial X} = f.$$

Applying the maximum principle again, we have $u \leqslant \phi f(p)$ in X. Hence

$$S\left(\frac{n}{2} + \gamma\right) f\bigg|_p = 2^{-2\gamma} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)} \lambda_1 f(p) \leqslant 0 \Rightarrow \lambda_1 \geqslant 0.$$

If $\hat{h} = \mathrm{e}^{2\varphi}\hat{g}$ and $\tilde{f} = \mathrm{e}^{(\frac{n}{2}-\gamma)\varphi} f$, then

$$\oint_{\partial X} f P^{\hat{h}}_{2\gamma} f \, dS_{\hat{h}} = \oint_{\partial X} \tilde{f} P^{\hat{g}}_{2\gamma} \tilde{f} \, dS_{\hat{g}}.$$

The sign of the bottom spectrum does not depend on the choice of the boundary representative. □

Actually, in the nonnegative case, by [3,8], we know that the first eigenvalue vanishes if and only if the curvature vanishes (see [3, Theorem 7.3]).

Lemma 2.4. *Suppose that* $\mathrm{Spec}(\Delta_+) > \frac{n^2}{4} - \gamma^2$ *for some* $\gamma \in (0,1)$ *and* $\hat{g}$ *is a boundary representative such that* $Q_{2\gamma}^{\hat{g}} \geqslant 0$. *Then* $P_{2\gamma}^{\hat{g}} \geqslant 0$. *In addition,* $\mathrm{Ker}(P_{2\gamma}^{\hat{g}}) \neq \{0\}$ *if and only if* $Q_{2\gamma}^{\hat{g}} \equiv 0$.

2.1 Proof of Theorem 1.1

If $Y_2(\partial X, [\hat{g}]) < 0$, then the strict inequality in (1.2) holds automatically. So we only need to consider the case $Y_2(\partial X, [\hat{g}]) \geqslant 0$. In this case, $\mathrm{Spec}(\Delta_+) \geqslant \frac{n^2}{4}$ by [21]. Let $\hat{g}$ be the boundary representative such that $Y_1(\partial X, [\hat{g}])$ is achieved. Then

$$Q_1^{\hat{g}} = \frac{2}{n-1} Y_1(\partial X, [\hat{g}]) \mathrm{Vol}(\partial X, \hat{g})^{-\frac{1}{n}}$$

is a constant. Let ρ be the adapted defining function in the sense of [3]. Then $\bar{g} = \rho^2 g_+$ satisfies $R_{\bar{g}} = 0$. Here, ρ is uniquely determined by $\hat{g}$. Hence by Lemma 2.1, $H_{\bar{g}} = nQ_1^{\hat{g}}$ is also a constant and

$$\begin{aligned}
Y_2(\partial X, [\hat{g}]) &\leqslant \frac{n-2}{4(n-1)} \bigg(\oint_{\partial X} R_{\hat{g}} dS_{\hat{g}} \bigg) \mathrm{Vol}(\partial X, \hat{g})^{\frac{2}{n}-1} \\
&= \frac{n(n-2)}{4} \bigg(\oint_{\partial X} (Q_1^{\hat{g}})^2 dS_{\hat{g}} - \frac{2}{n(n-1)^2} \int_X \rho |E_{\bar{g}}|^2 dV_{\bar{g}} \bigg) \mathrm{Vol}(\partial X, \hat{g})^{\frac{2}{n}-1} \\
&\leqslant \frac{n(n-2)}{(n-1)^2} (Y_1(\partial X, [\hat{g}]))^2.
\end{aligned}$$

This proves the inequality (1.2). If the equality holds, then $E_{\bar{g}} = 0$ and $Y_2(\partial X, [\hat{g}])$ is attained by $\hat{g}$. Hence $R_{\hat{g}}$ is also a constant. By Lemma 2.2, (X, g_+) must be the hyperbolic space and $(\overline{X}, \bar{g})$ is the flat ball in $\mathbb{R}^{n+1}$.

2.2 Proof of Theorem 1.2

Notice that the bottom spectrum of $P_2^{\hat{g}}$ has the same sign with $Y_2(\partial X, [\hat{g}])$. If $Y_2(\partial X, [\hat{g}]) < 0$, then $\lambda_1(P_2) < 0$ and the strict inequality in (1.3) holds automatically. So we only need to deal with the case $Y_2(\partial X, [\hat{g}]) \geqslant 0$. In this case, $\mathrm{Spec}(\Delta_+) \geqslant \frac{n^2}{4}$ by [21] again. Let $\bar{g}$ be a conformally compactification such that $R_{\bar{g}} = 0$ and $H_{\bar{g}}$ is a constant. By Lemma 2.1, $Q_1^{\hat{g}} = H_{\bar{g}}/n$ is also a constant.

First, consider $Y_2(\partial X, [\hat{g}]) > 0$. Then by [14, Theorem 1.2], $P_1^{\hat{g}}$ has the positive spectrum and the positive Green function. Here, $Q_1^{\hat{g}}$ is a positive constant and $f \equiv 1$ is a positive eigenfunction corresponding to the eigenvalue $\frac{(n-1)}{2} Q_1^{\hat{g}}$. Hence $\lambda_1(P_1^{\hat{g}}) = \frac{(n-1)}{2} Q_1^{\hat{g}}$. Therefore,

$$\begin{aligned}
\lambda_1(P_2^{\hat{g}}) &\leqslant \frac{n-2}{4(n-1)} \bigg(\oint_{\partial X} R_{\hat{g}} dS_{\hat{g}} \bigg) \mathrm{Vol}(\partial X, \hat{g})^{-1} \\
&= \frac{n(n-2)}{4} \bigg(\oint_{\partial X} (Q_1^{\hat{g}})^2 dS_{\hat{g}} - \frac{2}{n(n-1)^2} \int_X \rho |E_{\bar{g}}|^2 dV_{\bar{g}} \bigg) \mathrm{Vol}(\partial X, \hat{g})^{-1} \\
&\leqslant \frac{n(n-2)}{(n-1)^2} (\lambda_1(P_1^{\hat{g}}))^2.
\end{aligned}$$

This proves the inequality (1.3). If the equality holds, then $E_{\bar{g}} = 0$. By Lemma 2.2, $(\overline{X}, \bar{g})$ is the flat ball in $\mathbb{R}^{n+1}$ and (X, g_+) must be the hyperbolic space.

Second, consider $Y_2(\partial X, [\hat{g}]) = 0$. Then $\lambda_1(P_2) = 0$. Let f be the first eigenfunction of $P_2^{\hat{g}}$. Then by the maximum principle of $P_2^{\hat{g}}$, we have $f > 0$. Take $\hat{h} = f^{\frac{4}{n-2}} \hat{g}$. Then the scalar curvature $R_{\hat{h}}$ vanishes. By Lemma 2.3, $Q_1^{\hat{h}} \geqslant 0$ and $\lambda_1(P_1^{\hat{h}}) \geqslant 0$. If $\lambda_1(P_1^{\hat{h}}) = 0$, then by [3, Theorem 7.3], $Q_1^{\hat{h}} \equiv 0$. Take $\bar{h}$ to be the adapted compactification of g_+ such that $\hat{h} = \bar{h}\,|_{\partial X}$. Then $(\overline{X}, \bar{h}; \partial X, \hat{h})$ satisfies

$$R_{\bar{h}} = R_{\hat{h}} = 0, \quad H_{\bar{h}} = 0.$$

By Lemma 2.1, this implies that $E_{\bar{h}} = 0$. Then by Lemma 2.2, $(\overline{X}, \bar{h})$ is a flat ball. However, this contradicts the vanishing mean curvature. So $\lambda_1(P_1^{\hat{h}})$ must be positive. Take $F = f^{-\frac{n-1}{n-2}} > 0$. Then

$$P_1^{\hat{h}}(F) = \frac{n-1}{2} Q_1^{\hat{g}} F^{\frac{n+1}{n-1}},$$

which implies

$$Q_1^{\hat{g}} = \frac{2}{n-1} \frac{\oint_{\partial X} F P_1^{\hat{h}} F dS_{\hat{h}}}{\oint_{\partial X} F^{\frac{2n}{n-1}} dS_{\hat{h}}} > 0.$$

By [3, Theorem 7.3] again, this implies that $\lambda_1(P_1^{\hat{g}}) > 0$. Hence the strict inequality of (1.3) holds. We finish the proof.

3 P_2 vs. P_4 on a closed compact Riemannian manifold (M, h)

In this section, we prove some comparison theorems between the 2nd and 4th order GJMS operators on a general smooth closed compact Riemannian manifold (M^n, h) $(n \geqslant 5)$. For simplicity, while there is no confusion about the background metric, we will omit the metric notation in the subscript or superscript. Recall that

$$P_2 = -\Delta + \frac{n-2}{2} J,$$
$$P_4 = (-\Delta)^2 + \delta((n-2)J - 4A)d + \frac{n-4}{2} Q_4,$$

where $\delta = \delta_h$ is the divergence operator: while it acts on 1-form ω,

$$\delta_h \omega = -\omega_{i;}{}^{i},$$

where ; denotes the covariant derivatives with respect to the metric h. Here, the curvature terms are

$$J = \frac{1}{2(n-1)} R,$$
$$E = \mathrm{Ric} - \frac{1}{n} Rh,$$
$$A = \frac{1}{n-2}(\mathrm{Ric} - Jh) = \frac{1}{(n-2)} E + Jh,$$
$$Q_4 = -\Delta J + \frac{n}{2} J^2 - 2|A|^2 = -\Delta J + \frac{(n+2)(n-2)}{2n} J^2 - \frac{2}{(n-2)^2} |E|^2.$$

3.1 Proof of Theorem 1.3

Without loss of generality, we assume that h is the representative such that $Y_2(M, [h])$ is achieved. Then J is constant and

$$Y_2(M, [h]) = \frac{n-2}{2} J \mathrm{Vol}(M, h)^{\frac{2}{n}},$$

where $\mathrm{Vol}(M, h)$ denotes the volume of (M, h). Then

$$Q_4 = \frac{(n+2)(n-2)}{2n} J^2 - \frac{2}{(n-2)^2} |E|^2.$$

By the definition of $Y_4(M, [h])$,

$$\begin{aligned} Y_4(M, [h]) &\leqslant \frac{n-4}{2} \left(\int_M Q_4 dV_h \right) \mathrm{Vol}(M, h)^{\frac{-(n-4)}{n}} \\ &= \frac{(n-4)(n+2)}{n(n-2)} (Y_2(M, [h]))^2 - \frac{n-4}{(n-2)^2} \left(\int_M |E_h|^2 dV_h \right) \mathrm{Vol}(M, h)^{\frac{-(n-4)}{n}} \\ &\leqslant \frac{(n-4)(n+2)}{n(n-2)} (Y_2(M, [h]))^2. \end{aligned}$$

So we have proved the inequality (1.4).

If the equality in (1.4) holds, then the two inequalities in the above formula become equalities. Hence $E=0$, i.e., (M,h) is Einstein and h achieves $Y_4(M,[h])$ and $Y_2(M,[h])$ together.

3.2 Proof of Corollary 1.4

While $-Y_2(\mathbb{S}^n,[g_{\mathbb{S}}]) < Y_2(M,[h]) \leqslant Y_2(\mathbb{S}^n,[g_{\mathbb{S}}])$, Theorem 1.3 directly implies that

$$Y_4(M,[h]) \leqslant Y_4(\mathbb{S}^n,[g_{\mathbb{S}}]).$$

If the equality holds, then

$$Y_4(\mathbb{S}^n,[g_{\mathbb{S}}]) = Y_4(M,[h]) \leqslant \frac{(n-4)(n+2)}{n(n-2)}(Y_2(M,[h]))^2 \leqslant \frac{(n-4)(n+2)}{n(n-2)}(Y_2(\mathbb{S}^n,[g_{\mathbb{S}}]))^2.$$

This forces all the middle inequalities to be equalities. Hence $Y_2(M,[h]) = Y_2(\mathbb{S}^n,[g_{\mathbb{S}}])$ and (M,h) is conformally equivalent to $(\mathbb{S}^n, g_{\mathbb{S}})$.

3.3 Proof of Theorem 1.5

Here, h has constant scalar curvature. Hence the first eigenvalue of P_2 is given by

$$\lambda_1(P_2) = \frac{n-2}{2}J.$$

Then

$$\begin{aligned}\lambda_2(P_4) \leqslant\ & \frac{n-4}{2}\bigg(\int_M Q_4 dV_h\bigg)\mathrm{Vol}(M,h)^{-1}\\ =\ & \frac{(n-4)(n+2)}{n(n-2)}(\lambda_1(P_2))^2 - \frac{n-4}{(n-2)^2}\bigg(\int_M |E|^2 dV_h\bigg)\mathrm{Vol}(M,h)^{-1}\\ \leqslant\ & \frac{(n-4)(n+2)}{n(n-2)}(\lambda_1(P_2))^2.\end{aligned}$$

So we have proved the inequality (1.5).

If the equality in (1.5) holds, then the two inequalities in the above formulae become equalities. Hence $E=0$, i.e., (M,h) is Einstein.

Conversely, if (M,h) is Einstein, then

$$P_4 = \bigg(-\Delta + \frac{n-2}{2}J\bigg)\bigg(-\Delta + \frac{(n-4)(n+2)}{2n}J\bigg).$$

If further $J \geqslant 0$, then

$$\lambda_1(P_4) = \frac{(n-4)(n-2)(n+2)}{4n}J^2 = \frac{(n-4)(n+2)}{n(n-2)}(\lambda_1(P_2))^2.$$

4 P_2 vs. P_6 on a closed compact Riemannian manifold (M,h)

In this section, we prove some comparison theorems between the 2nd and 6th order GJMS operators on a general smooth closed compact Riemannian manifold (M^n,h) $(n \geqslant 7)$. Similar to Section 3, while there is no confusion about the background metric, we omit the metric notation in the subscript or superscript. Recall the formulae for P_6 from [20, Theorem 10.2]:

$$P_6 = (-\Delta)^3 - \Delta\delta T_2 d - \delta T_2 d\Delta - \frac{n-2}{2}\Delta(J\Delta) - \delta T_4 d + \frac{n-6}{2}Q_6.$$

Here,

$$
\begin{aligned}
T_2 &= (n-2)J - 8A,\\
T_4 &= -\left(\frac{3}{4}(n-2)^2 - 4\right)J^2 + 4(n-4)|A|^2 + 8(n-2)JA\\
&\quad + (n-6)\Delta J - 48A^2 - \frac{16}{n-4}B,\\
Q_6 &= -\left(\frac{n}{2}+1\right)\Delta(J^2) + 4\Delta(|A|^2) - 8\delta(AdJ) + \Delta^2 J\\
&\quad - \frac{n-6}{2}J\Delta J - 4(n-6)J|A|^2 + \frac{(n-6)(n+6)}{4}J^3 - 3!2!2^5 v_6,
\end{aligned}
$$

where

$$
\begin{aligned}
v_6 &= -\frac{1}{48}[J^3 - 3J|A|^2 + 2\mathrm{tr}(A^3)] - \frac{1}{24(n-4)}\langle B, A\rangle,\\
B_{ij} &= \Delta A_{ij} - A_{ik;j}{}^{k} + W_{ikjl}A^{kl},\\
W_{ijkl} &= R_{ijkl} + (A_{jk}h_{il} + A_{il}h_{jk} - A_{ik}h_{jl} - A_{jl}h_{ik}).
\end{aligned}
$$

Notice that "tr" denotes the trace of a symmetric 2-tensor with respect to the metric h and $\langle\cdot,\cdot\rangle$ denotes the inner product with respect to metric h. By direct computations, we can also write Q_6 as

$$
\begin{aligned}
Q_6 &= -\left(\frac{n}{2}+1\right)\Delta(J^2) + 4\Delta(|A|^2) - 8\delta(AdJ) + \Delta^2 J\\
&\quad - \frac{n-6}{2}J\Delta J + \frac{n^2-4}{4}J^3 - 4nJ|A|^2 + 16\mathrm{tr}(A^3) + \frac{16}{n-4}\langle B, A\rangle.
\end{aligned}
$$

Lemma 4.1. *On (M,h), we have*

$$
\int_M \langle B, A\rangle dV_h = \int_M (-|\nabla A|^2 + |\delta A|^2 + 2W_{ikjl}A^{ij}A^{kl} - n\mathrm{tr}(A^3) + J|A|^2)dV_h, \tag{4.1}
$$

and hence

$$
\begin{aligned}
\int_M Q_6 dV_h = \int_M \bigg[&\left(\frac{n-6}{2} + \frac{16(n-1)}{(n-4)n}\right)|\nabla J|^2 - \frac{16}{(n-4)(n-2)^2}|\nabla E|^2\\
&+ \frac{32}{n-4}W_{ikjl}E^{ij}E^{lk} + \frac{(n^2-16)(n^2-4)}{4n^2}J^3\\
&- \frac{4n^2(n-4) - 16n + 192}{n(n-4)(n-2)^2}J|E|^2 - \frac{64}{(n-4)(n-2)^3}\mathrm{tr}(E^3)\bigg]dV_h.
\end{aligned} \tag{4.2}
$$

Proof. First, notice that

$$
\begin{aligned}
A_{ik;jl} &= A_{ik;lj} - R_{imjl}A_k{}^m - R_{kmjl}A_i{}^m\\
&\Rightarrow A_{ik;j}{}^{k} = A_{ik;}{}^{k}{}_{j} - R_{imjl}A^{ml} + R_{mj}A_i{}^m.
\end{aligned}
$$

So we have

$$
\begin{aligned}
A_{ik;j}{}^{k}A^{ij} &= A_{ik;}{}^{k}{}_{j}A_{ij} - W_{imjl}A^{ml}A^{ij} + (A_{mj}h_{il} + A_{il}h_{mj} - A_{ij}h_{ml} - A_{ml}h_{ij})A^{ml}A^{ij} + R_{mj}A_i{}^mA^{ij}\\
&= A_{ik;}{}^{k}{}_{j}A_{ij} - W_{imjl}A^{ml}A^{ij} + 2\mathrm{tr}(A^3) - 2J|A|^2 + (n-2)A_{mj}A_i{}^mA^{ij} + Jh_{mj}A_i{}^mA^{ij}\\
&= A_{ik;}{}^{k}{}_{j}A_{ij} - W_{imjl}A^{ml}A^{ij} + n\mathrm{tr}(A^3) - J|A|^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int_M \langle B, A\rangle dV_h &= \int_M (-|\nabla A|^2 + |\delta A|^2 + 2W_{ikjl}A^{ij}A^{lk} - n\mathrm{tr}(A^3) + J|A|^2)dV_h,\\
\int_M Q_6 dV_h &= \int_M \left(-\frac{n-6}{2}J\Delta J + \frac{n^2-4}{4}J^3 - 4nJ|A|^2 + 16\mathrm{tr}(A^3) + \frac{16}{n-4}\langle B, A\rangle\right)dV_h\\
&= \int_M \bigg[\frac{n-6}{2}|\nabla J|^2 - \frac{16}{n-4}|\nabla A|^2 + \frac{16}{n-4}|\delta A|^2 + \frac{32}{n-4}W_{ikjl}A^{ij}A^{lk}\\
&\quad + \frac{n^2-4}{4}J^3 - \left(4n - \frac{16}{n-4}\right)J|A|^2 - \frac{64}{n-4}\mathrm{tr}(A^3)\bigg]dV_h.
\end{aligned}
$$

Direct computation shows that

$$\begin{aligned}
&\mathrm{tr}(A^3) = \frac{1}{n^2}J^3 + \frac{3}{n(n-2)^2}J|E|^2 + \frac{1}{(n-2)^3}\mathrm{tr}(E^3),\\
&|A|^2 = \frac{1}{n}J^2 + \frac{1}{(n-2)^2}|E|^2,\\
&|\nabla A|^2 = \frac{1}{n}|\nabla J|^2 + \frac{1}{(n-2)^2}|\nabla E|^2,\\
&|\delta A|^2 = |\nabla J|^2.
\end{aligned}$$

Hence we have (4.2). □

Proposition 4.2. *Suppose that (M^n,h) $(n \geqslant 7)$ is a smooth closed compact Riemannian manifold, locally conformally flat and of positive Yamabe type. If h is a Yamabe metric and satisfies*

$$\mathrm{tr}(E^3) + CJ|E|^2 \geqslant 0, \quad \textit{where } C < C_1(n) = \frac{(n-2)(n^3-4n^2-4n+48)}{16n}, \tag{4.3}$$

then

$$\begin{aligned}
&Y_6(M,[h]) \leqslant \frac{(n^2-16)(n-6)(n+2)}{n^2(n-2)^2}(Y_2(M,[h]))^3,\\
&\lambda_1(P_6) \leqslant \frac{(n^2-16)(n-6)(n+2)}{n^2(n-2)^2}\lambda_1(P_2)^3.
\end{aligned}$$

If any equality holds, then (M,h) is Einstein.

Proof. Since h is the Yamabe metric, we have J is a positive constant and

$$Y_2(M,[h]) = \frac{n-2}{2}J\mathrm{Vol}(M,h)^{\frac{2}{n}}, \quad \lambda_1(P_2) = \frac{n-2}{2}J.$$

In addition, h is locally conformally flat, and by (4.2),

$$\begin{aligned}
\int_M Q_6 dV_h = \int_M \bigg[&-\frac{16}{(n-4)(n-2)^2}|\nabla E|^2 + \frac{(n^2-16)(n^2-4)}{4n^2}J^3\\
&-\frac{4n^2(n-4)-16n+192}{n(n-4)(n-2)^2}J|E|^2 - \frac{64}{(n-4)(n-2)^3}\mathrm{tr}(E^3)\bigg] dV_h.
\end{aligned} \tag{4.4}$$

While $C < C_1(n)$, we have some $\epsilon > 0$ such that

$$\begin{aligned}
Y_6(M,[h]) &\leqslant \frac{n-6}{2}\bigg(\int_M Q_6 dV_h\bigg)\mathrm{Vol}(M,h)^{\frac{6-n}{n}}\\
&\leqslant \frac{(n^2-16)(n-6)(n+2)}{n^2(n-2)^2}(Y_2(M,[h]))^3\\
&\quad - \frac{8(n-6)}{(n-4)(n-2)^2}\bigg(\int_M (|\nabla E|^2 dV_h + \epsilon J|E|^2) dV_h\bigg)\mathrm{Vol}(M,h)^{\frac{6-n}{n}}\\
&\leqslant \frac{(n^2-16)(n-6)(n+2)}{n^2(n-2)^2}(Y_2(M,[h]))^3.
\end{aligned}$$

If the equalities hold, then $E=0$ and hence (M,h) is Einstein. Moreover, h achieves the minimizers of both $Y_6(M,[h])$ and $Y_2(M,[h])$.

A similar proof leads to the inequality

$$\lambda_1(P_6) \leqslant \frac{n-6}{2}\bigg(\int_M Q_6 dV_h\bigg)\mathrm{Vol}(M,h)^{-1} \leqslant \frac{(n^2-16)(n-6)(n+2)}{n^2(n-2)^2}\lambda_1(P_2)^3.$$

In addition, the characterisation of the equality case follows as above. □

Proposition 4.3. *Suppose that (M^n, h) $(n \geqslant 7)$ is a smooth closed compact Riemannian manifold, locally conformally flat and of positive Yamabe type. If h is a Yamabe metric satisfying one of the following*:

(a) $Q_4 \geqslant 0$;

(b) *A is semi-positive,*

then the condition (4.3) *holds.*

Proof. Here, J is a positive constant. Notice that since E is trace free, we have

$$|\mathrm{tr}(E^3)| \leqslant \frac{n-2}{\sqrt{n(n-1)}}|E|^3.$$

(a) If $Q_4 \geqslant 0$ holds, then

$$\frac{(n+2)(n-2)}{2n}J^2 - \frac{2}{(n-2)^2}|E|^2 \geqslant 0.$$

Therefore,

$$|\mathrm{tr}(E^3)| \leqslant \frac{n-2}{\sqrt{n(n-1)}}|E|^3 \leqslant \frac{(n-2)^2\sqrt{(n+2)(n-2)}}{2n\sqrt{n-1}}J|E|^2 < C_1(n)J|E|^2.$$

(b) If A is semi-positive, then all the eigenvalues of A satisfy $\mu_i \geqslant 0$. Hence,

$$|A|^2 = \sum_{i=1}^n \mu_i^2 \leqslant \left(\sum_{i=1}^n \mu_i\right)^2 = J^2.$$

Since $|A|^2 = \frac{1}{n}J^2 + \frac{1}{(n-2)^2}|E|^2$, we have

$$|E| \leqslant (n-2)\sqrt{\frac{n-1}{n}}J.$$

Therefore,

$$|\mathrm{tr}(E^3)| \leqslant \frac{n-2}{\sqrt{n(n-1)}}|E|^3 \leqslant \frac{(n-2)^2}{n}J|E|^2 < C_1(n)J|E|^2.$$

This completes the proof. □

The above conclusions are based on the assumption that (M, h) is locally conformally flat. If (M, h) is not locally conformally flat, we have the following proposition.

Proposition 4.4. *Suppose that (M^n, h) $(n \geqslant 7)$ is a smooth compact closed manifold of positive Yamabe type. If h is a Yamabe metric satisfying*

$$|W| + \frac{2|E|}{(n-2)^2\sqrt{n(n-1)}} < C_2(n)J, \quad \textit{where } C_2(n) = \frac{n^3-4n^2-4n+48}{8n(n-2)^2}, \tag{4.5}$$

then

$$Y_6(M,[h]) \leqslant \frac{(n^2-16)(n-6)(n+2)}{n^2(n-2)^2}(Y_2(M,[h]))^3,$$

$$\lambda_1(P_6) \leqslant \frac{(n^2-16)(n-6)(n+2)}{n^2(n-2)^2}\lambda_1(P_2)^3.$$

If any equality holds, then (M, h) is Einstein.

Proof. Since h is a Yamabe metric, we have that J is a positive constant and

$$Y_2(M,[h]) = \frac{n-2}{2}J\mathrm{Vol}(M,h)^{\frac{2}{n}}.$$

By (4.2) and (4.5), there is some $\epsilon > 0$ such that

$$\begin{aligned}Y_6(M,[h]) &\leqslant \frac{n-6}{2}\left(\int_M Q_6 dV_h\right)\mathrm{Vol}(M,h)^{\frac{6-n}{n}}\\ &\leqslant \frac{(n^2-16)(n-6)(n+2)}{n^2(n-2)^2}(Y_2(M,[h]))^3\\ &\quad - \frac{8(n-6)}{(n-4)(n-2)^2}\left(\int_M (|\nabla E|^2 dV_h + \epsilon J|E|^2)dV_h\right)\mathrm{Vol}(M,h)^{\frac{6-n}{n}}\\ &\leqslant \frac{(n^2-16)(n-6)(n+2)}{n^2(n-2)^2}(Y_2(M,[h]))^3.\end{aligned}$$

If the equalities hold, then $E = 0$ and hence (M,h) is Einstein. Moreover, h achieves the minimizers of both $Y_6(M,[h])$ and $Y_2(M,[h])$.

A similar proof leads to the inequality

$$\lambda_1(P_6) \leqslant \frac{n-6}{2}\left(\int_M Q_6 dV_h\right)\mathrm{Vol}(M,h)^{-1} \leqslant \frac{(n^2-16)(n-6)(n+2)}{n^2(n-2)^2}\lambda_1(P_2)^3.$$

In addition, the characterisation of the equality case follows as above. □

Proof of Theorem 1.6. This is directly from Proposition 4.4. □

Proof of Corollary 1.7. This is similar to the proof of Corollary 1.4. □

Acknowledgements The first author was supported by National Natural Science Foundation of China (Grant Nos. 11871331 and 11571233). The second author was supported by National Natural Science Foundation of China (Grant No. 11871331). The first author thanks Professor Sun-Yung Alice Chang for helpful discussions on this topic.

References

1 Almaraz S. Convergence of scalar-flat metrics on manifolds with boundary under a Yamabe-type flow. J Differential Equations, 2015, 259: 2626–2694

2 Beckner W. Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. Ann of Math (2), 1993, 138: 213–243

3 Case J, Chang S-Y A. On the fractional GJMS operators. Comm Pure Appl Math, 2016, 69: 1017–1061

4 Chen D, Wang F, Zhang X. Eigenvalue estimate of the Dirac operator and rigidity of Poincaré-Einstein metrics. Math Z, 2019, 293: 485–502

5 Chen S-Y. Conformal deformation to scalar flat metrics with constant mean curvature on the boundary in higher dimensions. arXiv:0912.1302v2, 2009

6 Chen X, Lai M, Wang F. Escobar-Yamabe compactifications for Poincaré-Einstein manifolds and rigidity theorems. Adv Math, 2019, 343: 16–35

7 Chruściel P, Delay E, Lee J, et al. Boundary regularity of conformally compact Einstein metrics. J Differential Geom, 2005, 69: 111–136

8 del Mar González M, Qing J. Fractional conformal Laplacians and fractional Yamabe problems. Anal PDE, 2013, 6: 1535–1576

9 Escobar J. Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary. Ann of Math (2), 1992, 136: 1–50 (with an addendum, Ann of Math (2), 1994, 139: 749–750)

10 Escobar J. Conformal metrics with prescribed mean curvature on the boundary. Calc Var Partial Differential Equations, 1996, 4: 559–592

11 Graham C R. Volume renormalization for singular Yamabe metrics. Proc Amer Math Soc, 2017, 145: 1781–1792

12 Graham C R, Jenne R, Mason L J, et al. Conformally invariant powers of the Laplacian, I: Existence. J Lond Math Soc (2), 1992, 46: 557–565

13 Graham C R, Zworski M. Scattering matrix in conformal geometry. Invent Math, 2003, 152: 89–118

14 Guillarmou C, Qing J. Spectral characterization of Poincaré-Einstein manifolds with infinity of positive Yamabe type. Int Math Res Not IMRN, 2010, 9: 1720–1740

15 Gursky M J, Han Q. Non-existence of Poincaré-Einstein manifolds with prescribed conformal infinity. Geom Funct Anal, 2017, 27: 863–879

16 Gursky M J, Hang F, Lin Y-J. Riemannian manifolds with positive Yamabe invariant and Paneitz operator. Int Math Res Not IMRN, 2016, 2016: 1348–1367

17 Gursky M J, Malchiodi A. A strong maximum principle for the Paneitz operator and a non-local flow for the Q-curvature. J Eur Math Soc JEMS, 2015, 17: 2137–2173

18 Hang F, Yang P. Sign of Green's function of Paneitz operators and the Q curvature. Int Math Res Not IMRN, 2015, 2015: 9775–9791

19 Joshi M, Sá Barreto A. Inverse scattering on asymptotically hyperbolic manifolds. Acta Math, 2000, 184: 41–86

20 Juhl A. Explicit formulas for GJMS-operators and Q-curvatures. Geom Funct Anal, 2013, 23: 1278–1370

21 Lee J. The spectrum of an asymptotically hyperbolic Einstein manifold. Comm Anal Geom, 1995, 3: 253–271

22 Marques F. Conformal deformations to scalar-flat metrics with constant mean curvature on the boundary. Comm Anal Geom, 2007, 15: 381–405

23 Mayer M, Ndiaye C B. Barycenter technique and the Riemann mapping problem of Escobar. J Differential Geom, 2017, 107: 519–560

24 Mazzeo R, Melrose R. Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature. J Funct Anal, 1987, 75: 260–310

25 Raulot S. Scalar flat compactifications of Poincaré-Einstein manifolds and applications. arXiv:1909.08274, 2019

26 Witten E, Yau S T. Connectedness of the boundary in the AdS/CFT correspondence. Adv Theor Math Phys, 1999, 3: 1635–1655