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Nonlinear stability of rarefaction waves for a viscous radiative and reactive gas with large initial perturbation

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Abstract We investigate the time-asymptotically nonlinear stability of rarefaction waves to the Cauchy problem of a one-dimensional compressible Navier-Stokes type system for a viscous, compressible, radiative and reactive gas, where the constitutive relations for the pressure p, the specific internal energy e, the specific volume v, the absolute temperature θ, and the specific entropy s are given by $p = R\theta/v + a\theta^4/3$, $e = C_v\theta + av\theta^4$, and $s = C_v \ln\theta + 4av\theta^3/3 + R\ln v$ with $R > 0$, $C_v > 0$ and $a > 0$ being the perfect gas constant, the specific heat and the radiation constant, respectively. For such a specific gas motion, a somewhat surprising fact is that, generally speaking, the pressure $\widetilde{p}(v, s)$ is not a convex function of the specific volume v and the specific entropy s. Even so, we show in this paper that the rarefaction waves are time-asymptotically stable for large initial perturbation provided that the radiation constant a and the strength of the rarefaction waves are sufficiently small. The key point in our analysis is to deduce the positive lower and upper bounds on the specific volume and the absolute temperature, which are uniform with respect to the space and the time variables, but are independent of the radiation constant a.

Keywords viscous radiative and reactive gas, rarefaction waves, nonlinear stability, large initial perturbation

MSC(2020) 35D35, 35Q10, 35Q35, 76D03

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1 Introduction

In this paper, we investigate the large-time behavior of global, strong, large-amplitude solutions to the Cauchy problem of a one-dimensional compressible Navier-Stokes type system for a viscous radiative and reactive gas. The model is described as follows (see [3, 31, 32, 47]):

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$$
\begin{aligned}
& v_t - u_x = 0, \\
& u_t + p(v,\theta)_x = \left(\frac{\mu u_x}{v}\right)_x, \\
& \left(e + \frac{u^2}{2}\right)_t + (up(v,\theta))_x = \left(\frac{\mu u u_x}{v}\right)_x + \left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x + \lambda\phi z, \\
& z_t = \left(\frac{d z_x}{v^2}\right)_x - \phi z.
\end{aligned}
\tag{1.1}
$$

Here, $x \in \mathbb{R}$ is the Lagrangian space variable, and $t \in \mathbb{R}^+$ is the time variable. The unknown quantities are the specific volume $v = v(t,x)$, the velocity $u = u(t,x)$, the absolute temperature $\theta = \theta(t,x)$, and the mass fraction of the reactant $z = z(t,x)$. The positive constants d and λ stand for the species diffusion coefficient and the difference in the heat between the reactant and the product, respectively. According to the Arrhenius law [4, 47], the reaction rate function $\phi = \phi(\theta)$ is given by

$$
\phi(\theta) = K\theta^{\beta} \exp\left(-\frac{A}{\theta}\right), \tag{1.2}
$$

where positive constants K and A represent the coefficients of the rates of the reactant and the activation energy, respectively. Besides, β is a non-negative number.

Due to the Stefan-Boltzmann radiative law [40, 47], the pressure p and the specific internal energy e consist of a fourth-order term radiative part in the absolute temperature θ as well as the perfect polytropic contribution

$$
p(v,\theta) = \frac{R\theta}{v} + \frac{a\theta^4}{3}, \quad e(v,\theta) = C_v\theta + av\theta^4, \tag{1.3}
$$

where the positive constants R and C_v are the perfect gas constant and the specific heat capacity at constant volume, respectively. Specifically, as shown in [26, 40], $C_v = \frac{3}{2}R$ for the radiative gas. $a > 0$ is the radiation constant which measures the amount of heat that is emitted by a blackbody, which absorbs all of the radiant energy that hits it, and will emit all the radiant energy. Moreover, we have (see [26, 40])

$$
a = \frac{4\sigma}{c} = \frac{8\pi^5 k_B^4}{15c^3h^3}, \tag{1.4}
$$

where σ is the Stefan-Boltzmann constant, c is the speed of light, k_B is the Boltzmann constant, and h is the Planck constant. Numerically,

$$
a = 7.5657 \times 10^{-16}\mathrm{Jm^{-3}K^{-4}}.
$$

In general, compared with the perfect gas constant R and the specific heat C_v, the radiation constant a is much smaller.

On the other hand, one can conclude from (1.3) and the second law of thermodynamics that

$$
s(v,\theta) = C_v \ln\theta + \frac{4}{3}av\theta^3 + R\ln v. \tag{1.5}
$$

If one takes $a = 0$, then the above constitutive relations for the five thermodynamic variables p, v, θ, s and e given by (1.3) and (1.5) reduce to the equations of state for ideal polytropic gases. If $a > 0$, and we choose (v,θ) or (v,s) as independent variables and write $(p,e,s) = (p(v,\theta), e(v,\theta), s(v,\theta))$ or $(p,e,\theta) = (\tilde{p}(v,s), \tilde{e}(v,s), \tilde{\theta}(v,s))$, respectively, then after cumbersome calculations, we can deduce that (see [26] for details)

$$
\begin{aligned}
\frac{\partial^2\tilde{p}(v,s)}{\partial v^2} = \frac{1}{s_\theta^3}\bigg[& \frac{C_vR^3 + 3C_v^2R^2 + 2C_v^3R}{v^3\theta^2} + \frac{(40aC_vR^2 + 28aC_v^2R - 8aR^3)\theta}{v^2} \\
& + \frac{(496a^2C_vR + 192a^2R^2)\theta^4}{3v} + \frac{(640a^3C_v + 7488a^3R)\theta^7}{27} + \frac{1792a^4v\theta^{10}}{27}\bigg],
\end{aligned}
\tag{1.6}
$$

$$
\frac{\partial^2\tilde{p}(v,s)}{\partial s^2} = \frac{(\tilde{\theta}_s)^2}{s_\theta}\left[\frac{C_vR}{v\theta^2} + \left(\frac{16aC_v}{3} - 8aR\right)\theta + \frac{16a^2v\theta^4}{3}\right] \tag{1.7}
$$

and

$$\frac{\partial^2\tilde{p}(v,s)}{\partial s^2}\frac{\partial^2\tilde{p}(v,s)}{\partial v^2}-\left(\frac{\partial^2\tilde{p}(v,s)}{\partial v\partial s}\right)^2$$
$$=\frac{(\tilde{\theta}_s)^2}{s_\theta^2}\left[\frac{C_vR^3+C_v^2R^2}{\theta^2v^4}+\frac{(32aC_v^2R-52aC_vR^2-24aR^3)\theta}{3v^3}\right.$$
$$\left.+\frac{(448a^2C_vR-1200a^2R^2)\theta^4}{9v^2}-\frac{320a^3R\theta^7}{9v}-\frac{256a^4\theta^{10}}{9}\right]. \tag{1.8}$$

From (1.6)–(1.8), it is easy to see that $\tilde{p}(v,s)$ is a convex function of v and s for the ideal polytropic gas, while if $a>0$, it is not clear whether $\tilde{p}(v,s)$ is a convex function of v and s or not.

As in [32, 47, 48], we also assume that the bulk viscosity μ is a positive constant and the thermal conductivity $\kappa=\kappa(v,\theta)$ takes the form

$$\kappa(v,\theta)=\kappa_1+\kappa_2 v\theta^b, \tag{1.9}$$

where κ_1, κ_2 and b are both positive constants. Furthermore, the system (1.1) is supplemented with the initial data

$$(v(0,x),u(0,x),\theta(0,x),z(0,x))=(v_0(x),u_0(x),\theta_0(x),z_0(x)) \tag{1.10}$$

for $x\in\mathbb{R}$, which is assumed to satisfy the far-field condition

$$\lim_{|x|\to\infty}(v_0(x),u_0(x),\theta_0(x),z_0(x))=(v_\pm,u_\pm,\theta_\pm,0). \tag{1.11}$$

Here, $v_\pm>0$, $u_\pm$ and $\theta_\pm>0$ are prescribed constants.

The problem on the global solvability and the precise description of the large-time behavior of the global solutions constructed for the initial value problem and the initial-boundary value problems of the systems (1.1)–(1.3), (1.9) and (1.10) is a hot topic in the field of nonlinear partial differential equations and many results have been obtained recently. A complete literature in this direction is beyond the scope of this paper and to go directly to the main points of the present paper, in what follows we only review some former results which are closely related to our main results.

• For the multidimensional case, there have been some results concerning the global existence, the uniqueness and the large time behavior of spherically (cylindrically) symmetric solutions to the systems (1.1)–(1.3) and (1.9)–(1.11) (see [44,49,53] for the bounded concentric annular domain case and see [28] for the exterior domain case). Here, the asymptotics of the global solutions constructed in [44,49,53] and [28], as in [16,17] and [5,32], are constant equilibrium states $(v_\infty,u_\infty,\theta_\infty,0)$ of (1.1) satisfying $v_\infty>0,\theta_\infty>0$, which are uniquely determined by the initial data for the corresponding initial-boundary value problem in a bounded domain and by the far fields of the initial data for the case in an exterior domain.

• For the one-dimensional initial-boundary value problem in the interval $[0,1]$, the existence and the uniqueness of global classical solutions was established in [3] for the following initial-boundary value problem:

$$\begin{aligned}&(v(0,x),u(0,x),\theta(0,x),z(0,x))=(v_0(x),u_0(x),\theta_0(x),z_0(x)),\quad x\in(0,1),\\&u(t,x)=0,\quad x=0,1,\quad t\geqslant 0,\\&(\theta_x(t,x),z_x(t,x))=(0,0),\quad x=0,1,\quad t\geqslant 0,\end{aligned} \tag{1.12}$$

while for the initial-boundary value problem

$$\begin{aligned}&(v(0,x),u(0,x),\theta(0,x),z(0,x))=(v_0(x),u_0(x),\theta_0(x),z_0(x)),\quad x\in(0,1),\\&\sigma(t,x)\equiv -p(v(t,x),\theta(t,x))+\frac{\mu u_x(t,x)}{v(t,x)}=-p_e,\quad x=0,1,\quad t\geqslant 0,\\&(\theta_x(t,x),z_x(t,x))=(0,0),\quad x=0,1,\quad t\geqslant 0\end{aligned} \tag{1.13}$$

for some positive constant $p_e > 0$, similar global solvability results were obtained in [31, 43, 47, 48]. Moreover, it is shown in [16, 17] that the asymptotics of the global solutions constructed above can be exactly described by $(1, 0, \theta_\infty, 0)$ with θ_∞ being a positive constant uniquely determined by

$$C_v\theta_\infty + a\theta_\infty^4 = \int_0^1 \left(\frac{1}{2}|u_0(x)|^2 + C_v\theta_0(x) + av_0(x)|\theta_0(x)|^4 + \lambda z_0(x)\right)dx$$

for the initial-boundary value problems (1.1)–(1.3), (1.9), (1.10), (1.12) and $(v_\infty, 0, \theta_\infty, 0)$ with v_∞ and θ_∞ being positive constants uniquely determined by

$$\begin{aligned}
&\frac{R\theta_\infty}{v_\infty} + \frac{a}{3}\theta_\infty^4 = p_e,\\
&C_v\theta_\infty + av_\infty\theta_\infty^4 p_e v_\infty = \int_0^1 \left(\frac{1}{2}|u_0(x)|^2 + C_v\theta_0(x) + av_0(x)|\theta_0(x)|^4 + \lambda z_0(x) + p_e v_0(x)\right)dx
\end{aligned}$$

for the initial-boundary value problems (1.1)–(1.3), (1.9), (1.10) and (1.13), respectively. Note that since $\int_0^1 v(t,x)dx$ is conserved for the initial-boundary value problems (1.1)–(1.3), (1.9), (1.10) and (1.12), while $\int_0^1 u(t,x)dx$ is conserved for the initial-boundary value problems (1.1)–(1.3), (1.9), (1.10) and (1.13), one can thus assume without loss of generality that $\int_0^1 v_0(x)dx = 1$ for the initial-boundary value problems (1.1)–(1.3), (1.9), (1.10), (1.12) and $\int_0^1 u_0(x)dx = 0$ for the initial-boundary value problems (1.1)–(1.3), (1.9), (1.10) and (1.13).

• For the Cauchy problems (1.1)–(1.3) and (1.9)–(1.11), the existence of a unique global solution was established very recently in [27, 32] for the case when the far fields $(v_\pm, u_\pm, \theta_\pm)$ of the initial data $(v_0(x), u_0(x), \theta_0(x))$ are equal, i.e., $(v_-, u_-, \theta_-) = (v_+, u_+, \theta_+)$ (see also [5] for the case with temperature-dependent viscosity and [29] for the case with density-dependent viscosity). Here, since $(v_-, u_-, \theta_-) = (v_+, u_+, \theta_+)$, the asymptotics of the global solutions constructed in [5, 32] are exactly the far fields $(v_\pm, u_\pm, \theta_\pm, 0)$ of the initial data $(v_0(x), u_0(x), \theta_0(x), z_0(x))$. The asymptotic stability of 1-rarefaction waves to the systems $(1.1)_1$–$(1.1)_3$ $(z = 0)$, (1.2), (1.3), (1.10) and (1.11) without viscosity $(\mu = 0)$ under the small perturbation was studied in [25]. Recently, Liao [26] has studied nonlinear stability of rarefaction waves for the systems (1.1)–(1.3) and (1.9)–(1.11) when the viscosity μ takes the following form:

$$\mu = \mu(v,\theta) = h(v)\theta^\alpha, \quad h(v) \sim \begin{cases} v^{-\ell_1}, & v \to 0^+, \\ v^{\ell_2}, & v \to \infty, \end{cases} \quad v|h'(v)|^2 \leqslant Ch^3(v). \tag{1.14}$$

Here, $h(v)$ is a smooth function of v for $v > 0$ and α, ℓ_1, and ℓ_2 are positive constants. It should be pointed out that (1.14) cannot cover the case when μ is a positive constant even when α goes to zero.

The main purpose of this manuscript is to study the nonlinear stability of rarefaction waves for the systems (1.1)–(1.3) and (1.9)–(1.11) with constant viscosity $(\mu \equiv C)$ under the large initial perturbation. For the Cauchy problems (1.1)–(1.3) and (1.9)–(1.11), if the far fields $(v_\pm, u_\pm, \theta_\pm)$ of the initial data $(v_0(x), u_0(x), \theta_0(x))$ are not equal, i.e.,

$$(v_-, u_-, \theta_-) \neq (v_+, u_+, \theta_+),$$

the asymptotics of the global solutions should be nontrivial and is expected to be described by the unique global entropy solution $(V^r(x/t), U^r(x/t), \Theta^r(x/t), 0)$ of the resulting Riemann problem of the corresponding compressible Euler equations

$$\begin{aligned}
&v_t - u_x = 0,\\
&u_t + p(v,\theta)_x = 0,\\
&\left(e + \frac{u^2}{2}\right)_t + (up(v,\theta))_x = 0,\\
&z_t = 0
\end{aligned} \tag{1.15}$$

with Riemann data

$$(v(0,x),u(0,x),\theta(0,x),z(0,x)) = (v_0^r(x),u_0^r(x),\theta_0^r(x),z_0^r(x)) = \begin{cases} (v_-,u_-,\theta_-,0), & x<0,\\ (v_+,u_+,\theta_+,0), & x>0. \end{cases} \tag{1.16}$$

In fact, it is expected (see [19, 20, 33–37, 45] and the references cited therein) that if the unique global entropy solution

$$(V^r(x/t),U^r(x/t),\Theta^r(x/t),0)$$

of the Riemann problem (1.15) and (1.16) consists of rarefaction waves

$$(V^{R_i}(x/t),U^{R_i}(x/t),\Theta^{R_i}(x/t),0)$$

of the i-th family ($i=1,3$), shock waves

$$(V^{S_i}(x/t),U^{S_i}(x/t),\Theta^{S_i}(x/t),0)$$

of the i-th family ($i=1,3$), contact discontinuity

$$(V^{CD}(x/t),U^{CD}(x/t),\Theta^{CD}(x/t),0)$$

of the second family, and/or their superpositions, then the large time behavior of the global solution

$$(v(t,x),u(t,x),\theta(t,x),z(t,x))$$

of the Cauchy problems (1.1)–(1.3) and (1.9)–(1.11) is expected to be well-described by the rarefaction wave

$$(V^{R_i}(x/t),U^{R_i}(x/t),\Theta^{R_i}(x/t),0)$$

of the i-th family ($i=1,3$), the viscous shock profile

$$(V^{VSW_i}(x-s_it),U^{VSW_i}(x-s_it),\Theta^{VSW_i}(x-s_it),0)$$

of the i-th family ($i=1,3$) under the suitable shift, the viscous contact discontinuity wave

$$(V^{\mathrm{VCD}}(t,x),U^{\mathrm{VCD}}(t,x),\Theta^{\mathrm{VCD}}(t,x),0)$$

of the second family, and/or their superpositions.

As in [2, 26], it will be convenient to consider the following equations for the entropy s and the absolute temperature θ:

$$s_t = \left(\frac{\kappa(v,\theta)\theta_x}{v\theta}\right)_x + \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2} + \frac{\mu u_x^2}{v\theta} + \frac{\lambda\phi z}{\theta} \tag{1.17}$$

and

$$\theta_t + \frac{\theta p_\theta u_x}{e_\theta} = \frac{1}{e_\theta}\left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x + \frac{\mu u_x^2}{v e_\theta} + \frac{\lambda\phi z}{e_\theta}, \tag{1.18}$$

where $p_\theta := \frac{\partial p(v,\theta)}{\partial\theta} = \frac{R}{v} + \frac{4}{3}a\theta^3$ and $e_\theta := \frac{\partial e(v,\theta)}{\partial\theta} = C_v + 4av\theta^3$.

From now on, we will consider $(1.1)_1$, $(1.1)_2$, (1.17) and $(1.1)_4$ with the initial data

$$(v(t,x),u(t,x),s(t,x),z(t,x))\,|_{t=0} = (v_0(x),u_0(x),s_0(x),z_0(x)) \to (v_\pm,u_\pm,s_\pm,0) \quad \text{as } x\to\pm\infty. \tag{1.19}$$

Here, $v_\pm>0$, $u_\pm$, $s_\pm := C_v\ln\theta_\pm + \frac{4}{3}av_\pm\theta_\pm^3 + R\ln v_\pm$ are constants and

$$s_0(x) := C_v\ln\theta_0(x) + \frac{4}{3}av_0(x)\theta_0(x)^3 + R\ln v_0(x).$$

Moreover, we assume that $s_+=s_-=\bar{s}$ for considering the expansion waves to (1.1).

It is well known that (1.1) can be approximated by the Riemann problem of the following equations:

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t + (\tilde{p}(v,s))_x = 0, \\ &s_t = \frac{\lambda\phi z}{\theta}, \\ &z_t = -\phi z \end{aligned} \tag{1.20}$$

with Riemann data

$$(v(0,x),u(0,x),s(0,x),z(0,x)) = (v_0^R(x),u_0^R(x),s_0^R(x),z_0^R(x)) = \begin{cases} (v_-,u_-,s_-,0), & x<0, \\ (v_+,u_+,s_+,0), & x>0. \end{cases} \tag{1.21}$$

The solutions of the Riemann problems (1.20)–(1.21) have two characteristics, which leads to two families of expansion (rarefaction) waves: the 1-rarefaction wave $(V_1^R(\frac{x}{t}),U_1^R(\frac{x}{t}),\bar{s},0)$ and the 3-rarefaction wave $(V_3^R(\frac{x}{t}),U_3^R(\frac{x}{t}),\bar{s},0)$. We define the regime

$$\begin{aligned} \mathbb{R}_1(v_-,u_-,\bar{s},0) &= \left\{(v,u,s,z)\,\middle|\, u = u_- + \int_{v_-}^{v}\sqrt{-\tilde{p}_\xi(\xi,\bar{s})}d\xi, u \geqslant u_-, s=\bar{s}, z=0\right\}, \\ \mathbb{R}_3(v_m,u_m,\bar{s},0) &= \left\{(v,u,s,z)\,\middle|\, u = u_m - \int_{v_m}^{v}\sqrt{-\tilde{p}_\xi(\xi,\bar{s})}d\xi, u \geqslant u_m, s=\bar{s}, z=0\right\}, \end{aligned}$$

and further assume that there exists a unique constant state $(v_m,u_m)\in\mathbb{R}^2 (v_m>0)$, which satisfies $(v_m,u_m)\in\mathbb{R}_1(v_-,u_-)$ and $(v_+,u_+)\in\mathbb{R}_3(v_m,u_m)$. Then the unique weak solution $(V^R(\frac{x}{t}),U^R(\frac{x}{t}), S^R(\frac{x}{t}),0)$ to the system (1.20)–(1.21) is characterized by

$$\begin{aligned} &\left(V^R\left(\frac{x}{t}\right),U^R\left(\frac{x}{t}\right),S^R\left(\frac{x}{t}\right),0\right) \\ &\quad = \left(V_1^R\left(\frac{x}{t}\right)+V_3^R\left(\frac{x}{t}\right)-v_m, U_1^R\left(\frac{x}{t}\right)+U_3^R\left(\frac{x}{t}\right)-u_m,\bar{s},0\right) \end{aligned} \tag{1.22}$$

with $(V_i^R(\frac{x}{t}),U_i^R(\frac{x}{t}),S^R(\frac{x}{t}),0)$ $(i=1,3)$ satisfying the following equations:

$$\begin{aligned} &S^R\left(\frac{x}{t}\right) = \bar{s}, \\ &U_1^R\left(\frac{x}{t}\right) - \int_1^{V_1^R(\frac{x}{t})}\sqrt{-\tilde{p}_\xi(\xi,\bar{s})}d\xi = u_- - \int_1^{v_-}\sqrt{-\tilde{p}_\xi(\xi,\bar{s})}d\xi, \\ &\lambda_{1x}\left(V_1^R\left(\frac{x}{t}\right),\bar{s}\right) > 0, \\ &\lambda_1(v,s) = -\sqrt{-\tilde{p}_v(v,s)}, \\ &U_3^R\left(\frac{x}{t}\right) + \int_1^{V_3^R(\frac{x}{t})}\sqrt{-\tilde{p}_\xi(\xi,\bar{s})}d\xi = u_m + \int_1^{v_m}\sqrt{-\tilde{p}_\xi(\xi,\bar{s})}d\xi, \\ &\lambda_{3x}\left(V_3^R\left(\frac{x}{t}\right),\bar{s}\right) > 0, \\ &\lambda_3(v,s) = \sqrt{-\tilde{p}_v(v,s)}. \end{aligned} \tag{1.23}$$

To construct the approximate waves $(V(t,x),U(t,x),S(t,x),0)$, we begin with the following Burger's equation (see [37]). Let $\omega_i(t,x)$ $(i=1,3)$ be the unique global smooth solution to the Cauchy problem

$$\begin{aligned} &\omega_{it} + \omega_i\omega_{ix} = 0, \\ &\omega_i(t,x)\,|_{t=0} = \omega_{i0}(x) = \frac{w_{i+}+w_{i-}}{2} + \frac{w_{i+}-w_{i-}}{2}K_q\int_0^{\epsilon x}(1+y^2)^{-q}dy, \end{aligned} \tag{1.24}$$

where $q > \frac{3}{2}$, $K_q = (\int_0^{+\infty}(1+y^2)^{-q}dy)^{-1}, \epsilon > 0$ is a positive constant to be determined later, and

$$
\begin{aligned}
\omega_{1-} &= \lambda_1(v_-,\bar{s}) = -\sqrt{-\tilde{p}_v(v_-,\bar{s})},\\
\omega_{1+} &= \lambda_1(v_m,\bar{s}) = -\sqrt{-\tilde{p}_v(v_m,\bar{s})},\\
\omega_{3-} &= \lambda_3(v_m,\bar{s}) = \sqrt{-\tilde{p}_v(v_m,\bar{s})},\\
\omega_{3+} &= \lambda_3(v_+,\bar{s}) = \sqrt{-\tilde{p}_v(v_+,\bar{s})}.
\end{aligned}
$$

Then, by setting

$$\epsilon = \delta = |v_- - v_+| + |u_- - u_+|,$$

the approximate rarefaction waves $(V(t,x),U(t,x),S(t,x),0)$ are defined by

$$
\begin{aligned}
&(V(t,x),U(t,x),S(t,x),0)\\
&\quad = (V_1(t+1,x)+V_3(t+1,x)-v_m, U_1(t+1,x)+U_3(t+1,x)-u_m,\bar{s},0),
\end{aligned}
\tag{1.25}
$$

where $(V_i(t,x),U_i(t,x))$ $(i=1,3)$ satisfy

$$
\begin{aligned}
&\lambda_i(V_i(t,x),\bar{s}) = \omega_i(t,x), \quad i=1,3,\\
&\lambda_1(v,s) = -\sqrt{-\tilde{p}_v(v,s)},\\
&\lambda_3(v,s) = \sqrt{-\tilde{p}_v(v,s)},\\
&U_1(t,x) = u_- + \int_{v_-}^{V_1(t,x)} \sqrt{-\tilde{p}_\xi(\xi,\bar{s})}d\xi,\\
&U_3(t,x) = u_m - \int_{v_m}^{V_3(t,x)} \sqrt{-\tilde{p}_\xi(\xi,\bar{s})}d\xi,
\end{aligned}
\tag{1.26}
$$

and $\Theta(t,x)$ is given by

$$\Theta(t,x) = \tilde{\theta}(V(t,x),\bar{s}).$$

Furthermore, if we denote the strength of the rarefaction waves by

$$\delta = |v_- - v_+| + |u_- - u_+|,$$

then our main result is the following stability theorem.

Theorem 1.1. *Suppose that*

- *the parameters b and β are assumed to satisfy*

$$b > 6, \quad 0 \leqslant \beta < b+3;$$

- *there exist positive constants $0 < \underline{V} \leqslant 1$, $\overline{V} > 1$, $0 < \underline{\Theta} \leqslant 1$, $\overline{\Theta} > 1$, which do not depend on the strength of the rarefaction wave δ and the radiation constant a, such that*

$$
\begin{aligned}
2\underline{V} \leqslant v_0(x) \leqslant \frac{1}{2}\overline{V},\\
2\underline{V} \leqslant V(t,x) \leqslant \frac{1}{2}\overline{V},\\
2\underline{\Theta} \leqslant \theta_0(x) \leqslant \frac{1}{2}\overline{\Theta},\\
2\underline{\Theta} \leqslant \Theta(t,x) \leqslant \frac{1}{2}\overline{\Theta}
\end{aligned}
$$

hold for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$(v_0(x)-V(0,x), u_0(x)-U(0,x), \theta_0(x)-\Theta(0,x), z_0(x)) \in H^1(\mathbb{R}),$$

$$\frac{\partial^2(u_0(x)-U(0,x))}{\partial x^2}\in L^2(\mathbb{R}),\quad z_0(x)\in L^1(\mathbb{R}),$$
$$0\leqslant z_0(x)\leqslant 1,\quad \forall\, x\in\mathbb{R}$$

and

$$H_0:=\|(v_0(x)-V(0,x),u_0(x)-U(0,x),\theta_0(x)-\Theta(0,x),z_0(x))\|_{H^1(\mathbb{R})}$$

together with $v_\pm, u_\pm$ *and* $\theta_\pm$ *being assumed to be independent of* δ *and* a.

Then there exist positive constants δ_0 *and* a_0*, which depend only on* $\underline{V}$*,* $\underline{\Theta}$ *and* H_0*, such that when*

$$0<\delta\leqslant\delta_0,\quad 0<a\leqslant a_0,\tag{1.27}$$

the systems (1.1)–(1.3) *and* (1.9)–(1.11) *admit a unique global solution* $(v(t,x),u(t,x),\theta(t,x),z(t,x))$ *which satisfies*

$$C_1^{-1}\leqslant v(t,x)\leqslant C_1,$$
$$C_2^{-1}\leqslant \theta(t,x)\leqslant C_2,$$
$$0\leqslant z(t,x)\leqslant 1$$

for all $(t,x)\in[0,\infty)\times\mathbb{R}$ *and*

$$\begin{aligned}&\sup_{0\leqslant t<\infty}\|(v-V,u-U,\theta-\Theta,z)(t)\|_{H^1(\mathbb{R})}^2\\&\quad+\int_0^\infty(\|\partial_x(v-V)(\tau)\|_{L^2(\mathbb{R})}^2+\|(\partial_x(u-U),\partial_x(\theta-\Theta),\partial_x z)(\tau)\|_{H^1(\mathbb{R})}^2)d\tau\\&\quad\leqslant C.\end{aligned}\tag{1.28}$$

Here, C_1*,* C_2 *and* C *are some positive constants depending only on* $\underline{V}$*,* $\underline{\Theta}$ *and* H_0*.*

Moreover, it holds that

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\{|(v(t,x)-V^R(t,x),u(t,x)-U^R(t,x),s(t,x)-\overline{s},z(t,x))|\}=0.\tag{1.29}$$

Remark 1.2. Here are some remarks concerning Theorem 1.1.

- Note that the result in [25] focuses on the case when $\mu\equiv 0$ and $\kappa(v,\theta)\equiv$ constant. As pointed out before, the initial perturbation between the initial data and the approximation solution in [25] needs to be sufficiently small. Besides, an additional stability condition should also be imposed on the state of the specific volume $v(t,x)$ and the temperature $\theta(t,x)$ at the far field (see [25, (1.14)]). Compared with the result obtained in [25], the result in this paper is the first one concerning the stability analysis of viscous wave patterns of (1.1)–(1.3) and (1.9)–(1.11) with constant viscosity under the large initial perturbation. Moreover, we do not need to impose the above additional stability condition in our study. Furthermore, our method in this paper can also be applied to Navier-Stokes equations when thermodynamic variables satisfy the equations of state for ideal polytropic gases ($\lambda=0$, $a=0$).
- We emphasize that the result in [26] cannot include the case when $\mu\equiv C$. Besides, the methods to deduce the uniform lower and upper bounds on the specific volume $v(t,x)$ and the absolute temperature $\theta(t,x)$ in our paper are also different from those developed in [26].
- It is interesting to study the global nonlinear stability of viscous shock waves, viscous contact waves, and some of their superpositions for (1.1)–(1.3) and (1.9)–(1.11) in the future and such problems are under our current study.

As we can see in the analysis performed in [20, 37–39, 41] and from the estimate (2.10) obtained in Lemma 2.6 of this paper, $\tilde{p}(v,s)$ is a convex function of v and s plays an essential role in deducing the nonlinear stability of rarefaction waves of the one-dimensional compressible Navier-Stokes type equations. We note, however, that, from (1.6)–(1.8), it is not clear whether $\tilde{p}(v,s)$ is a convex function of v and s or not for the case when the radiation constant $a>0$. To overcome such a difficulty, our main observation is that if both the specific volume v and the absolute temperature θ are bounded from the above and below by some positive constants independent of the radiation constant a, then one can choose a sufficiently small such that $\tilde{p}(v,s)$ is a convex function of v and s in the regime for v and θ under consideration. It is worth pointing out that in the proof of Theorem 1.1, the smallness assumption we imposed on the radiation constant a is used only to ensure that $\tilde{p}(v,s)$ is convex with respect to (v,s) in the regime for v and θ under our consideration and we do not use such a smallness assumption elsewhere to control certain nonlinear terms involved. The main purpose of such an analysis is that once we can impose some other assumptions to guarantee that $\tilde{p}(v,s)$ is convex with respect to (v,s) in the regime for v and θ under our consideration, then we can deduce that a similar result holds accordingly.

Our next result shows that, if in addition to using the smallness of a to guarantee that $\tilde{p}(v,s)$ is convex with respect to (v,s) in the regime for v and θ under our consideration, we also use such an assumption to control certain nonlinear terms involved, then we can get a similar stability result but with less restrictions on the ranges of the parameters b and β, which includes the most physically interesting radiation case $b=3$ (see [17]).

Theorem 1.3. *Under similar assumptions imposed on the initial data $(v_0(x),u_0(x),\theta_0(x),z_0(x))$ and the radiation constant a, a similar stability result still holds when $b>2$, $0\leqslant\beta<b+3$.*

In order to deduce the main results of this paper, the key points in our analysis are the following:

• The first is to deduce the uniform positive lower and upper bounds on the specific volume $v(t,x)$ and the absolute temperature $\theta(t,x)$.

• The second is to show that the above bounds on the specific volume $v(t,x)$ and the absolute temperature $\theta(t,x)$ are independent of the radiation constant, since only in this case, we can choose $a>0$ sufficiently small such that $\tilde{p}(v,s)$ is a convex function of v and s.

We are now in a position to state our main ideas to overcome the above difficulties, especially on the way to yield the uniform upper bound on the absolute temperature $\theta(t,x)$. To this end, we first recall that for the case when $a=0$ and $\kappa_2=0$ in (1.3), (1.5) and (1.9), which is the equations of a viscous heat-conductive ideal polytropic gas with constant nondegenerate transport coefficients, the nonlinear stability of some basic wave patterns with large initial perturbation is obtained in [13, 50, 51] for the whole range of the adiabatic exponent $\gamma>1$. The method used in [13, 50, 51] to deduce the upper bound on the absolute temperature $\theta(t,x)$ is motivated by [24], which relies on the following Sobolev inequality:

$$\|\theta(t)-1\|^2_{L^\infty(\mathbb{R})}\leqslant C\|\theta(t)-1\|_{L^2(\mathbb{R})}\|\theta_x(t)\|_{L^2(\mathbb{R})}\leqslant C(1+\|\theta\|_{L^\infty([0,T]\times\mathbb{R})}).$$

However, such a method loses its power for the case $\kappa_2\neq0$ since some nonlinear terms caused by the thermal conductivity

$$\kappa(v,\theta)=\kappa_1+\kappa_2 v\theta^b$$

cannot be controlled properly when we deduce the estimate on $\|\theta_x(t)\|_{L^2(\mathbb{R})}$ by employing the argument developed in [24].

To overcome such a difficulty, for the case

$$(v_-,u_-,\theta_-)=(v_+,u_+,\theta_+)=:(v_\infty,u_\infty,\theta_\infty),$$

i.e., for the case when the far fields of the initial data $(v_0(x),u_0(x),\theta_0(x))$ are equal, the argument developed in [32] is to introduce the following auxiliary functions:

$$\begin{aligned}
\widetilde{X}(t)&:=\int_0^t\int_{\mathbb{R}}(1+\theta^{b+3}(s,x))\theta_t^2(s,x)dxds,\\
\widetilde{Y}(t)&:=\max_{s\in(0,t)}\int_{\mathbb{R}}(1+\theta^{2b}(s,x))\theta_x^2(s,x)dx,\\
\widetilde{Z}(t)&:=\max_{s\in(0,t)}\int_{\mathbb{R}}u_{xx}^2(s,x)dx,\\
\widetilde{W}(t)&:=\int_0^t\int_{\mathbb{R}}u_{xt}^2(s,x)dxds
\end{aligned}\tag{1.30}$$

and then try to deduce certain estimates between them by employing the structure of (1.1)–(1.3) and (1.9) under our consideration, from which one can deduce the desired upper bound on the absolute temperature $\theta(t,x)$. A key point in the analysis there is that the basic energy estimates based on the entropy $\widetilde{\eta}(v,u,\theta,v_\infty,u_\infty,\theta_\infty)$ normalized around the constant state $(v,u,\theta)=(v_-,u_-,\theta_-)$,

$$\begin{aligned}&\widetilde{\eta}(v,u,\theta,v_\infty,u_\infty,\theta_\infty)\\&\quad=C_v\theta_\infty\Phi\left(\frac{\theta}{\theta_\infty}\right)+R\theta_\infty\Phi\left(\frac{v}{v_\infty}\right)+\frac{1}{2}(u-u_\infty)^2+\frac{av}{3}(\theta-\theta_\infty)^2(3\theta^2+2\theta_\infty\theta+\theta_\infty^2),\\&\Phi(x)=x-\ln x-1\end{aligned}$$

can yield an $L^4_{\mathrm{loc}}(\mathbb{R})$-estimate on $\theta(t,x)$. From such an estimate, one can get by employing the argument developed in [23] that (see [32, (2.53)])

$$\|\theta(t)\|_{L^\infty(\mathbb{R})}\lesssim 1+\widetilde{Y}(t)^{\frac{1}{2b+6}} \tag{1.31}$$

and the estimate (1.31) plays an essential role in [32] to deduce the upper bound of $\theta(t,x)$.

But for the case considered in this paper, $(v_-,u_-,\theta_-)\neq(v_+,u_+,\theta_+)$, since, as we pointed out before, we need to use the smallness of the radiation constant a to ensure that $\tilde{p}(v,s)$ is a convex function of v and s, although we can still construct a convex entropy $\eta(v,u,\theta;V,U,\Theta)$ normalized around the profile $(v,u,\theta)=(V(t,x),U(t,x),\Theta(t,x))$,

$$\eta(v,u,\theta;V,U,\Theta)=C_v\Theta\Phi\left(\frac{\theta}{\Theta}\right)+R\Theta\Phi\left(\frac{v}{V}\right)+\frac{1}{2}(u-U)^2+\frac{av(\theta-\Theta)^2}{3}(3\theta^2+2\theta\Theta+\Theta^2), \tag{1.32}$$

to yield a similar estimate (see (2.10) obtained in Lemma 2.6) to guarantee that the estimate we obtained on $\theta(t,x)$ does not depend on a, we can only use the boundedness of $\int_{\mathbb{R}}\Phi(\frac{\theta}{\Theta})dx$. Moreover, the construction of the auxiliary functions $X(t),Y(t)$ and $Z(t)$ should also be modified accordingly as follows:

$$\begin{aligned}X(t)&:=\int_0^t\int_{\mathbb{R}}(1+\theta^b(s,x))\chi_t^2(s,x)dxds,\\Y(t)&:=\sup_{s\in(0,t)}\int_{\mathbb{R}}(1+\theta^{2b}(s,x))\chi_x^2(s,x)dx,\\Z(t)&:=\sup_{s\in(0,t)}\int_{\mathbb{R}}\psi_{xx}^2(s,x)dx,\end{aligned} \tag{1.33}$$

where $\psi(t,x)=u(t,x)-U(t,x)$ and $\chi(t,x)=\theta(t,x)-\Theta(t,x)$.

A consequence of the above modifications is that instead of the estimate (1.31), one has (see the estimate (4.1) in Lemma 4.1)

$$\|\theta(t)\|_{L^\infty(\mathbb{R})}\lesssim 1+Y(t)^{\frac{1}{2b+3}}. \tag{1.34}$$

The above changes make it harder to deduce the upper bound of $\theta(t,x)$, especially to yield a nice bound on the term I_{17} in (4.6) which cannot be controlled by exploiting the method used in [32] to estimate the corresponding term, i.e., the term I_8 in [32].

Our strategy to overcome the above difficulties can be summarized as follows:

• The smallness of the strength of the rarefaction waves is made full use of to control the nonlinear terms originating from the nonlinearities of equations, the interactions of rarefaction waves from different families and the interaction between the solutions and the rarefaction waves.

• The specific volume $v(t,x)$ is shown to be uniformly bounded from below and above with respect to space and time variables through delicate analysis based on the basic energy estimate and the cut-off technique used by [18,32]. It is worth emphasizing that the positive lower and upper bounds we derived are independent of δ and a.

• Motivated by [22,32], we introduce the auxiliary functions $X(t)$, $Y(t)$ and $Z(t)$ defined by (1.33) to derive the desired upper bound of $\theta(t,x)$, especially to yield a nice estimate on the term I_{17} given in (4.6).

To this end, we first derive bounds on $\|\varphi_x(t)\|_{L^2(\mathbb{R})}$ and $\int_0^t\int_{\mathbb{R}}\chi_{xx}^2 dxd\tau$ in terms of $\|\theta\|_{L^\infty([0,T]\times\mathbb{R})}$ as in Lemmas 3.5 and 3.6. Then by using Sobolev's inequality and Lemma 3.6, the term

$$\int_0^t\int_{\mathbb{R}}(1+\theta^b)\psi_x^4 dxd\tau$$

can be estimated as follows:

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}(1+\theta^b)\psi_x^4 dxd\tau &\leqslant C(1+\|\theta\|^b_{L^\infty([0,T]\times\mathbb{R})})\int_0^t\|\psi_x\|^2_{L^\infty(\mathbb{R})}\|\psi_x\|^2_{L^2(\mathbb{R})}d\tau\\ &\leqslant C(1+\|\theta\|^b_{L^\infty([0,T]\times\mathbb{R})})\int_0^t\|\psi_x\|^3_{L^2(\mathbb{R})}\|\psi_{xx}\|_{L^2(\mathbb{R})}d\tau\\ &\leqslant C(1+\|\theta\|^{b+3}_{L^\infty([0,T]\times\mathbb{R})})\left(\int_0^t\|\psi_x\|^2_{L^2(\mathbb{R})}d\tau\right)^{\frac12}\left(\int_0^t\|\psi_{xx}\|^2_{L^2(\mathbb{R})}d\tau\right)^{\frac12}\\ &\leqslant C(1+\|\theta\|^{b+5}_{L^\infty([0,T]\times\mathbb{R})}).\end{aligned}\tag{1.35}$$

Note that we do not need to introduce the additional function $W(t)$ as in [32] (see [32, (2.51) and (2.70)]).

Finally, we point out that there are a lot of results concerning the stability analysis of viscous wave patterns of the 1D compressible Navier-Stokes equations. We refer to [6,19,33,46] for the viscous shock wave, [2,9,34,37–39,41] for the rarefaction wave, [7,12,14,15,35] for the viscous contact wave, and [8,10,11,13] for the superpositions of the above three wave patterns. For more references in this direction, please refer to [4,21,30,42,45,50–52] and the references therein.

The rest of this paper is organized as follows. We first give some basic energy estimates and some properties of the smooth approximation of the rarefaction wave solutions in Section 2. In Section 3, we derive the uniform-in-time lower and upper bounds of the specific volume $v(t,x)$ which are also independent of δ and a. Then the uniform-in-time, δ and a independent upper bound of the absolute temperature $\theta(t,x)$ will be obtained in Section 4. Furthermore, a local-in-time lower bound on the absolute temperature will be deduced in Section 5. The proofs of our main results are given in Section 6. Note that although the lower bound on the absolute temperature $\theta(t,x)$ obtained in Section 5 depends on time t, it is sufficient to prove the main theorem in this paper by combining these *a priori* estimates with the continuation argument introduced in [32].

Notations. In what follows, C represents a generic positive constant, which is independent of t, δ, a and x but may depend on $v_\pm$, $u_\pm$, $\theta_\pm$, $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ and H_0. Notice that the value of it may change from line to line. $C_i(\cdot,\cdot)$ $(i\in\mathbb{Z}_+)$ stands for some generic constants depending only on the quantities listed in the parentheses and ϵ denotes some small positive constant.

For two quantities B and B', if there is a generic positive constant $C>0$ independent of t, δ, a and x such that $B\leqslant CB'$, we take the note $B\lesssim B'$, while $B\sim B'$ means that $B\lesssim B'$ and $B'\lesssim B$. Moreover, for two functions $f(x)$ and $g(x)$, $f(x)\sim g(x)$ as $x\to x_0$ means that there exists a generic positive constant $C>0$ which is independent of t, δ, a and x but may depend on $v_\pm$, $u_\pm$, $\theta_\pm$, $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ and H_0 such that

$$C^{-1}f(x)\leqslant g(x)\leqslant Cf(x)$$

in a neighborhood of x_0. $H^l(\mathbb{R})$ $(l\geqslant 0)$ denotes the usual Sobolev space with the standard norm $\|\cdot\|_l$, and for brevity, we take $\|\cdot\|:=\|\cdot\|_0$ to denote the usual L^2-norm. For $1\leqslant p\leqslant+\infty$, $f(x)\in L^p(\mathbb{R})$, $\|f\|_{L^p}=(\int_{\mathbb{R}}|f(x)|^p dx)^{\frac1p}$. It is easy to see that $\|f\|_{L^2}=\|\cdot\|$. Finally, $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_\infty$ are used to denote $\|\cdot\|_{L^\infty(\mathbb{R})}$ and $\|\cdot\|_{L^\infty([0,t]\times\mathbb{R})}$, respectively.

2 Preliminaries

First of all, (1.1), (1.23) and (1.25) tell us that $(V(t,x),U(t,x),S(t,x),0)$ solves the following problem:

$$V_t-U_x=0,$$

$$
\begin{aligned}
&U_t + p(V,\Theta)_x = g(V,\Theta)_x,\\
&\left(e(V,\Theta) + \frac{U^2}{2}\right)_t + (Up(V,\Theta))_x = q(V,\Theta),\\
&\Theta_t + \frac{\Theta p_\Theta(V,\Theta)}{e_\Theta(V,\Theta)}U_x = r(V,\Theta),\\
&S_t = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
g(V,\Theta) &= p(V,\Theta) - p(V_1,\Theta_1) - p(V_3,\Theta_3) - p(v_m,\theta_m),\\
q(V,\Theta) &= (e(V,\Theta) - e(V_1,\Theta_1) - e(V_3,\Theta_3))_t + \left(\frac{U^2}{2} - \frac{U_1^2}{2} - \frac{U_3^2}{2}\right)_t\\
&\quad + (Up(V,\Theta) - U_1 p(V_1,\Theta_1) - U_3 p(V_3,\Theta_3))_x,\\
r(V,\Theta) &= \frac{\Theta p_\Theta(V,\Theta)}{e_\Theta(V,\Theta)}U_x - \frac{\Theta_1 p_\Theta(V_1,\Theta_1)}{e_\Theta(V_1,\Theta_1)}U_{1x} - \frac{\Theta_3 p_\Theta(V_3,\Theta_3)}{e_\Theta(V_3,\Theta_3)}U_{3x},
\end{aligned}
$$

and $\theta_m = \widetilde{\theta}(v_m,\bar{s})$.

Due to the fact that $\omega_0(x)$ is strictly increasing, we can deduce the following lemma (see [2,26]).

Lemma 2.1. *For each $i \in \{1,3\}$, the Cauchy problem* (1.24) *admits a unique global smooth solution $\omega_i(t,x)$ which satisfies the following properties*:

(i) $\omega_- < \omega_i(t,x) < \omega_+$, $\omega_{ix}(t,x) > 0$ *for each* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

(ii) *For any p with $1 \leqslant p \leqslant \infty$, there exists a constant $C_{p,q}$, depending only on p and q, such that*

$$
\begin{aligned}
&\|\omega_{ix}(t)\|_{L^P}^p \leqslant C_{p,q}\min\{\epsilon^{p-1}\tilde{\omega}_i^p, \tilde{\omega}_i t^{-p+1}\},\\
&\|\omega_{ixx}(t)\|_{L^P}^p \leqslant C_{p,q}\min\{\epsilon^{2p-1}\tilde{\omega}_i^p, \epsilon^{(p-1)(1-\frac{1}{2q})}\tilde{\omega}_i^{-\frac{p-1}{2q}} t^{-p-\frac{p-1}{2q}}\}.
\end{aligned}
$$

(iii) *If $0 < \omega_{i-} (< \omega_{i+})$ and q is suitably large, then*

$$
\begin{aligned}
&|\omega_i(t,x) - \omega_{i-}| \leqslant C\tilde{\omega}_i(1+(\epsilon x)^2)^{-\frac{q}{3}}(1+(\epsilon\omega_{i-}t)^2)^{-\frac{q}{3}}, \quad x \leqslant 0,\\
&|\omega_{ix}(t,x)| \leqslant C\epsilon\tilde{\omega}_i(1+(\epsilon x)^2)^{-\frac{1}{2}}(1+(\epsilon\omega_{i+}t)^2)^{-\frac{q}{2}}, \quad x \leqslant 0.
\end{aligned}
$$

(iv) *If $(\omega_{i-}) < \omega_{i+} \leqslant 0$ and q is suitably large, then*

$$
\begin{aligned}
&|\omega_i(t,x) - \omega_{i+}| \leqslant C\tilde{\omega}_i(1+(\epsilon x)^2)^{-\frac{q}{3}}(1+(\epsilon\omega_{i-}t)^2)^{-\frac{q}{3}}, \quad x \leqslant 0,\\
&|\omega_{ix}(t,x)| \leqslant C\epsilon\tilde{\omega}_i(1+(\epsilon x)^2)^{-\frac{1}{2}}(1+(\epsilon\omega_{i+}t)^2)^{-\frac{q}{2}}, \quad x \leqslant 0.
\end{aligned}
$$

(v) $\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}|\omega_i(t,x) - \omega_i^R(\frac{x}{t})| = 0$.

Here, $\tilde{\omega}_i = \omega_{i+} - \omega_{i-} > 0$ and $\omega_i^R(\frac{x}{t})$ is the unique rarefaction wave solution of the corresponding Riemann problem of $(1.14)_1$, *i.e.,*

$$
\omega_i^R(\xi) = \begin{cases}
\omega_{i-}, & \xi \leqslant \omega_{i-},\\
\xi, & \omega_{i-} \leqslant \xi \leqslant \omega_{i+},\\
\omega_{i+}, & \xi \geqslant \omega_{i+}.
\end{cases}
$$

Owing to Lemma 2.1, (1.25), and (1.26), we can conclude the following lemma (see [2,26]).

Lemma 2.2. *By letting $\epsilon = \delta$, $q = 2$, the smooth approximations $(V(t,x), U(t,x), \Theta(t,x), 0)$ constructed in* (1.25) *and* (1.26) *have the following properties*:

(i) $V_t(t,x) = U_x(t,x) > 0$ *for each* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

(ii) *For any p with $1 \leqslant p \leqslant \infty$ there exists a constant C_p, depending only on p, such that*

$$
\begin{aligned}
&\|(V_x, U_x, \Theta_x)(t)\|_{L^p}^p \leqslant C_p\min\{\delta^{2p-1}, \delta(t+1)^{-p+1}\},\\
&\|(V_{xx}, U_{xx}, \Theta_{xx})(t)\|_{L^p}^p \leqslant C_p\min\{\delta^{3p-1}, \delta^{\frac{p-1}{2}}(t+1)^{-\frac{5p-1}{4}}\}.
\end{aligned}
$$

It is obvious that $\|V_x(t)\|_{L^2}^2$ *is not integrable with respect to* t. *However, we can get for any* $r>0$ *and* $p>1$ *that*

$$\int_0^\infty \|(V_x,U_x,\Theta_x)(t)\|_{L^{2+r}}^{2+r}dt \leqslant C(r)\delta,$$
$$\int_0^\infty \|(V_{xx},U_{xx},\Theta_{xx})(t)\|_{L^p}dt \leqslant C(p)\left(\frac{1}{\delta}\right)^{-\frac{1}{4}(1-\frac{1}{p})}.$$

(iii) *For each* $p\geqslant 1$,

$$\|(g(V,\Theta)_x, r(V,\Theta), q(V,\Theta))(t)\|_{L^p} \leqslant C(p)\delta^{\frac{2}{3}}(t+1)^{-\frac{4}{3}}.$$

Especially,

$$\int_0^\infty \|(g(V,\Theta)_x, r(V,\Theta), q(V,\Theta))(t)\|_{L^p}dt \leqslant C(p)\left(\frac{1}{\delta}\right)^{-\frac{1}{3}}.$$

(iv) $\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}|(V(t,x),U(t,x),\Theta(t,x))-(V^R(\frac{x}{t}),U^R(\frac{x}{t}),\Theta^R(\frac{x}{t}))|=0$.

(v) $|(V_t(t,x),U_t(t,x),\Theta_t(t,x))|\leqslant O(1)|(V_x(t,x),U_x(t,x),\Theta_x(t,x))|$.

Setting

$$\begin{aligned}&(\varphi(t,x),\psi(t,x),\chi(t,x),\xi(t,x))\\&\quad=(v(t,x)-V(t,x),u(t,x)-U(t,x),\theta(t,x)-\Theta(t,x),s(t,x)-\bar{s}),\end{aligned}\tag{2.1}$$

we can deduce that $(\varphi(t,x),\psi(t,x),\chi(t,x),\xi(t,x),z(t,x))$ satisfies

$$\begin{aligned}&\varphi_t-\psi_x=0,\\&\psi_t+[p(v,\theta)-p(V,\Theta)]_x=\mu\left(\frac{u_x}{v}\right)_x-g(V,\Theta)_x,\\&\chi_t+\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}\psi_x=\frac{1}{e_\theta(v,\theta)}\left(\frac{\mu u_x^2}{v}+\left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x+\lambda\phi z\right)\\&\qquad\qquad-\left(\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}-\frac{\Theta p_\Theta(V,\Theta)}{e_\Theta(V,\Theta)}\right)U_x-r(V,\Theta),\\&\xi_t=\frac{\mu u_x^2}{v\theta}+\left(\frac{\kappa(v,\theta)\theta_x}{v\theta}\right)_x+\frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}+\frac{\lambda\phi z}{\theta},\\&z_t=\left(\frac{dz_x}{v^2}\right)_x-\phi z\end{aligned}\tag{2.2}$$

with the initial data

$$\begin{aligned}&(\varphi(t,x),\psi(t,x),\chi(t,x),\xi(t,x),z(t,x))\,|_{t=0}\\&\quad=(\varphi_0(x),\psi_0(x),\chi_0(x),\xi_0(x),z_0(x))\\&\quad:=(v_0(x)-V(0,x),u_0(x)-U(0,x),\theta_0(x)-\Theta(0,x),s_0(x)-\bar{S},z_0(x)).\end{aligned}\tag{2.3}$$

On the other hand, it is easy to see that $\eta(v,u,\theta;V,U,\Theta)$ defined by (1.32) is a convex entropy of the system (1.1) around the smooth rarefaction wave profile $(V(t,x),U(t,x),\Theta(t,x),0)$ which solves

$$\begin{aligned}&\eta_t(v,u,\theta,V,U,\Theta)+((p(v,\theta)-p(V,\Theta))\psi)_x+\left(\frac{\mu\Theta\psi_x^2}{v\theta}+\frac{\kappa(v,\theta)\Theta\chi_x^2}{v\theta^2}\right)\\&\quad+(\tilde{p}(v,s)-\tilde{p}(V,\bar{s})-\tilde{p_v}(V,\bar{s})\varphi-\tilde{p_s}(V,\bar{s})\xi)U_x+\frac{\lambda\phi z\Theta}{\theta}\\&\quad=\left(\frac{\mu\psi\psi_x}{v}+\frac{\kappa(v,\theta)\chi\chi_x}{v\theta}\right)_x+\left\{\frac{2\mu U_x\chi\psi_x}{v\theta}-\frac{\mu U_x\psi\varphi_x}{v^2}-\frac{\kappa(v,\theta)\Theta_x\chi\varphi_x}{v^2\theta}+\frac{\kappa(v,\theta)\Theta_x\chi\chi_x}{v\theta^2}\right\}\\&\qquad+\left(\frac{\mu\psi U_{xx}}{v}+\frac{\kappa(v,\theta)\chi\Theta_{xx}}{v\theta}\right)+\left(\frac{\mu U_x^2\chi}{v\theta}-\frac{\mu U_xV_x\psi}{v^2}-\frac{\kappa(v,\theta)V_x\Theta_x\chi}{v^2\theta}\right)\end{aligned}$$

$$- q(V,\Theta) - g(V,\Theta)_x\psi + g(V,\Theta)_x U - r(V,\Theta)\xi + \lambda\phi z + \frac{\kappa_x(v,\theta)\chi\Theta_x}{v\theta}. \tag{2.4}$$

We first give the set of functions $X(0,T;M_1,M_2)$ for which we seek the solutions of (2.2)–(2.3) as follows:

$$X(0,T;M_1,M_2) := \left\{ (\varphi,\psi,\chi,z)(t,x) \;\middle|\; \begin{array}{l} (\varphi,\psi,\chi)(t,x) \in C(0,T;H^1(\mathbb{R})), \\ (\psi_x,\chi_x,z_x)(t,x) \in L^2(0,T;H^1(\mathbb{R})), \\ \psi_{xx}(t,x) \in L^2(\mathbb{R}), \\ \varphi_x(t,x) \in L^2(0,T;L^2(\mathbb{R})), \\ M_1^{-1} \leqslant V(t,x)+\varphi(t,x) \leqslant M_1,\ \forall\,(t,x)\in[0,T]\times\mathbb{R}, \\ M_2^{-1} \leqslant \Theta(t,x)+\chi(t,x) \leqslant M_2,\ \forall\,(t,x)\in[0,T]\times\mathbb{R}, \\ z(t,x) \in C(0,T;H^1(\mathbb{R})\cap L^1(\mathbb{R})), \\ 0 \leqslant z(t,x) \leqslant 1 \end{array} \right\}.$$

Here, $0 < T \leqslant +\infty$, M_1 and M_2 are some positive constants.

For the local solvability of the Cauchy problems (2.2) and (2.3) in the above set of functions, one has the following lemma.

Lemma 2.3 (Local existence). *Under the assumptions listed in Theorem* 1.1*, there exists a sufficiently small positive constant* t_1*, which depends only on* $\|(\varphi_0,\psi_0,\chi_0,z_0)\|_1$*,* $\underline{V}$*,* $\overline{V}$*,* $\underline{\Theta}$ *and* $\overline{\Theta}$*, such that the Cauchy problems* (2.2) *and* (2.3) *admit a unique smooth solution*

$$(\varphi(t,x),\psi(t,x),\chi(t,x),z(t,x)) \in X(0,t_1;M_1',M_2')$$

which satisfies

$$\begin{cases} 0 < (M_1')^{-1} \leqslant \varphi(t,x)+V(t,x) \leqslant M_1', \\ 0 < (M_2')^{-1} \leqslant \phi(t,x)+\Theta(t,x) \leqslant M_2', \\ 0 \leqslant z(t,x) \leqslant 1 \end{cases}$$

for all $(t,x)\in[0,t_1]\times\mathbb{R}$ *and*

$$\sup_{t\in[0,t_1]} \{\|(\varphi,\psi,\chi,z)(t)\|_1\} \leqslant 2\|(\varphi_0,\psi_0,\chi_0,z_0)\|_1.$$

Suppose that such a local solution

$$(\varphi(t,x),\psi(t,x),\chi(t,x),z(t,x))$$

constructed in Lemma 2.3 has been extended to the time step $t = T > t_1$ and satisfies the *a priori* assumption

$$0 < M_1^{-1} \leqslant v(t,x) \leqslant M_1, \quad 0 < M_2^{-1} \leqslant \theta(t,x) \leqslant M_2 \tag{2.5}$$

for all $x\in\mathbb{R}, 0\leqslant t\leqslant T$ and some generic positive constants M_1 and M_2 (without loss of generality, we assume in the rest of this manuscript that $M_1 \geqslant 1$ and $M_2 \geqslant 1$). What we want to do next is to deduce some energy type estimates in terms of $\|(\varphi_0,\psi_0,\chi_0,z_0)\|_1$, $\underline{V}$, $\overline{V}$, $\underline{\Theta}$ and $\overline{\Theta}$, but independent of M_1 and M_2. Throughout this paper, we assume δ (the strength of the rarefaction waves) and the radiation constant a are small enough such that

$$\delta M_1^{100} M_2^{100+100b} \ll 1, \tag{2.6}$$

$$a M_1^{100} M_2^{100+100b} \ll 1. \tag{2.7}$$

The following lemma guarantees that $\tilde{p}(v,s)$ is a convex function with respect to (v,s). In fact, from (1.6)–(1.8), the *a priori* assumption (2.5), and the assumption (2.7) imposed on the radiation constant a, we can get the following lemma.

Lemma 2.4. *Suppose that*

$$(\varphi(t,x),\psi(t,x),\chi(t,x),z(t,x))\in X(0,T;M_1,M_2)$$

is a solution to the Cauchy problems (2.2) *and* (2.3) *defined on the strip* $\Pi_T:=[0,T]\times\mathbb{R}$ *and satisfying the a priori assumption* (2.5)*. Then* $\tilde{p}(v,s)$ *is convex with respect to* v *and* s *provided that* $a>0$ *is sufficiently small such that* (2.7) *holds. Consequently, we have*

$$\tilde{p}(v,s)-\tilde{p}(V,\bar{s})-\tilde{p_v}(V,\bar{s})\varphi-\tilde{p_s}(V,\bar{s})\xi\geqslant 0.$$

Remark 2.5. To ensure that we can find sufficiently small positive constants δ_0 and a_0 such that the assumptions (2.6) and (2.7) hold for all $0<\delta\leqslant\delta_0, 0<a\leqslant a_0$, a sufficient condition is to show that the positive lower and upper bounds on $v(t,x)$ and $\theta(t,x)$ depend only on $\|(\varphi_0,\psi_0,\chi_0,z_0)\|_1$, $\underline{V}$, $\overline{V}$, $\underline{\Theta}$ and $\overline{\Theta}$, but are independent of M_1, M_2, δ and a.

Now we give the following lemma concerning the basic energy estimates about the solution

$$(\varphi(t,x),\psi(t,x),\chi(t,x),z(t,x)),$$

which will be frequently used later on.

Lemma 2.6 (Basic energy estimates)**.** *In addition to the conditions stated in Lemma* 2.4*, we assume further that* (2.6) *holds. Then we have for all* $0\leqslant t\leqslant T$ *that*

$$\int_{\mathbb{R}} z(t,x)dx+\int_0^t\int_{\mathbb{R}}\phi(\tau,x)z(\tau,x)dxd\tau\lesssim 1, \tag{2.8}$$

$$\int_{\mathbb{R}} z^2(t,x)dx+\int_0^t\int_{\mathbb{R}}\left(\frac{d}{v^2}z_x^2+\phi z^2\right)(\tau,x)dxd\tau\lesssim 1, \tag{2.9}$$

$$\begin{aligned}&\int_{\mathbb{R}}\eta(t,x)dx+\int_0^t\int_{\mathbb{R}}\left(\frac{\mu\Theta\psi_x^2}{v\theta}+\frac{\kappa(v,\theta)\Theta\chi_x^2}{v\theta^2}\right)(\tau,x)dxd\tau+\int_0^t\int_{\mathbb{R}}\left(\frac{\lambda\Theta\phi z}{\theta}\right)(\tau,x)dxd\tau\\&\quad+\int_0^t\int_{\mathbb{R}}[(\tilde{p}(v,s)-\tilde{p}(V,\bar{s})-\tilde{p_v}(V,\bar{s})\varphi-\tilde{p_s}(V,\bar{s})\xi)U_x](\tau,x)dxd\tau\lesssim 1.\end{aligned} \tag{2.10}$$

Proof. The estimates (2.8) and (2.9) follow directly from $(1.1)_4$ and integrations by parts. As for (2.10), we have by integrating (2.4) with respect to t and x over $(0,t)\times\mathbb{R}$ that

$$\begin{aligned}&\int_{\mathbb{R}}\eta(t,x)dx+\int_0^t\int_{\mathbb{R}}\left(\frac{\mu\Theta\psi_x^2}{v\theta}+\frac{\kappa(v,\theta)\Theta\chi_x^2}{v\theta^2}\right)+\int_0^t\int_{\mathbb{R}}\frac{\lambda\Theta\phi z}{\theta}\\&\quad+\int_0^t\int_{\mathbb{R}}(\tilde{p}(v,s)-\tilde{p}(V,\bar{s})-\tilde{p_v}(V,\bar{s})\varphi-\tilde{p_s}(V,\bar{s})\xi)U_x\\&\quad=:\int_{\mathbb{R}}\eta_0dx+\sum_{j=1}^5 I_j.\end{aligned} \tag{2.11}$$

By virtue of Lemma 2.2, the *a priori* assumption (2.5), (2.6), (2.8), and Cauchy-Schwarz inequality, I_j $(j=1,2,3,4,5)$ can be bounded as follows:

$$\begin{aligned}I_1&=\int_0^t\int_{\mathbb{R}}\left(\frac{2\mu U_x\chi\psi_x}{v\theta}-\frac{\mu U_x\psi\varphi_x}{v^2}-\frac{\kappa(v,\theta)\Theta_x\chi\varphi_x}{v^2\theta}+\frac{\kappa(v,\theta)\Theta_x\chi\chi_x}{v\theta^2}\right)\\&\leqslant\delta^{\frac14}\left(\int_0^t\left\|\frac{\varphi_x}{v}(\tau)\right\|^2d\tau+\int_0^t\int_{\mathbb{R}}\left(\frac{\mu\Theta\psi_x^2}{v\theta}+\frac{\kappa(v,\theta)\Theta\chi_x^2}{v\theta^2}\right)\right)\\&\quad+\delta^{-\frac14}C\left(\int_0^t\|U_x\|_{L^\infty}^{\frac12}\|U_x\|_{L^\infty}^{\frac32}\int_{\mathbb{R}}\left(\frac{\psi}{v}\right)^2dxd\tau+\int_0^t\int_{\mathbb{R}}\left(\frac{\mu U_x^2\chi^2}{v\Theta\theta}+\frac{\kappa(v,\theta)\chi^2\Theta_x^2}{v\theta^2\Theta}\right)\right.\\&\quad\left.+\int_0^t\|\Theta_x\|_{L^\infty}^2\int_{\mathbb{R}}\left(\frac{\kappa(v,\theta)\chi}{v\theta}\right)^2dxd\tau\right)\end{aligned}$$

$$
\begin{aligned}
&\lesssim \delta^{\frac{1}{4}}\left(\int_0^t\left\|\frac{\varphi_x}{v}\right\|^2 d\tau+\int_0^t\int_{\mathbb{R}}\left(\frac{\mu\Theta\psi_x^2}{v\theta}+\frac{\kappa(v,\theta)\Theta\chi_x^2}{v\theta^2}\right)\right)\\
&\quad+\delta^{\frac{3}{4}}C(M_1^2+M_1M_2+M_1^2M_2^2(1+M_1M_2^b)^2\\
&\quad+M_1M_2^2(1+M_1M_2^b))\int_0^t(1+\tau)^{-\frac{3}{2}}\|(\psi,\chi)(\tau)\|^2 d\tau, \qquad (2.12)
\end{aligned}
$$

$$
\begin{aligned}
I_2&=\int_0^t\int_{\mathbb{R}}\left(\frac{\mu U_{xx}\psi}{v}+\frac{\kappa(v,\theta)\Theta_{xx}\chi}{v\theta}\right)\\
&\lesssim \int_0^t\left(\|U_{xx}\|+\|\Theta_{xx}\|+\|U_{xx}\|\left\|\frac{\psi}{v}\right\|^2+\|\Theta_{xx}\|\left\|\frac{\kappa(v,\theta)\chi}{v\theta}\right\|^2\right)d\tau\\
&\lesssim \delta^{\frac{1}{8}}+\delta^{\frac{1}{4}}(M_1^2+M_1^2M_2^2(1+M_1^2M_2^{2b}))\int_0^t(1+\tau)^{-\frac{9}{8}}\|(\psi,\chi)(\tau)\|^2 d\tau, \qquad (2.13)
\end{aligned}
$$

$$
\begin{aligned}
I_3&=\int_0^t\int_{\mathbb{R}}\left(\frac{\mu U_x^2\chi}{v\theta}-\frac{\mu U_x\psi V_x}{v^2}-\frac{\kappa(v,\theta)\Theta_x\chi V_x}{v^2\theta}\right)\\
&\lesssim \int_0^t\int_{\mathbb{R}}\left(|U_x|^{\frac{5}{2}}+|V_x|^{\frac{5}{2}}+\left|\frac{V_x\chi}{v^2}\right|^{\frac{5}{3}}+\left|\frac{U_x\chi}{v\theta}\right|^{\frac{5}{3}}+\left|\frac{\kappa(v,\theta)\Theta_x\chi}{v^2\theta}\right|^{\frac{5}{3}}\right)\\
&\lesssim \int_0^t\|(U_x,V_x,\Theta_x)\|_{L^{\frac{5}{2}}}^{\frac{5}{2}}d\tau+\int_0^t\int_{\mathbb{R}}\left(|V_x|^{\frac{3}{2}}\left|\frac{\psi}{v^2}\right|^2+|U_x|^{\frac{3}{2}}\left|\frac{\chi}{v\theta}\right|^2+|\Theta_x|^{\frac{3}{2}}\left|\frac{\kappa(v,\theta)\chi}{v^2\theta}\right|^2\right)\\
&\lesssim \delta+\delta^{\frac{1}{2}}(M_1^4+M_1^2M_2^2+M_1^4M_2^2(1+M_1^2M_2^{2b}))\int_0^t(1+\tau)^{-\frac{5}{4}}\|(\psi,\chi)(\tau)\|^2 d\tau, \qquad (2.14)
\end{aligned}
$$

$$
\begin{aligned}
I_4&=\int_0^t\int_{\mathbb{R}}(-q(V,\Theta)-g(V,\Theta)_x\psi+g(V,\Theta)_xU-r(V,\Theta)\xi)\\
&\lesssim \int_0^t(\|q(V,\Theta)\|_{L^1}+\|g(V,\Theta)_x\|_{L^1}+\|g(V,\Theta)_x\|+\|r(V,\Theta)\|+\|g(V,\Theta)_x\|\|\psi\|^2+\|r(V,\Theta)\|\|\xi\|^2)d\tau\\
&\lesssim \delta^{\frac{1}{3}}+\delta^{\frac{2}{3}}\int_0^t(1+\tau)^{-\frac{4}{3}}\|(\psi,\xi)(\tau)\|^2 d\tau \qquad (2.15)
\end{aligned}
$$

and

$$
\begin{aligned}
I_5&=\int_0^t\int_{\mathbb{R}}\left(\lambda\phi z+\frac{\kappa_x(v,\theta)\chi\Theta_x}{v\theta}\right)\\
&\lesssim 1+\int_0^t\int_{\mathbb{R}}\left(\frac{\theta^b|\varphi_x\Theta_x\chi|}{v\theta}+\frac{\theta^b|V_x\Theta_x\chi|}{v\theta}+\theta^{b-2}|\Theta_x\chi_x\chi|+\theta^{b-2}|\Theta_x^2\chi|\right)\\
&\lesssim 1+\delta^{\frac{1}{4}}\int_0^t\int_{\mathbb{R}}\left(\left|\frac{\varphi_x}{v}\right|^2+\frac{\kappa(v,\theta)\Theta\chi_x^2}{v\theta^2}\right)+\int_0^t\int_{\mathbb{R}}\left(|V_x|^{\frac{5}{2}}+|\Theta_x|^{\frac{5}{2}}+\left|\frac{\theta^{b-1}\chi\Theta_x}{v}\right|^{\frac{5}{3}}+|\theta^{b-2}\chi\Theta_x|^{\frac{5}{3}}\right)\\
&\quad+\delta^{-\frac{1}{4}}\int_0^t\int_{\mathbb{R}}\left(\theta^{2b-2}|\Theta_x\chi|^2+\frac{v\theta^{2b-2}\chi^2|\Theta_x|^2}{\Theta\kappa(v,\theta)}\right)\\
&\lesssim 1+\delta^{\frac{1}{4}}\left(1+\int_0^t\left\|\frac{\varphi_x}{v}(\tau)\right\|^2 d\tau+\int_0^t\int_{\mathbb{R}}\frac{\kappa(v,\theta)\Theta\chi_x^2}{v\theta^2}\right)\\
&\quad+(\delta^{\frac{3}{4}}M_2^{2b-2}+\delta^{\frac{1}{2}}M_1^2M_2^{2b-2}+\delta^{\frac{3}{4}}M_2^{b-2}+\delta^{\frac{1}{2}}M_2^{2b-4})\int_0^t[(1+\tau)^{-\frac{3}{2}}+(1+\tau)^{-\frac{5}{4}}]\|\chi(\tau)\|^2 d\tau. \qquad (2.16)
\end{aligned}
$$

Combining (2.6), (2.11)–(2.16) and using Gronwall's inequality, we can deduce that

$$
\begin{aligned}
&\int_{\mathbb{R}}\eta(t,x)dx+\int_0^t\int_{\mathbb{R}}\left(\frac{\mu\Theta\psi_x^2}{v\theta}+\frac{\kappa(v,\theta)\Theta\chi_x^2}{v\theta^2}\right)+\int_0^t\int_{\mathbb{R}}\frac{\phi z}{\theta}\\
&\quad+\int_0^t\int_{\mathbb{R}}(\tilde{p}(v,s)-\tilde{p}(V,\bar{s})-\tilde{p_v}(V,\bar{s})\varphi-\tilde{p_s}(V,\bar{s})\xi)U_x\\
&\qquad\lesssim 1+\delta^{\frac{1}{4}}M_1M_2\int_0^t\int_{\mathbb{R}}\frac{\theta\varphi_x^2}{v^3}. \qquad (2.17)
\end{aligned}
$$

Now we turn to estimate the term $\int_0^t\int_{\mathbb{R}}\frac{\theta\varphi_x^2}{v^3}$. For this purpose, we multiply $(2.2)_2$ by $\frac{\varphi_x}{v}$ to deduce that

$$\begin{aligned}&\left[\frac{\mu}{2}\left(\frac{\varphi_x}{v}\right)^2-\frac{\varphi_x\psi}{v}\right]_t+\frac{R\theta\varphi_x^2}{v^3}+\left(\frac{\psi\psi_x}{v}\right)_x\\&=\left[\frac{\psi_x^2}{v}+\frac{p_\theta(v,\theta)\varphi_x\chi_x}{v}\right]+\left\{\frac{(p_v(v,\theta)-p_V(V,\Theta))V_x\varphi_x}{v}+\frac{(p_\theta(v,\theta)-p_\Theta(V,\Theta))\Theta_x\varphi_x}{v}\right\}\\&\quad+\left[\frac{U_x\psi\varphi_x}{v^2}-\frac{V_x\psi\psi_x}{v^2}+\frac{\mu V_x\psi_x\varphi_x}{v^3}\right]+\left[\frac{\mu V_xU_x\varphi_x}{v^3}-\frac{\mu U_{xx}\varphi_x}{v^2}\right]+\frac{g(V,\Theta)_x\varphi_x}{v}.\end{aligned}\tag{2.18}$$

Then we integrate (2.18) over $(0,t)\times\mathbb{R}$ to derive

$$\begin{aligned}&\left\|\frac{\varphi_x}{v}(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}\\&\lesssim 1+\|\psi(t)\|^2+\underbrace{\int_0^t\int_{\mathbb{R}}\left[\frac{\psi_x^2}{v}+\frac{p_\theta(v,\theta)\varphi_x\chi_x}{v}\right]}_{I_6}\\&\quad+\underbrace{\int_0^t\int_{\mathbb{R}}\left\{\frac{(p_v(v,\theta)-p_V(V,\Theta))V_x\varphi_x}{v}+\frac{(p_\theta(v,\theta)-p_\Theta(V,\Theta))\Theta_x\varphi_x}{v}\right\}}_{I_7}\\&\quad+\underbrace{\int_0^t\int_{\mathbb{R}}\left[\frac{U_x\psi\varphi_x}{v^2}-\frac{v_x\psi\psi_x}{v^2}+\frac{\mu V_x\psi_x\varphi_x}{v^3}\right]}_{I_8}+\underbrace{\int_0^t\int_{\mathbb{R}}\left[\frac{\mu V_xU_x\varphi_x}{v^3}-\frac{\mu U_{xx}\varphi_x}{v^2}\right]}_{I_9}\\&\quad+\underbrace{\int_0^t\int_{\mathbb{R}}\frac{g(V,\Theta)_x\varphi_x}{v}}_{I_{10}}.\end{aligned}\tag{2.19}$$

Now we turn to estimate I_j $(j=6,\ldots,10)$ term by term. In fact, we have from Lemma 2.2, *a priori* assumption (2.5) and Cauchy-Schwarz inequality that

$$I_6\leqslant\frac{1}{10}\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}+C\left(\|\theta\|_\infty\int_0^t\int_{\mathbb{R}}\frac{\mu\Theta\psi_x^2}{v\theta}+\left\|\frac{\theta^2p_\theta^2(v,\theta)}{\kappa(v,\theta)p_v(v,\theta)}\right\|_\infty\int_0^t\int_{\mathbb{R}}\frac{\kappa(v,\theta)\Theta\chi_x^2}{v\theta^2}\right)\tag{2.20a}$$

$$\leqslant\frac{1}{10}\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}+C\left(M_2\int_0^t\int_{\mathbb{R}}\frac{\mu\Theta\psi_x^2}{v\theta}+(M_2+M_1^2M_2^7)\int_0^t\int_{\mathbb{R}}\frac{\kappa(v,\theta)\Theta\chi_x^2}{v\theta^2}\right),\tag{2.20b}$$

$$\begin{aligned}I_7&\leqslant\frac{1}{10}\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}+C\int_0^t\int_{\mathbb{R}}\left(\frac{(p_v(v,\theta)-p_V(V,\Theta))^2V_x^2}{v(-p_v(v,\theta))}+\frac{(p_\theta(v,\theta)-p_\Theta(V,\Theta))^2\Theta_x^2}{v(-p_v(v,\theta))}\right)\\&\leqslant\frac{1}{10}\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}+C\int_0^t\int_{\mathbb{R}}\left(\left(\frac{\chi^2}{v^3\theta}+\frac{\varphi^2}{v^3\theta}+\frac{\varphi^2}{v\theta}\right)V_x^2\right.\\&\quad\left.+\left(\frac{\varphi^2}{vV^2\theta}+v\chi^2\theta^3+v\theta\Theta^2\chi^2+\frac{v\Theta^4\chi^2}{\theta}\right)\Theta_x^2\right)\\&\leqslant\frac{1}{10}\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}+\delta C(M_1^3M+M_1M_2^3)\int_0^t(1+\tau)^{-\frac{3}{2}}\|(\varphi,\chi)(\tau)\|^2d\tau,\end{aligned}\tag{2.21}$$

$$\begin{aligned}I_8&\leqslant\frac{1}{10}\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}+C\int_0^t\int_{\mathbb{R}}\left(\frac{U_x^2\psi^2}{v^3(-p_v(v,\theta))}+\frac{V_x^2\psi^2}{v^3(-p_v(v,\theta))}+\frac{V_x^2\psi_x^2}{v^5(-p_v(v,\theta))}\right)\\&\leqslant\frac{1}{10}\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}+\delta^4CM_1^3\int_0^t\int_{\mathbb{R}}\frac{\mu\Theta\psi_x^2}{v\theta}+\delta CM_1M_2\int_0^t(1+\tau)^{-\frac{3}{2}}\|\psi(\tau)\|^2d\tau,\end{aligned}\tag{2.22}$$

$$\begin{aligned}I_9&\leqslant\frac{1}{10}\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}+C\int_0^t\int_{\mathbb{R}}\left(\frac{U_x^2V_x^2}{v^5(-p_v(v,\theta))}+\frac{U_{xx}^2}{v^3(-p_v(v,\theta))}\right)\\&\leqslant\frac{1}{10}\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}+C\left(M_1^3M_2\int_0^t\int_{\mathbb{R}}(|U_x|^{\frac{5}{2}}+|V_x|^{10})+M_1M_2\int_0^t\int_{\mathbb{R}}U_{xx}^2\right)\end{aligned}$$

$$\leqslant \frac{1}{10}\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}+C(M_1M_2\delta^{\frac{1}{2}}+M_1^3M_2\delta) \tag{2.23}$$

and

$$\begin{aligned} I_{10} &\leqslant \frac{1}{10}\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}+C\int_0^t\int_{\mathbb{R}}\frac{(g(V,\Theta)_x)^2}{v(-p_v(v,\theta))} \\ &\leqslant \frac{1}{10}\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}+CM_1M_2\delta^{\frac{4}{3}}. \end{aligned} \tag{2.24}$$

Plugging (2.20b)–(2.24) into (2.19), we have

$$\begin{aligned} &\left\|\left(\frac{\varphi_x}{v}\right)(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3} \\ &\quad\lesssim 1+\|\psi(t)\|^2+(M_1^2M_2^7+\delta^4M_1^3M_2)\int_0^t\int_{\mathbb{R}}\left(\frac{\mu\Theta\psi_x^2}{v\theta}+\frac{\kappa(v,\theta)\Theta\chi_x^2}{v\theta^2}\right) \\ &\qquad+\delta(M_1^3M_2+M_1M_2^3)\int_0^t(1+\tau)^{-\frac{3}{2}}\|(\varphi,\psi,\chi)(\tau)\|^2d\tau. \end{aligned} \tag{2.25}$$

Having obtained (2.17) and (2.25), we can deduce (2.10) immediately by the assumption (2.6) and Gronwall's inequality. □

By repeating the argument developed in [1], we can deduce the pointwise bounds of $z(t,x)$. Here we omit the proof for brevity.

Lemma 2.7. *Under the conditions listed in Lemma* 2.6*, we have for all* $(t,x)\in[0,T]\times\mathbb{R}$ *that*

$$0\leqslant z(t,x)\leqslant 1. \tag{2.26}$$

3 Uniform bounds for the specific volume

The main purpose of this section is to deduce the uniform-in-time pointwise bounds for the specific volume $v(t,x)$ for the Cauchy problems (2.2) and (2.3), which do not depend on δ and a. To this end, we first give the following lemma, which is a consequence of (2.10) and Jensen's inequality.

Lemma 3.1. *Under the conditions listed in Lemma* 2.6*, we have that for all* $k\in\mathbb{Z}$ *and* $t\in[0,T]$ *there exist* $a_k(t),b_k(t)\in\Omega_k:=[-k-1,k+1]$ *such that*

$$\int_{\Omega_k}v(t,x)dx\sim 1,\quad \int_{\Omega_k}\theta(t,x)dx\sim 1,\quad v(t,a_k(t))\sim 1,\quad \theta(t,b_k(t))\sim 1. \tag{3.1}$$

The next lemma is concerned with a rough estimate on $\theta(t,x)$ in terms of the entropy dissipation rate functional

$$V(t)=\int_{\mathbb{R}}\left(\frac{\mu\Theta\psi_x^2}{v\theta}+\frac{\kappa(v,\theta)\Theta\chi_x^2}{v\theta^2}\right)(t,x)dx.$$

Lemma 3.2. *Under the conditions listed in Lemma* 2.6*, we have that for* $0\leqslant m\leqslant\frac{b+1}{2}$ *and each* $x\in\mathbb{R}$ (*without loss of generality, we can assume that* $x\in\Omega_k$ *for some* $k\in\mathbb{Z}$),

$$|\theta^m(t,x)-\theta^m(t,b_k(t))|\lesssim V^{\frac{1}{2}}(t)+1 \tag{3.2}$$

holds for $0\leqslant t\leqslant T$ *and consequently,*

$$|\theta(t,x)|^{2m}\lesssim 1+V(t),\quad x\in\overline{\Omega}_k,\quad 0\leqslant t\leqslant T. \tag{3.3}$$

Proof. From (1.9), we have

$$|\theta^m(t,x)-\theta^m(t,b_k(t))|$$

$$
\begin{aligned}
&\lesssim \int_{\Omega_k} |\theta^{m-1}(\Theta_x+\chi_x)|dx \\
&\lesssim \left(\int_{\Omega_k} \frac{v\theta^{2m}}{1+v\theta^b}dx\right)^{\frac{1}{2}} \left(\int_{\Omega_k} \left(\frac{\mu\Theta\psi_x^2}{v\theta} + \frac{\kappa(v,\theta)\Theta\chi_x^2}{v\theta^2}\right)dx\right)^{\frac{1}{2}} + \int_{\Omega_k} \theta^{m-1}|\Theta_x|dx \\
&\lesssim V^{\frac{1}{2}}(t) + M_2^{\frac{b-1}{2}}\delta^2 \\
&\lesssim V^{\frac{1}{2}}(t) + 1. \qquad (3.4)
\end{aligned}
$$

It is worth pointing out that we have used the assumption $0 \leqslant m \leqslant \frac{b+1}{2}$, boundedness of Ω_k, (2.5), (2.6), and (3.1) in deriving the above inequality. □

The next lemma will give a local representation of $v(t,x)$ by using the following cut-off function $\varphi(x) \in W^{1,\infty}(\mathbb{R})$:

$$
\varphi(x) = \begin{cases} 1, & x \leqslant k+1, \\ k+2-x, & k+1 \leqslant x \leqslant k+2, \\ 0, & x \geqslant k+2. \end{cases} \qquad (3.5)
$$

Lemma 3.3. *Under the assumptions stated in Theorem* 1.1*, we have that for each* $0 \leqslant t \leqslant T$,

$$
v(t,x) = B(t,x)Q(t) + \frac{1}{\mu}\int_0^t \frac{B(t,x)Q(t)v(\tau,x)p(\tau,x)}{B(\tau,x)Q(\tau)}d\tau, \quad x \in \overline{\Omega}_k. \qquad (3.6)
$$

Here

$$
\begin{aligned}
&B(t,x) := v_0(x)\exp\left\{\frac{1}{\mu}\int_x^{\infty}(u_0(y)-u(t,y))\varphi(y)dy\right\}, \\
&Q(t) := \exp\left\{\frac{1}{\mu}\int_0^t \int_{k+1}^{k+2} \sigma(\tau,y)\right\}, \qquad (3.7) \\
&\sigma := -p(v,\theta) + \frac{\mu u_x}{v}.
\end{aligned}
$$

With the above presentation in hand, we can deduce uniform-in-time pointwise bounds of $v(t,x)$ by repeating the argument used in [32], and we omit the proof for brevity.

Lemma 3.4. *Assume that the conditions listed in Lemma* 2.6 *hold. Then there exists a positive constant* C_1 *which depends only on* $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ *and* H_0*, but is independent of* δ *and* a*, such that*

$$
C_1^{-1} \leqslant v(t,x) \leqslant C_1, \quad \forall\,(t,x) \in [0,T]\times\mathbb{R}. \qquad (3.8)
$$

The following lemma concerns the estimate on the term $\|\varphi_x(t)\|^2$, which will be frequently used later on.

Lemma 3.5. *Under the assumptions listed in Lemma* 2.6*, we have that for any* $0 \leqslant t \leqslant T$,

$$
\|\varphi_x(t)\|^2 + \int_0^t \int_{\mathbb{R}} \theta(\tau,x)\varphi_x^2(\tau,x)dxd\tau \lesssim 1 + \|\theta\|_\infty. \qquad (3.9)
$$

Proof. In light of (2.10), (2.20a) and (3.8), we can conclude that

$$
\begin{aligned}
I_6 &\leqslant \frac{1}{10}\int_0^t \int_{\mathbb{R}} \frac{R\theta\varphi_x^2}{v^3} + C\left(\|\theta\|_\infty + \left\|\frac{\theta^2 p_\theta^2(v,\theta)}{\kappa(v,\theta)p_v(v,\theta)}\right\|_\infty\right) \\
&\leqslant \frac{1}{10}\int_0^t \int_{\mathbb{R}} \frac{R\theta\varphi_x^2}{v^3} + C(1 + \|\theta\|_\infty + \|\theta\|_\infty^{(7-b)_+}). \qquad (3.10)
\end{aligned}
$$

Here, $(7-b)_+ := \max\{0, 7-b\}$.

Then inserting (3.10), (2.21)–(2.24) into (2.19), we can get (3.9) by employing (2.6) and the assumption $b > 6$. □

The next lemma pays attention to the estimate on the term

$$\int_0^t \|\psi_{xx}(\tau)\|^2 d\tau,$$

which will be useful in deducing the upper bound of $\theta(t,x)$.

Lemma 3.6. *Under the conditions listed in Lemma* 2.6, *we have that for any* $0 \leqslant t \leqslant T$,

$$\|\psi_x(t)\|^2 + \int_0^t \|\psi_{xx}(\tau)\|^2 d\tau \lesssim 1 + \|\theta\|_\infty^3. \tag{3.11}$$

Proof. We multiply $(2.2)_2$ by $-\psi_{xx}$ to get

$$\begin{aligned}
&\partial_t\left(\frac{\psi_x^2}{2}\right) + \frac{\mu\psi_{xx}^2}{v} - (\psi_t\psi_x)_x \\
&\quad = (p(v,\theta) - p(V,\Theta))_x\psi_{xx} - \frac{\mu\psi_{xx}U_{xx}}{v} + g(V,\Theta)_x\psi_{xx} \\
&\qquad + \frac{\mu(\psi_x\varphi_x\psi_{xx} + \psi_x V_x\psi_{xx} + U_x\varphi_x\psi_{xx} + V_x U_x\psi_{xx})}{v^2}.
\end{aligned} \tag{3.12}$$

Integrating (3.12) with respect to t and x over $(0,t)\times\mathbb{R}$, we utilize Cauchy's inequality, Sobolev's inequality, Lemma 2.2, (3.8) and (3.9) to find that

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{\psi_x^2}{2} dx + \int_0^t \int_{\mathbb{R}} \frac{\mu\psi_{xx}^2}{v} \\
&\leqslant \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu\psi_{xx}^2}{v} + C(\epsilon)\int_0^t \int_{\mathbb{R}} [(1+a^2\theta^6)|\chi_x|^2 + |(V_x,\Theta_x)|^2|\varphi|^2 + |\theta\varphi_x|^2 + \chi^2|V_x|^2 \\
&\quad + a^2(1+\theta^4)|\Theta_x|^2|\chi|^2 + U_{xx}^2 + |g(V,\Theta)_x|^2 + \psi_x^2\varphi_x^2 + \psi_x^2 V_x^2 + U_x^2\varphi_x^2 + V_x^2 U_x^2] \\
&\leqslant \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu\psi_{xx}^2}{v} + C(\epsilon)\left(1 + \|\theta\|_\infty^{(8-b)_+} + \|\theta\|_\infty^2 + \int_0^t \|\psi_x\|\|\psi_{xx}\|\|\varphi_x\|^2 d\tau\right) \\
&\leqslant 2\epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu\psi_{xx}^2}{v} + C(\epsilon)\left(1 + \|\theta\|_\infty^{(8-b)_+} + \|\theta\|_\infty^2 + (1+\|\theta\|_\infty^2)\int_0^t \int_{\mathbb{R}} \frac{\psi_x^2}{\theta}\cdot\theta\right) \\
&\leqslant 2\epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu\psi_{xx}^2}{v} + C(\epsilon)(1 + \|\theta\|_\infty^{(8-b)_+} + \|\theta\|_\infty^3).
\end{aligned} \tag{3.13}$$

If we choose $\epsilon > 0$ small enough and use the assumption $b > 6$, we can obtain (3.11) immediately. □

4 A uniform upper bound of the absolute temperature

Now we are in a position to derive an estimate on the upper bound of $\theta(t,x)$. To this end, we recall the definitions of the auxiliary functions $X(t), Y(t)$ and $Z(t)$ defined by (1.33), and then try to deduce certain estimates among them by employing the special structure of the system (2.2).

Our first result is to show that $\|\theta(t)\|_{L^\infty}$, $\|\psi(t)\|$ and $\|\psi_x(t)\|_{L^\infty}$ can be controlled by $Y(t)$ and $Z(t)$, respectively.

Lemma 4.1. *Under the conditions listed in Lemma* 2.6, *we have that for all* $0 \leqslant t \leqslant T$,

$$\|\theta(t)\|_{L^\infty} \lesssim 1 + Y(t)^{\frac{1}{2b+3}}, \tag{4.1}$$

$$\sup_{\tau\in(0,t)} \|\psi_x(\tau)\|^2 \lesssim 1 + Z(t)^{\frac{1}{2}}, \quad \|\psi_x(t)\|_{L^\infty} \lesssim 1 + Z(t)^{\frac{3}{8}}. \tag{4.2}$$

Proof. We assume that $x \in [-k-1, k+1]$ for some $k \in \mathbb{Z}$ and $x \geqslant b_k(t)$ and observe that

$$(\theta(t,x) - \Theta(t,x))^{2b+3}$$

$$
\begin{aligned}
&= (\theta(t,b_k(t)) - \Theta(t,b_k(t)))^{2b+3} + \int_{b_k(t)}^{x} (2b+3)(\theta(t,y)-\Theta(t,y))^{2b+2}\chi_x(t,y)dy \\
&\lesssim 1 + \|(\theta-\Theta)(t)\|_{L^\infty}^{\frac{2b+3}{2}} \left[\int_{-k-1}^{k+1}\left(1+\Phi\left(\frac{\theta}{\Theta}\right)\right)dx\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}}(\theta-\Theta)^{2b}\chi_x^2dx\right]^{\frac{1}{2}} \\
&\lesssim 1 + \|(\theta-\Theta)(t)\|_{L^\infty}^{\frac{2b+3}{2}} Y^{\frac{1}{2}}(t).
\end{aligned}
$$

Then applying Cauchy's inequality, we can obtain (4.1).

The estimate (4.2) is a consequence of Gagliardo-Nirenberg and Sobolev inequalities. This completes the proof of Lemma 4.1. □

Our next result shows that $X(t)$ and $Y(t)$ can be bounded by $Z(t)$.

Lemma 4.2. *Under the conditions listed in Lemma* 2.6*, we have that for* $0 \leqslant t \leqslant T$,

$$
X(t) + Y(t) \lesssim 1 + Z(t)^{\frac{6b+9}{12b+4}}. \tag{4.3}
$$

Proof. In the same manner as [22, 32], we set

$$
\mathbb{K}(v,\theta) = \int_0^\theta \frac{\kappa(v,\xi)}{v}d\xi = \frac{\kappa_1\theta}{v} + \frac{\kappa_2\theta^{b+1}}{b+1}. \tag{4.4}
$$

Then we can deduce that

$$
\begin{aligned}
&\mathbb{K}_t(v,\theta) = \mathbb{K}_v(v,\theta)\psi_x + \mathbb{K}_\theta(v,\theta)\chi_t + \mathbb{K}_v(v,\theta)U_x + \mathbb{K}_\theta(v,\theta)\Theta_t, \\
&\mathbb{K}_x(v,\theta) = \mathbb{K}_v(v,\theta)\varphi_x + \mathbb{K}_\theta(v,\theta)\chi_x + \mathbb{K}_v(v,\theta)V_x + \mathbb{K}_\theta(v,\theta)\Theta_x, \\
&\mathbb{K}_{xt}(v,\theta) = (\mathbb{K}_\theta(v,\theta)\chi_x)_t + [\mathbb{K}_{vv}(v,\theta)(\psi_x+U_x) + \mathbb{K}_{v\theta}(v,\theta)(\chi_t+\Theta_t)]\varphi_x + \mathbb{K}_v(v,\theta)\psi_{xx} \\
&\qquad + [\mathbb{K}_{vv}(v,\theta)(\psi_x+U_x) + \mathbb{K}_{v\theta}(v,\theta)(\chi_t+\Theta_t)]V_x + \mathbb{K}_v(v,\theta)U_{xx} \\
&\qquad + [\mathbb{K}_{v\theta}(v,\theta)(\psi_x+U_x) + \mathbb{K}_{\theta\theta}(v,\theta)(\chi_t+\Theta_t)]\Theta_x + \mathbb{K}_\theta(v,\theta)\Theta_{xt}, \\
&|\mathbb{K}_v(v,\theta)| + |\mathbb{K}_{vv}(v,\theta)| \lesssim \theta, \quad |\mathbb{K}_\theta(v,\theta)| \lesssim 1+\theta^b, \quad |\mathbb{K}_{v\theta}(v,\theta)| \lesssim 1, \quad |\mathbb{K}_{\theta\theta}(v,\theta)| \lesssim \theta^{b-1}.
\end{aligned} \tag{4.5}
$$

Hereafter, for simplicity of presentation, we use $\mathbb{K}, p, e, P$ and E to denote $\mathbb{K}(v,\theta), p(v,\theta), e(v,\theta), p(V,\Theta)$ and $e(V,\Theta)$, respectively.

We multiply $(2.2)_3$ by $\mathbb{K}_t$ and integrate the resulting identity with respect to t and x over $(0,t)\times\mathbb{R}$ to find that

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}} e_\theta\mathbb{K}_\theta\chi_t^2 + \int_0^t\int_{\mathbb{R}}(\mathbb{K}_\theta\chi_x)(\mathbb{K}_\theta\chi_x)_t - \int_0^t\int_{\mathbb{R}}\mathbb{K}_\theta\mathbb{K}_{v\theta}U_x\chi_x^2 \\
&\quad = \underbrace{\int_0^t\int_{\mathbb{R}}(\mathbb{K}_\theta\mathbb{K}_{\theta\theta}\chi_t\Theta_x^2 + \mathbb{K}_\theta^2\chi_t\Theta_{xx} + \mathbb{K}_\theta\mathbb{K}_{v\theta}\chi_t\Theta_xV_x)}_{I_{11}} \\
&\qquad + \underbrace{\int_0^t\int_{\mathbb{R}}\left\{\mathbb{K}_\theta\mathbb{K}_{\theta\theta}\chi_x^2\Theta_t + \mathbb{K}_\theta\mathbb{K}_{v\theta}\chi_t\varphi_x\Theta_x - \mathbb{K}_\theta\mathbb{K}_{v\theta}\chi_x\chi_tV_x - \left(\theta p_\theta - \frac{e_\theta\Theta P_\Theta}{E_\Theta}\right)U_x\mathbb{K}_\theta\chi_t\right\}}_{I_{12}} \\
&\qquad + \underbrace{\int_0^t\int_{\mathbb{R}}\mathbb{K}_\theta\mathbb{K}_{v\theta}\chi_x^2\psi_x}_{I_{13}} - \underbrace{\int_0^t\int_{\mathbb{R}}\mathbb{K}_\theta\mathbb{K}_{v\theta}\chi_t\varphi_x\chi_x}_{I_{14}} \\
&\qquad + \underbrace{\int_0^t\int_{\mathbb{R}}(\lambda\phi z - e_\theta r(v,\theta))\mathbb{K}_\theta\chi_t}_{I_{15}} - \underbrace{\int_0^t\int_{\mathbb{R}}\theta p_\theta\mathbb{K}_\theta\psi_x\chi_t}_{I_{16}} + \underbrace{\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2\mathbb{K}_\theta\chi_t}{v}}_{I_{17}}.
\end{aligned} \tag{4.6}
$$

Firstly, we find that

$$
\int_0^t\int_{\mathbb{R}} e_\theta\mathbb{K}_\theta\chi_t^2 \gtrsim \int_0^t\int_{\mathbb{R}}(1+a\theta^3)(1+\theta^b)\chi_t^2 \gtrsim X(t) \tag{4.7}
$$

and

$$\int_0^t \int_{\mathbb{R}} (\mathbb{K}_\theta \chi_x)(\mathbb{K}_\theta \chi_x)_t = \frac{1}{2}\int_{\mathbb{R}} (\mathbb{K}_\theta \chi_x)^2(t,x)dx - \frac{1}{2}\int_{\mathbb{R}} (\mathbb{K}_\theta \chi_x)^2(0,x)dx$$
$$\gtrsim Y(t) - C. \tag{4.8}$$

We now estimate I_k $(k = 11, 12, \ldots, 17)$ term by term. For the term I_{11}, it follows from Lemma 2.2, (2.6) and (4.5) that

$$\begin{aligned} I_{11} &\leqslant \epsilon X(t) + C(\epsilon)\int_0^t \int_{\mathbb{R}} [(1+\theta^{3b-2})|\Theta_x|^4 + (1+\theta^{3b})|\Theta_{xx}|^2 + (1+\theta^b)|\Theta_x|^2|V_x|^2] \\ &\leqslant \epsilon X(t) + C(\epsilon)[(1+M_2^{3b-2})\delta + (1+M_2^{3b})\delta^{\frac{1}{2}} + (1+M_2^b)\delta] \\ &\leqslant \epsilon X(t) + C(\epsilon). \end{aligned} \tag{4.9}$$

After a simple calculation, we can deduce from Lemma 3.4 that

$$\left|\frac{\theta p_\theta}{e_\theta} - \frac{\Theta P_\Theta}{E_\Theta}\right| \lesssim |\varphi| + |\chi|. \tag{4.10}$$

On the other hand, by using Taylor's formula, we can deduce for $0 < \omega < 1$ that

$$\int_{\mathbb{R}} \chi^2 dx = 2\int_{\mathbb{R}} \Phi\left(\frac{\theta}{\Theta}\right)(\omega\Theta + (1-\omega)\theta)^2 dx \lesssim 1 + M_2^2. \tag{4.11}$$

Thus we can obtain from Lemma 2.2, (2.6), (2.10), (3.9), (4.5), (4.10) and (4.11) that

$$\begin{aligned} I_{12} &\leqslant \epsilon X(t) + C(\epsilon)\int_0^t \int_{\mathbb{R}} \bigg[(1+\theta^b)|\varphi_x|^2|\Theta_x|^2 + \frac{(1+\theta^b)\Theta|\chi_x|^2}{v\theta^2}\cdot\theta^2|V_x|^2 \\ &\quad + (1+\theta^{b+6})(\varphi^2+\chi^2)|U_x|^2\bigg] \\ &\quad + C\|\Theta_t\|_{L^\infty}\int_0^t \int_{\mathbb{R}} \frac{(1+\theta^b)\Theta|\chi_x|^2}{v\theta^2}\cdot(1+\theta^{b+1}) \\ &\leqslant \epsilon X(t) + C(\epsilon)[(1+M_2^{b+1})\delta + M_2^2\delta^4 + (1+M_2^{b+8})\delta + \delta^2(1+M_2^{b+1})] \\ &\leqslant \epsilon X(t) + C(\epsilon). \end{aligned} \tag{4.12}$$

Moreover, we get by combining the estimates (2.10), (4.1), (4.2) and (4.5) that

$$\begin{aligned} I_{13} &\lesssim \int_0^t \int_{\mathbb{R}} \frac{(1+\theta^b)\Theta|\chi_x|^2}{v\theta^2}\cdot|\psi_x|\theta^2 \lesssim (1+Y(t)^{\frac{2}{2b+3}})(1+Z(t)^{\frac{3}{8}}) \\ &\leqslant \epsilon Y(t) + C(\epsilon)(1+Z(t)^{\frac{6b+9}{16b+8}}). \end{aligned} \tag{4.13}$$

By employing Sobolev's inequality, (3.9), and (2.10), we find that

$$\begin{aligned} I_{14} &\leqslant \epsilon X(t) + C(\epsilon)\int_0^t \left\|\frac{\kappa(v,\theta)\chi_x}{v}\right\|_{L^\infty}^2 \|\varphi_x\|^2 d\tau \\ &\leqslant \epsilon X(t) + C(\epsilon)(1+\|\theta\|_\infty)\int_0^t \int_{\mathbb{R}} \left|\frac{\kappa(v,\theta)\chi_x}{v}\right|\left|\left(\frac{\kappa(v,\theta)\chi_x}{v}\right)_x\right| \\ &\leqslant \epsilon X(t) + C(\epsilon)(1+\|\theta\|_\infty)\left(\int_0^t \int_{\mathbb{R}} \theta^2\kappa\left|\left(\frac{\kappa(v,\theta)\chi_x}{v}\right)_x\right|^2\right)^{\frac{1}{2}}\left(\int_0^t \int_{\mathbb{R}} \frac{\kappa(v,\theta)\Theta\chi_x^2}{v\theta^2}\right)^{\frac{1}{2}} \\ &\leqslant \epsilon X(t) + C(\epsilon)(1+Y(t)^{\frac{1}{2b+3}})\underbrace{\left(\int_0^t \int_{\mathbb{R}} (1+\theta^{b+2})\left|\left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x - \left(\frac{\kappa(v,\theta)\Theta_x}{v}\right)_x\right|^2\right)^{\frac{1}{2}}}_{J^{\frac{1}{2}}}. \end{aligned} \tag{4.14}$$

In view of $(2.2)_3$, Lemma 2.2, (2.6), (2.8), (2.10), (4.2) and (4.10), one has

$$\begin{aligned}
J &\lesssim \int_0^t \int_{\mathbb{R}} \bigg[(1+\theta^{b+2})e_\theta^2\chi_t^2 + (1+\theta^{b+2})\theta^2 p_\theta^2\psi_x^2 + (1+\theta^{b+2})e_\theta^2\bigg(\frac{\theta p_\theta}{e_\theta} - \frac{\Theta P_\Theta}{E_\Theta}\bigg)^2 U_x^2 \\
&\quad + (1+\theta^{b+2})\psi_x^4 + (1+\theta^{b+2})U_x^4 + (1+\theta^{b+2})\phi^2 z^2 + (1+\theta^{b+2})e_\theta^2 r^2(V,\Theta)\bigg] \\
&\quad + \underbrace{\int_0^t \int_{\mathbb{R}} (1+\theta^{b+2})\bigg|\bigg(\frac{\kappa(v,\theta)\Theta_x}{v}\bigg)_x\bigg|^2}_{J^a} \\
&\lesssim J^a + \int_0^t \int_{\mathbb{R}} \bigg[(1+\theta^{b+8})\chi_t^2 + \frac{\mu\Theta\psi_x^2}{v\theta}\cdot(1+\theta^{b+11}) + (1+\theta^{b+8})(\chi^2+\varphi^2)U_x^2\bigg] \\
&\quad + (1+\|\theta\|_\infty^{b+3})\|\psi_x\|_{L^\infty}^2 \int_0^t \int_{\mathbb{R}} \frac{\mu\Theta\psi_x^2}{v\theta} + (1+M_2^{b+8})\delta^{\frac{4}{3}} \\
&\quad + (1+M_2^{b+2})\delta + (1+\|\theta\|_\infty^{b+\beta+2})\int_0^t \int_{\mathbb{R}} \phi z^2 \\
&\lesssim 1 + X(t)(1+Y(t)^{\frac{8}{2b+3}}) + Y(t)^{\frac{b+11}{2b+3}} + (1+Y(t)^{\frac{b+3}{2b+3}})(1+Z(t)^{\frac{3}{4}}) + Y(t)^{\frac{b+\beta+2}{2b+3}} + J^a
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
J^a &\lesssim \int_0^t \int_{\mathbb{R}} (1+\theta^{b+2})[\theta^{2b}\varphi_x^2\Theta_x^2 + \theta^{2b}V_x^2\Theta_x^2 + \theta^{2b-2}\chi_x^2\Theta_x^2 + \theta^{2b-2}\Theta_x^4 + \Theta_{xx}^2 + \theta^{2b}\Theta_{xx}^2 \\
&\quad + (1+\theta^{2b})\Theta_x^2(\varphi_x^2+V_x^2)] \\
&\lesssim (1+\|\theta\|_\infty^{3b+2})\int_0^t \|\Theta_x\|_{L^\infty}^2\|\varphi_x\|^2 d\tau + (1+\|\theta\|_\infty^{3b+2})\int_0^t\int_{\mathbb{R}} V_x^2\Theta_x^2 \\
&\quad + \int_0^t\int_{\mathbb{R}} \frac{(1+\theta^b)\Theta|\chi_x|^2}{v\theta^2}\cdot(1+\theta^{2b+2})\Theta_x^2 \\
&\quad + (1+\|\theta\|_\infty^{3b})\int_0^t\int_{\mathbb{R}} \Theta_x^4 + (1+\|\theta\|_\infty^{3b+2})\int_0^t\int_{\mathbb{R}} \Theta_{xx}^2 \\
&\lesssim (1+M_2^{3b+3})(\delta^{\frac{1}{2}}+\delta^4) \lesssim 1.
\end{aligned} \tag{4.16}$$

Thus we can conclude from (4.15)–(4.16) that

$$J \lesssim 1 + X(t)(1+Y(t)^{\frac{8}{2b+3}}) + Y(t)^{\frac{b+11}{2b+3}} + Z(t)^{\frac{3}{4}} + Y(t)^{\frac{b+3}{2b+3}}Z(t)^{\frac{3}{4}} + Y(t)^{\frac{b+\beta+2}{2b+3}}. \tag{4.17}$$

Plugging (4.17) into (4.14), we have

$$I_{14} \leqslant \epsilon(X(t)+Y(t)) + C(\epsilon)(1+Z(t)^{\frac{6b+9}{12b+4}}). \tag{4.18}$$

Here, we have used the facts that

$$b > \frac{19}{4} \quad \text{and} \quad 0 \leqslant \beta < \min\{3b+2, 5b-10\}.$$

As for the term I_{15}, from Lemma 2.2, (2.9) and the assumption $0 \leqslant \beta < b+3$ we have

$$\begin{aligned}
I_{15} &\lesssim \int_0^t\int_{\mathbb{R}} [(1+\theta^b)\phi z + (1+\theta^b)(1+a\theta^3)|r(V,\Theta)|]|\chi_t| \\
&\leqslant \epsilon X(t) + C(\epsilon)\int_0^t\int_{\mathbb{R}} [(1+\theta^b)\phi^2 z^2 + (1+\theta^{b+6})|r(V,\Theta)|^2] \\
&\leqslant \epsilon X(t) + C(\epsilon)(1+\|\theta\|_\infty^{b+\beta} + \delta^{\frac{4}{3}}(1+M_2^{b+6})) \\
&\leqslant \epsilon(X(t)+Y(t)) + C(\epsilon).
\end{aligned} \tag{4.19}$$

For the term I_{16}, we employ (4.5) and the assumption $b > 6$ to find that

$$\begin{aligned}
I_{16} &\lesssim \int_0^t \int_{\mathbb{R}} (1+\theta^b)\theta(1+a\theta^3)|\psi_x\chi_t| \\
&\leqslant \epsilon X(t) + C(\epsilon)\int_0^t \int_{\mathbb{R}} (1+\theta^{b+9})\frac{\psi_x^2}{\theta} \\
&\leqslant \epsilon X(t) + C(\epsilon)(1+Y(t)^{\frac{b+9}{2b+3}}) \\
&\leqslant \epsilon(X(t)+Y(t)) + C(\epsilon). \qquad (4.20)
\end{aligned}$$

It suffices to bound the term I_{17}. To this end, we conclude from Lemma 2.2 and (4.5) that

$$I_{17} \leqslant \epsilon X(t) + C(\epsilon)\int_0^t \int_{\mathbb{R}} (1+\theta^b)(\psi_x^4 + U_x^4) \leqslant \epsilon X(t) + C(\epsilon)\left(1 + \int_0^t \int_{\mathbb{R}} (1+\theta^b)\psi_x^4\right). \qquad (4.21)$$

Then by virtue of Sobolev's inequality, Lemma 3.6 and (2.10), we can get

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}} (1+\theta^b)\psi_x^4 &\lesssim (1+\|\theta\|_\infty^b)\int_0^t \|\psi_x\|_{L^\infty}^2 \|\psi_x\|^2 d\tau \\
&\lesssim (1+\|\theta\|_\infty^b)\int_0^t \|\psi_x\|^3\|\psi_{xx}\| d\tau \\
&\lesssim (1+\|\theta\|_\infty^{b+3})\left(\int_0^t \|\psi_x\|^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|\psi_{xx}\|^2 d\tau\right)^{\frac{1}{2}} \\
&\lesssim 1 + \|\theta\|_\infty^{b+5}. \qquad (4.22)
\end{aligned}$$

Thus the combination of (4.21), (4.22) and the assumption $b > 2$ gives

$$I_{17} \leqslant \epsilon(X(t)+Y(t)) + C(\epsilon). \qquad (4.23)$$

By substituting (4.7)–(4.23) into (4.6) and by choosing $\epsilon > 0$ small enough, it yields (4.3). □

The next lemma tells us that $Z(t)$ can be controlled by $X(t)$ and $Y(t)$.

Lemma 4.3. *Under the conditions listed in Lemma* 2.6, *we have that for all* $0 \leqslant t \leqslant T$,

$$Z(t) \lesssim 1 + X(t) + Y(t) + Z(t)^{\frac{6b+9}{8b+8}}. \qquad (4.24)$$

Proof. We differentiate $(2.2)_2$ with respect to t and multiply the resulting identity by ψ_t to derive

$$\begin{aligned}
&\left(\frac{\psi_t^2}{2}\right)_t + \frac{\mu\psi_{xt}^2}{v} + \left[\left((p-P)_t + g(V,\Theta)_t - \mu\left(\frac{u_x}{v}\right)_t\right)\psi_t\right]_x \\
&= \left[\frac{\mu(\psi_x+U_x)^2}{v^2} - \frac{\mu U_{xt}}{v} + (p-P)_t + g(V,\Theta)_t\right]\psi_{xt}. \qquad (4.25)
\end{aligned}$$

Integrating the above identity with respect to t and x over $(0,t)\times\mathbb{R}$, one has

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{\psi_t^2}{2}dx + \int_0^t \int_{\mathbb{R}} \frac{\mu\psi_{tx}^2}{v} \\
&= \underbrace{\int_0^t \int_{\mathbb{R}} \left(\frac{\mu\psi_{tx}\psi_x^2 + \mu\psi_{tx}U_x^2 + 2\mu\psi_{tx}\psi_x U_x}{v^2} - \frac{\mu\psi_{tx}U_{tx}}{v} + g(V,\Theta)_t\psi_{tx}\right)}_{I_{18}} \\
&\quad + \int_{\mathbb{R}} \frac{\psi_{0t}^2}{2}dx + \underbrace{\int_0^t \int_{\mathbb{R}} (p-P)_t\psi_{tx}}_{I_{19}}. \qquad (4.26)
\end{aligned}$$

It suffices to estimate the terms I_k $(k = 18, 19)$. For this purpose, we compute from (2.6), (2.10) and (3.8) that

$$\begin{aligned} I_{18} &\leqslant \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu \psi_{tx}^2}{v} + C(\epsilon) \int_0^t \int_{\mathbb{R}} \left(\frac{\psi_x^2}{\theta} \cdot (\theta \psi_x^2 + \theta U_x^2) + U_x^4 + U_{xt}^2 + |g(V,\Theta)_t|^2 \right) \\ &\leqslant \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu \psi_{tx}^2}{v} + C(\epsilon)((1 + Y(t)^{\frac{1}{2b+3}})(1 + Z(t)^{\frac{3}{4}}) + \delta + \delta^4 M_2) \\ &\leqslant \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu \psi_{tx}^2}{v} + C(\epsilon)(1 + Y(t) + Z(t)^{\frac{6b+9}{8b+8}}). \end{aligned} \tag{4.27}$$

Moreover, it is easy to see that

$$|(p-P)_t|^2 \lesssim (1 + a^2\theta^6)\chi_t^2 + |\Theta_t|^2(\varphi^2 + \chi^2(1+\theta^4)) + \chi^2\psi_x^2 + \chi^2 U_x^2 + \psi_x^2 + U_x^2\varphi^2. \tag{4.28}$$

Then it follows from Lemma 2.2 and (2.10) that

$$\begin{aligned} I_{19} &\leqslant \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu \psi_{tx}^2}{v} + C(\epsilon) \int_0^t \int_{\mathbb{R}} |(p-P)_t|^2 \\ &\leqslant \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu \psi_{tx}^2}{v} + C(\epsilon) \bigg(X(t) + \int_0^t \|\Theta_t\|_{L^\infty}^2 (\|\varphi\|^2 + (1 + \|\theta\|_{L^\infty}^4)\|\chi\|^2) d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}} \frac{\psi_x^2}{\theta} \cdot (1 + \theta^3) + \int_0^t \|U_x\|_{L^\infty}^2 d\tau \bigg) \\ &\leqslant \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu \psi_{tx}^2}{v} + C(\epsilon)(1 + X(t) + Y(t) + \delta(1 + M_2^6)) \\ &\leqslant \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu \psi_{tx}^2}{v} + C(\epsilon)(1 + X(t) + Y(t)). \end{aligned} \tag{4.29}$$

By choosing $\epsilon > 0$ small enough, the combination of (4.26)–(4.29) and (3.8) shows

$$\|\psi_t\|^2 + \int_0^t \|\psi_{tx}(\tau)\|^2 d\tau \lesssim 1 + X(t) + Y(t) + Z(t)^{\frac{6b+9}{8b+8}}. \tag{4.30}$$

Now we are in a position to yield an estimate on $\|\psi_{xx}(t)\|$. Firstly, $(2.2)_2$ tells us that

$$\psi_{xx} = \frac{\varphi_x \psi_x + \varphi_x U_x + V_x \psi_x + V_x U_x}{v} - U_{xx} + \frac{v}{\mu}[\psi_t + (p(v,\theta) - P(V,\Theta))_x + g(V,\Theta)_x]. \tag{4.31}$$

On the other hand, Lemma 2.2, (2.6), (3.9), (4.1) and (4.11) show that

$$\begin{aligned} \|(p-P)_x\|^2 &\lesssim \int_{\mathbb{R}} [(1 + a^2\theta^6)|\chi_x|^2 + |(V_x,\Theta_x)|^2|\varphi|^2 + (1+\theta^4)|(V_x,\Theta_x)|^2|\chi|^2 + |\theta\varphi_x|^2] dx \\ &\lesssim Y(t) + (1 + \|\theta\|_{L^\infty}^2)(1 + \|\theta\|_{L^\infty}) + \delta^4(1 + M_2^6) \\ &\lesssim 1 + Y(t). \end{aligned} \tag{4.32}$$

Thus one can conclude from Lemma 3.6, (3.9), (4.2), (4.30)–(4.32) and Young's inequality that

$$\begin{aligned} \int_{\mathbb{R}} \psi_{xx}^2 dx &\lesssim \int_{\mathbb{R}} (\varphi_x^2\psi_x^2 + \varphi_x^2 U_x^2 + \psi_x^2 V_x^2 + V_x^2 U_x^2 + U_{xx}^2 + \psi_t^2 + |(p(v,\theta) - P(V,\Theta))_x|^2 + |(g(V,\Theta))_x|^2) dx \\ &\lesssim 1 + X(t) + Y(t) + Z(t)^{\frac{6b+9}{8b+8}} + \|\psi_x\|_{L^\infty}^2 \|\varphi_x\|^2 + \|U_x\|_{L^\infty}^2 \|\varphi_x\|^2 + \|V_x\|_{L^\infty}^2 \|\psi_x\|^2 \\ &\quad + \|V_x\|_{L^\infty}^2 \|U_x\|^2 + \|U_{xx}\|^2 + \|(g(V,\Theta))_x\|^2 \\ &\lesssim 1 + X(t) + Y(t) + Z(t)^{\frac{6b+9}{8b+8}} + \delta^4(1 + M_2^3) + (1 + Y(t)^{\frac{1}{2b+3}})(1 + Z(t)^{\frac{3}{4}}) \\ &\lesssim 1 + X(t) + Y(t) + Z(t)^{\frac{6b+9}{8b+8}}. \end{aligned} \tag{4.33}$$

We thus get the estimate (4.24) by using the definition of $Z(t)$. □

We can deduce that $Y(t) \lesssim 1$ by combining Lemmas 4.1–4.3. Then the desired upper bound on the absolute temperature $\theta(t,x)$ follows from (4.1) immediately. Moreover, we can infer from Lemma 2.1 to Lemma 4.3 as follows.

Lemma 4.4. *Under the conditions listed in Lemma* 2.6*, there exists a positive constant* C_2 *which depends only on* $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ *and* H_0*, such that*

$$\theta(t,x) \leqslant C_2, \quad \forall\,(t,x) \in [0,T] \times \mathbb{R}. \tag{4.34}$$

Moreover, we have that for $0 \leqslant t \leqslant T$*,*

$$\sup_{0\leqslant t<\infty} \|(\varphi,\psi,\chi,z,\varphi_x,\psi_x,\psi_t,\chi_x,\psi_{xx})(t)\|^2 + \int_0^t \|(\sqrt{\theta}\varphi_x,\psi_x,\chi_t,\chi_x,\psi_{xx},\psi_{xt},z_x)(\tau)\|^2 d\tau \lesssim 1 \tag{4.35}$$

and

$$\int_0^t \|\psi_x(\tau)\|^4_{L^4(\mathbb{R})} d\tau \lesssim 1, \quad \|\psi_x\|_{L^\infty([0,T]\times\mathbb{R})} \lesssim 1. \tag{4.36}$$

The next lemma gives nice bounds on the terms $\int_0^t \|\chi_{xx}(\tau)\|^2 d\tau$ and $\|z_x(t)\|^2$, whose proof is similar to Lemma 4.5 developed in [26]. Thus we omit the proof for brevity.

Lemma 4.5. *Under the conditions listed in Lemma* 2.6*, we can get that for* $0 \leqslant t \leqslant T$*,*

$$\|\chi_x(t)\|^2 + \int_0^t \|\chi_{xx}(\tau)\|^2 d\tau \lesssim 1 \tag{4.37}$$

and

$$\|z_x(t)\|^2 + \int_0^t \|z_{xx}(\tau)\|^2 d\tau \lesssim 1. \tag{4.38}$$

5 A local-in-time lower bound on the absolute temperature

The following lemma will give a local-in-time lower bound on $\theta(t,x)$. In fact, we can deduce the lemma by repeating the argument developed in [32].

Lemma 5.1. *Under the conditions stated in Lemma* 2.6*, for each* $0 \leqslant s \leqslant t \leqslant T$ *and* $x \in \mathbb{R}$ *we have the following estimate*:

$$\theta(t,x) \geqslant \frac{C \min_{x\in\mathbb{R}}\{\theta(s,x)\}}{1+(t-s)\min_{x\in\mathbb{R}}\{\theta(s,x)\}} \tag{5.1}$$

holds for some positive constant C *which depends only on* $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ *and* H_0*.*

6 The proof of main results

With the above preparations in hand, we now turn to prove our main results.

We first prove Theorem 1.1. To this end, suppose that

$$(\varphi(t,x),\psi(t,x),\chi(t,x),z(t,x)) \in X(0,T;M_1,M_2)$$

is a solution to the Cauchy problems (2.2) and (2.3) defined on the strip $\Pi_T := [0,T]\times\mathbb{R}$ and satisfying the *a priori* assumption (2.5). Then if the assumptions listed in Theorem 1.1 hold true and $\delta > 0$ and $a > 0$ are chosen sufficiently small such that (2.6) and (2.7) hold, we can get from Lemmas 2.6, 3.4, 4.4 and 5.1 that

$$\begin{aligned}
&0 \leqslant z(t,x) \leqslant 1, \quad \forall\,(t,x) \in [0,T]\times\mathbb{R},\\
&C_1^{-1} \leqslant v(t,x) \leqslant C_1, \quad \forall\,(t,x) \in [0,T]\times\mathbb{R},\\
&\theta(t,x) \leqslant C_2, \quad \forall\,(t,x) \in [0,T]\times\mathbb{R},\\
&\theta(t,x) \geqslant \frac{C_3 \min_{x\in\mathbb{R}}\{\theta(s,x)\}}{1+(t-s)\min_{x\in\mathbb{R}}\{\theta(s,x)\}}, \quad \forall\,(t,x) \in [s,t]\times\mathbb{R}
\end{aligned} \tag{6.1}$$

hold for some positive constants C_i $(i = 1, 2, 3)$ which depend only on $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ and H_0.

By having obtained (6.1), Theorem 1.1 can be proved by combining the local solvability result Lemma 2.3 with the continuation argument introduced in [32, 52] and we omit the details for brevity.

Recall that in the proof of Theorem 1.1, the assumption on the smallness of the radiation constant a is only used in Lemma 2.4 to guarantee that $\tilde{p}(v, s)$ is convex with respect to v and s, and we do not use such a smallness assumption elsewhere to control certain nonlinear terms involved. As explained in Section 1, the very reason for such an analysis is that once we can impose some other assumptions to guarantee the convexity of $\tilde{p}(v, s)$ with respect to (v, s) in the regime for v and s under our consideration, then one can deduce that a similar stability result holds accordingly.

The main purpose of Theorem 1.3 is to show that if we use the smallness of a to control the involved nonlinear terms, then we can relax the assumptions we imposed on the parameters b and β while the similar stability result still holds. For this purpose, we only need to re-estimate those terms related to the radiation constant a, since the terms can be estimated in the same way as in the proof of Theorem 1.1.

First of all, we treat the term $\|\varphi_x(t)\|^2$. By using (2.7), (2.10), (2.20a) and (3.8), I_6 can be re-estimated as

$$\begin{aligned} I_6 &\leqslant \frac{1}{10}\int_0^t\int_{\mathbb{R}} \frac{R\theta\varphi_x^2}{v^3} + C\left(\|\theta\|_\infty + \left\|\frac{\theta^2(\frac{R}{v} + \frac{4a\theta^3}{3})^2}{\kappa(v,\theta)p_v(v,\theta)}\right\|_\infty\right) \\ &\leqslant \frac{1}{10}\int_0^t\int_{\mathbb{R}} \frac{R\theta\varphi_x^2}{v^3} + C(1 + \|\theta\|_\infty). \end{aligned} \tag{6.2}$$

Inserting (6.2), (2.21)–(2.24) into (2.19) and employing (3.8), we can infer

$$\|\varphi_x(t)\|^2 + \int_0^t\int_{\mathbb{R}} \theta\varphi_x^2 \lesssim 1 + \|\theta\|_\infty. \tag{6.3}$$

Now we deal with the term $\int_0^t \|\psi_{xx}(\tau)\|^2 d\tau$. By virtue of (2.7) and (2.10), we have

$$\int_0^t\int_{\mathbb{R}} (1 + a^2\theta^6)|\chi_x|^2 \lesssim \int_0^t\int_{\mathbb{R}} \frac{\kappa(v,\theta)\Theta\chi_x^2}{v\theta^2}\cdot\frac{\theta^2}{1+\theta^b} \lesssim 1 + \|\theta\|_\infty^{(2-b)_+}. \tag{6.4}$$

Plugging (6.4) into (3.13) and utilizing (6.3), we deduce

$$\begin{aligned} \|\psi_x(t)\|^2 + \int_0^t\int_{\mathbb{R}} \frac{\mu\psi_{xx}^2}{v} &\leqslant 2\epsilon\int_0^t\int_{\mathbb{R}} \frac{\mu\psi_{xx}^2}{v} + C(\epsilon)(1 + \|\theta\|_\infty^{(2-b)_+} + \|\theta\|_\infty^3) \\ &\leqslant 2\epsilon\int_0^t\int_{\mathbb{R}} \frac{\mu\psi_{xx}^2}{v} + C(\epsilon)(1 + \|\theta\|_\infty^3). \end{aligned} \tag{6.5}$$

By choosing $\epsilon > 0$ small enough, we can see (3.11) still holds true without imposing any condition on the parameter b.

On the other hand, (2.7) tells us that

$$\int_0^t\int_{\mathbb{R}} (1 + \theta^{b+2})e_\theta^2\chi_t^2 \lesssim \int_0^t\int_{\mathbb{R}} (1 + \theta^{b+2})\chi_t^2 \lesssim X(t)(1 + Y(t)^{\frac{2}{2b+3}}) \tag{6.6}$$

and

$$\int_0^t\int_{\mathbb{R}} (1 + \theta^{b+2})\theta^2 p_\theta^2\psi_x^2 \lesssim \int_0^t\int_{\mathbb{R}} \frac{\mu\Theta\psi_x^2}{v\theta}\cdot(1 + \theta^{b+5}) \lesssim 1 + Y(t)^{\frac{b+5}{2b+3}}. \tag{6.7}$$

Then (4.15), (4.16), (6.6) and (6.7) imply that

$$J \lesssim 1 + X(t)(1 + Y(t)^{\frac{2}{2b+3}}) + Y(t)^{\frac{b+5}{2b+3}} + Z(t)^{\frac{3}{4}} + Y(t)^{\frac{b+3}{2b+3}} Z(t)^{\frac{3}{4}} + Y(t)^{\frac{b+\beta+2}{2b+3}}. \tag{6.8}$$

We utilize (4.14), (6.8), the assumption $b > \frac{5}{6}$ and $0 \leqslant \beta < 3b + 2$ to derive (4.18).

Meanwhile, it follows from (2.7), (4.5) and the fact $b > 0$ that

$$\begin{aligned} I_{16} &\lesssim \int_0^t \int_{\mathbb{R}} (1+\theta^{b+1})|\psi_x \chi_t| \\ &\leqslant \epsilon X(t) + C(\epsilon) \int_0^t \int_{\mathbb{R}} (1+\theta^{b+3}) \frac{\psi_x^2}{\theta} \\ &\leqslant \epsilon X(t) + C(\epsilon)(1 + Y(t)^{\frac{b+3}{2b+3}}) \\ &\leqslant \epsilon (X(t) + Y(t)) + C(\epsilon). \end{aligned} \tag{6.9}$$

We can exploit the same method developed in Section 4 to estimate the other terms. Here, we need the condition $0 \leqslant \beta < b + 3$ to bound the term I_{15} and $b > 2$ to bound the term I_{17}. By repeating the argument used to prove Theorem 1.1, we can complete the proof of Theorem 1.2.

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References

1 Chen G-Q. Global solutions to the compressible Navier-Stokes equations for a reacting mixture. SIAM J Math Anal, 1992, 23: 609–634

2 Duan R, Liu H-X, Zhao H-J. Nonlinear stability of rarefaction waves for the compressible Navier-Stokes equations with large initial perturbation. Trans Amer Math Soc, 2009, 361: 453–493

3 Ducomet B. A model of thermal dissipation for a one-dimensional viscous reactive and radiative gas. Math Methods Appl Sci, 1999, 22: 1323–1349

4 Ducomet B, Zlotnik A. On the large-time behavior of 1D radiative and reactive viscous flows for higher-order kinetics. Nonlinear Anal, 2005, 63: 1011–1033

5 He L, Liao Y-K, Wang T, et al. One-dimensional viscous radiative gas with temperature dependent viscosity. Acta Math Sci Ser B Engl Ed, 2018, 38: 1515–1548

6 He L, Tang S-J, Wang T. Stability of viscous shock waves for the one-dimensional compressible Navier-Stokes equations with density-dependent viscosity. Acta Math Sci Ser B Engl Ed, 2016, 36: 34–48

7 Hong H. Global stability of viscous contact wave for 1-D compressible Navier-Stokes equations. J Differential Equations, 2012, 252: 3482–3505

8 Huang B-K, Liao Y-K. Global stability of combination of viscous contact wave with rarefaction wave for compressible Navier-Stokes equations with temperature-dependent viscosity. Math Models Methods Appl Sci, 2017, 27: 2321–2379

9 Huang B-K, Wang L-S, Xiao Q-H. Global nonlinear stability of rarefaction waves for compressible Navier-Stokes equations with temperature and density dependent transport coefficients. Kinet Relat Models, 2016, 3: 469–514

10 Huang F-M, Li J, Matsumura A. Asymptotic stability of combination of viscous contact wave with rarefaction waves for one-dimensional compressible Navier-Stokes system. Arch Ration Mech Anal, 2010, 197: 89–116

11 Huang F-M, Matsumura A. Stability of a composite wave of two viscous shock waves for the full compressible Navier-Stokes equation. Comm Math Phys, 2009, 289: 841–861

12 Huang F-M, Matsumura A, Xin Z-P. Stability of contact discontinuities for the 1-D compressible Navier-Stokes equations. Arch Ration Mech Anal, 2006, 179: 55–77

13 Huang F-M, Wang T. Stability of superposition of viscous contact wave and rarefaction waves for compressible Navier-Stokes system. Indiana Univ Math J, 2016, 65: 1833–1875

14 Huang F-M, Xin Z-P, Yang T. Contact discontinuity with general perturbations for gas motions. Adv Math, 2008, 219: 1246–1297

15 Huang F-M, Zhao H-J. On the global stability of contact discontinuity for compressible Navier-Stokes equations. Rend Semin Mat Univ Padova, 2003, 109: 283–305

16 Jiang J, Zheng S-M. Global solvability and asymptotic behavior of a free boundary problem for the one-dimensional viscous radiative and reactive gas. J Math Phys, 2012, 53: 1–33

17 Jiang J, Zheng S-M. Global well-posedness and exponential stability of solutions for the viscous radiative and reactive gas. Z Angew Math Phys, 2014, 65: 645–686

18 Jiang S. Large-time behavior of solutions to the equations of a one-dimensional viscous polytropic ideal gas in unbounded domains. Comm Math Phys, 1999, 200: 181–193

19 Kawashima S, Matsumura A. Asymptotic stability of travelling wave solutions of systems for one-dimensional gas motion. Comm Math Phys, 1985, 101: 97–127

20 Kawashima S, Matsumura A, Nishihara K. Asymptotic behaviour of solutions for the equations of a viscous heat-conductive gas. Proc Japan Acad Ser A Math Sci, 1986, 62: 249–252

21 Kawashima S, Nakamura T, Nishibata S, et al. Stationary waves to viscous heat-conductive gases in half-space: Existence, stability and convergence rate. Math Models Methods Appl Sci, 2010, 20: 2201–2235

22 Kawohl B. Global existence of large solutions to initial-boundary value problems for a viscous, heat-conducting, one-dimensional real gas. J Differential Equations, 1985, 58: 76–103

23 Kazhikhov A-V, Shelukhin V-V. Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas. J Appl Math Mech, 1977, 41: 273–282

24 Li J, Liang Z-L. Some uniform estimates and large-time behavior for one-dimensional compressible Navier-Stokes system in unbounded domains with large data. Arch Ration Mech Anal, 2016, 220: 1195–1208

25 Li K-Q, Wang W-K, Yang X-F. Asymptotic stability of rarefaction waves to a radiation hydrodynamic limit model. J Differential Equations, 2020, 269: 1693–1717

26 Liao Y-K. Global stability of rarefaction waves for a viscous radiative and reactive gas with temperature-dependent viscosity. Nonlinear Anal Real World Appl, 2020, 53: 103056

27 Liao Y-K. Remarks on the Cauchy problem of the one-dimensional viscous radiative and reactive gas. Acta Math Sci Ser B Engl Ed, 2020, 40: 1020–1034

28 Liao Y-K, Wang T, Zhao H-J. Global spherically symmetric flows for a viscous radiative and reactive gas in an exterior domain. J Differential Equations, 2019, 266: 6459–6506

29 Liao Y K, Xu Z D, Zhao H J. Cauchy problem of the one-dimensional compressible viscous radiative and reactive gas with degenerate density dependent viscosity (in Chinese). Sci Sin Math, 2019, 49: 175–194

30 Liao Y-K, Zhang S-X. Global solutions to the one-dimensional compressible Navier-Stokes equation with radiation. J Math Anal Appl, 2018, 461: 1009–1052

31 Liao Y-K, Zhao H-J. Global solutions to one-dimensional equations for a self-gravitating viscous radiative and reactive gas with density-dependent viscosity. Commun Math Sci, 2017, 15: 1423–1456

32 Liao Y-K, Zhao H-J. Global existence and large-time behavior of solutions to the Cauchy problem of one-dimensional viscous radiative and reactive gas. J Differential Equations, 2018, 265: 2076–2120

33 Liu T-P. Shock waves for compressible Navier-Stokes equations are stable. Comm Pure Appl Math, 1986, 39: 565–594

34 Liu T-P, Xin Z-P. Nonlinear stability of rarefaction waves for compressible Navier-Stokes equations. Comm Math Phys, 1988, 118: 451–465

35 Liu T-P, Xin Z-P. Pointwise decay to contact discontinuities for systems of viscous conservation laws. Asian J Math, 1997, 1: 34–84

36 Liu T-P, Zeng Y-N. Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws. Mem Amer Math Soc, 1997, 125: 1–120

37 Matsumura A, Nishihara K. Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas. Japan J Appl Math, 1986, 3: 1–13

38 Matsumura A, Nishihara K. Global stability of the rarefaction waves of a one-dimensional model system for compressible viscous gas. Comm Math Phys, 1992, 144: 325–335

39 Matsumura A, Nishihara K. Global asymptotics toward the rarefaction wave for solutions of viscous p-system with boundary effect. Quart Appl Math, 2000, 58: 69–83

40 Mihalas D, Mihalas B-W. Foundations of Radiation Hydrodynamics. New York: Oxford University Press, 1984

41 Nishihara K, Yang T, Zhao H-J. Nonlinear stability of strong rarefaction waves for compressible Navier-Stokes equations. SIAM J Math Anal, 2004, 35: 1561–1597

42 Qin X-H, Wang Y. Stability of wave patterns to the inflow problem of full compressible Navier-Stokes equations. SIAM J Math Anal, 2009, 41: 2057–2087

43 Qin Y-M, Hu G-L, Wang T-G, et al. Remarks on global smooth solutions to a 1D self-gravitating viscous radiative and reactive gas. J Math Anal Appl, 2013, 408: 19–26

44 Qin Y-M, Zhang J-L, Su X, et al. Global existence and exponential stability of spherically symmetric solutions to a compressible combustion radiative and reactive gas. J Math Fluid Mech, 2016, 18: 415–461

45 Smoller J. Shock Waves and Reaction-Diffusion Equations, 2nd ed. Grundlehren der Mathematischen Wissenschaften, vol. 258. New York: Springer-Verlag, 1994

46 Tang S-J, Zhang L. Nonlinear stability of viscous shock waves for one-dimensional nonisentropic compressible Navier-Stokes equations with a class of large initial perturbation. Acta Math Sci Ser B Engl Ed, 2018, 38: 973–1000

47 Umehara M, Tani A. Global solution to the one-dimensional equations for a self-gravitating viscous radiative and reactive gas. J Differential Equations, 2007, 234: 439–463

48 Umehara M, Tani A. Global solvability of the free-boundary problem for one-dimensional motion of a self-gravitating viscous radiative and reactive gas. Proc Japan Acad Ser A Math Sci, 2008, 84: 123–128

49 Umehara M, Tani A. Temporally global solution to the equations for a spherically symmetric viscous radiative and reactive gas over the rigid core. Anal Appl (Singap), 2008, 6: 183–211

50 Wan L, Wang T, Zhao H-J. Asymptotic stability of wave patterns to compressible viscous and heat-conducting gases in the half space. J Differential Equations, 2016, 261: 5949–5991

51 Wan L, Wang T, Zou Q-Y. Stability of stationary solutions to the outflow problem for full compressible Navier-Stokes equations with large initial perturbation. Nonlinearity, 2016, 29: 1329–1354

52 Wang T, Zhao H-J. One-dimensional compressible heat-conducting gas with temperature-dependent viscosity. Math Models Methods Appl Sci, 2016, 26: 2237–2275

53 Zhang J-L. Remarks on global existence and exponential stability of solutions for the viscous radiative and reactive gas with large initial data. Nonlinearity, 2017, 30: 1221–1261