

On local strong and classical solutions to the three-dimensional barotropic compressible Navier-Stokes equations with vacuum

Xiangdi Huang

*Institute of Mathematics, Academy of Mathematics and Systems Science,
Chinese Academy of Sciences, Beijing 100190, China*

Email: xduang@amss.ac.cn

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Abstract We consider the local well-posedness of strong and classical solutions to the three-dimensional barotropic compressible Navier-Stokes equations with density containing vacuum initially. We first prove the local existence and uniqueness of the strong solutions, where the initial compatibility condition proposed by Cho et al. (2004), Cho and Kim (2006) and Choe and Kim (2003) is removed in a suitable sense. Then the continuous dependence of strong solutions on the initial data is derived under an additional compatibility condition. Moreover, for the initial data satisfying some additional regularity and the compatibility condition, the strong solution is proved to be a classical one.

Keywords compressible Navier-Stokes equations, vacuum, strong solutions, classical solutions

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1 Introduction and main results

We consider the three-dimensional barotropic compressible Navier-Stokes equations which read as follows:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u, \end{cases} \quad (1.1)$$

where $t \geq 0$, $x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3$, $\rho = \rho(x, t)$, $u = (u_1(x, t), u_2(x, t), u_3(x, t))$ and $P = P(\rho)$ represent, respectively, the density, the velocity and the pressure. The constant viscosity coefficients μ and λ satisfy the physical hypothesis

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0. \quad (1.2)$$

Let $\Omega \subset \mathbb{R}^3$ be either a smooth bounded domain or the whole space \mathbb{R}^3 . We impose the following initial and boundary conditions on (1.1):

$$\rho(x, 0) = \rho_0(x), \quad \rho u(x, 0) = m_0(x), \quad x \in \Omega \quad (1.3)$$

and

$$\begin{cases} u(x, t) = 0, & x \in \partial\Omega, & \text{if } \Omega \subset\subset \mathbb{R}^3, \\ (\rho, u)(x, t) \rightarrow (\rho_\infty, 0), & \text{as } |x| \rightarrow \infty, & \text{if } \Omega = \mathbb{R}^3 \end{cases} \tag{1.4}$$

with constant $\rho_\infty \geq 0$.

It is important to investigate the well-posedness of strong solutions for compressible Navier-Stokes equations.

As long as the initial density is away from vacuum, the local well-posedness theory of the problem (1.1) is established in [22] and [19, 21], respectively. In 1980s, Matsumura and Nishida [18] proved the existence of global classical solutions when the initial data are close to some positive constants. Besides, it is shown by Hoff [8, 9] that the system will admit at least one global weak solution with strictly positive initial density and temperature for discontinuous initial data.

Things become more complicated when the density is allowed to vanish. In 1990s, the major breakthrough is due to Lions [16, 17] (then improved by Feireisl et al. [5, 6]), where the global existence of weak solutions with finite energy without any size restriction on the initial data can be proved under the condition that the exponent γ is suitably large. Later, Hoff [10], Hoff and Santos [11] and Hoff and Tsyganov [12] obtained a new type of global weak solutions with small energy. Considering the strong or classical solutions with vacuum, Cho et al. [2], Cho and Kim [3], Choe and Kim [4] and Salvi and Straškraba [20] obtained the local existence and uniqueness of strong and classical solutions for three-dimensional bounded or unbounded domains and for two-dimensional bounded ones. It should be noted that the results in [2–4, 20] are derived under some additional compatibility conditions (see (1.9) in the below). More precisely, they required that $g \in L^2(\Omega)$ or $g \in H^1(\Omega)$ in (1.9) for the strong or classical solutions, respectively. In this direction, a natural question arises whether one can remove or relax the initial compatibility conditions with nonnegative density in a suitable sense. Indeed, this is the aim of this paper, i.e., we establish the local existence of strong solutions without the initial compatibility condition.

Before stating the main results, we first explain the notations and conventions used throughout this paper. For $1 \leq r \leq \infty$ and $k \geq 1$, the standard Lebesgue and Sobolev spaces are defined as follows:

$$\begin{cases} L^r = L^r(\Omega), & W^{k,r} = W^{k,r}(\Omega), & H^k = W^{k,2}, \\ D_0^1 = \begin{cases} H_0^1(\Omega) & \text{for bounded } \Omega \subset \mathbb{R}^3, \\ \{f \in L^6 \mid \nabla f \in L^2\} & \text{for } \Omega = \mathbb{R}^3. \end{cases} \end{cases}$$

The first main result of this paper is the following Theorem 1.2 concerning the local existence of strong solutions whose definition is as follows.

Definition 1.1. If all the derivatives involved in (1.1) for (ρ, u) are regular distributions, and the equations (1.1) hold almost everywhere in $\Omega \times (0, T)$, then (ρ, u) is called a strong solution to (1.1).

Theorem 1.2. Assume that $P = P(\cdot) \in C^1[0, \infty)$. For some $3 < q < 6$ and $\rho_\infty \geq 0$, assume that the initial data (ρ_0, m_0) satisfy

$$\rho_0 \geq 0, \quad \rho_0 - \rho_\infty \in L^{\tilde{p}} \cap D^1 \cap W^{1,q}, \quad u_0 \in D_0^1 \tag{1.5}$$

and

$$m_0 = \rho_0 u_0, \tag{1.6}$$

where

$$\tilde{p} \triangleq \begin{cases} \frac{3}{2} & \text{for } \Omega = \mathbb{R}^3 \text{ and } \rho_\infty = 0, \\ 2, & \text{otherwise.} \end{cases} \tag{1.7}$$

Then there exists a positive time $T_0 > 0$ such that the problems (1.1)–(1.4) have a unique strong solution (ρ, u) on $\Omega \times (0, T_0]$ satisfying that

$$\begin{cases} \rho - \rho_\infty \in C([0, T_0]; L^{\tilde{p}} \cap D^1 \cap W^{1,q}), \\ \nabla u, \sqrt{t}\nabla^2 u, \sqrt{t}\sqrt{\rho}u_t, t\nabla u_t \in L^\infty(0, T_0; L^2), \\ t\nabla u \in L^\infty(0, T_0; W^{1,q}), \quad \sqrt{\rho}u_t, \sqrt{t}\nabla u_t \in L^2(\Omega \times (0, T_0)). \end{cases} \tag{1.8}$$

Furthermore, if in addition to (1.5) and (1.6), (ρ_0, u_0) satisfies the compatibility conditions

$$-\mu\Delta u_0 - (\mu + \lambda)\nabla\operatorname{div}u_0 + \nabla P(\rho_0) = \rho_0^{1/2}g, \tag{1.9}$$

for $g \in L^2$, (ρ, u) also satisfies

$$\begin{cases} \nabla u \in L^\infty(0, T_0; H^1), & \sqrt{t}\nabla u \in L^\infty(0, T_0; W^{1,q}), \\ \sqrt{\rho}u_t, \sqrt{t}\nabla u_t \in L^\infty(0, T_0; L^2), & \nabla u_t \in L^2(\Omega \times (0, T_0)). \end{cases} \tag{1.10}$$

Next, the following Corollary 1.3 whose proof is similar to that of [4, Theorem 3] gives the continuous dependence of the solution on the data provided (1.9) holds.

Corollary 1.3. *For each $i = 1, 2$, let (ρ_i, u_i) be the local strong solution to the problems (1.1)–(1.4) with the initial data (ρ_{0i}, u_{0i}) satisfying (1.5), (1.6), and the compatibility condition (1.9) with $g = g_i$. Moreover, assume that (ρ_{0i}, u_{0i}) satisfies*

$$\|\rho_{0i} - \rho_\infty\|_{L^{\bar{p}} \cap D^1 \cap W^{1,q}} + \|\nabla u_{0i}\|_{H^1} + \|g_i\|_{L^2} \leq K. \tag{1.11}$$

Then there exist a small time T_0 and a positive constant C depending only on T_0 and K such that

$$\begin{aligned} & \sup_{0 \leq t \leq T_0} (\|\rho_1^{1/2}(u_1 - u_2)\|_{L^2}^2 + \|\rho_1 - \rho_2\|_{L^{\bar{p}}}^2) + \int_0^{T_0} \|\nabla(u_1 - u_2)\|_{L^2}^2 ds \\ & \leq C\|\rho_{01}^{1/2}(u_{01} - u_{02})\|_{L^2}^2 + C\|\rho_{01} - \rho_{02}\|_{L^{\bar{p}}}^2. \end{aligned} \tag{1.12}$$

Finally, if the initial data (ρ_0, m_0) satisfy some additional regularity and compatibility conditions, the local strong solution (ρ, u) obtained by Theorem 1.2 becomes a classical one.

Theorem 1.4. *Assume that $P(\rho)$ satisfies either*

$$P(\cdot) \in C^2[0, \infty) \tag{1.13}$$

or

$$P(\rho) = A\rho^\gamma \quad (A > 0, \gamma > 1). \tag{1.14}$$

In addition to (1.5), (1.6) and (1.9), assume further that

$$\nabla^2 \rho_0, \nabla^2 P(\rho_0) \in L^2 \cap L^q. \tag{1.15}$$

Then, in addition to (1.8) and (1.10), the strong solution (ρ, u) obtained by Theorem 1.2 satisfies

$$\begin{cases} \nabla^2 \rho, \nabla^2 P(\rho) \in C([0, T_0]; L^2 \cap L^q), \\ \nabla u \in L^2(0, T_0; H^2), & \sqrt{t}\nabla u \in L^\infty(0, T_0; H^2), \\ t\nabla u \in L^\infty(0, T_0; W^{2,q}), & \sqrt{t}\nabla u_t \in L^2(0, T_0; H^1), \\ t\nabla u_t \in L^\infty(0, T_0; H^1), & tu_{tt} \in L^2(0, T_0; D_0^1), \\ t\sqrt{\rho}u_{tt} \in L^\infty(0, T_0; L^2), & \sqrt{t}\sqrt{\rho}u_{tt} \in L^2(0, T_0; L^2). \end{cases} \tag{1.16}$$

A few remarks are in order.

Remark 1.5. To obtain the local existence and uniqueness of strong solutions, in Theorem 1.2, we need the compatibility condition (1.6) which is much weaker than those of [2–4, 20] where not only (1.6) but also (1.9) is needed. Moreover, the strong solutions obtained in Theorem 1.2 are somewhat more regular than those in [2–4] when $t > 0$. In this sense, we successfully remove the compatibility condition required in [2–4, 20]. Indeed, as indicated by Li and Liang [14], after they are motivated by our results which appeared as [11] there, they continue to consider the 2D Cauchy problem of the density-dependent compressible Navier-Stokes equations. Moreover, some of the results of Li and Xin [15] were also partially based on ours which appeared as [18] there.

Remark 1.6. After obtaining the existence result in Theorem 1.2, we show the continuous dependence of the solution on the data in Corollary 1.3, provided that the initial data satisfy the compatibility condition (1.9). Indeed, Theorem 1.2 and Corollary 1.3 tell us how the compatibility condition (1.9) plays its role when we discuss the local well-posedness of strong solutions to the problems (1.1)–(1.4) with vacuum.

Remark 1.7. For the local existence of classical solutions obtained in Theorem 1.4, we only need the initial data satisfying the compatibility condition (1.9) for some $g \in L^2$ which is in sharp contrast to Cho and Kim [3], where the compatibility condition (1.9) is needed for $g \in H^1$. This means that our Theorem 1.4 essentially weakens those assumptions on the compatibility condition in [3].

We now comment on the analysis of this paper. First, we will consider the approximating system for the initial density strictly away from vacuum, whose local existence theory has been shown in Lemma 2.1. By employing some basic ideas due to Hoff [8,9] and careful analysis, we succeed in deriving the uniform *a priori* estimates on the density and velocity which are independent of the lower bound of the density. To do this, the key issue is to get the uniform upper bound of the density without requiring the additional compatibility condition (1.9). Indeed, this is achieved by deriving the time weighted estimates on $\|\sqrt{\rho}u_t\|_{L^2}$ and $\|\nabla u_t\|_{L^2}$ (see Lemma 3.3), which are crucial for bounding the L^1L^∞ -norm of ∇u and thus getting the uniform upper bound of the density. Then with the desired estimates on solutions at hand, we will apply the standard compact arguments which show that the limit is exactly the strong solutions of the original one. Finally, for the initial data satisfying some additional regularity and compatibility conditions, the standard arguments will be used to obtain the higher order estimates of the solutions which are needed to guarantee the local strong solution to be a classical one.

We shall briefly describe the structure of this article. Some fundamental lemmas will be exhibited in Section 2. To get the local existence and uniqueness of strong and classical solutions, we establish some *a priori* estimates in Sections 3 and 4. Consequently, we arrive at the results of Theorems 1.2 and 1.4 in Section 5.

2 Preliminaries

First, in this section and the following two, we define

$$\Omega_R = \begin{cases} \Omega & \text{for bounded } \Omega \subset \mathbb{R}^3, \\ B_R \triangleq \{x \in \mathbb{R}^3 \mid |x| < R\} & \text{for } \Omega = \mathbb{R}^3 \end{cases} \quad (2.1)$$

and

$$L^p = L^p(\Omega_R), \quad W^{k,p} = W^{k,p}(\Omega_R), \quad H^k = W^{k,2}$$

for $p \geq 1$ and positive integer k .

Then for the initial density strictly away from vacuum, the following local existence theory can be shown by similar arguments to that in [2–4, 22].

Lemma 2.1. *Assume that $P(\cdot) \in C^3[0, \infty)$ and that the initial data (ρ_0, m_0) satisfy*

$$0 < \delta \leq \rho_0, \quad \rho_0 \in H^3, \quad u_0 \in H_0^1 \cap H^3, \quad m_0 = \rho_0 u_0.$$

Then there exists a small time $T_ > 0$ such that the problems (1.1)–(1.4) admit a unique classical solution*

(ρ, u) on $\Omega_R \times (0, T_*]$ satisfying

$$\begin{cases} \rho \in C([0, T_*]; H^3), & u \in C([0, T_*]; H_0^1 \cap H^3) \cap L^2(0, T_*; H^4), \\ u_t \in L^\infty(0, T_*; H_0^1) \cap L^2(0, T_*; H^2), & \sqrt{\rho}u_{tt} \in L^2(0, T_*; L^2), \\ \sqrt{t}u \in L^\infty(0, T_*; H^4), & \sqrt{t}u_t \in L^\infty(0, T_*; H^2), \quad \sqrt{t}u_{tt} \in L^2(0, T_*; H^1), \\ \sqrt{t}\sqrt{\rho}u_{tt} \in L^\infty(0, T_*; L^2), & tu_t \in L^\infty(0, T_*; H^3), \\ tu_{tt} \in L^\infty(0, T_*; H^1) \cap L^2(0, T_*; H^2), & t\sqrt{\rho}u_{ttt} \in L^2(0, T_*; L^2), \\ t^{3/2}u_{tt} \in L^\infty(0, T_*; H^2), & t^{3/2}u_{ttt} \in L^2(0, T_*; H^1), \\ t^{3/2}\sqrt{\rho}u_{ttt} \in L^\infty(0, T_*; L^2). \end{cases}$$

Next, the following well-known Gagliardo-Nirenberg inequality will be used later frequently (see [13]).

Lemma 2.2 (Gagliardo-Nirenberg inequality [13]). *For $p \in [2, 6]$, $q \in (1, \infty)$ and $r \in (3, \infty)$, there exists some generic constant $C > 0$ independent of R such that for $f \in H_0^1(\Omega_R)$ and $g \in L^q(\Omega_R) \cap W^{1,r}(\Omega_R)$,*

$$\|f\|_{L^p}^p \leq C \|f\|_{L^2}^{(6-p)/2} \|\nabla f\|_{L^2}^{(3p-6)/2}, \tag{2.2}$$

$$\|g\|_{L^\infty} \leq C \|g\|_{L^q} + C \|g\|_{L^q}^{q(r-3)/(3r+q(r-3))} \|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}. \tag{2.3}$$

Finally, we state the following L^p -bounds for the weak solutions to the Lamé system with the Dirichlet boundary conditions:

$$\begin{cases} -\mu\Delta v - (\mu + \lambda)\nabla\operatorname{div}v = F, & x \in \Omega_R, \\ v = 0, & x \in \partial\Omega_R. \end{cases} \tag{2.4}$$

Lemma 2.3 (See [1, 2]). *For $p > 1$ and $k \geq 0$, there exists a positive constant C independent of R such that*

$$\|\nabla^{k+2}v\|_{L^p(\Omega_R)} \leq C \|F\|_{W^{k,p}(\Omega_R)} \tag{2.5}$$

for every solution $v \in W_0^{1,p}(\Omega_R)$ of (2.4).

3 A priori estimates (I)

Let Ω_R and (ρ_0, m_0) be as in Lemma 2.1 and (ρ, u) the solution to the problems (1.1)–(1.4) on $\Omega_R \times (0, T_*]$ obtained by Lemma 2.1. For $q \in (3, 6)$, we define

$$\psi(t) \triangleq 1 + \|\nabla u\|_{L^2} + \|\rho - \rho_\infty\|_{L^{\bar{p}} \cap D^1 \cap W^{1,q}}. \tag{3.1}$$

Then the main aim of this section is to derive the following key *a priori* estimate on ψ .

Proposition 3.1. *For $q \in (3, 6)$, there exist positive constants T_0 and M both depending only on $\mu, \lambda, P, q, \rho_\infty, \psi(0)$ and Ω but independent of R such that*

$$\sup_{0 \leq t \leq T_0} (\psi(t) + t(\|\nabla^2 u\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2)) + t^2(\|\nabla u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^q}^2) + \int_0^{T_0} t\|\nabla u_t\|_{L^2}^2 dt \leq M. \tag{3.2}$$

To prove Proposition 3.1, we begin with the following L^2 -bound for ∇u .

Lemma 3.2. *There exist positive constants $\alpha = \alpha(q) > 1$ such that*

$$\begin{aligned} & \sup_{0 \leq s \leq t} (\|\nabla u\|_{L^2}^2 + \|P - P(\rho_\infty)\|_{L^2}^2) + \int_0^t \|\sqrt{\rho}u_t\|_{L^2}^2 ds \\ & \leq C + C \int_0^t M_P(\psi)\psi^\alpha ds, \end{aligned} \tag{3.3}$$

where and in this section,

$$M_P(\psi) \triangleq 1 + \max_{0 \leq s \leq \psi} (|P(s)| + |P'(s)|), \quad (3.4)$$

and C denotes a generic positive constant depending only on $\mu, \lambda, P, q, \rho_\infty, \psi(0)$ and Ω but independent of R .

Proof. First, multiplying the equation (1.1)₂ by u_t and integrating the resulting equations by parts, we have

$$\begin{aligned} & \frac{d}{dt} \int ((\mu + \lambda)(\operatorname{div} u)^2 + \mu |\nabla u|^2) dx + \int \rho |u_t|^2 dx \\ & \leq C \int \rho |u|^2 |\nabla u|^2 dx + 2 \int (P - P(\rho_\infty)) \operatorname{div} u_t dx, \end{aligned} \quad (3.5)$$

where, in this section and the next, we define

$$\int \cdot dx = \int_{\Omega_R} \cdot dx.$$

Then on the one hand, the Gagliardo-Nirenberg inequality implies that

$$\begin{aligned} \int \rho |u|^2 |\nabla u|^2 dx & \leq \|\rho\|_{L^\infty} \|u\|_{L^6}^2 \|\nabla u\|_{L^3}^2 \\ & \leq C \|\rho\|_{L^\infty} \|\nabla u\|_{L^2}^3 \|\nabla u\|_{H^1} \\ & \leq C \psi^\alpha \|\nabla^2 u\|_{L^2} + C \psi^\alpha, \end{aligned} \quad (3.6)$$

where (and in what follows) $\alpha = \alpha(q) > 1$. Note that u is a solution of the following elliptic system:

$$\begin{cases} -\mu \Delta u - (\mu + \lambda) \nabla \operatorname{div} u = -\rho(u_t + u \cdot \nabla u) - \nabla P, & x \in \Omega_R, \\ u = 0, & x \in \partial \Omega_R. \end{cases} \quad (3.7)$$

Applying Lemma 2.3 to (3.7) yields

$$\begin{aligned} \|\nabla^2 u\|_{L^2} & \leq C(\|\rho(u_t + u \cdot \nabla u)\|_{L^2} + \|\nabla P\|_{L^2}) \\ & \leq C \psi^{1/2} \|\sqrt{\rho} u_t\|_{L^2} + C M_P(\psi) \psi^\alpha + \frac{1}{2} \|\nabla^2 u\|_{L^2}, \end{aligned}$$

where in the second inequality we have used (3.6). This implies

$$\|\nabla^2 u\|_{L^2} + \|\rho(u_t + u \cdot \nabla u)\|_{L^2} \leq C \psi^{1/2} \|\sqrt{\rho} u_t\|_{L^2} + C M_P(\psi) \psi^\alpha. \quad (3.8)$$

On the other hand, we deduce from the Sobolev inequality that

$$\begin{aligned} & 2 \int (P - P(\rho_\infty)) \operatorname{div} u_t dx \\ & = 2 \frac{d}{dt} \int (P - P(\rho_\infty)) \operatorname{div} u dx - 2 \int P'(\rho) \rho_t \operatorname{div} u dx \\ & \leq 2 \frac{d}{dt} \int (P - P(\rho_\infty)) \operatorname{div} u dx + C M_P(\psi) \psi^2, \end{aligned} \quad (3.9)$$

where we have used

$$\|\rho_t\|_{L^2} \leq C \|u\|_{L^6} \|\nabla \rho\|_{L^3} + C \|\rho\|_{L^\infty} \|\nabla u\|_{L^2} \leq C \psi^2, \quad (3.10)$$

due to (1.1)₁.

Substituting (3.6), (3.8) and (3.9) into (3.5) and using Cauchy's inequality lead to

$$\frac{d}{dt} \int ((\mu + \lambda)(\operatorname{div} u)^2 + \mu |\nabla u|^2 - 2(P - P(\rho_\infty)) \operatorname{div} u) dx + \int \rho |u_t|^2 dx$$

$$\begin{aligned} &\leq C\psi^\alpha \|\rho^{1/2}u_t\|_{L^2} + CM_P(\psi)\psi^\alpha \\ &\leq \frac{1}{2}\|\rho^{1/2}u_t\|_{L^2}^2 + CM_P(\psi)\psi^\alpha. \end{aligned} \tag{3.11}$$

Finally, it follows from (3.10) that

$$\begin{aligned} \frac{d}{dt}\|P - P(\rho_\infty)\|_{L^2}^2 &\leq C \int |P - P(\rho_\infty)| |P'(\rho)| |\rho_t| dx \\ &\leq CM_P(\psi)\psi^\alpha, \end{aligned} \tag{3.12}$$

which together with (3.11) gives (3.3) and finishes the proof of Lemma 3.2. \square

Lemma 3.3. *It holds that*

$$\sup_{0 \leq s \leq t} s \int \rho |u_t|^2 dx + \int_0^t s \|\nabla u_t\|_{L^2}^2 ds \leq C \exp \left\{ C \int_0^t M_P^2(\psi)\psi^\alpha ds \right\}. \tag{3.13}$$

Proof. Differentiating (1.1)₂ with respect to t gives

$$\begin{aligned} &-\mu \Delta u_t - (\mu + \lambda) \nabla \operatorname{div} u_t \\ &= -\rho u_{tt} - \rho u \cdot \nabla u_t - \rho_t (u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u - \nabla P_t. \end{aligned} \tag{3.14}$$

Multiplying (3.14) by u_t , we obtain after using integration by parts and (1.1)₁ that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 dx + \int ((\mu + \lambda)(\operatorname{div} u_t)^2 + \mu |\nabla u_t|^2) dx \\ &= -2 \int \rho u \cdot \nabla u_t \cdot u_t dx - \int \rho u \cdot \nabla (u \cdot \nabla u \cdot u_t) dx \\ &\quad - \int \rho u_t \cdot \nabla u \cdot u_t dx + \int P_t \operatorname{div} u_t dx \\ &\leq C \int \rho |u| |u_t| (|\nabla u_t| + |\nabla u|^2 + |u| |\nabla^2 u|) dx + C \int \rho |u|^2 |\nabla u| |\nabla u_t| dx \\ &\quad + C \int \rho |u_t|^2 |\nabla u| dx + C \int |P_t| |\operatorname{div} u_t| dx \triangleq \sum_i^4 J_i. \end{aligned} \tag{3.15}$$

We estimate each term on the right-hand side of (3.15) as follows.

First, it follows from Holder's and the Gagliardo-Nirenberg inequalities that

$$\begin{aligned} J_1 &\leq C \|\rho\|_{L^\infty}^{1/2} \|u\|_{L^6} \|\rho^{1/2}u_t\|_{L^2}^{1/2} \|\rho^{1/2}u_t\|_{L^6}^{1/2} \|\nabla u_t\|_{L^2} \\ &\quad + C \|\rho\|_{L^\infty} \|u\|_{L^6} \|u_t\|_{L^6} \|\nabla u\|_{L^3}^2 + C \|\rho\|_{L^\infty} \|u\|_{L^6}^2 \|u_t\|_{L^6} \|\nabla^2 u\|_{L^2} \\ &\leq C\psi^\alpha \|\rho^{1/2}u_t\|_{L^2}^{1/2} \|\nabla u_t\|_{L^2}^{3/2} + C\psi^\alpha \|\nabla u_t\|_{L^2} \|\nabla u\|_{H^1} \\ &\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C(\varepsilon)\psi^\alpha (1 + \|\nabla^2 u\|_{L^2}^2 + \|\rho^{1/2}u_t\|_{L^2}^2) \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} J_2 + J_3 &\leq C \|\rho\|_{L^\infty} \|u\|_{L^6}^2 \|\nabla u\|_{L^6} \|\nabla u_t\|_{L^2} + C \|\nabla u\|_{L^2} \|\sqrt{\rho}u_t\|_{L^6}^{3/2} \|\sqrt{\rho}u_t\|_{L^2}^{1/2} \\ &\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C(\varepsilon)\psi^\alpha \|\nabla^2 u\|_{L^2}^2 + C(\varepsilon)\psi^\alpha \|\rho^{1/2}u_t\|_{L^2}^2. \end{aligned} \tag{3.17}$$

Next, it follows from (3.10) that

$$\begin{aligned} J_4 &\leq C \|P'(\rho)\|_{L^\infty} \|\rho_t\|_{L^2} \|\nabla u_t\|_{L^2} \\ &\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C(\varepsilon)M_P^2(\psi)\psi^\alpha. \end{aligned} \tag{3.18}$$

Substituting (3.16)–(3.18) into (3.15) and choosing ε suitably small lead to

$$\frac{d}{dt} \int \rho |u_t|^2 dx + \int ((\mu + \lambda)(\operatorname{div} u_t)^2 + \mu |\nabla u_t|^2) dx$$

$$\begin{aligned} &\leq C\psi^\alpha(1 + \|\rho^{1/2}u_t\|_{L^2}^2 + \|\nabla^2u\|_{L^2}^2) \\ &\leq C\psi^\alpha\|\rho^{1/2}u_t\|_{L^2}^2 + CM_P^2(\psi)\psi^\alpha, \end{aligned} \tag{3.19}$$

where in the last inequality one has used (3.8).

Finally, multiplying (3.19) by t , we obtain (3.13) after using Gronwall's inequality and (3.3). The proof of Lemma 3.3 is completed. \square

Lemma 3.4. *It holds that*

$$\sup_{0 \leq s \leq t} \|\rho - \rho_\infty\|_{L^{\bar{p}} \cap D^1 \cap W^{1,q}} \leq C \exp \left\{ C \int_0^t M_P^2(\psi)\psi^\alpha ds \right\}. \tag{3.20}$$

Proof. First, using (1.1)₁, we have

$$\frac{d}{dt} \|\rho - \rho_\infty\|_{L^{\bar{p}}} \leq C\psi^\alpha. \tag{3.21}$$

Next, differentiating (1.1)₁ with respect to x_i and multiplying the resulting equation by $r|\partial_i\rho|^{r-2}\nabla\rho$ with $r \in [2, q]$, we obtain after integration by parts that

$$\begin{aligned} \frac{d}{dt} \|\nabla\rho\|_{L^r} &\leq C(\|\nabla u\|_{L^\infty} \|\nabla\rho\|_{L^r} + \|\rho\|_{L^\infty} \|\nabla^2u\|_{L^r}) \\ &\leq C\psi(\|\nabla u\|_{L^\infty} + \|\nabla^2u\|_{L^r}). \end{aligned} \tag{3.22}$$

Taking $r = 2, q$ in (3.22) and using the Gagliardo-Nirenberg inequality, we have

$$\frac{d}{dt} \|\nabla\rho\|_{L^2 \cap L^q} \leq C(1 + \|\nabla^2u\|_{L^2 \cap L^q})\psi^\alpha,$$

which together with (3.21) yields (3.20) provided we show that

$$\int_0^t \|\nabla^2u\|_{L^2 \cap L^q}^{p_0} ds \leq C \exp \left\{ C \int_0^t M_P^2(\psi)\psi^\alpha ds \right\} \tag{3.23}$$

for

$$p_0 \triangleq \frac{9q - 6}{10q - 12} \in \left(1, \frac{7}{6}\right).$$

Indeed, applying Lemma 2.3 to (3.7) yields that

$$\begin{aligned} \|\nabla^2u\|_{L^q} &\leq C\|\rho u_t\|_{L^q} + C\|\rho u \cdot \nabla u\|_{L^q} + C\|\nabla P\|_{L^q} \\ &\leq C\|\rho u_t\|_{L^2}^{\frac{6-q}{2q}} \|\rho u_t\|_{L^6}^{\frac{3q-6}{2q}} + C\|\rho\|_{L^\infty} \|u\|_{L^\infty} \|\nabla u\|_{L^q} + CM_P(\psi)\psi^\alpha \\ &\leq C\psi^\alpha \|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}} \|\nabla u_t\|_{L^2}^{\frac{3q-6}{2q}} + C\psi^\alpha \|\nabla u\|_{H^1}^{\frac{3}{2}} + CM_P(\psi)\psi^\alpha \\ &\leq C\psi^\alpha \|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}} \|\nabla u_t\|_{L^2}^{\frac{3q-6}{2q}} + C\psi^\alpha \|\sqrt{\rho}u_t\|_{L^2}^{\frac{3}{2}} + CM_P^{\frac{3}{2}}(\psi)\psi^\alpha, \end{aligned} \tag{3.24}$$

where in the last inequality one has used (3.8). Combining this with (3.8), (3.3) and (3.13) shows that

$$\begin{aligned} \int_0^t \|\nabla^2u\|_{L^2 \cap L^q}^{p_0} ds &\leq C \int_0^t \psi^\alpha s^{-p_0/2} (s\|\rho^{1/2}u_t\|_{L^2}^2)^{\frac{6-q}{4q}p_0} (s\|\nabla u_t\|_{L^2}^2)^{\frac{3q-6}{4q}p_0} ds \\ &\quad + C \int_0^t \|\rho^{1/2}u_t\|_{L^2}^2 ds + C \int_0^t M_P^{3/2}(\psi)\psi^\alpha ds \\ &\leq C \exp \left\{ C \int_0^t M_P^2(\psi)\psi^\alpha ds \right\} \int_0^t (\psi^\alpha + s^{-\frac{31q^2+12q-36}{26q^2+48q-72}} + s\|\nabla u_t\|_{L^2}^2) ds \\ &\quad + C \exp \left\{ C \int_0^t M_P^2(\psi)\psi^\alpha ds \right\} \end{aligned}$$

$$\leq C \exp \left\{ C \int_0^t M_P^2(\psi) \psi^\alpha ds \right\},$$

which proves (3.23) and finishes the proof of Lemma 3.4. □

Now, we are in a position to prove Proposition 3.1.

Proof of Proposition 3.1. It follows from (3.3) and (3.20) that

$$\psi(t) \leq C_1 \exp \left\{ C_2 \int_0^t M_P^2(\psi) \psi^\alpha ds \right\}.$$

Since $\psi(0) < \tilde{M} \triangleq C_1$, standard arguments yield that for $T_0 \triangleq \min\{1, [C_2 M_P^2(\tilde{M}) \tilde{M}^\alpha]^{-1}\}$,

$$\sup_{0 \leq t \leq T_0} \psi(t) \leq \tilde{M}, \tag{3.25}$$

which together with (3.8) and (3.13) gives

$$\sup_{0 \leq t \leq T_0} t(\|\nabla^2 u\|_{L^2}^2 + \|\sqrt{\rho} u_t\|_{L^2}^2) + \int_0^{T_0} (t\|\nabla u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2) dt \leq C. \tag{3.26}$$

Next, multiplying (3.14) by $u_{tt} + u \cdot \nabla u_t$ and integrating the resulting equation by parts lead to

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (\mu |\nabla u_t|^2 + (\lambda + \mu) (\operatorname{div} u_t)^2) dx + \int \rho |u_{tt} + u \cdot \nabla u_t|^2 dx \\ &= \frac{d}{dt} \left(- \int \rho_t u \cdot \nabla u \cdot u_t dx - \frac{1}{2} \int \rho_t |u_t|^2 dx + \int P_t \operatorname{div} u_t dx \right) \\ &+ \int \rho_{tt} u \cdot \nabla u \cdot u_t dx + \int \rho_t (u \cdot \nabla u)_t \cdot u_t dx \\ &+ \frac{1}{2} \int (\rho_{tt} + \operatorname{div}(u \rho_t)) |u_t|^2 dx - \int \rho_t u \cdot \nabla u \cdot (u \cdot \nabla u_t) dx \\ &- \int \rho u_t \cdot \nabla u \cdot (u_{tt} + u \cdot \nabla u_t) dx - \mu \int \partial_i u_t \partial_i u \cdot \nabla u_t dx \\ &+ \frac{\mu}{2} \int \operatorname{div} u |\nabla u_t|^2 dx - (\mu + \lambda) \int \operatorname{div} u_t \nabla u \cdot \nabla u_t dx \\ &+ \frac{\mu + \lambda}{2} \int \operatorname{div} u (\operatorname{div} u_t)^2 dx - \int P_{tt} \operatorname{div} u_t dx \\ &+ \int P_t \operatorname{div}(u \cdot \nabla u_t) dx \triangleq \frac{d}{dt} I_0 + \sum_{i=1}^{11} I_i. \end{aligned} \tag{3.27}$$

We estimate each I_i ($i = 0, \dots, 11$) as follows.

First, it follows from (1.1)₁, (3.25) and (3.8) that

$$\begin{aligned} |I_0| &= \left| - \frac{1}{2} \int \rho_t |u_t|^2 dx - \int \rho_t u \cdot \nabla u \cdot u_t dx + \int P_t \operatorname{div} u_t dx \right| \\ &\leq C \left| \int \operatorname{div}(\rho u) |u_t|^2 dx \right| + C \|\rho_t\|_{L^2} \|u\|_{L^6} \|\nabla u\|_{L^6} \|u_t\|_{L^6} \\ &\quad + C \|P_t\|_{L^2} \|\nabla u_t\|_{L^2} \\ &\leq C \int \rho |u| |u_t| |\nabla u_t| dx + C(1 + \|\nabla u\|_{H^1}^2) \|\nabla u_t\|_{L^2} \\ &\leq C \|u\|_{L^6} \|\rho^{1/2} u_t\|_{L^2}^{1/2} \|\nabla u_t\|_{L^2}^{3/2} + C(1 + \|\nabla u\|_{H^1}) \|\nabla u_t\|_{L^2} \\ &\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C(\varepsilon) \|\rho^{1/2} u_t\|_{L^2}^2 + C, \end{aligned} \tag{3.28}$$

where in the third inequality we have used

$$\|\rho_t\|_{L^2} + \|P_t\|_{L^2} \leq C \|u\|_{L^6} (\|\nabla \rho\|_{L^3} + \|\nabla P\|_{L^3}) + C \|\nabla u\|_{L^2} \leq C. \tag{3.29}$$

Next, using (1.1)₁ and (3.25), we have

$$\|\rho_t\|_{L^2 \cap L^q} + \|P_t\|_{L^2 \cap L^q} \leq C\|\nabla u\|_{H^1}, \quad (3.30)$$

which together with (1.1)₁ and (3.25) yields that

$$\begin{aligned} |I_1| &= \left| \int \rho_{tt} u \cdot \nabla u \cdot u_t dx \right| \\ &= \left| \int (\rho_t u + \rho u_t) \cdot \nabla (u \cdot \nabla u \cdot u_t) dx \right| \\ &\leq C\|\rho_t u + \rho u_t\|_{L^3} (\|\nabla(u \cdot \nabla u)\|_{L^2} \|u_t\|_{L^6} + \|u \cdot \nabla u\|_{L^6} \|\nabla u_t\|_{L^2}) \\ &\leq C(\|\nabla u\|_{H^1}^2 + \|\rho^{1/2} u_t\|_{L^2}^{1/2} \|\nabla u_t\|_{L^2}^{1/2}) \|\nabla u\|_{H^1}^2 \|\nabla u_t\|_{L^2} \\ &\leq C\|\nabla u\|_{H^1}^2 \|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{H^1}^6 + C\|\rho^{1/2} u_t\|_{L^2}^2 \|\nabla u\|_{H^1}^2 \end{aligned} \quad (3.31)$$

and that

$$\begin{aligned} |I_2| &= \left| \int \rho_t (u \cdot \nabla u)_t \cdot u_t dx \right| \\ &\leq C\|\rho_t\|_{L^3} \|(u \cdot \nabla u)_t\|_{L^2} \|u_t\|_{L^6} \\ &\leq C\|\nabla u\|_{H^1}^2 \|\nabla u_t\|_{L^2}^2. \end{aligned} \quad (3.32)$$

Since (1.1)₁ implies $\rho_{tt} + \operatorname{div}(u\rho_t) = -\operatorname{div}(\rho u_t)$, we have

$$\begin{aligned} |I_3| &= \frac{1}{2} \left| \int \rho u_t \cdot \nabla |u_t|^2 dx \right| \\ &\leq C\|\rho^{1/2} u_t\|_{L^2}^{1/2} \|u_t\|_{L^6}^{1/2} \|u_t\|_{L^6} \|\nabla u_t\|_{L^2} \\ &\leq C\|\rho^{1/2} u_t\|_{L^2}^{1/2} \|\nabla u_t\|_{L^2}^{5/2} \\ &\leq C\|\nabla u_t\|_{L^2}^2 (t\|\nabla u_t\|_{L^2}^2 + \|\rho^{1/2} u_t\|_{L^2}^2 + t^{-1/2}). \end{aligned} \quad (3.33)$$

Next, Hölder's inequality gives

$$\begin{aligned} |I_4| &= \left| \int \rho_t u \cdot \nabla u \cdot (u \cdot \nabla u_t) dx \right| \\ &\leq C\|\rho_t\|_{L^3} \| |u|^2 |\nabla u \|_{L^6} \|\nabla u_t\|_{L^2} \\ &\leq C\|\nabla u\|_{H^1}^2 \|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{H^1}^6, \end{aligned} \quad (3.34)$$

$$\begin{aligned} |I_5| &= \left| \int \rho u_t \cdot \nabla u \cdot (u_{tt} + u \cdot \nabla u_t) dx \right| \\ &\leq C\|\rho^{1/2} (u_{tt} + u \cdot \nabla u_t)\|_{L^2} \|\rho^{1/2} u_t\|_{L^3} \|\nabla u\|_{L^6} \\ &\leq \frac{1}{2} \|\rho^{1/2} (u_{tt} + u \cdot \nabla u_t)\|_{L^2}^2 + C\|\rho^{1/2} u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla u\|_{H^1}^2 \end{aligned} \quad (3.35)$$

and

$$\sum_{i=6}^9 |I_i| \leq C\|\nabla u_t\|_{L^2}^2 \|\nabla u\|_{L^\infty}. \quad (3.36)$$

Finally, direct calculations together with (3.30) lead to

$$\begin{aligned} &|I_{10} + I_{11}| \\ &= \left| \int P_{tt} \operatorname{div} u_t dx - \int P_t \operatorname{div} (u \cdot \nabla u_t) dx \right| \\ &= \left| \int P_{tt} \operatorname{div} u_t dx - \int P_t u \cdot \nabla \operatorname{div} u_t dx - \int P_t \nabla u \cdot \nabla u_t dx \right| \end{aligned}$$

$$\begin{aligned}
 &= \left| \int (P_{tt} + u \cdot \nabla P_t) \operatorname{div} u_t dx + \int P_t \operatorname{div} u \operatorname{div} u_t dx - \int P_t \nabla u \cdot \nabla u_t dx \right| \\
 &\leq C \int (|P_t| |\nabla u| |\nabla u_t| + |\nabla u_t|^2 + |u_t| |\nabla P| |\nabla u_t|) dx \\
 &\leq C (\|P_t\|_{L^3} \|\nabla u\|_{H^1} + \|\nabla P\|_{L^3} \|u_t\|_{L^6}) \|\nabla u_t\|_{L^2} + C \|\nabla u_t\|_{L^2}^2 \\
 &\leq C \|\nabla u\|_{H^1}^2 \|\nabla u_t\|_{L^2} + C \|\nabla u_t\|_{L^2}^2,
 \end{aligned} \tag{3.37}$$

where in the last inequality, we have used

$$P_{tt} + u \cdot \nabla P_t = -(\gamma P_t \operatorname{div} u + \gamma P \operatorname{div} u_t + u_t \cdot \nabla P), \tag{3.38}$$

due to (1.1)₁.

Putting all the estimates (3.31)–(3.37) into (3.27) and choosing ε suitably small, we have

$$\begin{aligned}
 &\Psi'(t) + \int \rho |u_{tt} + u \cdot \nabla u_t|^2 dx \\
 &\leq C \|\nabla u_t\|_{L^2}^2 (t \|\nabla u_t\|_{L^2}^2 + \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla u\|_{L^\infty} + \|\nabla u\|_{H^1}^2 + t^{-1/2}) \\
 &\quad + C \|\nabla u\|_{H^1}^6 + C \|\sqrt{\rho} u_t\|_{L^2}^2 \|\nabla u\|_{H^1}^2 + C,
 \end{aligned} \tag{3.39}$$

where

$$\Psi(t) \triangleq \mu \|\nabla u_t\|_{L^2}^2 + (\mu + \lambda) \|\operatorname{div} u_t\|_{L^2}^2 - 2I_0$$

satisfies

$$\frac{\mu}{2} \|\nabla u_t\|_{L^2}^2 - C \|\sqrt{\rho} u_t\|_{L^2}^2 - C \leq \Psi(t) \leq C \|\nabla u_t\|_{L^2}^2 + C \|\sqrt{\rho} u_t\|_{L^2}^2 + C, \tag{3.40}$$

owing to (3.28). Hence, multiplying (3.39) by t^2 , we obtain after using Gronwall's inequality, (3.40), (3.25) and (3.26) that

$$\sup_{0 \leq t \leq T_0} t^2 \|\nabla u_t\|_{L^2}^2 + \int_0^{T_0} t^2 \|\rho^{1/2} u_{tt}\|_{L^2}^2 dt \leq C, \tag{3.41}$$

where we have used the following simple fact that

$$\int \rho |u|^2 |\nabla u_t|^2 dx \leq C \|\nabla u\|_{H^1}^2 \|\nabla u_t\|_{L^2}^2. \tag{3.42}$$

Combining (3.41) and (3.24)–(3.26) gives (3.2) and completes the proof of Proposition 3.1. \square

Corollary 3.5. *Assume that (ρ_0, u_0) satisfies (1.9) with some $g \in L^2$. Then there exists some positive constant \tilde{C} depending only on $\mu, \lambda, P, q, \rho_\infty, \psi(0), \|\nabla u_0\|_{H^1}, \|g\|_{L^2}$ and Ω if $\Omega_R = \Omega$ such that*

$$\sup_{0 \leq t \leq T_0} (\|\nabla u\|_{H^1} + \|\sqrt{\rho} u_t\|_{L^2} + t(\|\nabla u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^q}^2)) + \int_0^{T_0} \|\nabla u_t\|_{L^2}^2 dt \leq \tilde{C}. \tag{3.43}$$

Proof. Taking into account on the compatibility condition (1.9), we can define

$$\rho^{1/2} u_t(x, t = 0) = -g - \rho_0^{1/2} u_0 \cdot \nabla u_0,$$

which together with (3.19), (3.2) and Gronwall's inequality yields

$$\sup_{0 \leq t \leq T_0} \int \rho |u_t|^2 dx + \int_0^{T_0} \|\nabla u_t\|_{L^2}^2 dt \leq \tilde{C}. \tag{3.44}$$

It thus follows from this, (3.8) and (3.2) that

$$\sup_{0 \leq t \leq T_0} \|\nabla u\|_{H^1} \leq \tilde{C}, \tag{3.45}$$

which combined with (3.39), (3.40), (3.44) and (3.42) gives

$$\sup_{0 \leq t \leq T_0} t \|\nabla u_t\|_{L^2}^2 + \int_0^{T_0} t \|\rho^{1/2} u_{tt}\|_{L^2}^2 dt \leq \tilde{C}. \tag{3.46}$$

Combining this, (3.44), (3.45) and (3.24) gives (3.43) and completes the proof of Corollary 3.5. \square

4 *A priori estimates (II)*

This section will show some higher order estimates of the solutions with the initial data satisfying the additional compatibility condition (1.9) and further regularity assumptions (1.15). In this section, the generic positive constant C depends only on $\mu, \lambda, P, q, \rho_\infty, \|\nabla u_0\|_{H^1}, \|\rho_0 - \rho_\infty\|_{L^{\tilde{p}} \cap D^1 \cap W^{1,q}}, \|\nabla^2 \rho_0\|_{L^2 \cap L^q}, \|\nabla^2 P(\rho_0)\|_{L^2 \cap L^q}$ and $\|g\|_{L^2}$.

Lemma 4.1. *It holds that*

$$\sup_{0 \leq t \leq T_0} (\|\nabla \rho\|_{H^1} + \|\nabla P\|_{H^1} + \|\rho_t\|_{H^1} + \|P_t\|_{H^1} + t\|\nabla u\|_{H^2}^2) \leq C. \tag{4.1}$$

Proof. It follows from (1.1)₁ and (3.2) that

$$\begin{aligned} & \frac{d}{dt} (\|\nabla^2 P\|_{L^2} + \|\nabla^2 \rho\|_{L^2}) \\ & \leq C(1 + \|\nabla u\|_{L^\infty}) (\|\nabla^2 P\|_{L^2} + \|\nabla^2 \rho\|_{L^2}) + C\|\nabla^2 u\|_{H^1}. \end{aligned} \tag{4.2}$$

Applying Lemma 2.3 to (3.7) shows

$$\begin{aligned} \|\nabla^2 u\|_{H^1} & \leq C(\|\rho(u_t + u \cdot \nabla u)\|_{H^1} + \|\nabla P\|_{H^1}) \\ & \leq C + C\|\nabla u_t\|_{L^2} + C\|\nabla^2 P\|_{L^2}, \end{aligned} \tag{4.3}$$

where in the second inequality we have used (3.2), (3.8) and the following simple fact:

$$\begin{aligned} \|\nabla(\rho(u_t + u \cdot \nabla u))\|_{L^2} & \leq \|\nabla \rho\|_{L^3} \|u_t\|_{L^6} + \|\rho \nabla u_t\|_{L^2} + \|\rho |\nabla u|^2\|_{L^2} \\ & \quad + \|\nabla \rho\|_{L^3} \|u\|_{L^6} \|\nabla u\|_{L^2} + \|\rho |u| \nabla^2 u\|_{L^2} \\ & \leq C\|\nabla \rho\|_{L^3} \|u_t\|_{L^6} + C\|\nabla u_t\|_{L^2} + C\|\nabla u\|_{H^1}^2 \\ & \quad + C\|u\|_{L^\infty} (\|\nabla \rho\|_{L^3} \|\nabla u\|_{L^6} + C\|\nabla^2 u\|_{L^2}) \\ & \leq C + C\|\nabla u_t\|_{L^2} \end{aligned} \tag{4.4}$$

due to (3.2) and (3.43). Using (4.2), (4.3), (3.43) and Gronwall's inequality, one obtains

$$\sup_{0 \leq t \leq T_0} (\|\nabla^2 \rho\|_{L^2} + \|\nabla^2 P\|_{L^2} + t\|\nabla^2 u\|_{H^1}^2) \leq C. \tag{4.5}$$

Finally, applying ∇ to (1.1)₁, we have

$$\nabla P_t + u \cdot \nabla \nabla P + \nabla u \cdot \nabla P + \gamma \nabla P \operatorname{div} u + \gamma P \nabla \operatorname{div} u = 0,$$

which together with (4.5), (3.2) and (3.43) yields

$$\|\nabla P_t\|_{L^2} \leq C\|u\|_{L^\infty} \|\nabla^2 P\|_{L^2} + C\|\nabla u\|_{L^6} \|\nabla P\|_{L^3} + C\|\nabla^2 u\|_{L^2} \leq C. \tag{4.6}$$

Similarly, one has

$$\|\nabla \rho_t\|_{L^2} \leq C.$$

Combining this with (3.2), (3.29), (4.6) and (4.5) gives (4.1) and completes the proof of Lemma 4.1. \square

Lemma 4.2. *It holds that*

$$\sup_{0 \leq t \leq T_0} (\|\nabla^2 \rho\|_{L^q} + \|\nabla^2 P\|_{L^q}) \leq C. \tag{4.7}$$

Proof. First, similar to (4.2), one has

$$\begin{aligned} & (\|\nabla^2 \rho\|_{L^q} + \|\nabla^2 P\|_{L^q})_t \\ & \leq C(1 + \|\nabla u\|_{L^\infty}) (\|\nabla^2 \rho\|_{L^q} + \|\nabla^2 P\|_{L^q}) + C\|\nabla^2 u\|_{W^{1,q}}. \end{aligned} \tag{4.8}$$

Applying Lemma 2.3 to (3.7) gives

$$\begin{aligned} \|\nabla^2 u\|_{W^{1,q}} &\leq C\|\rho(u_t + u \cdot \nabla u)\|_{W^{1,q}} + C\|\nabla P\|_{W^{1,q}} \\ &\leq C\|\rho(u_t + u \cdot \nabla u)\|_{L^2} + C\|\nabla(\rho(u_t + u \cdot \nabla u))\|_{L^q} \\ &\quad + C\|\nabla P\|_{L^2} + C\|\nabla^2 P\|_{L^q} \\ &\leq C + C\|\nabla^2 P\|_{L^q} + C\|\nabla(\rho(u_t + u \cdot \nabla u))\|_{L^q}, \end{aligned} \tag{4.9}$$

due to (3.8), (3.2) and (3.43). For the last term of (4.9), it follows from the Gagliardo-Nirenberg inequality, (3.2), (3.43), (3.24), (4.1) and (4.3) that

$$\begin{aligned} &\|\nabla(\rho(u_t + u \cdot \nabla u))\|_{L^q} \\ &\leq C\|\nabla\rho\|_{L^{6q/(6-q)}}(\|u_t\|_{L^6} + \|u\|_{L^\infty}\|\nabla u\|_{L^6}) + C\|\nabla(u_t + u \cdot \nabla u)\|_{L^q} \\ &\leq C(1 + \|\nabla^2\rho\|_{L^q})(1 + \|\nabla u_t\|_{L^2}) + C\|\nabla u_t\|_{L^q} \\ &\quad + C\|\nabla u\|_{H^1}\|\nabla u\|_{H^2} + C\|u\|_{L^\infty}\|\nabla^2 u\|_{L^q} \\ &\leq C(1 + \|\nabla^2\rho\|_{L^q})(1 + \|\nabla u_t\|_{L^2}) + C\|\nabla u_t\|_{L^q}. \end{aligned} \tag{4.10}$$

Then, applying Lemma 2.3 to (3.14), we have

$$\begin{aligned} \|\nabla^2 u_t\|_{L^2} &\leq C\|\rho u_{tt} + \rho_t u_t + \rho_t u \cdot \nabla u + \rho u_t \cdot \nabla u + \rho u \cdot \nabla u_t + \nabla P_t\|_{L^2} \\ &\leq C(\|\rho u_{tt}\|_{L^2} + \|\rho_t\|_{L^3}\|u_t\|_{L^6} + \|\rho_t\|_{L^3}\|u\|_{L^\infty}\|\nabla u\|_{L^6}) \\ &\quad + C(\|u_t\|_{L^6}\|\nabla u\|_{L^3} + \|u\|_{L^\infty}\|\nabla u_t\|_{L^2} + \|\nabla P_t\|_{L^2}) \\ &\leq C\|\rho^{1/2}u_{tt}\|_{L^2} + C\|\nabla u_t\|_{L^2} + C, \end{aligned} \tag{4.11}$$

where in the last inequality we have used (3.43), (3.2), (3.30) and (4.1). Combining this with (3.43) and (3.46) shows

$$\begin{aligned} \int_0^{T_0} \|\nabla u_t\|_{L^q} dt &\leq C \int_0^{T_0} \|\nabla u_t\|_{L^2}^{(6-q)/(2q)} \|\nabla u_t\|_{H^1}^{3(q-2)/(2q)} dt \\ &\leq C + C \int_0^{T_0} t^{-1/2} (t\|\rho^{1/2}u_{tt}\|_{L^2}^2)^{3(q-2)/(4q)} dt \\ &\leq C + C \int_0^{T_0} (t^{-2q/(q+6)} + t\|\rho^{1/2}u_{tt}\|_{L^2}^2) dt \leq C. \end{aligned} \tag{4.12}$$

Finally, putting (4.9) and (4.10) into (4.8) and using Gronwall's inequality, (3.43) and (4.12), we obtain (4.7) and complete the proof of Lemma 4.2. \square

Lemma 4.3. *It holds that*

$$\sup_{0 \leq t \leq T_0} t(\|\nabla^3 u\|_{L^q} + \|\nabla u_t\|_{H^1} + \|\sqrt{\rho}u_{tt}\|_{L^2}) + \int_0^{T_0} t^2 \|\nabla u_{tt}\|_{L^2}^2 dt \leq C. \tag{4.13}$$

Proof. We claim that

$$\sup_{0 \leq t \leq T_0} t^2 \|\sqrt{\rho}u_{tt}\|_{L^2}^2 + \int_0^{T_0} t^2 \|\nabla u_{tt}\|_{L^2}^2 dt \leq C, \tag{4.14}$$

which together with (3.43) and (4.11) yields that

$$\sup_{0 \leq t \leq T_0} t \|\nabla u_t\|_{H^1} \leq C. \tag{4.15}$$

It thus follows from this, (4.9), (4.10) and (4.7) that

$$\sup_{0 \leq t \leq T_0} t \|\nabla^3 u\|_{L^q} \leq C. \tag{4.16}$$

Combining (4.14)–(4.16) yields (4.13).

Now, it remains to prove (4.14). In fact, differentiating (3.14) with respect to t leads to

$$\begin{aligned} & \rho u_{ttt} + \rho u \cdot \nabla u_{tt} - \mu \Delta u_{tt} - (\mu + \lambda) \nabla \operatorname{div} u_{tt} \\ &= 2 \operatorname{div}(\rho u) u_{tt} + \operatorname{div}(\rho u)_t u_t - 2(\rho u)_t \cdot \nabla u_t - (\rho_{tt} u + 2\rho_t u_t) \cdot \nabla u \\ & \quad - \rho u_{tt} \cdot \nabla u - \nabla P_{tt}. \end{aligned} \tag{4.17}$$

Multiplying (4.17) by u_{tt} and integrating the resulting equation by parts yield

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \rho |u_{tt}|^2 dx + \int (\mu |\nabla u_{tt}|^2 + (\mu + \lambda) (\operatorname{div} u_{tt})^2) dx \\ &= -4 \int \rho u \cdot \nabla u_{tt} \cdot u_{tt} dx - \int (\rho u)_t \cdot [\nabla(u_t \cdot u_{tt}) + 2\nabla u_t \cdot u_{tt}] dx \\ & \quad - \int (\rho_{tt} u + 2\rho_t u_t) \cdot \nabla u \cdot u_{tt} dx - \int \rho u_{tt} \cdot \nabla u \cdot u_{tt} dx \\ & \quad + \int P_{tt} \operatorname{div} u_{tt} dx \triangleq \sum_{i=1}^5 K_i. \end{aligned} \tag{4.18}$$

Using (3.2), (3.43) and (4.1), we can estimate each K_i ($i = 1, \dots, 5$) as follows:

$$\begin{aligned} |K_1| &\leq C \|\rho^{1/2} u_{tt}\|_{L^2} \|\nabla u_{tt}\|_{L^2} \|u\|_{L^\infty} \\ &\leq \varepsilon \|\nabla u_{tt}\|_{L^2}^2 + C(\varepsilon) \|\rho^{1/2} u_{tt}\|_{L^2}^2, \end{aligned} \tag{4.19}$$

$$\begin{aligned} |K_2| &\leq C(\|\rho u_t\|_{L^3} + \|\rho_t u\|_{L^3})(\|u_{tt}\|_{L^6} \|\nabla u_t\|_{L^2} + \|\nabla u_{tt}\|_{L^2} \|u_t\|_{L^6}) \\ &\leq C(\|\rho^{1/2} u_t\|_{L^2}^{1/2} \|u_t\|_{L^6}^{1/2} + \|\rho_t\|_{L^6} \|u\|_{L^6}) \|\nabla u_{tt}\|_{L^2} \|\nabla u_t\|_{L^2} \\ &\leq C(\|\nabla u_t\|_{L^2} + 1) \|\nabla u_{tt}\|_{L^2} \|\nabla u_t\|_{L^2} \\ &\leq \varepsilon \|\nabla u_{tt}\|_{L^2}^2 + C(\varepsilon) \|\nabla u_t\|_{L^2}^4 + C(\varepsilon), \end{aligned} \tag{4.20}$$

$$\begin{aligned} |K_3| &\leq C(\|\rho_{tt}\|_{L^2} \|u\|_{L^\infty} \|\nabla u\|_{L^3} + \|\rho_t\|_{L^6} \|u_t\|_{L^6} \|\nabla u\|_{L^2}) \|u_{tt}\|_{L^6} \\ &\leq \varepsilon \|\nabla u_{tt}\|_{L^2}^2 + C(\varepsilon) \|\rho_{tt}\|_{L^2}^2 + C(\varepsilon) \|\nabla u_t\|_{L^2}^2 \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} |K_4| + |K_5| &\leq C \|\rho u_{tt}\|_{L^2} \|\nabla u\|_{L^3} \|u_{tt}\|_{L^6} + C \|P_{tt}\|_{L^2} \|\nabla u_{tt}\|_{L^2} \\ &\leq \varepsilon \|\nabla u_{tt}\|_{L^2}^2 + C(\varepsilon) \|\rho^{1/2} u_{tt}\|_{L^2}^2 + C(\varepsilon) \|P_{tt}\|_{L^2}^2. \end{aligned} \tag{4.22}$$

Substituting (4.19)–(4.22) into (4.18) and choosing ε suitably small, we have

$$\begin{aligned} & \frac{d}{dt} \|\rho^{1/2} u_{tt}\|_{L^2}^2 + \mu \|\nabla u_{tt}\|_{L^2}^2 \\ & \leq C \|\rho^{1/2} u_{tt}\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^4 + C + C \|\rho_{tt}\|_{L^2}^2 + C \|P_{tt}\|_{L^2}^2. \end{aligned} \tag{4.23}$$

Finally, it follows from (3.38), (4.1) and (3.44) that

$$\begin{aligned} \int_0^{T_0} \|P_{tt}\|_{L^2}^2 ds &\leq C \int_0^{T_0} (\|u\|_{L^\infty} \|\nabla P_t\|_{L^2} + \|P_t\|_{L^6} \|\nabla u\|_{L^3})^2 dx \\ & \quad + C \int_0^{T_0} (\|\nabla u_t\|_{L^2} + \|u_t\|_{L^6} \|\nabla P\|_{L^3})^2 dt \leq C. \end{aligned} \tag{4.24}$$

Similarly, one has

$$\int_0^{T_0} \|\rho_{tt}\|_{L^2}^2 dt \leq C. \tag{4.25}$$

Multiplying (4.23) by t^2 and using (3.43), (3.46), (4.24) and (4.25), we obtain (4.14) and finish the proof of Lemma 4.3. □

5 Proofs of Theorems 1.2 and 1.4

To prove Theorems 1.2 and 1.4, we will only deal with the case where Ω is bounded. Since for the Cauchy problem, all the *a priori* estimates obtained in Sections 3 and 4 are independent of the radius R , one can use the standard domain expansion technique to treat the whole space case; please refer to [17] and the references therein.

Proof of Theorem 1.2. Let (ρ_0, u_0) be as in Theorem 1.2. For $\delta > 0$, we choose $0 \leq \hat{\rho}_0^\delta \in C^\infty(\Omega)$ and $u_0^\delta \in C_0^\infty(\Omega)$ satisfying

$$\lim_{\delta \rightarrow 0} (\|\hat{\rho}_0^\delta - \rho_0\|_{W^{1,q}} + \|u_0^\delta - u_0\|_{H^1}) = 0. \tag{5.1}$$

Then in terms of Lemma 2.1, the problems (1.1)–(1.4) with the initial data $(\hat{\rho}_0^\delta + \delta, (\hat{\rho}_0^\delta + \delta)u_0^\delta)$ have a unique smooth solution (ρ^δ, u^δ) on $\Omega \times [0, T_\delta]$ for some $T_\delta > 0$. Moreover, Proposition 3.1 shows that there exist two positive constants T_0 and M independent of δ such that (3.2) holds for (ρ^δ, u^δ) . More precisely, it holds that

$$\begin{aligned} & \sup_{0 \leq t \leq T_0} (\|\nabla u\|_{L^2} + \|\rho\|_{H^1 \cap W^{1,q}} + \|P(\rho)\|_{H^1 \cap W^{1,q}} + t(\|\nabla^2 u\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2)) \\ & + \sup_{0 \leq t \leq T_0} (t^2(\|\nabla u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^q}^2)) + \int_0^{T_0} t \|\nabla u_t\|_{L^2}^2 dt \leq M, \end{aligned} \tag{5.2}$$

$$\begin{aligned} & \sup_{0 \leq t \leq T_0} (\|\rho^\delta\|_{W^{1,q}} + \|\rho_t^\delta\|_{L^2} + \|u^\delta\|_{H^1} + t^{1/2}\|\nabla^2 u^\delta\|_{L^2} + \|\rho^\delta u^\delta\|_{H^1}) \\ & + \int_0^{T_0} (\|\nabla^2 u^\delta\|_{L^q}^{p_0} + t\|\nabla u_t^\delta\|_{L^2}^2 + t\|\nabla^2 u^\delta\|_{L^q}^2 + \|\nabla^2 u\|_{L^2}^2 + \|(\rho^\delta u^\delta)_t\|_{L^2}^2) dt \leq \bar{C}, \end{aligned} \tag{5.3}$$

where \bar{C} is independent of δ . With all the estimate (5.2) at hand, we find that the sequence (ρ^δ, u^δ) converges, up to the extraction of subsequences, to some limit (ρ, u) in the obvious weak sense, i.e., as $\delta \rightarrow 0$, we have

$$\rho^\delta \rightarrow \rho \quad \text{in } L^\infty(0, T_0; L^\infty), \tag{5.4}$$

$$\rho^\delta \rightharpoonup \rho \quad \text{weakly } * \text{ in } L^\infty(0, T_0; W^{1,q}), \tag{5.5}$$

$$u^\delta \rightharpoonup u \quad \text{weakly } * \text{ in } L^\infty(0, T_0; H^1), \tag{5.6}$$

$$\nabla^2 u^\delta \rightharpoonup \nabla^2 u \quad \text{weakly in } L^{p_0}(0, T_0; L^q) \cap L^2(\Omega \times (0, T_0)), \tag{5.7}$$

$$t^{1/2}\nabla^2 u^\delta \rightharpoonup t^{1/2}\nabla^2 u \quad \text{weakly in } L^2(0, T_0; L^q), \tag{5.8}$$

$$t^{1/2}\nabla u_t^\delta \rightharpoonup t^{1/2}\nabla u_t \quad \text{weakly in } L^2(\Omega \times (0, T_0)), \tag{5.9}$$

$$\rho^\delta u^\delta \rightarrow \rho u \quad \text{in } L^\infty(0, T_0; L^2). \tag{5.10}$$

Then by letting $\delta \rightarrow 0$, it follows from (5.4)–(5.10) that (ρ, u) is a strong solution of (1.1)–(1.4) on $\Omega \times (0, T_0]$ satisfying (1.8). The proof of the existence part of Theorem 1.2 is finished.

It only remains to prove the uniqueness of the strong solutions satisfying (1.8). Indeed, we will use the method which is due to Germain [7]. Let (ρ, u) and $(\bar{\rho}, \bar{u})$ be two strong solutions satisfying (1.8) with the same initial data. Subtracting the mass equation for (ρ, u) and $(\bar{\rho}, \bar{u})$, we have

$$H_t + \bar{u} \cdot \nabla H + H \operatorname{div} \bar{u} + \rho \operatorname{div} U + U \cdot \nabla \rho = 0 \tag{5.11}$$

with

$$H \triangleq \rho - \bar{\rho}, \quad U \triangleq u - \bar{u}.$$

For $3/2 \leq r \leq 2$, multiplying (5.11) by $rH|H|^{r-2}$ and integrating the resulting equation by parts lead to

$$\begin{aligned} \frac{d}{dt} \|H\|_{L^r}^r & \leq C \int \operatorname{div} \bar{u} |H|^r dx + C \int \rho |\nabla U| |H|^{r-1} dx + C \int |U| |\nabla \rho| |H|^{r-1} dx \\ & \leq C \|\nabla \bar{u}\|_{L^\infty} \|H\|_{L^r}^r + C(\|\rho\|_{L^{\frac{2r}{2-r}}} + \|\nabla \rho\|_{L^{\frac{6r}{6-r}}}) \|\nabla U\|_{L^2} \|H\|_{L^r}^{r-1} \end{aligned}$$

$$\leq C\|\nabla\bar{u}\|_{L^\infty}\|H\|_{L^r}^r + C\|\nabla U\|_{L^2}\|H\|_{L^r}^{r-1}, \tag{5.12}$$

where one has used $\rho \in H^1 \cap W^{1,q}$. This together with Gronwall's inequality and (3.2) gives

$$\|H\|_{L^r} \leq C \int_0^t \|\nabla U\|_{L^2} ds \quad \text{for } \frac{3}{2} \leq r \leq 2. \tag{5.13}$$

Next, subtracting the momentum equations for (ρ, u) and $(\bar{\rho}, \bar{u})$, we have

$$\begin{aligned} &\rho U_t + \rho u \cdot \nabla U - \mu \Delta U - (\mu + \lambda) \nabla(\operatorname{div} U) \\ &= -\rho U \cdot \nabla \bar{u} - H(\bar{u}_t + \bar{u} \cdot \nabla \bar{u}) - \nabla(P(\rho) - P(\bar{\rho})). \end{aligned} \tag{5.14}$$

Multiplying (5.14) by U and integrating the resulting equations by parts lead to

$$\begin{aligned} &\frac{d}{dt} \int \rho |U|^2 dx + 2\mu \int |\nabla U|^2 dx \\ &\leq C\|\nabla\bar{u}\|_{L^\infty} \int \rho |U|^2 dx + C \int |H||U|(|\bar{u}_t| + |\bar{u}||\nabla\bar{u}|) dx \\ &\quad + C\|P(\rho) - P(\bar{\rho})\|_{L^2} \|\operatorname{div} U\|_{L^2} \\ &\leq C\|\nabla\bar{u}\|_{L^\infty} \int \rho |U|^2 dx + C\|H\|_{L^{3/2}} \|U\|_{L^6} \|\bar{u}_t\|_{L^6} \\ &\quad + C\|H\|_{L^2} \|U\|_{L^6} \|\bar{u}\|_{L^6} \|\nabla\bar{u}\|_{L^6} + C\|H\|_{L^2} \|\nabla U\|_{L^2} \\ &\leq C\|\nabla\bar{u}\|_{L^\infty} \int \rho |U|^2 dx + C(1 + \|\nabla\bar{u}_t\|_{L^2} + \|\nabla^2\bar{u}\|_{L^2}) \|\nabla U\|_{L^2} \int_0^t \|\nabla U\|_{L^2} ds \\ &\leq C\|\nabla\bar{u}\|_{L^\infty} \int \rho |U|^2 dx + C(1 + t\|\nabla\bar{u}_t\|_{L^2} + t\|\nabla^2\bar{u}\|_{L^2}) \int_0^t \|\nabla U\|_{L^2}^2 ds + \mu \|\nabla U\|_{L^2}^2 \\ &\leq C(1 + t\|\nabla\bar{u}_t\|_{L^2}^2 + \|\nabla\bar{u}\|_{L^\infty}) \left(\int \rho |U|^2 dx + \int_0^t \|\nabla U\|_{L^2}^2 dt \right) + \mu \|\nabla U\|_{L^2}^2 \end{aligned} \tag{5.15}$$

owing to (3.2) and (5.13). This together with Gronwall's inequality and (3.2) gives $U(x, t) = 0$ for almost everywhere $(x, t) \in \Omega \times (0, T_0)$. Then (5.13) implies that $H(x, t) = 0$ for almost everywhere $(x, t) \in \Omega \times (0, T_0)$. The proof of Theorem 1.2 is completed. \square

Proof of Theorem 1.4. Let (ρ_0, u_0) be as in Theorem 1.4. We construct $\rho_0^\delta = \hat{\rho}_0^\delta + \delta$, where $0 \leq \hat{\rho}_0^\delta \in C_0^\infty(\Omega)$ satisfies (5.1) and

$$\nabla^2 \hat{\rho}_0^\delta \rightarrow \nabla^2 \rho_0, \quad \nabla^2 P(\hat{\rho}_0^\delta) \rightarrow \nabla^2 P(\rho_0) \quad \text{in } L^2 \cap L^q, \quad \text{as } \delta \rightarrow 0.$$

Thus, we have

$$\begin{cases} \rho_0^\delta \rightarrow \rho_0 & \text{in } W^{1,q}(\Omega), \\ \nabla^2 \rho_0^\delta \rightarrow \nabla^2 \rho_0 & \text{in } L^2 \cap L^q, \\ \nabla^2 P(\rho_0^\delta) \rightarrow \nabla^2 P(\rho_0) & \text{in } L^2 \cap L^q, \end{cases} \quad \text{as } \delta \rightarrow 0. \tag{5.16}$$

Then we consider the unique smooth solution u_0^δ of the following elliptic problem:

$$\begin{cases} -\mu \Delta u_0^\delta - (\mu + \lambda) \nabla \operatorname{div} u_0^\delta + \nabla P(\rho_0^\delta) = \sqrt{\rho_0^\delta} g^\delta & \text{in } \Omega, \\ u_0^\delta = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.17}$$

where $g^\delta = g * j_\delta$ with j_δ being the standard mollifying kernel of width δ .

Subtracting the equations (1.9) and (5.17) gives

$$\begin{cases} -\mu \Delta (u_0^\delta - u_0) - (\mu + \lambda) \nabla \operatorname{div} (u_0^\delta - u_0) = F & \text{in } \Omega, \\ u_0^\delta - u_0 = 0 & \text{on } \partial\Omega \end{cases} \tag{5.18}$$

with

$$F \triangleq -\nabla(P(\rho_0^\delta) - P(\rho_0)) + \sqrt{\rho_0^\delta} g^\delta - \sqrt{\rho} g.$$

Multiplying (5.18) by $u_0^\delta - u_0$, we obtain after integration by parts that

$$\begin{aligned} \|\nabla(u_0^\delta - u_0)\|_{L^2} &\leq C\|P(\rho_0^\delta) - P(\rho_0)\|_{L^2} + C\|\sqrt{\rho_0^\delta} - \sqrt{\rho_0}\|_{L^3} + C\|g^\delta - g\|_{L^2} \\ &\rightarrow 0, \quad \text{as } \delta \rightarrow 0, \end{aligned} \quad (5.19)$$

due to (5.1) and (5.16). Moreover, Lemma 2.3 combined with (5.18) yields that

$$\begin{aligned} \|\nabla^2(u_0^\delta - u_0)\|_{L^2} &\leq C\|\nabla P(\rho_0^\delta) - \nabla P(\rho_0)\|_{L^2} + C\|\sqrt{\rho_0^\delta} - \sqrt{\rho_0}\|_{L^\infty} + C\|g^\delta - g\|_{L^2} \\ &\rightarrow 0, \quad \text{as } \delta \rightarrow 0, \end{aligned} \quad (5.20)$$

owing to (5.1) and (5.16).

For the problems (1.1)–(1.4) with the initial data $(\rho_0^\delta, u_0^\delta)$ satisfying (5.1) and (5.16)–(5.17), Lemma 2.1 shows that there exists a classical solution (ρ^δ, u^δ) on $\Omega \times [0, T_0]$. Moreover, we deduce from (3.2) and Lemmas 4.1–4.3 that the sequence (ρ^δ, u^δ) converges weakly, up to the extraction of subsequences, to some limit (ρ, u) satisfying (1.8), (1.12) and (1.16). Moreover, standard arguments yield that (ρ, u) in fact is a classical solution to the problems (1.1)–(1.4). The proof of Theorem 1.4 is completed. \square

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