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Non-commutative Rényi entropic uncertainty principles

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Abstract In this paper, we calculate the norm of the string Fourier transform on subfactor planar algebras and characterize the extremizers of the inequalities for parameters $0 < p, q \leqslant \infty$. Furthermore, we establish Rényi entropic uncertainty principles for subfactor planar algebras.

Keywords Rényi entropy, uncertainty principles, Fourier transform, subfactors

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1 Introduction

A fundamental result in quantum mechanics is Heisenberg's uncertainty principle for position and momentum. By using the Shannon entropy of the measurement, the Hirschman-Beckner uncertainty principle was established [3, 12]. The Rényi entropy introduced by Rényi [29] generalized the Shannon entropy. In 2006, Bialynicki-Birula [4] showed Rényi entropic uncertainty principles for position and momentum and also for a pair of complementary observables in N-level systems. The Rényi entropy has been used for quantum entanglement [5, 9], quantum communication protocols [8, 28], quantum correlation [21], and quantum measurement [2], etc. The Rényi entropy has applications in biology, linguistics, economics, and computer sciences as well. The max-entropy, the min-entropy and the collision entropy are important in quantum mechanics and they can be considered as special limits of the Rényi entropy.

In 1936, Murray and von Neumann [24] introduced von Neumann algebras and factors to investigate the connections between mathematics and quantum mechanics. A subfactor is an inclusion of factors $\mathcal{N} \subset \mathcal{M}$ and its index δ^2 describes the relative size of the two factors. Jones [17] gave a surprising classification of the indices of subfactors. The index of a subfactor generalizes the order of a group, but it could be a non-integer which has been considered as a quantum dimension in various ways. Subfactor theory turns out to be a natural framework to study quantum symmetries appeared in statistical physics, conformal field theory and topological quantum field theory (see [6]).

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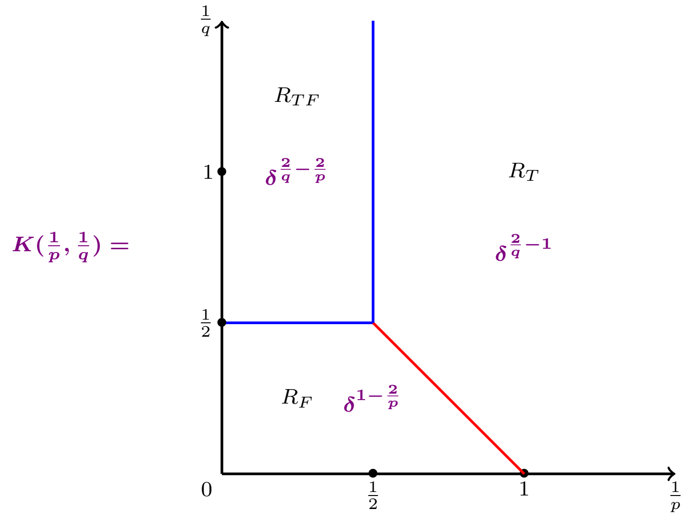

Figure 1 (Color online) The norm of the SFT

In [14], Jiang et al. proved various uncertain principles for subfactors in terms of Jones' planar algebras [18], including the Donoho-Stark uncertainty principle for the max-entropy, the Hirschman-Beckner uncertainty principle for the von Neumann entropy, and Hardy's uncertainty principle.

In the noncommutative case, the Rényi entropy of order p is defined by

$$h_p(x) = \frac{p}{1-p}\log\|x\|_p, \quad p \in (0,1)\cup(1,\infty),$$

where x is an operator in a von Neumann algebra with a trace tr and

$$\|x\|_p = (\mathrm{tr}_2(|x|^p))^{1/p}, \quad p \in (0,\infty).$$

When $p \geqslant 1$, $\|x\|_p$ is called the p-norm of x. It is natural to ask whether Rényi entropic uncertainty principles hold for subfactor planar algebras. In this paper, we answer this question positively.

To establish Rényi entropic uncertainty principles, we calculate the norm of the string Fourier transform $\mathfrak{F}_s$ (SFT) on subfactor planar algebras. We divide the first quadrant into three regions R_T, R_F and R_{TF} as illustrated in Figure 1, and let K be a function on $[0,\infty)^2$ given by

$$K\left(\frac{1}{p},\frac{1}{q}\right) = \begin{cases} \delta^{1-2/p} & \text{for } \left(\frac{1}{p},\frac{1}{q}\right) \in R_F, \\ \delta^{2/q-1} & \text{for } \left(\frac{1}{p},\frac{1}{q}\right) \in R_T, \\ \delta^{2/q-2/p} & \text{for } \left(\frac{1}{p},\frac{1}{q}\right) \in R_{TF}. \end{cases} \tag{1.1}$$

Theorem 1.1 (See Propositions 3.2, 3.5 and 3.10 and Theorem 3.13). *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra. Let $0 < p,q \leqslant \infty$ and $x \in \mathscr{P}_{2,\pm}$. Then*

$$K\left(\frac{1}{p},\frac{1}{q}\right)^{-1}\|x\|_p \leqslant \|\mathfrak{F}_s(x)\|_q \leqslant K\left(\frac{1}{p},\frac{1}{q}\right)\|x\|_p. \tag{1.2}$$

Theorem 1.2 (Rényi entropic uncertainty principles: Proposition 4.1). *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra. For any $x \in \mathscr{P}_{2,\pm}$ with $\|x\|_2 = 1$, $0 < p,q < \infty$, we have*

$$\left(\frac{1}{p}-\frac{1}{2}\right)h_{p/2}(|x|^2) + \left(\frac{1}{2}-\frac{1}{q}\right)h_{q/2}(|\mathfrak{F}_s(x)|^2) \geqslant -\log K\left(\frac{1}{p},\frac{1}{q}\right).$$

We also prove a second Rényi entropic uncertainty principles.

Theorem 1.3 (Rényi entropic uncertainty principles: Theorem 4.5). *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra. Let $x \in \mathscr{P}_{2,\pm}$ be such that $\|x\|_2 = 1$. Then for any $1/p + 1/q \geqslant 1$, we have*

$$h_{p/2}(|x|^2) + h_{q/2}(|\mathfrak{F}_s(x)|^2) \geqslant \left(-1 + \frac{2}{2-p} + \frac{2}{2-q} \right) \log \delta^2.$$

The Donoho-Stark uncertainty principle and the Hirschman-Beckner uncertainty principle can be recovered as limits of the second Rényi entropic uncertainty principles (see Corollary 4.7).

We characterize the extremizers of the inequality (1.2) for the three regions, the four critical lines and the two critical points illustrated in Figure 1 (see Table 1 for the nine characterizations and Figure 2).

Table 1 Characterization of the extremizers

Regions	Extremizers (up to scale)
$\frac{1}{p}+\frac{1}{q}>1, \frac{1}{p}>\frac{1}{2}$	Trace-one projections
$\frac{1}{p}+\frac{1}{q}=1, \frac{1}{2}<\frac{1}{p}<1$	Bishifts of biprojections
$\frac{1}{p}=1, \frac{1}{q}=0$	Extremal elements
$\frac{1}{p}=\frac{1}{2}, \frac{1}{q}=\frac{1}{2}$	$\mathscr{P}_{2,\pm}$
$\frac{1}{p}+\frac{1}{q}<1, 0<\frac{1}{q}<\frac{1}{2}$	Fourier transform of trace-one projections
$\frac{1}{q}=0, 0\leqslant\frac{1}{p}<1$	Extremal unitary elements
$\frac{1}{q}=\frac{1}{2}, 0\leqslant\frac{1}{p}<\frac{1}{2}$	Unitary elements
$\frac{1}{q}>\frac{1}{2}, \frac{1}{p}=\frac{1}{2}$	Fourier transform of unitary elements
$\frac{1}{q}>\frac{1}{2}, \frac{1}{p}<\frac{1}{2}$	Biunitary elements if exist

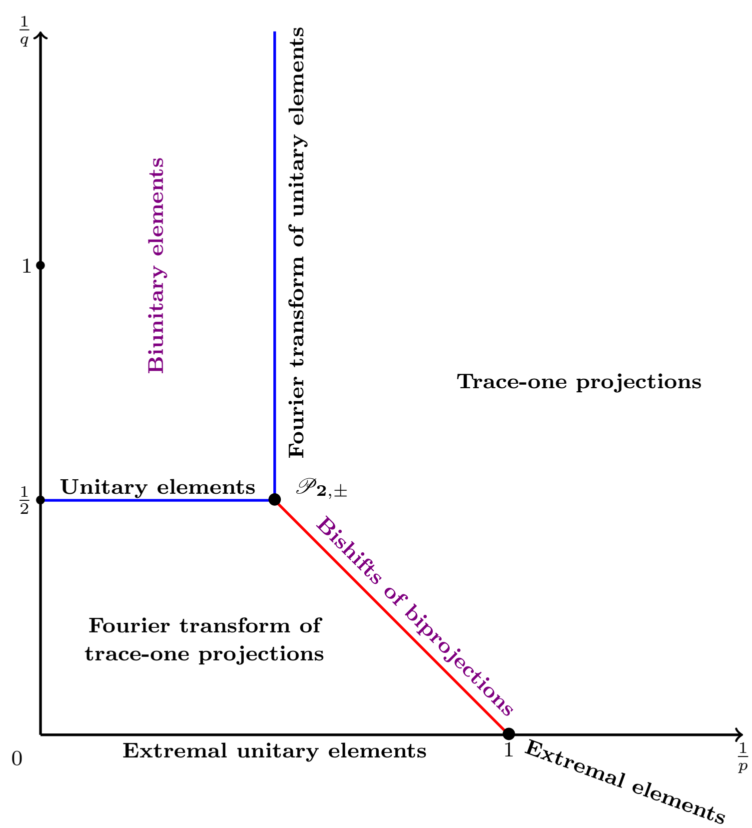

Figure 2 (Color online) Extremizers

For the special case $1/p + 1/q = 1$, we recover the Hausdorff-Young inequality and the characterization of the extremizers as bi-shifts of biprojections in [14].

If $\mathscr{P}_\bullet$ is the planar algebra of a group G crossed product subfactor, then the functions on G are given by the 2-box space $\mathscr{P}_{2,+}$ and the representations of G are characterized by the dual space $\mathscr{P}_{2,-}$. The string Fourier transform $\mathfrak{F}_s$ coincides with the classical Fourier transform. In this way, we recover the results of Gilbert and Rzeszotnik [7] on the norm of the Fourier transform on finite abelian groups for $1 \leqslant p, q \leqslant \infty$. The notions in Table 1 generalize time basis, frequency basis, wave packets, biunimodular functions etc in [7]. We could not find in the published literature the full results summarized in Table 1 for the case of finite non-abelian groups. For a general subfactor, both $\mathscr{P}_{2,+}$ and $\mathscr{P}_{2,-}$ could be highly non-commutative. All finite Kac algebras can be realized by the 2-box spaces $\mathscr{P}_{2,\pm}$ of planar algebras [19]. The first Rényi entropic uncertainty principles for locally compact quantum groups [20] was studied in [15], and the Donoho-Stark uncertainty principles and Hirschman-Beckner uncertainty principles were also obtained.

The rest of this paper is organized as follows. In Section 2, we recall some results in [14, 16] on the Fourier analysis for subfactor planar algebras. In Section 3, we calculate the norm of the Fourier transform on subfactor planar algebras and find all extremizers. In Section 4, we prove Rényi entropic uncertainty principles for subfactor planar algebras.

2 Preliminaries

We refer the reader to [18] for the definition of subfactor planar algebras and keep the notations in [16]. Suppose $\mathscr{P}_\bullet = \{\mathscr{P}_{n,\pm}\}_{n\geqslant 0}$ is a subfactor planar algebra. Denote by δ the square root of the Jones index. The n-box space $\mathscr{P}_{n,\pm}$ is a finite dimensional C^*-algebra. Denote by $\mathcal{Z}(\mathscr{P}_{n,\pm})$ the center of the C^*-algebra $\mathscr{P}_{n,\pm}$. Let tr_n be the (un-normalized) Markov trace on $\mathscr{P}_{n,\pm}$. Denote by e_1 the Jones projection in $\mathscr{P}_{2,\pm}$. The convolution (or coproduct) of $x, y \in \mathscr{P}_{2,\pm}$ is denoted by $x * y$. The string Fourier transform (SFT) $\mathfrak{F}_s$ from $\mathscr{P}_{2,\pm}$ onto $\mathscr{P}_{2,\mp}$ is the clockwise 1-click rotation. The notation SFT was introduced in [13] to distinguish from the quantum Fourier transform appeared in quantum information. The algebraic formulation of the SFT goes back to the work of Ocneanu [25]. Its analytic properties were studied in [22].

For any x in $\mathscr{P}_{2,\pm}$, we denote by $\mathcal{R}(x)$ the range projection of x, $\mathcal{S}(x)$ the trace of $\mathcal{R}(x)$, $H(|x|^2)$ the von Neumann entropy of $|x|$, namely,

$$H(|x|) = -\mathrm{tr}_2(|x| \log |x|).$$

A projection B in $\mathscr{P}_{2,\pm}$ is a biprojection if $\mathfrak{F}_s(B)$ is a multiple of a projection. A projection x is a left shift of a biprojection B if $\mathrm{tr}_2(B) = \mathrm{tr}_2(x)$ and $x * B = \frac{\mathrm{tr}_2(B)}{\delta} x$. A projection x is a right shift of a biprojection B if $\mathrm{tr}_2(B) = \mathrm{tr}_2(x)$ and $B * x = \frac{\mathrm{tr}_2(B)}{\delta} x$. In [14], it is shown that a left shift of a biprojection is a right shift of a biprojection, where the two biprojections may be different. A projection x in $\mathscr{P}_{2,\pm}$ is a trace-one projection if $\mathrm{tr}_2(x) = 1$. By the results in [26, Proposition 1.9], we have that a trace-one projection is a central minimal projection in $\mathscr{P}_{2,\pm}$. Moreover, trace-one projections are left shifts of the Jones projection e_1.

For a biprojection in $\mathscr{P}_{2,\pm}$, we denote by $\tilde{B}$ the range projection of $\mathfrak{F}_s(B)$. A nonzero element x in $\mathscr{P}_{2,\pm}$ is a bi-shift of a biprojection B if there exists a right shift B_g of the biprojection B and a right shift $\tilde{B}_h$ of the biprojection $\tilde{B}$ and an element y in $\mathscr{P}_{2,\pm}$ such that $x = \mathfrak{F}_s(\tilde{B}_h) * (yB_g)$. By the results in [14], there are actually eight forms of a bishift of a biprojection. A unitary element $u \in \mathscr{P}_{2,\pm}$ is biunitary if $\mathfrak{F}_s(u)$ is a unitary. Biunitary elements generalize biunimodular functions for finite abelian groups.

In Fourier analysis, the Hausdorff-Young inequality for locally compact abelian groups was studied by Hardy and Littlewood [10], Hewitt and Hirschman [11], Babenko [1], Beckner [3] and Russo [30], etc. Their results completely characterize the extremizers of the Hausdorff-Young inequality. In [14], Jiang et al. proved Plancherel's formula $\|\mathfrak{F}_s(x)\|_2 = \|x\|_2$ and the Hausdorff-Young inequality for subfactor planar algebras,

$$\|\mathfrak{F}_s(x)\|_q \leqslant \delta^{1-2/p}\|x\|_p, \quad x \in \mathscr{P}_{2,\pm}, \quad \frac{1}{p}+\frac{1}{q}=1, \quad 1 \leqslant p \leqslant 2.$$

An element x in $\mathscr{P}_{2,\pm}$ is extremal if $\|\mathfrak{F}_s(x)\|_\infty = \delta^{-1}\|x\|_1$. When $\mathscr{P}_\bullet$ is a group subfactor planar algebra arising from a finite abelian group, there is an explicit expression for extremal elements (see, for examples, [7]). In general, we have the following characterization:

Proposition 2.1 (See [14, Corollary 6.12 and Theorem 6.13]). *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra and $w \in \mathscr{P}_{2,\pm}$. If $\mathfrak{F}_s^{-1}(w)$ is extremal, then wQ is a bishift of a biprojection, where Q is the spectral projection of $|w|$ with the spectrum $\|w\|_\infty$.*

Proposition 2.2 (See [14, Main Theorems 1 and 2]). *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra. Then for any $x \in \mathscr{P}_{2,\pm}$, we have $\mathcal{S}(\mathfrak{F}_s(x))\mathcal{S}(x) \geqslant \delta^2$. Moreover, $\mathcal{S}(\mathfrak{F}_s(x))\mathcal{S}(x) = \delta^2$ if and only if x is a bishift of a biprojection.*

Jiang et al. [16] completely characterized the extremizers of the Hausdorff-Young inequality.

Proposition 2.3 (See [16, Theorem 1.4]). *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra. Let x be nonzero in $\mathscr{P}_{2,\pm}$. Then the following are equivalent*:

(1) $\|\mathfrak{F}_s(x)\|_{\frac{p}{p-1}} = \delta^{1-2/p}\|x\|_p$ *for some* $1 < p < 2$;

(2) $\|\mathfrak{F}_s(x)\|_{\frac{p}{p-1}} = \delta^{1-2/p}\|x\|_p$ *for all* $1 \leqslant p \leqslant 2$;

(3) *x is a bi-shift of a biprojection.*

The 2-box space $\mathscr{P}_{2,\pm}$ is a direct sum of matrix algebras. The following proposition is a consequence of Hölder's inequality on matrix algebras.

Proposition 2.4 (Hölder's inequality). *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra. Then for any $x, y \in \mathscr{P}_{2,\pm}$, we have*

$$\|xy\|_r \leqslant \|x\|_p\|y\|_q, \quad \frac{1}{r}+1 = \frac{1}{p}+\frac{1}{q}, \quad 0 < r, p, q \leqslant \infty.$$

(1) *If $r = 1$, $1 < p < \infty$, then we have $\|xy\|_1 = \|x\|_p\|y\|_q$ if and only if $\frac{|x|^p}{\|x\|_p^p} = \frac{|y^*|^q}{\|y\|_q^q}$.*

(2) *If $r = 1$, $p = \infty$, then $\|xy\|_1 = \|x\|_\infty\|y\|_1$ if and only if the spectral projection of $|x|$ corresponding to $\|x\|_\infty$ contains the projection $\mathcal{R}(y)$.*

Definition 2.5. Suppose $\mathscr{P}_\bullet$ is a subfactor planar algebra. For any $0 < p, q \leqslant \infty$, the norm $C_{p,q}$ of $\mathfrak{F}_s$ on $\mathscr{P}_{2,\pm}$ is defined to be

$$C_{p,q} = \sup_{\|x\|_p=1} \|\mathfrak{F}_s(x)\|_q.$$

Proposition 2.3 shows that $C_{p,q} = \delta^{1-2/p}$, when $1/p + 1/q = 1$ and $1 \leqslant p \leqslant 2$. We compute $C_{p,q}$ for $0 < p, q \leqslant \infty$ in Section 3. We refer the readers to [14, 16, 23, 27] for other interesting inequalities on subfactor planar algebras and on non-commutative L^p spaces. For example, Young's inequality has been established for subfactor planar algebras in [14]:

$$\|x * y\|_r \leqslant \delta^{-1}\|x\|_p\|y\|_q, \quad \frac{1}{r}+1 = \frac{1}{p}+\frac{1}{q}, \quad 1 \leqslant p, q, r \leqslant \infty.$$

It would be interesting to compute $C_{p,q,r} := \sup_{\|x\|_p=1, \|y\|_q=1} \|x * y\|_r$, for general parameters p, q and r.

3 The norm of the Fourier transform

In this section, we calculate the norm $C_{p,q}$ of the SFT. We will deal with three different cases corresponding to the three regions in Figure 1. Precisely,

$$R_F := \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \in [0,\infty]^2 : \frac{1}{p}+\frac{1}{q} \leqslant 1, \frac{1}{q} \leqslant \frac{1}{2} \right\},$$

$$R_T := \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \in [0,\infty]^2 : \frac{1}{p} + \frac{1}{q} \geqslant 1, \frac{1}{p} \geqslant \frac{1}{2} \right\},$$
$$R_{TF} := \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \in [0,\infty]^2 : \frac{1}{p} \leqslant \frac{1}{2}, \frac{1}{q} \geqslant \frac{1}{2} \right\}.$$

Remark 3.1. In the finite abelian group case [7], the regions R_F, R_T and R_{TF} correspond to the frequency basis, the time basis, and the time-frequency basis, respectively.

Let K be a function on $[0,\infty)^2$ defined in Equation (1.1). Then $\log K$ is an affine function in each of the three regions.

Proposition 3.2. *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra. If $(1/p, 1/q) \in R_F$, then $C_{p,q} = \delta^{1-2/p}$. Moreover, the following statements are equivalent*:

(1) $\|\mathfrak{F}_s(x)\|_q = \delta^{1-2/p}\|x\|_p$ *for some* $p > 0, q > 0$ *with* $1/p + 1/q < 1$, $0 < 1/q < 1/2$;

(2) $\|\mathfrak{F}_s(x)\|_q = \delta^{1-2/p}\|x\|_p$ *for all* $p > 0, q > 0$ *with* $1/p + 1/q \leqslant 1$, $1/q \leqslant 1/2$;

(3) $\mathfrak{F}_s(x)$ *is a multiple of a trace-one projection.*

Proof. Let q' be a real number such that $1/q + 1/q' = 1$. Then $1/p < 1 - 1/q = 1/q'$. By Propositions 2.3 and 2.4,

$$\|\mathfrak{F}_s(x)\|_q \leqslant \delta^{1-2/q'}\|x\|_{q'} \leqslant \delta^{1-2/q'}\|x\|_p\|1\|_{\frac{pq'}{p-q'}}$$
$$= \delta^{1-2/q'}\|x\|_p\delta^{2(1/q'-1/p)} = \delta^{1-2/p}\|x\|_p. \tag{3.1}$$

(1)⇒(3). Since Inequality (3.1) becomes an equality, we have

$$\|\mathfrak{F}_s(x)\|_q = \delta^{1-2/q'}\|x\|_{q'}, \tag{3.2}$$
$$\|x\|_{q'} = \|x\|_p\|1\|_{\frac{pq'}{p-q'}}. \tag{3.3}$$

Note that $2 < q < \infty$. Applying Proposition 2.3 to Equation (3.2), we see that x is a bishift of a biprojection. Applying Proposition 2.4 to Equation (3.3), we see that $|x|$ is a multiple of 1. By Proposition 2.2, we have $\mathcal{S}(\mathfrak{F}_s(x)) = \frac{\delta^2}{\mathcal{S}(x)} = 1$ and $\mathcal{R}(\mathfrak{F}_s(x))$ is a trace-one projection. Hence $\mathfrak{F}_s(x)$ is a multiple of a trace-one projection.

(3)⇒(2). Suppose $\mathfrak{F}_s(x)$ is a trace-one projection. Then $\|\mathfrak{F}_s(x)\|_q = 1$ and $1 \leqslant p \leqslant 2$, x is a bishift of a biprojection as $\mathfrak{F}_s(x)$, so by Proposition 2.3(2) $\|x\|_p = \delta^{2/p-1}$. Hence $\|\mathfrak{F}_s(x)\|_q = \delta^{1-2/p}\|x\|_p$. This also indicates that $C_{p,q} = \delta^{1-2/p}$.

(2)⇒(1). It is obvious. □

Remark 3.3. In the proof of Proposition 3.2, when the inequality becomes equality and $q = \infty$, we have

$$\|\mathfrak{F}_s(x)\|_\infty = \delta^{-1}\|x\|_1, \quad \|x\|_1 = \|x\|_p\|1\|_{\frac{p}{p-1}}, \quad p \neq 1.$$

We obtain that x is a multiple of an extremal unitary element. Note that if $\mathscr{P}_\bullet$ is a group subfactor planar algebra raising from a finite abelian group, then x is a multiple of a character.

Remark 3.4. For the finite abelian group case, the extremizers form a basis which is a frequency basis. But for the noncommutative case, the extremal unitaries do not form a basis in general.

Proposition 3.5. *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra and $(1/p, 1/q) \in R_{TF}$. Then $C_{p,q} \leqslant \delta^{2/q-2/p}$. Moreover, if there exists a biunitary in $\mathscr{P}_{2,\pm}$, the following statements are equivalent*:

(1) $\|\mathfrak{F}_s(x)\|_q = \delta^{2/q-2/p}\|x\|_p$ *for some* $p > 0, q > 0$ *with* $1/p < 1/2$, $1/q > 1/2$;

(2) $\|\mathfrak{F}_s(x)\|_q = \delta^{2/q-2/p}\|x\|_p$ *for all* $p > 0, q > 0$ *with* $1/p \leqslant 1/2$, $1/q \geqslant 1/2$;

(3) x *is a multiple of a biunitary.*

If there is a biunitary in $\mathscr{P}_{2,\pm}$, we have $C_{p,q} = \delta^{2/q-2/p}$.

Proof. For any $1/q \geqslant 1/2$ and $1/p \leqslant 1/2$, we have

$$\|\mathfrak{F}_s(x)\|_q \leqslant \|\mathfrak{F}_s(x)\|_2\|1\|_{\frac{2q}{2-q}} = \delta^{2/q-1}\|x\|_2$$

$$\leqslant \delta^{2/q-1}\|x\|_p\|1\|_{\frac{2p}{p-2}} = \delta^{2/q-2/p}\|x\|_p. \tag{3.4}$$

(1)⇒(3). Since Inequality (3.4) becomes equality, we have $\|\mathfrak{F}_s(x)\|_q = \|\mathfrak{F}_s(x)\|_2\|1\|_{\frac{2q}{2-q}}$ and $\|x\|_2 = \|x\|_p\|1\|_{\frac{2p}{p-2}}$. Therefore x is a multiple of a biunitary.

(3)⇒(2). Suppose x is a biunitary in $\mathscr{P}_{2,\pm}$. Then $\|x\|_p = \delta^{2/p}$ and $\|\mathfrak{F}_s(x)\|_q = \delta^{2/q}$. Hence $\|\mathfrak{F}_s(x)\|_q = \delta^{2/q-2/p}\|x\|_p$. This indicates that $C_{p,q} = \delta^{2/q-2/p}$ if there exists a biunitary in $\mathscr{P}_{2,\pm}$.

(2)⇒(1). It is obvious. □

Remark 3.6. By [7] if a function generates a time-frequency basis the it is unimodular (i.e., biunitary), but the converse is false. Now for the finite abelian group case, by [7, Theorem 4.7] there always exists a function which generates a time-frequency basis. In general, there might not exist a biunitary element.

Remark 3.7. Suppose $\mathscr{P}_\bullet = \mathscr{P}^{\mathbb{Z}_n}$ is the group subfactor planar algebra arising from the group $\mathbb{Z}_n$. It is shown in [7, Theorem 4.5] that $u \in \mathscr{P}_{2,\pm}$ generates a time-frequency basis if and only if

$$u(k) = \exp\left(\frac{2\pi \mathrm{i}}{n}(\lambda k^2 + \mu k)\right), \quad k \in \mathbb{Z}_n, \quad n \text{ odd},$$
$$u(k) = \exp\left(\frac{2\pi \mathrm{i}}{n}\left(\frac{\lambda}{2}k^2 + \mu k\right)\right), \quad k \in \mathbb{Z}_n, \quad n \text{ even},$$

where $\lambda, \mu \in \mathbb{Z}_n$ and λ relatively prime to n.

Remark 3.8. Suppose $TL(\delta)$ is the Temperley-Lieb subfactor planar algebra. Then $x \in TL(\delta)$ is a biunitary element if and only if

$$x = 1 - e_1 + \left(1 - \frac{\delta^2}{2} \pm \mathrm{i}\frac{\delta\sqrt{4-\delta^2}}{2}\right)e_1, \quad \delta \leqslant 2.$$

Lemma 3.9. *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra. Let $x \in \mathscr{P}_{2,\pm}$. Then for $0 < p \leqslant 1$, we have*

$$\delta^{2-2/p}\|x\|_p \leqslant \|x\|_1 \leqslant \|x\|_p;$$

for $1 \leqslant p \leqslant \infty$, we have

$$\|x\|_p \leqslant \|x\|_1 \leqslant \delta^{2-2/p}\|x\|_p.$$

Moreover, $\|x\|_p = \|x\|_1$, $p \neq 1$ if and only if x is a multiple of a trace-one projection.

Proof. It is enough to prove for the case $x = |x|$. Suppose that $|x| = \sum_k \lambda_k f_k \neq 0$, where $\{f_k\}_k$ is an orthogonal family of projections such that $\sum_k f_k = 1$ and $\lambda_k \geqslant 0$. Since $\mathrm{tr}_2(f_k) \geqslant 1$, we obtain the desired inequalities. □

Proposition 3.10. *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra and $(1/p, 1/q) \in R_T$. Then $C_{p,q} = \delta^{2/q-1}$. Moreover, the following statements are equivalent*:

(1) $\|\mathfrak{F}_s(x)\|_q = \delta^{2/q-1}\|x\|_p$ *for some $p > 0, q > 0$ with $1/p + 1/q > 1$, $1/p > 1/2$;*

(2) $\|\mathfrak{F}_s(x)\|_q = \delta^{2/q-1}\|x\|_p$ *for any $p > 0, q > 0$ with $1/p + 1/q \geqslant 1$, $1/p \geqslant 1/2$;*

(3) *x is a multiple of a trace-one projection.*

Proof. If $1/p \leqslant 1$, letting p' be such that $1/p + 1/p' = 1$, then we have $1/q \geqslant 1 - 1/p = 1/p'$ and

$$\|\mathfrak{F}_s(x)\|_q \leqslant \|\mathfrak{F}_s(x)\|_{p'}\|1\|_{\frac{p'q}{p'-q}} \leqslant \delta^{2/q-2/p'}\delta^{2/p'-1}\|x\|_p = \delta^{2/q-1}\|x\|_p. \tag{3.5}$$

If $1/p > 1$, $1/q \leqslant 1/2$, letting q' be such that $1/q + 1/q' = 1$, then by Lemma 3.9 we have

$$\|\mathfrak{F}_s(x)\|_q \leqslant \delta^{2/q-1}\|x\|_{q'} \leqslant \delta^{2/q-1}\|x\|_1 \leqslant \delta^{2/q-1}\|x\|_p. \tag{3.6}$$

If $1/p > 1$, $1/q > 1/2$, then by Lemma 3.9 we have

$$\|\mathfrak{F}_s(x)\|_q \leqslant \|\mathfrak{F}_s(x)\|_2\|1\|_{\frac{2q}{2-q}} = \delta^{2/q-1}\|x\|_2$$

$$\leqslant \delta^{2/q-1}\|x\|_1 \leqslant \delta^{2/q-1}\|x\|_p. \tag{3.7}$$

When Inequality (3.5) becomes equality, we have

$$\|\mathfrak{F}_s(x)\|_q = \|\mathfrak{F}_s(x)\|_{p'}\|1\|_{\frac{p'q}{p'-q}}, \tag{3.8}$$

$$\|\mathfrak{F}_s(x)\|_{p'} = \delta^{2/p'-1}\|x\|_p. \tag{3.9}$$

When $p \neq 1$, by Proposition 2.3 and Equation (3.9), we see that x is a bishift of a biprojection. By Equation (3.8) and Proposition 2.4, we have $|\mathfrak{F}_s(x)|$ is a multiple of 1. Then x is a left shift of the Jones projection e_1 from the argument in Proposition 3.2. When $p = 1$, we see that x is extremal and $\mathfrak{F}_s(x)$ is a multiple of a unitary element. By Proposition 2.1, we see that $\mathfrak{F}_s(x)$ is a bishift of a biprojection. Hence x is a multiple of a left shift of Jones' projection, i.e., x is a multiple of a trace-one projection.

When (3.6) becomes the equality, we have

$$\|\mathfrak{F}_s(x)\|_q = \delta^{2/q-1}\|x\|_{q'}, \quad \|x\|_1 = \|x\|_p.$$

Then by Lemma 3.9, we obtain that x is a multiple of a trace-one projection.

When (3.7) becomes the equality, we have

$$\|\mathfrak{F}_s(x)\|_q = \|\mathfrak{F}_s(x)\|_2\|1\|_{\frac{2q}{2-q}}, \quad \|x\|_1 = \|x\|_p = \|x\|_2.$$

Then by Lemma 3.9, we obtain that x is a multiple of a trace-one projection. □

Remark 3.11. For the finite abelian group case, the extremizers form a basis which is a time basis. But for the noncommutative case, the extremizers do not form a basis in general.

Proposition 3.12. *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra. Then*

(1) $\|\mathfrak{F}_s(x)\|_2 = \delta^{1-2/p}\|x\|_p$, $1/p < 1/2$ *if and only if* x *is a multiple of a unitary element*;

(2) $\|\mathfrak{F}_s(x)\|_q = \delta^{2/q-1}\|x\|_2$, $1/q > 1/2$ *if and only if* $\mathfrak{F}_s(x)$ *is a multiple of a unitary element.*

Proof. Since

$$\|\mathfrak{F}_s(x)\|_2 = \|x\|_2 \leqslant \|x\|_p\|1\|_{\frac{2p}{p-2}} = \delta^{1-2/p}\|x\|_p,$$

we see that $\|\mathfrak{F}_s(x)\|_2 = \delta^{1-2/p}\|x\|_p$, $1/p < 1/2$ if and only if x is a multiple of a unitary element.

Similarly, we have that $\|\mathfrak{F}_s(x)\|_q = \delta^{2/q-1}\|x\|_2$, $1/q > 1/2$ if and only if $\mathfrak{F}_s(x)$ is a multiple of a unitary element. □

Theorem 3.13. *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra. Then for any $0 < p, q \leqslant \infty$ we have $K(1/q, 1/p)^{-1}\|x\|_p \leqslant \|\mathfrak{F}_s(x)\|_q \leqslant K(1/p, 1/q)\|x\|_p$.*

Proof. Noting that

$$\|x\|_p = \|\mathfrak{F}_s^{-1}\mathfrak{F}_s(x)\|_p \leqslant K\left(\frac{1}{q}, \frac{1}{p}\right)\|\mathfrak{F}_s(x)\|_q,$$

we obtain $K(1/q, 1/p)^{-1}\|x\|_p \leqslant \|\mathfrak{F}_s(x)\|_q \leqslant K(1/p, 1/q)\|x\|_p$. □

Now the extremizers of the Fourier transform can be summarized as shown in Table 1. One can study the Tsallis entropy [31, 32] by using this inequality.

4 Rényi entropy uncertainty principles

In this section, we will show Rényi entropic uncertainty principles for subfactor planar algebras. First, we present the definition of the Rényi entropy for subfactor planar algebras $\mathscr{P}_\bullet$. For $p \in (0,1) \cup (1,\infty)$, we define the Rényi entropy of order p of x in $\mathscr{P}_{2,\pm}$ by

$$h_p(x) = \frac{p}{1-p}\log\|x\|_p.$$

Proposition 4.1 (The Rényi entropic uncertainty principle). *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra. Then for any nonzero $x \in \mathscr{P}_{2,\pm}$, we have*

$$\left(\frac{1}{p}-\frac{1}{2}\right)h_{p/2}(|x|^2)+\left(\frac{1}{2}-\frac{1}{q}\right)h_{q/2}(|\mathfrak{F}_s(x)|^2) \geqslant -\log K\left(\frac{1}{p},\frac{1}{q}\right).$$

Proof. For any $(1/p, 1/q) \in R_F$, we have $\|\mathfrak{F}_s(x)\|_q \leqslant \delta^{1-2/p}\|x\|_p$, i.e.,

$$\log\|\mathfrak{F}_s(x)\|_q \leqslant \left(1-\frac{2}{p}\right)\log\delta + \log\|x\|_p$$

and

$$\begin{aligned}
&\left(\frac{1}{p}-\frac{1}{2}\right)h_{p/2}(|x|^2)+\left(\frac{1}{2}-\frac{1}{q}\right)h_{q/2}(|\mathfrak{F}_s(x)|^2)\\
&=\left(\frac{1}{p}-\frac{1}{2}\right)\frac{\frac{p}{2}}{1-\frac{p}{2}}\log\||x|^2\|_{p/2}+\left(\frac{1}{2}-\frac{1}{q}\right)\frac{\frac{q}{2}}{1-\frac{q}{2}}\log\||\mathfrak{F}_s(x)|^2\|_{q/2}\\
&=\log\|x\|_p-\log\|\mathfrak{F}_s(x)\|_q\\
&\geqslant -\left(1-\frac{2}{p}\right)\log\delta.
\end{aligned}$$

The rest of the proposition can be obtained similarly. □

Remark 4.2. The minimizers of Rényi entropic uncertainty principles in Proposition 4.1 are the same as the extremizers for the inequalities in Theorem 3.13 given in Table 1.

Lemma 4.3. *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra. Let $x \in \mathscr{P}_{2,\pm}$ be such that $\|x\| \leqslant 1$. Then*

(1) $h_p(x) - \frac{1}{1-p}\log\delta^2$ *is a decreasing function with respect to p for $p \in (0,1)\cup(1,\infty)$;*
(2) $\lim_{p\to1}(h_p(x) - \frac{1}{1-p}\log\|x\|_1) = -\frac{\mathrm{tr}_2(|x|\log|x|)}{\|x\|_1}$;
(3) $\lim_{p\to0} h_p(x) = \log\mathcal{S}(x)$.

Proof. Note that

$$\frac{d}{dp}\left(h_p(x)-\frac{1}{1-p}\log\delta^2\right)=\frac{1}{(1-p)^2}\log\frac{\mathrm{tr}_2(|x|^p)}{\delta^2}+\frac{1}{1-p}\frac{\mathrm{tr}_2(|x|^p\log|x|)}{\mathrm{tr}_2(|x|^p)}.$$

By Jensen's inequality,

$$\begin{aligned}
\frac{d}{dp}\left(h_p(x)-\frac{1}{1-p}\log\delta^2\right)&=\frac{\delta^2}{p-1}\frac{\frac{\mathrm{tr}_2(|x|^p)}{\delta^2}\log\frac{\mathrm{tr}_2(|x|^p)}{\delta^2}-(p-1)\frac{\mathrm{tr}_2(|x|^p\log|x|)}{\delta^2}}{(p-1)\mathrm{tr}_2(|x|^p)}\\
&\leqslant\frac{1}{p-1}\frac{\mathrm{tr}_2(|x|^p\log|x|)}{(p-1)\mathrm{tr}_2(|x|^p)}.
\end{aligned}$$

When $\|x\| \leqslant 1$, we have $\log|x| \leqslant 0$ and $\frac{d}{dp}(h_p(x) - \frac{1}{1-p}\log\delta^2) < 0$. Hence $h_p(x) - \frac{1}{1-p}\log\delta^2$ is a decreasing function.

For the first limit, we have

$$\begin{aligned}
\lim_{p\to1}\left(h_p(x)-\frac{1}{1-p}\log\|x\|_1\right)&=\lim_{p\to1}\frac{\log\mathrm{tr}_2(|x|^p)-\log\mathrm{tr}_2(|x|)}{1-p}\\
&=-\frac{d}{dp}\log\mathrm{tr}_2(|x|^p)\Big|_{p=1}\\
&=-\frac{\mathrm{tr}_2(|x|^p\log|x|)}{\mathrm{tr}_2(|x|^p)}\Big|_{p=1}\\
&=-\frac{\mathrm{tr}_2(|x|\log|x|)}{\|x\|_1}.
\end{aligned}$$

For the second limit, we have

$$\lim_{p\to 0} h_p(x) = \lim_{p\to 0} \frac{1}{1-p}\log \mathrm{tr}_2(|x|^p) = \log \mathrm{tr}_2(\mathcal{R}(|x|)) = \log \mathcal{S}(x).$$

This completes the proof. □

Remark 4.4. Let x be nonzero in $\mathscr{P}_{2,\pm}$. When $p=0$, the entropy $h_0(x) = \lim_{p\to 0} h_p(x) = \log \mathcal{S}(x)$ is called the Hartley entropy or the max-entropy of x.

When $p=1$,

$$H(|x|) = \|x\|_1 \lim_{p\to 1}\left(h_p(x) - \frac{1}{1-p}\log \|x\|_1\right)$$

is the von Neumann entropy of x.

When $p=2$, the entropy $h_2(x)$ is called the Collision entropy of x.

When $p=\infty$, the entropy $h_\infty(x) = \lim_{p\to\infty} h_p(x) = -\log\|x\|_\infty$ is called the min-entropy of x.

Theorem 4.5 (The Rényi entropic uncertainty principle). *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra. Let $x \in \mathscr{P}_{2,\pm}$ be such that $\|x\|_2 = 1$. Then for any $1/p + 1/q \geqslant 1$, we have*

$$h_{p/2}(|x|^2) + h_{q/2}(|\mathfrak{F}_s(x)|^2) \geqslant \left(-1 + \frac{2}{2-p} + \frac{2}{2-q}\right)\log \delta^2.$$

Proof. Since $\|\mathfrak{F}_s(x)\|_{q_0} \leqslant \delta^{2/q_0 - 1}\|x\|_{q_0'}$ for any $q_0 > 2$, where $1/q_0 + 1/q_0' = 1$, we have

$$\log\|\mathfrak{F}_s(x)\|_{q_0} \leqslant \left(\frac{2}{q_0} - 1\right)\log\delta + \log\|x\|_{q_0'}.$$

Hence

$$\begin{aligned} h_{q_0'/2}(|x|^2) + h_{q_0/2}(|\mathfrak{F}_s(x)|^2) &= \frac{2q_0'}{2-q_0'}\log\|x\|_{q_0'} + \frac{2q_0}{2-q_0}\log\|\mathfrak{F}_s(x)\|_{q_0} \\ &= -\frac{2q_0}{2-q_0}\log\|x\|_{q_0'} + \frac{2q_0}{2-q_0}\log\|\mathfrak{F}_s(x)\|_{q_0} \\ &\geqslant -\frac{2q_0}{q_0-2}\left(1 - \frac{2}{q_0}\right)\log\delta = 2\log\delta. \end{aligned}$$

For each (p,q) with $1/p + 1/q \geqslant 1$, $0 < p < 2$, $0 < q$, we can find (q_0', q_0) as above such that $1/q_0 \leqslant 1/q$ and $1/q_0' \leqslant 1/p$. Since $\|x\|_2 = \|\mathfrak{F}_s(x)\|_2 = 1$, we have $\|x\| \leqslant 1$ and $\|\mathfrak{F}_s(x)\| \leqslant 1$. Then by decreasing in Lemma 4.3, we have

$$\begin{aligned} h_{p/2}(|x|^2) + h_{q/2}(|\mathfrak{F}_s(x)|^2) &\geqslant h_{q_0'/2}(|x|^2) + \frac{1}{1-\frac{p}{2}}\log\delta^2 - \frac{1}{1-\frac{q_0'}{2}}\log\delta^2 \\ &\quad + h_{q_0/2}(|\mathfrak{F}_s(x)|^2) + \frac{1}{1-\frac{q}{2}}\log\delta^2 - \frac{1}{1-\frac{q_0}{2}}\log\delta^2 \\ &\geqslant \log\delta^2 + \frac{2(p-q_0')}{(2-p)(2-q_0')}\log\delta^2 + \frac{2(q-q_0)}{(2-q)(2-q_0)}\log\delta^2. \end{aligned}$$

When $1/q < 1/2$, we can take $q_0 = q$, and then

$$\begin{aligned} h_{p/2}(|x|^2) + h_{q/2}(|\mathfrak{F}_s(x)|^2) &\geqslant 2\log\delta + \frac{2}{2-p}\log\delta^2 - \frac{2(q-1)}{q-2}\log\delta^2 \\ &= -2\log\delta + \left(\frac{2}{2-p} + \frac{2}{2-q}\right)\log\delta^2. \end{aligned}$$

When $1/p \leqslant 1$, we can take $q_0' = p$, and then $1/q \geqslant 1 - 1/p = 1/q_0$,

$$h_{p/2}(|x|^2) + h_{q/2}(|\mathfrak{F}_s(x)|^2) \geqslant 2\log\delta + \frac{2}{2-q}\log\delta^2 - \frac{2(p-1)}{p-2}\log\delta^2$$

$$= -2\log\delta + \left(\frac{2}{2-p} + \frac{2}{2-q}\right)\log\delta^2.$$

When $1/p > 1$, $1/q > 1/2$, we can take

$$q_0 = \frac{\frac{2}{p} + \frac{2}{q} - 3}{\frac{1}{p} - 1}, \quad q_0' = \frac{\frac{2}{p} + \frac{2}{q} - 3}{\frac{1}{p} + \frac{2}{q} - 2},$$

and we have

$$\begin{aligned} &h_{p/2}(|x|^2) + h_{q/2}(|\mathfrak{F}_s(x)|^2) \\ &\geqslant 2\log\delta + \frac{2}{2-p}\log\delta^2 + \frac{2}{2-q}\log\delta^2 - \frac{\frac{2}{p} + \frac{4}{q} - 4}{\frac{2}{q} - 1}\log\delta^2 + 2\frac{\frac{1}{p} - 1}{\frac{-2}{q} + 1}\log\delta^2 \\ &= -2\log\delta + \frac{2}{2-p}\log\delta^2 + \frac{2}{2-q}\log\delta^2. \end{aligned}$$

This completes the proof. □

Remark 4.6. Theorem 4.5 is simplified by the referee by correcting a computation mistake in the third part of the proof.

By using the second Rényi entropic uncertainty principles for subfactor planar algebras, we obtain the Donoho-Stark uncertainty principles and Hirschman-Beckner uncertainty principles again.

Corollary 4.7. *Suppose $\mathscr{P}_\bullet$ is an irreducible subfactor planar algebra. Let $x \in \mathscr{P}_{2,\pm}$ be such that $\|x\|_2 = 1$. Then $\mathcal{S}(x)\mathcal{S}(\mathfrak{F}_s(x)) \geqslant \delta^2$ and $H(|x|^2) + H(|\mathfrak{F}_s(x)|^2) \geqslant 2\log\delta$.*

Proof. Since $\|x\|_2 = 1$, we have $\|x\| \leqslant 1$. By Theorem 4.5, for p small enough, we have

$$h_{p/2}(|x|^2) + h_{p/2}(|\mathfrak{F}_s(x)|^2) \geqslant \left(-1 + \frac{2}{2-p} + \frac{2}{2-p}\right)\log\delta^2.$$

By Lemma 4.3, taking $p \to 0$, we have

$$\log\mathcal{S}(x) + \log\mathcal{S}(\mathfrak{F}_s(x)) \geqslant \log\delta^2,$$

i.e., $\mathcal{S}(x)\mathcal{S}(\mathfrak{F}_s(x)) \geqslant \delta^2$.

By Theorem 4.5, we have

$$h_{p/2}(|x|^2) + h_{p/2(p-1)}(|\mathfrak{F}_s(x)|^2) \geqslant 2\log\delta.$$

By Lemma 4.3, taking $p \to 2$, we have $H(|x|^2) + H(|\mathfrak{F}_s(x)|^2) \geqslant 2\log\delta$. □

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