SCIENCE CHINA
Mathematics

• ARTICLES •

December 2020 Vol. 63 No. 12: 2573–2594
https://doi.org/10.1007/s11425-019-1781-6

Compensated split-step balanced methods for nonlinear stiff SDEs with jump-diffusion and piecewise continuous arguments

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Received July 25, 2019; accepted September 17, 2020; published online September 29, 2020

Abstract This paper deals with numerical solutions of nonlinear stiff stochastic differential equations with jump-diffusion and piecewise continuous arguments. By combining compensated split-step methods and balanced methods, a class of compensated split-step balanced (CSSB) methods are suggested for solving the equations. Based on the one-sided Lipschitz condition and local Lipschitz condition, a strong convergence criterion of CSSB methods is derived. It is proved under some suitable conditions that the numerical solutions produced by CSSB methods can preserve the mean-square exponential stability of the corresponding analytical solutions. Several numerical examples are presented to illustrate the obtained theoretical results and the effectiveness of CSSB methods. Moreover, in order to show the computational advantage of CSSB methods, we also give a numerical comparison with the adapted split-step backward Euler methods with or without compensation and tamed explicit methods.

Keywords stiff stochastic differential equation, jump diffusion, piecewise continuous argument, compensated split-step balanced method, strong convergence, mean-square exponential stability

MSC(2010) 65C20, 60H35, 65L20

Citation: Xie Y, Zhang C J. Compensated split-step balanced methods for nonlinear stiff SDEs with jump-diffusion and piecewise continuous arguments. Sci China Math, 2020, 63: 2573–2594, https://doi.org/10.1007/s11425-019-1781-6

1 Introduction

Among stochastic differential equations (SDEs), there are a class of equations with jump-diffusion (JDSDEs). In the recent years, this class of equations have attracted an increasing interest due to their effectiveness in modeling some uncertainty problems in control science, biology, economics and other scientific and engineering fields (see [3, 6, 9, 10, 13, 18, 21, 35]). Nevertheless, it is difficult to obtain the explicit solutions of JDSDEs. Hence, developing various numerical methods to solve JDSDEs becomes an important topic (see [9, 10, 12–15, 17, 27, 28, 32, 39]). In the existing references, numerical methods for JDSDEs

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with delay have been concerned by some authors. For example, Higham and Kloeden [9] constructed the compensated split-step backward Euler (CSSBE) methods and studied their convergence and stability under the non-global Lipschitz condition, Wang et al. [28] proved that the semi-implicit Euler method is convergent with strong order $1/2$, Wang and Gan [32] presented the compensated stochastic θ-methods and pointed out that this kind of methods are mean-square A-stable when $1/2 \leqslant \theta \leqslant 1$, Jiang et al. [15] gave a convergence criterion of Taylor approximate solutions, and Li and Gan [17] derived the almost sure exponential stability conditions of the explicit and implicit Euler methods.

The above research for JDSDEs with delay devoted mainly to the case of constant delay. In fact, in many actual applications, there exist some more complicated cases of delay differential equations, where the delay argument changes with time. In [34], a detailed introduction to a class of delay differential equations with piecewise continuous arguments (PCAs) was given. For this class of equations, some numerical approaches have been presented (see [4, 7, 16, 30, 31, 33, 37, 38]). Recently, they have been extended to SDEs. For nonlinear SDEs with PCAs, the convergence criteria of the Euler-Maruyama method were derived under the different conditions in [23, 25, 40], split-step and compensated split-step θ-methods were constructed in [19, 20], and stochastic one-leg θ-methods and their convergence and stability were studied in [36].

Similar to deterministic ordinary differential equations, there are also the so-called stiff problems in stochastic ordinary and delay differential equations. In general, this type of problems cannot be solved by explicit methods due to the harsh requirement of the stability of explicit methods, which will lead to an unsuccessful computation or a large computational cost owing to that the computational stepsize is limited to very small. Hence, when solving a stochastic stiff problem, one usually considers some implicit methods with excellent stability, which has not any harsh restriction on the computational stepsize. In order to obtain the stochastic numerical methods with excellent stability, two techniques have been developed, namely, balanced technique (see [1, 8, 12, 13, 24, 27, 29]) and compensated split-step technique (see [9, 20, 26, 32]). The balanced technique provides a kind of balance between approximating stochastic terms in the numerical scheme and, in this way, one can find a numerical method with excellent stability suitable for the integration of stiff SDEs. On the other hand, the compensated split-step technique makes a significant improvement to the numerical stability by incorporating the compensated process into a split-step method. As we can see in the above references, both techniques have achieved a great deal of effect in numerically solving stiff SDEs. Motivated by these works, in the present paper, we will combine balanced technique and compensated split-step technique to derive a class of compensated split-step balanced methods to solve stiff SDEs with jump-diffusion and PCAs (JDPCAs).

The rest of this paper is organized as follows. In Section 2, some properties of analytical solutions are analyzed under the one-sided Lipschitz condition and local Lipschitz condition. In Section 3, the CSSB methods are suggested for solving stiff SDEs with JDPCAs and a strong convergence criterion of the methods is derived. In Section 4, it is proved under some suitable conditions that the numerical solutions produced by CSSB methods can preserve the mean-square exponential stability of the corresponding analytical solutions. Finally, in Section 5, several numerical examples of stiff SDEs with JDPCAs are given to illustrate the obtained theoretical results and the effectiveness of CSSB methods, where, in order to show the computational advantage of CSSB methods, a numerical comparison with the adapted split-step backward Euler methods with or without compensation and tamed explicit methods is also given.

2 The SDEs with JDPCAs

Let $\mathcal{L}^1([0,T];\mathbb{R}^d)$ be the family of all $\mathbb{R}^d$-valued measurable $\{\mathcal{F}_t\}$-adapted processes $f=\{f(t)\}_{0\leqslant t\leqslant T}$ with $\int_0^t |f(t)|dt<\infty$ with probability 1 (w.p.1.), where $|\cdot|$ denotes the norm induced by the standard inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^d$, and $\mathcal{L}^2([0,T];\mathbb{R}^{d\times m})$ the family of all $\mathbb{R}^{d\times m}$-matrix-value measurable $\{\mathcal{F}_t\}$-adapted processes $f=\{f(t)\}_{0\leqslant t\leqslant T}$ with $\int_0^t |f(t)|^2dt<\infty$ w.p.1. For a given matrix A, we define its trace norm $|A|=\sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$ and the operator norm $\|A\|=\sup\{|Ax|:|x|=1\}$.

Moreover, we assume that $W(t)=(W^1(t),W^2(t),\ldots,W^m(t))^{\mathrm{T}}$ is an m-dimensional Brownian motion, $N(t)$ is a scalar Poisson process with intensity $\lambda>0$, and they are all defined on the complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant 0},\mathrm{P})$ and independent with each other, where the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ is increasing and right-continuous and $\mathcal{F}_0$ contains all P-null sets. Consider the following problems of SDEs with JDPCAs:

$$\begin{cases} dx(t)=f(x(t^-),x(\lfloor t^-\rfloor))dt+g(x(t^-),x(\lfloor t^-\rfloor))dW(t)+h(x(t^-),x(\lfloor t^-\rfloor))dN(t), \quad t\in[0,T],\\ x(0)=x_0, \end{cases} \tag{2.1}$$

where $x(t^-)=\lim_{s\to t^-}x(s)$, $\lfloor\cdot\rfloor$ denotes the greatest-integer function, $f:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$, $g:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^{d\times m}$ and $h:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ are three given functions with $\{f(x(t^-),x(\lfloor t^-\rfloor))\}_{0\leqslant t\leqslant T}\in\mathcal{L}^1([0,T];\mathbb{R}^d)$, $\{h(x(t^-),x(\lfloor t^-\rfloor))\}_{0\leqslant t\leqslant T}\in\mathcal{L}^1([0,T];\mathbb{R}^d)$ and $\{g(x(t^-),x(\lfloor t^-\rfloor))\}_{0\leqslant t\leqslant T}\in\mathcal{L}^2([0,T];\mathbb{R}^{d\times m})$, and x_0 is a $\mathcal{F}_0$- measurable $\mathbb{R}^d$-value random variable with $\mathrm{E}|x_0|^p\leqslant N_p<\infty$ for all $p>0$.

Definition 2.1. An $\mathbb{R}^d$-valued stochastic process $\{x(t)\}_{0\leqslant t\leqslant T}$ is called a solution of (2.1) if, $x(t)$ is left-continuous and $\mathcal{F}_t$-adapted, $\{f(x(t^-),x(\lfloor t^-\rfloor))\}_{0\leqslant t\leqslant T}\in\mathcal{L}^1([0,T];\mathbb{R}^d)$, $\{h(x(t^-),x(\lfloor t^-\rfloor))\}_{0\leqslant t\leqslant T}\in\mathcal{L}^1([0,T];\mathbb{R}^d)$, $\{g(x(t^-),\,x(\lfloor t^-\rfloor))\}_{0\leqslant t\leqslant T}\in\mathcal{L}^2([0,T];\mathbb{R}^{d\times m})$ and (2.1) holds for all $t\in[0,T]$ w.p.1.

For the subsequent analysis, we introduce the following conditions:

$\mathcal{A}_1$ (The local Lipschitz condition). There are constants R, $L_R>0$ such that

$$\begin{aligned} &|f(x_1,y_1)-f(x_2,y_2)|^2\vee|g(x_1,y_1)-g(x_2,y_2)|^2\vee|h(x_1,y_1)-h(x_2,y_2)|^2\\ &\quad\leqslant L_R(|x_1-x_2|^2+|y_1-y_2|^2),\quad \forall\, x_1,x_2,y_1,y_2\in\mathbb{R}^d, \end{aligned} \tag{2.2}$$

where $|x_1|\vee|x_2|\vee|y_1|\vee|y_2|\leqslant R$.

$\mathcal{A}_2$. There exist constants γ_i $(i=1,2,\ldots,6)$ such that

$$\langle x_1-x_2,f(x_1,y)-f(x_2,y)\rangle\leqslant\gamma_1|x_1-x_2|^2,\quad\forall\, x_1,y_1,y\in\mathbb{R}^d, \tag{2.3}$$

$$|f(x,y_1)-f(x,y_2)|\leqslant\gamma_2|y_1-y_2|,\quad\forall\, x,y_1,y_2\in\mathbb{R}^d, \tag{2.4}$$

$$|g(x_1,y_1)-g(x_2,y_2)|^2\leqslant\gamma_3|x_1-x_2|^2+\gamma_4|y_1-y_2|^2,\quad\forall\, x_1,x_2,y_1,y_2\in\mathbb{R}^d, \tag{2.5}$$

$$|h(x_1,y_1)-h(x_2,y_2)|^2\leqslant\gamma_5|x_1-x_2|^2+\gamma_6|y_1-y_2|^2,\quad\forall\, x_1,x_2,y_1,y_2\in\mathbb{R}^d. \tag{2.6}$$

It follows from the condition $\mathcal{A}_2$ that

$$\langle x,f(x,y)\rangle\leqslant\left(\gamma_1+\frac{1}{2}\gamma_2+\frac{1}{2}\right)|x|^2+\frac{1}{2}\gamma_2|y|^2+\frac{1}{2}|f(0,0)|^2, \tag{2.7}$$

$$|g(x,y)|^2\leqslant 2\gamma_3|x|^2+2\gamma_4|y|^2+2|g(0,0)|^2, \tag{2.8}$$

$$|h(x,y)|^2\leqslant 2\gamma_5|x|^2+2\gamma_6|y|^2+2|h(0,0)|^2. \tag{2.9}$$

Moreover, it is well known that the compensated Poisson process $\widetilde{N}(t):=N(t)-\lambda t$ is a martingale with the following properties:

$$\mathrm{E}(\widetilde{N}(t+s)-\widetilde{N}(t))=0,\quad \mathrm{E}|\widetilde{N}(t+s)-\widetilde{N}(t)|^2=\lambda s.$$

Let $f_\lambda(x,y):=f(x,y)+\lambda h(x,y)$. Then (2.1) can be written in an equivalent form

$$\begin{cases} dx(t)=f_\lambda(x(t^-),x(\lfloor t^-\rfloor))dt+g(x(t^-),x(\lfloor t^-\rfloor))dW(t)+h(x(t^-),x(\lfloor t^-\rfloor))d\widetilde{N}(t),\\ x(0)=x_0. \end{cases} \tag{2.10}$$

Under the condition $\mathcal{A}_1$, for $x_1,x_2,y_1,y_2\in\mathbb{R}^d:\ |x_1|\vee|x_2|\vee|y_1|\vee|y_2|\leqslant R$, we have

$$|f_\lambda(x_1,y_1)-f_\lambda(x_2,y_2)|^2\leqslant 2L_R(1+\lambda^2)(|x_1-x_2|^2+|y_1-y_2|^2). \tag{2.11}$$

Under the condition $\mathcal{A}_2$, for all $x,y,x_1,x_2\in\mathbb{R}^d$, we have

$$\langle x_1-x_2,f_\lambda(x_1,y)-f_\lambda(x_2,y)\rangle\leqslant(\gamma_1+\lambda\sqrt{\gamma_5})|x_1-x_2|^2 \tag{2.12}$$

and

$$\langle x, f_\lambda(x,y)\rangle \leqslant \frac{1}{2}(\beta_1+1+\sqrt{2}\lambda)|x|^2+\frac{1}{2}\beta_4|y|^2+\frac{1}{2}|f(0,0)|^2+\frac{\sqrt{2}}{2}\lambda|h(0,0)|^2, \tag{2.13}$$

where $\beta_1=2\gamma_1+\gamma_2+2\sqrt{2}\lambda\sqrt{\gamma_5}+\sqrt{2}\lambda\sqrt{\gamma_6}$ and $\beta_4=\gamma_2+\sqrt{2}\lambda\sqrt{\gamma_6}$.

Under the above settings, a uniform moment estimate for the solution $x(t)$ of (2.1) can be stated as follows.

Theorem 2.2. *Assume that the conditions $\mathcal{A}_1$ and $\mathcal{A}_2$ hold. Then,* (2.1) *has a unique global solution $x(t)$ and, for all $p\geqslant 2$, there exists a constant H_0 depending on p and T such that*

$$\mathrm{E}\Big[\sup_{0\leqslant t\leqslant T}|x(t)|^p\Big]\leqslant H_0(1+\mathrm{E}|x_0|^p). \tag{2.14}$$

Proof. Since f, g and h satisfy local Lipschitz conditions, by Theorem 3.2 in Mao [21, Chapter 5], we know that there is a unique maximal local solution $x(t)$ on $\{(t,\omega)\in\mathbb{R}_+\times\Omega: 0\leqslant t<\rho_e\}$, where ρ_e is the explosion time related to the parameter ω. Define stopping times $\rho_R=\inf\{\rho_e>t\geqslant 0: |x(t)|\geqslant R\}$ and $\rho_\infty=\lim_{R\to\infty}\rho_R$. It is obvious that ρ_R increases as $R\to\infty$, and $\rho_\infty\leqslant\rho_e$ a.s. Hence, in the following, we need only to prove that $\rho_\infty=\infty$ a.s. Applying the Itô formula (see [5]) to (2.1) yields that

$$\begin{aligned}|x(t\wedge\rho_R)|^2 &= |x_0|^2+2\int_0^{t\wedge\rho_R}\langle x(s), f(x(s),x(\lfloor s\rfloor))\rangle+|g(x(s),x(\lfloor s\rfloor))|^2ds\\&\quad+2\int_0^{t\wedge\rho_R}x(s^-)^{\mathrm{T}}g(x(s^-),x(\lfloor s^-\rfloor))dW(s)\\&\quad+\lambda\int_0^{t\wedge\rho_R}2\langle x(s),h(x(s),x(\lfloor s\rfloor))\rangle+|h(x(s),x(\lfloor s\rfloor))|^2ds\\&\quad+\int_0^{t\wedge\rho_R}2\langle x(s^-),h(x(s),x(\lfloor s^-\rfloor))\rangle+|h(x(s^-),x(\lfloor s^-\rfloor))|^2d\widetilde{N}(s).\end{aligned} \tag{2.15}$$

By (2.7)–(2.9), we further obtain

$$\begin{aligned}|x(t\wedge\rho_R)|^2 &\leqslant |x_0|^2+H\int_0^{t\wedge\rho_R}(1+|x(s)|^2+|x(\lfloor s\rfloor)|^2)ds+2\int_0^{t\wedge\rho_R}x(s^-)^{\mathrm{T}}g(x(s),x(\lfloor s^-\rfloor))dW(s)\\&\quad+\int_0^{t\wedge\rho_R}2\langle x(s^-),h(x(s^-),x(\lfloor s^-\rfloor))\rangle+|h(x(s^-),x(\lfloor s^-\rfloor))|^2d\widetilde{N}(s),\end{aligned} \tag{2.16}$$

where H is a generic constant depending on p ($\geqslant 2$) and T. Taking the power $\frac{p}{2}$ on both sides of (2.16) and using the common inequality $(a+b+c+d)^p\leqslant 4^{p-1}(a^p+b^p+c^p+d^p)$ $(a,b,c,d\in\mathbb{R})$ give that

$$\begin{aligned}&|x(t\wedge\rho_R)|^p\\&\quad\leqslant 4^{\frac{p}{2}-1}\bigg(|x_0|^p+H\int_0^{t\wedge\rho_R}(1+|x(s)|^p+|x(\lfloor s\rfloor)|^p)ds\bigg)+2^{\frac{p}{2}}\bigg|\int_0^{t\wedge\rho_R}x(s^-)^{\mathrm{T}}g(x(s),x(\lfloor s^-\rfloor))dW(s)\bigg|^{\frac{p}{2}}\\&\qquad+\bigg|\int_0^{t\wedge\rho_R}2\langle x(s^-),h(x(s^-),x(\lfloor s^-\rfloor))\rangle+|h(x(s^-),x(\lfloor s^-\rfloor))|^2d\widetilde{N}(s)\bigg|^{\frac{p}{2}}.\end{aligned} \tag{2.17}$$

It follows from the condition $\mathcal{A}_2$, the Burkholder-Davis-Gundy inequality (see [21]), Young's inequality $2ab\leqslant a^2+b^2$ $(a,b\in\mathbb{R})$ and Hölder's inequality that

$$\begin{aligned}&\mathrm{E}\bigg[\sup_{0\leqslant t\leqslant T}\bigg|\int_0^{t\wedge\rho_R}x(r^-)^{\mathrm{T}}g(x(r^-),x(\lfloor r^-\rfloor))dW(r)\bigg|^{\frac{p}{2}}\bigg]\\&\quad\leqslant H\mathrm{E}\bigg[\bigg[\int_0^{T\wedge\rho_R}|x(t)|^2|g(x(t),x(\lfloor t\rfloor))|^2dt\bigg]^{p/4}\bigg]\\&\quad\leqslant H\mathrm{E}\bigg[\bigg[\sup_{0\leqslant t\leqslant T}|x(t\wedge\rho_R)|^2\int_0^{T\wedge\rho_R}|g(x(t),x(\lfloor t\rfloor))|^2dt\bigg]^{p/4}\bigg]\end{aligned}$$

$$\begin{aligned}&\leqslant \frac{H}{2}\mathrm{E}\Big[\sup_{0\leqslant t\leqslant T}|x(t\wedge\rho_R)|^p\Big]+\frac{H}{2}T^{\frac{p}{2}-1}\mathrm{E}\int_0^{T\wedge\rho_R}|g(x(t),x(\lfloor t\rfloor))|^p dt\\&\leqslant H\mathrm{E}\Big[\sup_{0\leqslant t\leqslant T}|x(t\wedge\rho_R)|^p\Big]+H\mathrm{E}\int_0^T\Big(1+\sup_{0\leqslant r\leqslant t}|x(r\wedge\rho_R)|^p\Big)dt.\end{aligned}\tag{2.18}$$

With a similar proof for (2.18), we can deduce that

$$\begin{aligned}&\mathrm{E}\bigg[\sup_{0\leqslant t\leqslant T}\bigg|\int_0^{t\wedge\rho_R}\langle x(r^-),h(x(r^-),x(\lfloor r^-\rfloor))\rangle d\widetilde{N}(r)\bigg|^{\frac{p}{2}}\bigg]\\&\quad\leqslant H\mathrm{E}\Big[\sup_{0\leqslant t\leqslant T}|x(t\wedge\rho_R)|^p\Big]+H\int_0^T\Big(1+\sup_{0\leqslant r\leqslant t}|x(r\wedge\rho_R)|^p\Big)dt.\end{aligned}\tag{2.19}$$

Also, in terms of the Burkholder-Davis-Gundy inequality (see [21]) and the condition $\mathcal{A}_2$, we have

$$\begin{aligned}&\mathrm{E}\bigg[\sup_{0\leqslant t\leqslant T}\bigg|\int_0^{t\wedge\rho_R}|h(x(r^-),x(\lfloor r^-\rfloor))|^2 d\widetilde{N}(r)\bigg|^{\frac{p}{2}}\bigg]\\&\leqslant\lambda H\mathrm{E}\bigg[\int_0^{T\wedge\rho_R}|h(x(t),x(\lfloor t\rfloor))|^4dt\bigg]^{p/4}\leqslant H\mathrm{E}\bigg[\int_0^{T\wedge\rho_R}(1+|x(t)|^2+|x(\lfloor t\rfloor)|^2)^2dt\bigg]^{p/4}\\&\leqslant H\int_0^T\Big[1+\mathrm{E}\Big[\sup_{0\leqslant r\leqslant t}|x(r\wedge\rho_R)|^p\Big]\Big]dt.\end{aligned}\tag{2.20}$$

Inserting (2.18)–(2.20) into (2.17) yields that

$$\mathrm{E}\Big[\sup_{0\leqslant t\leqslant T}|x(t\wedge\rho_R)|^p\Big]\leqslant H\bigg[(1+\mathrm{E}[|x_0|^p])+\int_0^T\mathrm{E}\Big[\sup_{0\leqslant r\leqslant t}|x(r\wedge\rho_R)|^p\Big]dt\bigg].\tag{2.21}$$

An application of the Gronwall inequality (see [21]) derives that

$$\mathrm{E}\Big[\sup_{0\leqslant t\leqslant T}|x(t\wedge\rho_R)|^p\Big]\leqslant H(1+\mathrm{E}[|x_0|^p])\exp(HT).\tag{2.22}$$

This shows that

$$\mathrm{P}\{\rho_R\leqslant T\}R^p\leqslant H\,(1+\mathrm{E}[|x_0|^p])\exp(HT),$$

and thus $\lim_{R\to\infty}\mathrm{P}\{\rho_R\leqslant T\}=0$. Now $T\geqslant 0$ is arbitrary, and it holds that $\rho_\infty=\infty$ a.s. Hence, the existence and uniqueness of the global solution is proven. Moreover, the inequality (2.14) can be derived by letting $R\to\infty$ in (2.22). This completes the proof. □

3 The CSSB methods and their convergence

3.1 A class of CSSB methods

In the recent years, for solving stochastic stiff problems, some balanced methods have been proposed (see, e.g., [1,8,12,13,24,27,29]), where theoretical analyses were performed on the basis of the globally Lipschitz condition and the linear growth condition. However, as for stiff problems, their Lipschitz constants are very large in general. Hence, in [9], Higham and Kloeden considered the stochastic stiff problems with the one-sided Lipschitz condition on the drift item and, for this class of problems, presented the CSSBE methods and their strong convergence and stability results. Motivated by this research, in the present section, we construct a class of CSSB methods for the stochastic stiff delay problems (2.1) with JDPCAs and one-sided Lipschitz condition on the drift item.

Let $\Delta t=\frac{1}{Q}$ $(Q\in\mathbb{N})$, $t_n=n\Delta t$ $(n\in\mathbb{N}\cup\{0\})$ and

$$C_n=c_0(t_n,Y_n)\Delta t+\sum_{j=1}^m c_j(t_n,Y_n)|\Delta W_n^j|,$$

where Y_n is an approximation to $x(t_n)$, $\Delta W_n^j = W^j(t_{n+1}) - W^j(t_n)$ and $c_j(\cdot,\cdot)$ $(j = 0, 1, \ldots, m)$ denote a series of $d \times d$ matrix-valued functions on $[0, \infty] \times \mathbb{R}^d$, which are called *control functions* and required to satisfy the following condition (see [24]):

$\mathcal{A}_3$. There exists a constant $B > 0$ such that $|c_j(t, x)| \leqslant B$ $(j = 0, 1, \ldots, m)$ on $[0, \infty] \times \mathbb{R}^d$ and, for any real sequence of the form $\{\alpha_i : \alpha_0 \in [0, \Delta t],\ \alpha_i \geqslant 0,\ i = 0, 1, \ldots, m\}$ and all $(t, x) \in [0, \infty] \times \mathbb{R}^d$, the matrices

$$M(t, x) := \mathcal{I} + \alpha_0 c_0(t, x) + \sum_{j=1}^{m} \alpha_j c_j(t, x) \tag{3.1}$$

are invertible and satisfies $|M(t, x)^{-1}| \leqslant K < \infty$, in which $\mathcal{I}$ is the $d \times d$ identity matrix and K $(K > 0)$ is a given control constant.

Under the above settings, a class of CSSB methods for (2.10) can be suggested as follows:

$$\begin{cases} Y_n^* = Y_n + f_\lambda(Y_n^*, Y_{n-i_n})\Delta t, \\ Y_{n+1} = Y_n^* + g(Y_n^*, Y_{n-i_n})\Delta W_n + h(Y_n^*, Y_{n-i_n})\Delta\widetilde{N}_n + C_n(Y_n^* - Y_{n+1}), \end{cases} \tag{3.2}$$

where Y_n^* is the intermediate stage value, $Y_n \approx x(t_n)$, $\Delta W_n = W(t_{n+1}) - W(t_n)$, $\Delta\widetilde{N}_n = \widetilde{N}(t_{n+1}) - \widetilde{N}(t_n)$ and $i_n = \lfloor (t_n - \lfloor t_n \rfloor)/\Delta t \rfloor$. Write

$$Z_1(t) = \sum_{k=0}^{N} I_{[\Delta tk, \Delta t(k+1))} Y_k, \quad Z_2(t) = \sum_{k=0}^{N} I_{[\Delta tk, \Delta t(k+1))} Y_k^*, \quad Z_3(t) = \sum_{k=0}^{N} I_{[\Delta tk, \Delta t(k+1))} Y_{k-i_k}, \tag{3.3}$$

where $I_{[\Delta tk, \Delta t(k+1))}$ is the indicator function defined by

$$I_{[\Delta tk, \Delta t(k+1))} = \begin{cases} 1, & t \in [\Delta tk, \Delta t(k+1)), \\ 0, & \text{otherwise}. \end{cases}$$

By (3.2) and (3.3), a continuous time approximation scheme can be established as follows:

$$\begin{aligned} \hat{Y}(t) = Y_0 &+ \int_0^t f_\lambda(Z_2(s), Z_3(s))ds + \int_0^t (\mathcal{I} + C(\underline{s}, Z_1(s)))^{-1} g(Z_2(s), Z_3(s))dW(s) \\ &+ \int_0^t (\mathcal{I} + C(\underline{s}, Z_1(s)))^{-1} h(Z_2(s), Z_3(s))d\widetilde{N}(s), \end{aligned} \tag{3.4}$$

where $\underline{s} = \lfloor s/\Delta t \rfloor \Delta t$, $C(\underline{s}, Z_1(s)) = c_0(\underline{s}, Z_1(s))\Delta t + \sum_{j=1}^{m} c_j(\underline{s}, Z_1(s))|\Delta W^j_{\lfloor s/\Delta t \rfloor}|$, and it is easy to check that $\hat{Y}(t_n) = Y_n$ and $\hat{Y}(\lfloor t_n \rfloor) = Y_{n-i_n}$.

3.2 Convergence analysis of the CSSB methods

For studying the strong convergence of the methods, in the following, we first give some preparatory results.

Lemma 3.1. *Let the conditions $\mathcal{A}_2$ and $\mathcal{A}_3$ hold. Then, for each $p \geqslant 2$, there exist constants $H_1 = H_1(p, T)$ and $\Delta t_c \in (0, 1]$ such that*

$$\mathrm{E}\Big[\sup_{0 \leqslant n\Delta t \leqslant T} |Y_{n+1}|^{2p}\Big] \vee \mathrm{E}\Big[\sup_{0 \leqslant n\Delta t \leqslant T} |Y_n^*|^{2p}\Big] \vee \mathrm{E}\Big[\sup_{0 \leqslant t \leqslant T} |\hat{Y}(t)|^{2p}\Big] \leqslant H_1, \quad \Delta t \leqslant \Delta t_c. \tag{3.5}$$

Proof. It follows from (2.13) and the first equality of (3.2) that

$$\begin{aligned} |Y_n^*|^2 &= \langle Y_n^*, Y_n \rangle + \Delta t \langle Y_n^*, f_\lambda(Y_n^*, Y_{n-i_n}) \rangle \\ &\leqslant \frac{1}{2}|Y_n^*|^2 + \frac{1}{2}|Y_n|^2 + \frac{1}{2}(\beta_1 + 1 + \sqrt{2}\lambda)\Delta t |Y_n^*|^2 + \frac{1}{2}\Delta t \beta_4 |Y_{n-i_n}|^2 \\ &\quad + \frac{1}{2}\Delta t |f(0, 0)|^2 + \frac{\sqrt{2}}{2}\Delta t \lambda |h(0, 0)|^2, \quad \forall\, n > 0. \end{aligned} \tag{3.6}$$

Since anyone of the following two groups of conditions:

- $\beta_1+\sqrt{2}\lambda+1\leqslant 0$ and $\Delta t\leqslant \Delta t_c=1$;
- $\beta_1+\sqrt{2}\lambda+1>0$ and $0<\Delta t\leqslant \Delta t_c<1/(\beta_1+\sqrt{2}\lambda+1)$

implies that $1-(\beta_1+\sqrt{2}\lambda+1)\Delta t>0$, it holds by (3.6) that

$$|Y_n^*|^2\leqslant \frac{|Y_n|^2+\beta_4|Y_{n-i_n}|^2+\Delta t|f(0,0)|^2+\sqrt{2}\Delta t\lambda|h(0,0)|^2}{1-(\beta_1+1+\sqrt{2}\lambda)\Delta t},\quad 0<\Delta t\leqslant \Delta t_c. \tag{3.7}$$

Substituting (3.7) into the second equality of (3.2) yields that

$$\begin{aligned}|Y_{n+1}|^2\leqslant{}& |Y_n|^2+\frac{(\beta_1+\sqrt{2}\lambda+1)\Delta t}{1-(\beta_1+\sqrt{2}\lambda+1)\Delta t}|Y_n|^2\\ &+\frac{\beta_4\Delta t}{1-(\beta_1+\sqrt{2}\lambda+1)\Delta t}|Y_{n-i_n}|^2+\frac{\Delta t|f(0,0)|^2+\sqrt{2}\Delta t\lambda|h(0,0)|^2}{1-(\beta_1+\sqrt{2}\lambda+1)\Delta t}\\ &+K^2|g(Y_n^*,Y_{n-i_n})\Delta W_n|^2+K^2|h(Y_n^*,Y_{n-i_n})\Delta\widetilde{N}_n|^2\\ &+2\langle Y_n^*,(\mathcal{I}+C_n)^{-1}g(Y_n^*,Y_{n-i_n})\Delta W_n\rangle+2\langle Y_n^*,(\mathcal{I}+C_n)^{-1}h(Y_n^*,Y_{n-i_n})\Delta\widetilde{N}_n\rangle\\ &+2\langle(\mathcal{I}+C_n)^{-1}g(Y_n^*,Y_{n-i_n})\Delta W_n,(\mathcal{I}+C_n)^{-1}h(Y_n^*,Y_{n-i_n})\Delta\widetilde{N}_n\rangle.\end{aligned}\tag{3.8}$$

Let N be any given positive integer. Then by (3.8) we have

$$\begin{aligned}|Y_N|^2\leqslant{}& |Y_0|^2+\frac{(\beta_1+\sqrt{2}\lambda+1)\Delta t}{1-(\beta_1+\sqrt{2}\lambda+1)\Delta t}\sum_{j=0}^{N-1}|Y_j|^2\\ &+\frac{\beta_4\Delta t}{1-(\beta_1+\sqrt{2}\lambda+1)\Delta t}\sum_{j=0}^{N-1}|Y_{j-i_j}|^2+\frac{N\Delta t|f(0,0)|^2+N\sqrt{2}\Delta t\lambda|h(0,0)|^2}{1-(\beta_1+\sqrt{2}\lambda+1)\Delta t}\\ &+K^2\sum_{j=0}^{N-1}|g(Y_j^*,Y_{j-i_j})\Delta W_j|^2+2\sum_{j=0}^{N-1}\langle Y_j^*,(\mathcal{I}+C_j)^{-1}h(Y_j^*,Y_{j-i_j})\Delta\widetilde{N}_j\rangle\\ &+K^2\sum_{j=0}^{N-1}|h(Y_j^*,Y_{j-i_j})\Delta\widetilde{N}_j|^2+2\sum_{j=0}^{N-1}\langle Y_j^*,(\mathcal{I}+C_j)^{-1}g(Y_j^*,Y_{j-i_j})\Delta W_j\rangle\\ &+2\sum_{j=0}^{N-1}\langle(\mathcal{I}+C_j)^{-1}g(Y_j^*,Y_{j-i_j})\Delta W_n,(\mathcal{I}+C_j)^{-1}h(Y_j^*,Y_{j-i_j})\Delta\widetilde{N}_j\rangle.\end{aligned}\tag{3.9}$$

Taking the power p on both sides of (3.9), we further obtain that

$$\begin{aligned}|Y_N|^{2p}\leqslant{}& 9^{p-1}\Bigg[|Y_0|^{2p}+\bigg(\frac{(\beta_1+\sqrt{2}\lambda+1)\Delta t}{1-(\beta_1+\sqrt{2}\lambda+1)\Delta t}\bigg)^p\bigg(\sum_{j=0}^{N-1}|Y_j|^2\bigg)^p\\ &+\bigg(\frac{\beta_4\Delta t}{1-(\beta_1+\sqrt{2}\lambda+1)\Delta t}\bigg)^p\bigg(\sum_{j=0}^{N-1}|Y_{j-i_j}|^2\bigg)^p+\bigg(\frac{N\Delta t|f(0,0)|^2+N\sqrt{2}\Delta t\lambda|h(0,0)|^2}{1-(\beta_1+\sqrt{2}\lambda+1)\Delta t}\bigg)^p\\ &+K^{2p}N^{p-1}\sum_{j=0}^{N-1}|g(Y_j^*,Y_{j-i_j})\Delta W_j|^{2p}+2^p\bigg(\sum_{j=0}^{N-1}\langle Y_j^*,(\mathcal{I}+C_j)^{-1}h(Y_j^*,Y_{j-i_j})\Delta\widetilde{N}_j\rangle\bigg)^p\\ &+K^{2p}N^{p-1}\sum_{j=0}^{N-1}|h(Y_j^*,Y_{j-i_j})\Delta\widetilde{N}_j|^{2p}+2^p\bigg(\sum_{j=0}^{N-1}\langle Y_j^*,(\mathcal{I}+C_j)^{-1}g(Y_j^*,Y_{j-i_j})\Delta W_j\rangle\bigg)^p\\ &+2^P\bigg(\sum_{j=0}^{N-1}\langle(\mathcal{I}+C_j)^{-1}g(Y_j^*,Y_{j-i_j})\Delta W_n,(\mathcal{I}+C_j)^{-1}h(Y_j^*,Y_{j-i_j})\Delta\widetilde{N}_j\rangle\bigg)^p\Bigg].\end{aligned}\tag{3.10}$$

Next, we continue to estimate the various items of the right-hand side of (3.10). By the conditions $\mathcal{A}_2$–$\mathcal{A}_3$, (2.8) and (3.7), the following inequalities hold for any given positive integer $M \geqslant N$:

$$\begin{aligned}
&\mathrm{E}\bigg[\sup_{0\leqslant N\leqslant M}\sum_{j=0}^{N-1}|g(Y_j^*,Y_{j-i_j})\Delta W_j|^{2p}\bigg]\\
&= \mathrm{E}\sum_{j=0}^{M-1}|g(Y_j^*,Y_{j-i_j})\Delta W_j|^{2p}\\
&\leqslant 3^{p-1}m^p(\Delta t)^p\sum_{j=0}^{M-1}\mathrm{E}[(2\gamma_3)^p|Y_j^*|^{2p}+(2\gamma_4)^p|Y_{j-i_j}|^{2p}+2^p|g(0,0)|^{2p}]\\
&\leqslant 2^p3^{p-1}m^pM\mathrm{E}|g(0,0)|^{2p}(\Delta t)^p+(2\gamma_3)^p3^{p-1}m^p(\Delta t)^p\sum_{j=0}^{M-1}\mathrm{E}|Y_j^*|^{2p}\\
&\quad+(2\gamma_4)^p3^{p-1}m^p(\Delta t)^p\sum_{j=0}^{M-1}\mathrm{E}|Y_{j-i_j}|^{2p}\\
&\leqslant \widetilde{H_2}(\Delta t)^{p-1}\bigg(1+\Delta t\sum_{j=0}^{M-1}\mathrm{E}|Y_j^*|^{2p}+\Delta t\sum_{j=0}^{M-1}\mathrm{E}|Y_{j-i_j}|^{2p}\bigg),
\end{aligned}\tag{3.11}$$

where $\widetilde{H_2}$ denotes a positive constant independent of Δt. Similarly, there exists a constant $\widetilde{H_3} > 0$ independent of Δt such that

$$\begin{aligned}
&\mathrm{E}\bigg[\sup_{0\leqslant N\leqslant M}\sum_{j=0}^{N-1}|h(Y_j^*,Y_{j-i_j})\Delta\widetilde{N}_j|^{2p}\bigg]\\
&\quad\leqslant \widetilde{H_3}(\Delta t)^{p-1}\bigg(1+\Delta t\sum_{j=0}^{M-1}\mathrm{E}|Y_j^*|^{2p}+\Delta t\sum_{j=0}^{M-1}\mathrm{E}|Y_{j-i_j}|^{2p}\bigg).
\end{aligned}\tag{3.12}$$

Applying the Burkholder-Davis-Gundy inequality (see [21]), the conditions $\mathcal{A}_2$–$\mathcal{A}_3$ and (2.8) yields that

$$\begin{aligned}
&\mathrm{E}\bigg[\sup_{0\leqslant N\leqslant M}\bigg|\sum_{j=0}^{N-1}\langle Y_j^*,(\mathcal{I}+C_j)^{-1}g(Y_j^*,Y_{j-i_j})\Delta W_j\rangle\bigg|^p\bigg]\\
&\leqslant\bigg[\frac{p^{p+1}}{2(p-1)^{p-1}}\bigg]^{\frac{p}{2}}\mathrm{E}\bigg[\sum_{j=0}^{M-1}|Y_j^*|^2(\mathcal{I}+C_j)^{-2}|g(Y_j^*,Y_{j-i_j})|^2\Delta t\bigg]^{\frac{p}{2}}\\
&\leqslant\bigg[\frac{p^{p+1}}{2(p-1)^{p-1}}\bigg]^{\frac{p}{2}}(\Delta t)^{\frac{p}{2}}K^p(3M)^{\frac{p}{2}-1}\cdot\mathrm{E}\bigg[\sum_{j=0}^{M-1}|Y_j^*|^p((2\gamma_3)^{\frac{p}{2}}|Y_j^*|^p+(2\gamma_4)^{\frac{p}{2}}|Y_{j-i_j}|^p+2^{\frac{p}{2}}|g(0,0)|^p)\bigg]\\
&\leqslant\frac{1}{2}\bigg[\frac{p^{p+1}}{2(p-1)^{p-1}}\bigg]^{\frac{p}{2}}(\Delta t)^{\frac{p}{2}}K^p(3M)^{\frac{p}{2}-1}\\
&\quad\times\mathrm{E}\bigg[\sum_{j=0}^{M-1}(1+3(2\gamma_3)^p)|Y_j^*|^{2p}+3(2\gamma_4)^p|Y_{j-i_j}|^{2p}+32^p|g(0,0)|^{2p}\bigg]\\
&\leqslant\widetilde{H_4}\bigg(1+\Delta t\sum_{j=0}^{M-1}\mathrm{E}|Y_j^*|^{2p}+\Delta t\sum_{j=0}^{M-1}\mathrm{E}|Y_{j-i_j}|^{2p}\bigg),
\end{aligned}\tag{3.13}$$

where $\widetilde{H_4}$ denotes a positive constant independent of Δt. With the same arguments for (3.13), we also get

$$\mathrm{E}\bigg[\sup_{0\leqslant N\leqslant M}\bigg|\sum_{j=0}^{N-1}\langle Y_j^*,(\mathcal{I}+C_j)^{-1}h(Y_j^*,Y_{j-i_j})\Delta\widetilde{N}_j\rangle\bigg|^p\bigg]$$

$$\leqslant \widetilde{H_5}\bigg(1 + \Delta t\sum_{j=0}^{M-1}\mathrm{E}|Y_j^*|^{2p} + \Delta t\sum_{j=0}^{M-1}\mathrm{E}|Y_{j-i_j}|^{2p}\bigg), \tag{3.14}$$

where $\widetilde{H_5}$ denotes a positive constant independent of Δt. Moreover, in terms of the Cauchy-Schwarz inequality, Young's inequality $2ab \leqslant a^2 + b^2$ $(a, b \in \mathbb{R})$, the conditions $\mathcal{A}_2$–$\mathcal{A}_3$ and the estimations (3.11)–(3.12), the following inequalities hold:

$$\begin{aligned}
&\mathrm{E}\bigg[\sup_{0\leqslant N\leqslant M}\bigg|\sum_{j=0}^{N-1}\langle(\mathcal{I}+C_j)^{-1}g(Y_j^*, Y_{j-i_j})\Delta W_j, (\mathcal{I}+C_j)^{-1}h(Y_j^*, Y_{j-i_j})\Delta\widetilde{N}_j\rangle\bigg|^p\bigg]\\
&\quad\leqslant \mathrm{E}\bigg[\sup_{0\leqslant N\leqslant M}\bigg[\sum_{j=0}^{N-1}|(\mathcal{I}+C_j)^{-1}g(Y_J^*, Y_{j-i_j})\Delta W_j|^2\sum_{j=0}^{N-1}|(\mathcal{I}+C_j)^{-1}h(Y_j^*, Y_{j-i_j})\Delta\widetilde{N}_j|^2\bigg]^{\frac{p}{2}}\bigg]\\
&\quad\leqslant \frac{1}{2}\mathrm{E}\bigg[\sum_{j=0}^{M-1}|(\mathcal{I}+C_j)^{-1}g(Y_j^*, Y_{j-i_j})\Delta W_j|^2\bigg]^p + \frac{1}{2}\mathrm{E}\bigg[\sum_{j=0}^{M-1}|(\mathcal{I}+C_j)^{-1}h(Y_J^*, Y_{j-i_j})\Delta\widetilde{N}_j|^2\bigg]^p\\
&\quad\leqslant \frac{1}{2}M^{p-1}\mathrm{E}\bigg[\sum_{j=0}^{M-1}|(\mathcal{I}+C_j)^{-1}g(Y_j^*, Y_{j-i_j})\Delta W_j|^{2p}\bigg] + \frac{1}{2}M^{p-1}\mathrm{E}\bigg[\sum_{j=0}^{M-1}|(\mathcal{I}+C_j)^{-1}h(Y_J^*, Y_{j-i_j})\Delta\widetilde{N}_j|^{2p}\bigg]\\
&\quad\leqslant \widetilde{H_6}\bigg(1 + \Delta t\sum_{j=0}^{M-1}\mathrm{E}|Y_j^*|^{2p} + \Delta t\sum_{j=0}^{M-1}\mathrm{E}|Y_{j-i_j}|^{2p}\bigg),
\end{aligned}\tag{3.15}$$

where $\widetilde{H_6}$ denotes a positive constant independent of Δt. Substituting the inequalities (3.11)–(3.15) into (3.10) derives that

$$\mathrm{E}\Big[\sup_{0\leqslant N\leqslant M}|Y_N|^{2p}\Big] \leqslant \widetilde{H_7}\bigg(1 + \Delta t\sum_{j=0}^{M-1}\mathrm{E}\Big[\sup_{0\leqslant N\leqslant j}|Y_N|^{2p}\Big]\bigg), \tag{3.16}$$

where $\widetilde{H_7}$ denotes a positive constant independent of Δt. An application of the discrete-type Gronwall inequality (see [22]) to (3.16) shows that there is a constant $\widetilde{H}_1 > 0$ such that $\mathrm{E}[\sup_{n\Delta t\in[0,T]}|Y_n|^{2p}] \leqslant \widetilde{H}_1$. This, together with (3.7), implies that there is a constant $\widehat{H}_1 > 0$ such that

$$\mathrm{E}\Big[\sup_{n\Delta t\in[0,T]}|Y_n^*|^{2p}\Big] \leqslant \widehat{H}_1. \tag{3.17}$$

It follows from the continuous-time approximation solution (3.4) that

$$\begin{aligned}
\hat{Y}(t) &= Y_n + f_\lambda(Y_n^*, Y_{n-i_n})(t - n\Delta t) + (\mathcal{I}+C_n)^{-1}g(Y_n^*, Y_{n-i_n})(W(t) - W(t_n))\\
&\quad + (\mathcal{I}+C_n)^{-1}h(Y_n^*, Y_{n-i_n})(\widetilde{N}(t) - \widetilde{N}(t_n)).
\end{aligned}\tag{3.18}$$

Since by (3.2) $f_\lambda(Y_n^*, Y_{n-i_n}) = \frac{Y_n^* - Y_n}{\Delta t}$, (3.18) is equivalent to the following equality:

$$\begin{aligned}
\hat{Y}(t) &= (1-\alpha)Y_n + \alpha Y_n^* + (\mathcal{I}+C_n)^{-1}g(Y_n^*, Y_{n-i_n})(W(t) - W(t_n))\\
&\quad + (\mathcal{I}+C_n)^{-1}h(Y_n^*, Y_{n-i_n})(\widetilde{N}(t) - \widetilde{N}(t_n)), \quad \text{where } \alpha := \frac{t - n\Delta t}{\Delta t}.
\end{aligned}\tag{3.19}$$

This, as well as the condition $\mathcal{A}_3$, derives that

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T}|\hat{Y}(t)|^{2p}\\
&\quad\leqslant \sup_{0\leqslant n\Delta t\leqslant T}\sup_{0\leqslant s\leqslant \Delta t}\{|\hat{Y}(t_n+s)|^{2p} + K^{2p}|h(Y_n^*, Y_{n-i_n})(\widetilde{N}(t_n+s) - \widetilde{N}(t_n))|^{2p}\}\\
&\quad\leqslant \breve{H}_2\bigg[\sup_{0\leqslant n\Delta t\leqslant T}|Y_n|^{2p} + \sup_{0\leqslant n\Delta t\leqslant T}|Y_n^*|^{2p} + \sup_{0\leqslant s\leqslant \Delta t}\sum_{j=0}^{N}|g(Y_j^*, Y_{j-i_j})(W(t_j+s) - W(t_j))|^{2p}
\end{aligned}$$

$$+\sup_{0\leqslant s\leqslant\Delta t}\sum_{j=0}^{N}|h(Y_j^*,Y_{j-i_j})(\widetilde{N}(t_j+s)-\widetilde{N}(t_j))|^{2p}\bigg],\tag{3.20}$$

where $\breve{H}_2>0$ is a constant independent of Δt. Hence, by Doob's martingale inequality (see [21]), the condition $\mathcal{A}_2$ and (2.8), we obtain

$$\begin{aligned}\mathrm{E}\Big[\sup_{0\leqslant s\leqslant\Delta t}|g(Y_j^*,Y_{j-i_j})(W(t_j+s)-W(t_j))|^{2p}\Big]&\leqslant\left(\frac{2p}{2p-1}\right)^{2p}\mathrm{E}|g(Y_j^*,Y_{j-i_j})\Delta W_j|^{2p}\\&\leqslant\breve{H}_3\Delta t,\end{aligned}\tag{3.21}$$

where $\breve{H}_3>0$ is a constant independent of Δt. In a similar way, we can infer that

$$\mathrm{E}\Big[\sup_{0\leqslant s\leqslant\Delta t}|h(Y_j^*,Y_{j-i_j})(\widetilde{N}(t_j+s)-\widetilde{N}(t_j))|^{2p}\Big]\leqslant\breve{H}_4\Delta t,\tag{3.22}$$

where $\breve{H}_4>0$ is a constant independent of Δt. Substituting the inequalities (3.21) and (3.13) into (3.20) concludes that there exists a constant $\breve{H}_1>0$ such that $\mathrm{E}[\sup_{0\leqslant t\leqslant T}|\hat{Y}(t)|^{2p}]\leqslant\breve{H}_1$. This gives the desired result when setting $H_1=\widetilde{H}_1\vee\widehat{H}_1\vee\breve{H}_1$. □

Define the following stopping times:

$$\rho_R=\inf\{t\geqslant 0:|x(t)|\geqslant R\},\quad \tau_R=\inf\{t\geqslant 0:|\hat{Y}(t)|\geqslant R\text{ or }|Z_2(t)|\geqslant R\},\quad \sigma_R=\rho_R\wedge\tau_R.$$

Then we have the following lemma.

Lemma 3.2. *Let the condition $\mathcal{A}_1$ hold. Then there exists a constant $H_2>0$ such that*

$$\mathrm{E}[\mathrm{I}_{\{t\leqslant\sigma_R\}}|\hat{Y}(t)-Z_2(t)|^2]\leqslant H_2\Delta t,\quad \Delta t\leqslant 1,\quad t\in[0,T].\tag{3.23}$$

Proof. Since for any given $t\in[0,T]$ there is a nonnegative integer n such that $t\in[t_n,t_{n+1})$, it follows from (3.4) and the definition of $Z_2(t)$ that

$$\begin{aligned}|\hat{Y}(t)-Z_2(t)|&=|f_\lambda(Y_n^*,Y_{n-i_n})(t-t_{n+1})+(\mathcal{I}+C_n)^{-1}g(Y_n^*,Y_{n-i_n})(W(t)-W(t_n))\\&\quad+(\mathcal{I}+C_n)^{-1}h(Y_n^*,Y_{n-i_n})(\widetilde{N}(t)-\widetilde{N}(t_n))|.\end{aligned}\tag{3.24}$$

Thus,

$$\begin{aligned}&\mathrm{E}[\mathrm{I}_{\{s\leqslant\sigma_R\}}|\hat{Y}(t)-Z_2(t)|^2]\\&\quad\leqslant 3\Delta t^2\mathrm{E}[\mathrm{I}_{\{n\Delta t\leqslant\sigma_R\}}|f_\lambda(Y_n^*,Y_{n-i_n})|^2]+3K^2\Delta t\mathrm{E}[\mathrm{I}_{\{n\Delta t\leqslant\sigma_R\}}|g(Y_n^*,Y_{n-i_n})|^2]\\&\qquad+3K^2\lambda\Delta t\mathrm{E}[\mathrm{I}_{\{n\Delta t\leqslant\sigma_R\}}|h(Y_n^*,Y_{n-i_n})|^2].\end{aligned}\tag{3.25}$$

Also, under the condition $\mathcal{A}_1$, the following inequalities hold:

$$\mathrm{E}[\mathrm{I}_{\{n\Delta t\leqslant\sigma_R\}}|f_\lambda(Y_n^*,Y_{n-i_n})|^2]\leqslant 8(1+\lambda^2)L_RR^2+2\mathrm{E}|f_\lambda(0,0)|^2,\tag{3.26}$$
$$\mathrm{E}[\mathrm{I}_{\{n\Delta t\leqslant\sigma_R\}}|g(Y_n^*,Y_{n-i_n})|^2]\leqslant 4L_RR^2+2\mathrm{E}|g(0,0)|^2,\tag{3.27}$$
$$\mathrm{E}[\mathrm{I}_{\{n\Delta t\leqslant\sigma_R\}}|h(Y_n^*,Y_{n-i_n})|^2]\leqslant 4L_RR^2+2\mathrm{E}|h(0,0)|^2.\tag{3.28}$$

Let

$$H=12L_RR^2[2(1+\lambda^2)+K^2+\lambda K^2]+6\mathrm{E}|f_\lambda(0,0)|^2+6K^2\mathrm{E}|g(0,0)|^2+6\lambda K^2\mathrm{E}|h(0,0)|^2.$$

Then substituting inequalities (3.26)–(3.28) into (3.25) yields that

$$\mathrm{E}[\mathrm{I}_{\{s\leqslant\sigma_R\}}|\hat{Y}(t)-Z_2(t)|^2]\leqslant H_2(R)\Delta t.\tag{3.29}$$

This completes the proof. □

With the above arguments, a strong convergence theorem can be stated as follows.

Theorem 3.3. *Let the conditions $\mathcal{A}_1$–$\mathcal{A}_3$ hold. Then the continuous-time approximation solution $\hat{Y}(t)$ converges to the analytical solution of* (2.1) *in the mean-square sense, namely,*

$$\mathrm{E}\Big[\sup_{0\leqslant t\leqslant T}|\hat{Y}(t)-x(t)|^2\Big]\to 0,\quad \Delta t\to 0. \tag{3.30}$$

Proof. Write $e(t)=\hat{Y}(t)-x(t)$. It follows from (2.1) and (3.4) that

$$\begin{aligned} e(t)=&\int_0^t (f_\lambda(Z_2(s),Z_3(s))-f_\lambda(x(s),x(\lfloor s\rfloor)))ds\\ &+\int_0^t ((\mathcal{I}+C(\underline{s},Z_1(s)))^{-1}g(Z_2(s),Z_3(s))-g(x(s),x(\lfloor s\rfloor)))dWs\\ &+\int_0^t ((\mathcal{I}+C(\underline{s},Z_1(s)))^{-1}h(Z_2(s),Z_3(s))-h(x(s),x(\lfloor s\rfloor)))d\widetilde{N}_s. \end{aligned} \tag{3.31}$$

Applying the common inequality $(\sum_{i=1}^3 a_i)^2\leqslant 3\sum_{i=1}^3 a_i^2$ $(\forall\, a_i\in\mathbb{R})$, Hölder's inequality and the Burkholder-Davis-Gundy inequality (see [21]) to (3.31) yields that

$$\begin{aligned} &\mathrm{E}\Big[\sup_{0\leqslant s\leqslant t}|e(s\wedge\sigma_R)|^2\Big]\\ &\leqslant 3T\mathrm{E}\int_0^{t\wedge\sigma_R}|f_\lambda(Z_2(s),Z_3(s))-f_\lambda(x(s),x(\lfloor s\rfloor))|^2ds\\ &\quad+12\mathrm{E}\int_0^{t\wedge\sigma_R}|(\mathcal{I}+C(\underline{s},Z_1(s)))^{-1}g(Z_2(s),Z_3(s))-g(x(s),x(\lfloor s\rfloor))|^2ds\\ &\quad+12\lambda\mathrm{E}\int_0^{t\wedge\sigma_R}|(\mathcal{I}+C(\underline{s},Z_1(s)))^{-1}h(Z_2(s),Z_3(s))-h(x(s),x(\lfloor s\rfloor))|^2ds. \end{aligned} \tag{3.32}$$

Next, we estimate the last two items on the right-hand side of (3.32). Under the conditions $\mathcal{A}_1$ and $\mathcal{A}_3$, the inequality $(\sum_{i=1}^2 a_i)^2\leqslant 2\sum_{i=1}^2 a_i^2$ $(\forall\, a_i\in\mathbb{R})$ implies that

$$\begin{aligned} &\mathrm{E}\int_0^{t\wedge\sigma_R}|(\mathcal{I}+C(\underline{s},Z_1(s)))^{-1}g(Z_2(s),Z_3(s))-g(x(s),x(\lfloor s\rfloor))|^2ds\\ &=\mathrm{E}\int_0^{t\wedge\sigma_R}|(\mathcal{I}+C(\underline{s},Z_1(s)))^{-1}g(Z_2(s),Z_3(s))-(\mathcal{I}+C(\underline{s},Z_1(s)))^{-1}g(x(s),x(\lfloor s\rfloor))\\ &\quad+(\mathcal{I}+C(\underline{s},Z_1(s)))^{-1}g(x(s),x(\lfloor s\rfloor))-g(x(s),x(\lfloor s\rfloor))|^2ds\\ &\leqslant 2K^2\mathrm{E}\int_0^{t\wedge\sigma_R}|g(Z_2(s),Z_3(s))-g(x(s),x(\lfloor s\rfloor))|^2+|C(\underline{s},Z_1(s))|^2|g(x(s),x(\lfloor s\rfloor))|^2ds\\ &\leqslant 2K^2\mathrm{E}\int_0^{t\wedge\sigma_R}|g(Z_2(s),Z_3(s))-g(x(s),x(\lfloor s\rfloor))|^2ds\\ &\quad+4T\Delta tB^2(m+1)(\Delta t+m)(\mathrm{E}|g(0,0)|^2+2L_RR^2). \end{aligned} \tag{3.33}$$

Similarly, we have

$$\begin{aligned} &\mathrm{E}\int_0^{t\wedge\sigma_R}|(\mathcal{I}+C(\underline{s},Z_1(s)))^{-1}h(Z_2(s),Z_3(s))-h(x(s),x(\lfloor s\rfloor))|^2ds\\ &\leqslant 2K^2\mathrm{E}\int_0^{t\wedge\sigma_R}|h(Z_2(s),Z_3(s))-h(x(s),x(\lfloor s\rfloor))|^2ds\\ &\quad+4T\Delta tB^2(m+1)(\Delta t+m)(\mathrm{E}|h(0,0)|^2+2L_RR^2). \end{aligned} \tag{3.34}$$

Substituting the inequalities (3.33) and (3.34) into (3.32) and taking use of the condition $\mathcal{A}_1$ give that

$$\mathrm{E}\Big[\sup_{0\leqslant s\leqslant t}|e(s\wedge\sigma_R)|^2\Big]$$

$$\leqslant 48\Delta t K^2 B^2(1+m)^2 T\bigg\{(\mathrm{E}|g(0,0)|^2+2L_R R^2)+\lambda(\mathrm{E}|h(0,0)|^2+2L_R R^2)[6T(1+\lambda^2)L_R$$
$$+24K^2L_R+24\lambda K^2 L_R]\mathrm{E}\int_0^{t\wedge\sigma_R}(|Z_2(s)-x(s)|^2+|Z_3(s)-x(\lfloor s\rfloor)|^2)ds\bigg\}. \tag{3.35}$$

An application of the inequality $(\sum_{i=1}^2 a_i)^2\leqslant 2\sum_{i=1}^2 a_i^2$ ($\forall\, a_i\in\mathbb{R}$) and Lemma 3.4 derives that

$$\begin{aligned}\mathrm{E}\Big[\sup_{0\leqslant s\leqslant t}|e(s\wedge\sigma_R)|^2\Big]\leqslant{}& 2\Delta t T L_R H_2(R)[6T(1+\lambda^2)+24K^2+24\lambda K^2]\\ &+48\Delta t K^2B^2(1+m)^2T(\mathrm{E}|g(0,0)|^2+2L_RR^2)\\ &+48\Delta t\lambda K^2B^2(1+m)^2T(\mathrm{E}|h(0,0)|^2+2L_RR^2)\\ &+4L_R[6T(1+\lambda^2)+24K^2+24\lambda K^2]\int_0^t\mathrm{E}\Big[\sup_{0\leqslant r\leqslant s}|e(r\wedge\sigma_R)|^2\Big]ds.\end{aligned}\tag{3.36}$$

Thus, by the continuous Gronwall inequality (see [21]) we obtain that

$$\mathrm{E}\Big[\sup_{0\leqslant s\leqslant t}|e(s\wedge\sigma_R)|^2\Big]\leqslant\Delta t H_3(R,T)\exp(H_4(R,T)T), \tag{3.37}$$

where $H_3(R,T),H_4(R,T)>0$ are two constants depending on R and T. Moreover, with the condition $\mathcal{A}_2$, Lemma 3.1 and the similar argument in [11], the following equality holds:

$$\mathrm{E}\Big[\sup_{0\leqslant t\leqslant T}|e(t)|^2\Big]=\mathrm{E}\Big[\sup_{0\leqslant t\leqslant T}|e(t)|^2 1_{\{\tau_R>T,\rho_R>T\}}\Big]+\mathrm{E}\Big[\sup_{0\leqslant t\leqslant T}|e(t)|^2 1_{\{\tau_R\leqslant T\text{ or }\rho_R\leqslant T\}}\Big]. \tag{3.38}$$

Whereas, by Young's inequality

$$a^r b^{1-r}\leqslant ra+(1-r)b,\quad a,b>0,\quad r\in(0,1]$$

and Hölder's inequality, it holds that

$$\begin{aligned}\mathrm{E}\Big[\sup_{0\leqslant t\leqslant T}|e(t)|^2 1_{\{\tau_R\leqslant T\text{ or }\rho_R\leqslant T\}}\Big]\leqslant{}&\frac{2\delta}{p}\mathrm{E}\Big[\sup_{0\leqslant t\leqslant T}|e(t)|^p\Big]\\ &+\frac{p-2}{p\delta^{\frac{2}{p-2}}}\mathrm{P}(\tau_R\leqslant T\text{ or }\rho_R\leqslant T),\quad\forall\,\delta>0,\end{aligned}\tag{3.39}$$

where it is obvious that

$$\mathrm{P}(\tau_R\leqslant T\text{ or }\rho_R\leqslant T)\leqslant\mathrm{P}(\tau_R\leqslant T)+\mathrm{P}(\rho_R\leqslant T).$$

The facts $|x(\rho_R)|$, $|\hat{Y}(\tau_R)|$, $|Z_2(\tau_R)|\geqslant R$ imply that

$$\mathrm{P}(\rho_R\leqslant T)\leqslant\mathrm{E}\bigg(1_{\{\rho_R\leqslant T\}}\frac{|x(\rho_R)|^p}{R^p}\bigg)\leqslant\frac{1}{R^p}\mathrm{E}|x(\rho_R)|^p\leqslant\frac{H_0(1+\mathrm{E}|x_0|^p)}{R^p}. \tag{3.40}$$

Similarly, $\mathrm{P}(\tau_R\leqslant T)\leqslant\frac{H_1}{R^p}$. Also, under Theorem 2.2 and Lemma 3.1, we have

$$\mathrm{E}\Big[\sup_{0\leqslant t\leqslant T}|e(t)|^p\Big]\leqslant 2^{p-1}\mathrm{E}\Big[\sup_{0\leqslant t\leqslant T}|x(t)|^p+|\hat{Y}(t)|^p\Big]\leqslant 2^{p-1}[H_0(1+\mathrm{E}|x_0|^p)+H_1]. \tag{3.41}$$

Substituting the inequalities (3.40) and (3.41) into (3.39) yields that

$$\mathrm{E}\Big[\sup_{0\leqslant t\leqslant T}|e(t)|^2 1_{\{\tau_R\leqslant T\text{ or }\rho_R\leqslant T\}}\Big]\leqslant\frac{2^p[H_0(1+\mathrm{E}|x_0|^p)+H_1]\delta}{p}+\frac{p-2}{p\delta^{\frac{2}{p-2}}}\bigg[\frac{H_0(1+\mathrm{E}|x_0|^p)+H_1}{R^p}\bigg]. \tag{3.42}$$

A combination of the inequalities (3.37), (3.38) and (3.42) gives that

$$\mathrm{E}\Big[\sup_{0\leqslant t\leqslant T}|e(t)|^2\Big]\leqslant H_3(R,T)\exp(H_4(R,T)T)\Delta t+\frac{2^p[H_0(1+\mathrm{E}|x_0|^p)+H_1]\delta}{p}$$

$$+\frac{p-2}{p\delta^{\frac{2}{p-2}}}\left[\frac{H_0(1+\mathrm{E}|x_0|^p)+H_1}{R^p}\right].$$

While, for the right items of the above inequality, we have that, for any given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\frac{2^p[H_0(1+\mathrm{E}|x_0|^p)+H_1]\delta}{p} < \frac{\epsilon}{3};$$

for the above δ, there exists an $R > 0$ such that

$$\frac{p-2}{p\delta^{\frac{2}{p-2}}}\left[\frac{H_0(1+\mathrm{E}|x_0|^p)+H_1}{R^p}\right] < \frac{\epsilon}{3};$$

and for the above δ and R, there exists a $\Delta t > 0$ such that

$$H_3(R,T)\exp\left(H_4(R,T)T\right)\Delta t < \frac{\epsilon}{3}.$$

Therefore the theorem is proved. □

4 The analytical and numerical mean-square exponential stability

4.1 The analytical mean-square exponential stability

In this section, we deal with the analytical mean-square exponential stability of (2.1), whose definition is stated as follows.

Definition 4.1. The solution of (2.1) is said to be mean-square exponentially stable if, there exists a rate constant η such that

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathrm{E}|x(t)|^2 \leqslant -\eta \tag{4.1}$$

for any bounded initial value $x_0 \in \mathbb{R}^d$.

Without loss of generality, we assume that $f(0,0) = g(0,0) = h(0,0) = 0$. This condition implies that (2.1) with the initial value $x_0 = 0$ has the trivial $x(t) = 0$, and, under the condition $\mathcal{A}_2$, the following inequalities hold:

$$\langle x, f(x,y)\rangle \leqslant \left(\gamma_1+\frac{1}{2}\gamma_2\right)|x|^2+\frac{1}{2}\gamma_2|y|^2, \quad \langle x, f_\lambda(x,y)\rangle \leqslant \frac{1}{2}\widetilde{\beta}_1|x|^2+\frac{1}{2}\widetilde{\beta}_4|y|^2, \tag{4.2}$$

$$|g(x,y)|^2 \leqslant \gamma_3|x|^2+\gamma_4|y|^2, \quad |h(x,y)|^2 \leqslant \gamma_5|x|^2+\gamma_6|y|^2, \tag{4.3}$$

where $\widetilde{\beta}_1 = 2\gamma_1+\gamma_2+2\lambda\sqrt{\gamma_5}+\lambda\sqrt{\gamma_6}$, $\widetilde{\beta}_2 = \gamma_3+\lambda\gamma_5$, $\widetilde{\beta}_3 = \gamma_4+\lambda\gamma_6$ and $\widetilde{\beta}_4 = \gamma_2+\lambda\sqrt{\gamma_6}$.

The following lemma given by Baker and Buckwar [2] will play a key role in our stability analysis.

Lemma 4.2 (See [2]). *Let a, b and τ be given constants with $0 < b < a$ and $\tau \geqslant 0$. Suppose that the function υ: $[t_0-\tau,\infty)\to\mathbb{R}^+$ is continuous and its upper Dini-derivative $D^+\upsilon$ satisfies that*

$$D^+\upsilon(t) \leqslant -a\upsilon(t)+b\sup_{s\in[t-\tau,t]}\upsilon(s), \quad t\geqslant t_0. \tag{4.4}$$

Then

$$\upsilon(t) \leqslant \Big[\sup_{s\in[t_0-\tau,t_0]}\upsilon(s)\Big]\exp[-\upsilon^+(t-t_0)], \quad t\geqslant t_0, \tag{4.5}$$

where $\upsilon^+\in(0,a-b]$ is the zero of the function $l(\upsilon)=\upsilon-a+b\exp(\upsilon\tau)$.

With the above lemma, we have the following analytical stability criterion.

Theorem 4.3. *Suppose that the condition $\mathcal{A}_2$ holds and $\widetilde{\beta}_1+\widetilde{\beta}_2+\widetilde{\beta}_3+\widetilde{\beta}_4<0$. Then the analytical solution $x(t)$ of (2.1) is mean-square exponentially stable with $\eta\in(0,-(\widetilde{\beta}_1+\widetilde{\beta}_2+\widetilde{\beta}_3+\widetilde{\beta}_4)]$ being the zero of the function*

$$\mathcal{L}(\eta):=\eta+(\widetilde{\beta}_1+\widetilde{\beta}_2)+(\widetilde{\beta}_3+\widetilde{\beta}_4)\exp(\eta).$$

Proof. Applying the Itô formula (see [5]) to (2.1) yields that

$$\begin{aligned} d|x(t)|^2 &= (2\langle x(t),f(x(t),x(\lfloor t\rfloor))\rangle+|g(x(t),x(\lfloor t\rfloor))|^2)dt+2x(t^-)^{\mathrm{T}}g(x(t^-),x(\lfloor t^-\rfloor))dW_t \\ &\quad+(2\lambda\langle x(t),h(x(t),x(\lfloor t\rfloor))\rangle+\lambda|h(x(t),x(\lfloor t\rfloor))|^2)dt \\ &\quad+(|x(t^-)+h(x(t^-),x(\lfloor t^-\rfloor))|^2-|x(t^-)|^2)d\widetilde{N}(t). \end{aligned} \tag{4.6}$$

Thus, it follows from the condition $\mathcal{A}_2$, (4.2) and (4.3) that

$$d|x(t)|^2\leqslant[(2\gamma_1+\gamma_2+\gamma_3+2\lambda\sqrt{\gamma_5}+\lambda\sqrt{\gamma_6}+\lambda\gamma_5)|x(t)|^2+(\gamma_2+\gamma_4+\lambda\sqrt{\gamma_6}+\lambda\gamma_6)|x(\lfloor t\rfloor)|^2]dt+M_t, \tag{4.7}$$

where

$$M_t=2x(t^-)^{\mathrm{T}}g(x(t^-),x(\lfloor t^-\rfloor))dW_t+(|x(t^-)+h(x(t^-),x(\lfloor t^-\rfloor))|^2-|x(t^-)|^2)d\widetilde{N}(t)$$

is a martingale. Taking expectation on both sides of (4.7) derives that

$$\mathrm{E}|x(t+\delta)|^2\leqslant\mathrm{E}|x(t)|^2+\int_t^{t+\delta}\alpha_1\mathrm{E}|x(s)|^2ds+\int_t^{t+\delta}\alpha_2\mathrm{E}|x(\lfloor s\rfloor)|^2ds, \tag{4.8}$$

where $\alpha_1=\widetilde{\beta}_1+\widetilde{\beta}_2$ and $\alpha_2=\widetilde{\beta}_3+\widetilde{\beta}_4$. This shows that the upper Dini-derivative of $\mathrm{E}|x(t)|^2$ satisfies that

$$D^+\mathrm{E}|x(t)|^2\leqslant\alpha_1\mathrm{E}|x(t)|^2+\alpha_2\sup_{s\in[t-1,t]}\mathrm{E}|x(s)|^2,\quad t\geqslant 0. \tag{4.9}$$

An application of Lemma 4.2 to (4.9) infers that

$$\mathrm{E}|x(t)|^2\leqslant\mathrm{E}|x_0|^2\exp(-\eta t),\quad t\geqslant 0.$$

Hence the theorem is proven. □

4.2 The numerical mean-square exponential stability

This section focuses on the numerical mean-square exponential stability of the CSSB methods, whose definition is presented as follows.

Definition 4.4. A CSSB method (3.2) is said to be mean-square exponentially stable for any given stepsize $\Delta t>0$ if, there exists a rate constant $\eta_{\Delta t}$ such that

$$\limsup_{n\to\infty}\frac{1}{n\Delta t}\log\mathrm{E}|Y_n|^2\leqslant-\eta_{\Delta t} \tag{4.10}$$

for any bounded initial value $x_0\in\mathbb{R}^d$.

The following lemma given by Baker and Buckwar [2] will be very useful for our numerical stability analysis.

Lemma 4.5 (See [2]). *Suppose, for some fixed integer $Q\geqslant 0$, that $t_n=t_0+n\Delta t$ for some $\Delta t>0$ and $\upsilon_n, n\geqslant -Q$ is a sequence of real numbers that satisfies the relation*

$$\upsilon_{n+1}-\upsilon_n\leqslant-a_{\Delta t}\Delta t\upsilon_n+b_{\Delta t}\Delta t\max_{l\in[-Q,\ 0]}\upsilon_{n+l},\quad n,l\in\mathbb{N}, \tag{4.11}$$

where

$$0<b_{\Delta t}<a_{\Delta t},\quad 0<a_{\Delta t}\Delta t<1, \tag{4.12}$$

Then $\upsilon_n\leqslant\{\max_{l\in[-Q,\ 0]}\upsilon_l\}\exp[-\eta_{\Delta t}(t_n-t_0)]$, where $\eta_{\Delta t}>0$ and $\zeta_{\Delta t}=\exp(-\eta_{\Delta t}\Delta t)$ is the zero of the function $\mathfrak{R}(\zeta_{\Delta t};a_{\Delta t},b_{\Delta t})=\zeta_{\Delta t}^{Q+1}-(1-a_{\Delta t}\Delta t)\zeta_{\Delta t}^{Q}-b_{\Delta t}\Delta t$.

A numerical mean-square exponential stability criterion can be stated as follows.

Theorem 4.6. *Suppose that the conditions $\mathcal{A}_2$–$\mathcal{A}_3$ hold and*

$$\widetilde{\beta}_1 + K^2(\widetilde{\beta}_2 + \widetilde{\beta}_3) + \widetilde{\beta}_4 < 0. \tag{4.13}$$

Let $c_j = \operatorname{diag}\{c_j^1, c_j^2, \dots, c_j^d\}$ with $c_j^k \in \mathbb{R}$ $(0 \leqslant j \leqslant m; 1 \leqslant k \leqslant d)$. Then the CSSB method (3.2) for (2.10) is mean-square exponentially stable with $\lim_{\Delta t \to 0} \eta_{\Delta t}$ being the unique positive solution of the equation

$$\eta + \widetilde{\beta}_1 + K^2\widetilde{\beta}_2 + (K^2\widetilde{\beta}_3 + \widetilde{\beta}_4)\exp(\eta) = 0. \tag{4.14}$$

Proof. It follows from the condition $\mathcal{A}_2$, (3.2) and (4.2) that

$$|Y_n^*|^2 \leqslant \frac{|Y_n|^2 + \widetilde{\beta}_4 \Delta t |Y_{n-i_n}|^2}{1 - \widetilde{\beta}_1 \Delta t}. \tag{4.15}$$

Taking the square on both sides of (3.2) yields that

$$|Y_{n+1}|^2 = |Y_n^*|^2 + |(\mathcal{I} + C_n)^{-1} g(Y_n^*, Y_{n-i_n})\Delta W_n|^2 + |(\mathcal{I} + C_n)^{-1} h(Y_n^*, Y_{n-i_n})\Delta \widetilde{N}_n|^2 + M_n, \tag{4.16}$$

where

$$\begin{aligned} M_n &= 2\langle Y_n^*, (\mathcal{I} + C_n)^{-1} g(Y_n^*, Y_{n-i_n})\Delta W_n\rangle + 2\langle Y_n^*, (\mathcal{I} + C_n)^{-1} h(Y_n^*, Y_{n-i_n})\Delta \widetilde{N}_n\rangle \\ &\quad + 2\langle (\mathcal{I} + C_n)^{-1} g(Y_n^*, Y_{n-i_n})\Delta W_n, (\mathcal{I} + C_n)^{-1} h(Y_n^*, Y_{n-i_n})\Delta \widetilde{N}_n\rangle. \end{aligned}$$

Let $g_{ij} = g_{ij}(Y_n^*, Y_{n-i_n})$ and

$$g(Y_n^*, Y_{n-i_n}) = \begin{pmatrix} g_{11} & \cdots & \cdots & g_{1m} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ g_{d1} & \cdots & \cdots & g_{dm} \end{pmatrix}_{d\times m}.$$

Then

$$g(Y_n^*, Y_{n-i_n})\Delta W_n = \left[\sum_{i=1}^m g_{1i}\Delta W_n^i,\ \sum_{i=1}^m g_{2i}\Delta W_n^i, \dots, \sum_{i=1}^m g_{di}\Delta W_n^i\right]^{\mathrm{T}}.$$

Under the conditions $c_j = \operatorname{diag}\{c_j^1, c_j^2, \dots, c_j^d\}$ with $c_j^k \in \mathbb{R}$ $(0 \leqslant j \leqslant m; 1 \leqslant k \leqslant d)$, we have

$$\begin{aligned} &(\mathcal{I} + C_n)^{-1} \\ &\quad = \operatorname{diag}\left\{\frac{1}{1 + c_0^1\Delta t + \sum_{j=1}^m c_j^1 |\Delta W_n^j|}, \frac{1}{1 + c_0^2\Delta t + \sum_{j=1}^m c_j^2 |\Delta W_n^j|}, \dots, \frac{1}{1 + c_0^d\Delta t + \sum_{j=1}^m c_j^d |\Delta W_n^j|}\right\}. \end{aligned}$$

Thus, it holds that

$$\begin{aligned} &(\mathcal{I} + C_n)^{-1} g(Y_n^*, Y_{n-i_n})\Delta W_n \\ &\quad = \left[\frac{\sum_{i=1}^m g_{1i}\Delta W_n^i}{1 + c_0^1\Delta t + \sum_{j=1}^m c_j^1 |\Delta W_n^j|}, \frac{\sum_{i=1}^m g_{2i}\Delta W_n^i}{1 + c_0^2\Delta t + \sum_{j=1}^m c_j^2 |\Delta W_n^j|}, \dots, \frac{\sum_{i=1}^m g_{di}\Delta W_n^i}{1 + c_0^d\Delta t + \sum_{j=1}^m c_j^d |\Delta W_n^j|}\right]^{\mathrm{T}}. \end{aligned}$$

Also, for $1 \leqslant i \leqslant m$ and $1 \leqslant k \leqslant d$, a direct computation gives that

$$\mathrm{E}\left[\frac{\Delta W_n^i}{1 + c_0^k\Delta t + \sum_{j=1}^m c_j^k |\Delta W_n^j|}\right] = 0. \tag{4.17}$$

From this, we infer that

$$\mathrm{E}[\langle Y_n^*, (\mathcal{I} + C_n)^{-1} g(Y_n^*, Y_{n-i_n})\Delta W_n\rangle] = 0,$$

which implies $\mathrm{E}(M_n) = 0$. Hence, by taking expectation on both sides of (4.16) and using the condition $\mathcal{A}_3$, (4.3) and (4.15), we obtain

$$\begin{aligned}&\mathrm{E}|Y_{n+1}|^2\\&\leqslant \left[1 + \frac{\widetilde{\beta}_1\Delta t + \widetilde{\beta}_2 K^2\Delta t}{1-\widetilde{\beta}_1\Delta t}\right]\mathrm{E}|Y_n|^2 + \frac{(1+\widetilde{\beta}_2K^2\Delta t)\widetilde{\beta}_4\Delta t + (1-\widetilde{\beta}_1\Delta t)\widetilde{\beta}_3K^2\Delta t}{1-\widetilde{\beta}_1\Delta t}\mathrm{E}|Y_{n-i_n}|^2.\end{aligned} \tag{4.18}$$

Write $\upsilon_n = \mathrm{E}|Y_n|^2,\ \upsilon_{-j} = \upsilon_0\ (j = 1, 2, \ldots, Q),\ a_{\Delta t} = \frac{-\widetilde{\beta}_1-\widetilde{\beta}_2K^2}{1-\widetilde{\beta}_1\Delta t}$ and

$$b_{\Delta t} = \frac{(1+\widetilde{\beta}2K^2\Delta t)\widetilde{\beta}_4 + (1-\widetilde{\beta}_1\Delta t)\widetilde{\beta}_3K^2}{1-\widetilde{\beta}_1\Delta t}.$$

Then by (4.18) we obtain

$$\upsilon_{n+1} - \upsilon_n \leqslant -a_{\Delta t}\Delta t\upsilon_n + b_{\Delta t}\Delta t \max_{l\in[-Q,\ 0]} \upsilon_{n+l}. \tag{4.19}$$

In terms of Lemma 4.5, in the following, we need only to confirm (4.12). Firstly, we note that $a_{\Delta t}\Delta t < 1$ for all $\Delta t \in (0, +\infty)$. Secondly, when setting

$$\Delta t^* = \frac{-\widetilde{\beta}_1 - \widetilde{\beta}_2K^2 - \widetilde{\beta}_3K^2 - \widetilde{\beta}_4}{K^2(\widetilde{\beta}_2\widetilde{\beta}_4 - \widetilde{\beta}_1\widetilde{\beta}_3)}, \tag{4.20}$$

we have by (4.13) that $0 < b_\Delta < a_\Delta$ for all $\Delta \in (0, \Delta t^*)$. Therefore,

$$\mathrm{E}|Y_n|^2 \leqslant \mathrm{E}|Y_0|^2 \exp[-\eta_{\Delta t}(n\Delta t)], \quad \Delta t \in (0, \Delta t^*),$$

where $\eta_{\Delta t} > 0$, and $\zeta_{\Delta t} := \exp(-\eta_{\Delta t}\Delta t)$ is the zero of the function

$$\Re(\zeta_{\Delta t}; a_{\Delta t}, b_{\Delta t}) := \zeta_{\Delta t}^{Q+1} - (1 - a_{\Delta t}\Delta t)\zeta_{\Delta t}^Q - b_{\Delta t}\Delta t.$$

Introduce the function

$$M(\eta_{\Delta t}; a_{\Delta t}, b_{\Delta t}) := 1 - (1 - a_{\Delta t}\Delta t)\exp(\eta_{\Delta t}\Delta t) - b_{\Delta t}\Delta t \exp[\eta_{\Delta t}(1 + \Delta t)].$$

When $M(\eta_{\Delta t}; a_{\Delta t}, b_{\Delta t}) = 0$, it holds that

$$\Re(\zeta_{\Delta t}; a_{\Delta t}, b_{\Delta t}) = \zeta_{\Delta t}^{Q+1} - (1 - a_{\Delta t}\Delta t)\zeta_{\Delta t}^Q - b_{\Delta t}\Delta t = 0.$$

Write $\widehat{M}(\eta_{\Delta t}; a_{\Delta t}, b_{\Delta t}) = \Delta t^{-1}M(\eta_{\Delta t}; a_{\Delta t}, b_{\Delta t})$. Then, by $M(\eta_{\Delta t}; a_{\Delta t}, b_{\Delta t}) = 0$ and

$$\lim_{\Delta t\to 0} \widehat{M}(\eta_{\Delta t}; a_{\Delta t}, b_{\Delta t}) = \lim_{\Delta t\to 0}[\eta_{\Delta t} + (\widetilde{\beta}_1 + K^2\widetilde{\beta}_2) + (K^2\widetilde{\beta}_3 + \widetilde{\beta}_4)\exp(\eta_{\Delta t})] = 0,$$

$\widehat{M}(\eta_{\Delta t}; a_{\Delta t}, b_{\Delta t}) = 0$. This implies that $\lim_{\Delta t\to 0} \eta_{\Delta t}$ is the unique positive solution of (4.14). Therefore the theorem is proved. □

We remark that $\widetilde{\beta}_1 + K^2\widetilde{\beta}_2 + K^2\widetilde{\beta}_3 + \widetilde{\beta}_4 < 0$ when $\widetilde{\beta}_1 + \widetilde{\beta}_2 + \widetilde{\beta}_3 + \widetilde{\beta}_4 < 0$ and $0 < K \leqslant 1$. This, together with Theorems 4.3 and 4.6, shows that the CSSB methods (3.2) can preserve the mean-square exponentially stability whenever $0 < K \leqslant 1$, c_j $(j = 0, 1, \ldots, m)$ are a series of real diagonal matrices and the condition of Theorem 4.3 holds. Moreover, it can be observed from the conditions of Theorems 4.3 and 4.6 that the parameter γ_1 in the one-sided Lipschitz condition (2.3) should be a small negative number for ensuring the analytical and numerical mean-square exponential stability.

5 Numerical illustration

In this section, with some numerical experiments, we further illustrate the computational effectiveness and theoretical results of CSSB methods for stiff SDEs with JDPCAs. For this, we use $Y_n^{(i)}$ to denote the numerical approximation to the analytical solution $x^{(i)}(t_n)$, and introduce the formulae:

$$E(\mathcal{M}) = \frac{1}{10000}\sum_{i=1}^{10000}\max_{0\leqslant n\Delta t\leqslant T}|Y_n^{(i)} - x^{(i)}(t_n)|^2, \quad RC(t_n) = \frac{1}{t_n}\log\left(\frac{1}{10000}\sum_{i=1}^{10000}|Y_n^{(i)}|^2\right)$$

to characterize the mean-square error of a numerical method $\mathcal{M}$ on $[0,T]$ and the approximation of 2nd-moment Lyapunov exponent of the CSSB method at t_n, respectively. In order to show the computational advantages of CSSB methods, in the following numerical experiments, we will also present a comparison with the adapted split-step backward Euler (SSBE) method without compensation

$$\begin{cases} Y_n^* = Y_n + f(Y_n^*, Y_{n-i_n})\Delta t, \\ Y_{n+1} = Y_n^* + g(Y_n^*, Y_{n-i_n})\Delta W_n + h(Y_n^*, Y_{n-i_n})\Delta N_n, \end{cases} \tag{5.1}$$

the CSSBE method

$$\begin{cases} Y_n^* = Y_n + f_\lambda(Y_n^*, Y_{n-i_n})\Delta t, \\ Y_{n+1} = Y_n^* + g(Y_n^*, Y_{n-i_n})\Delta W_n + h(Y_n^*, Y_{n-i_n})\Delta \widetilde{N}_n, \end{cases} \tag{5.2}$$

and the tamed explicit (TE) method

$$Y_{n+1} = Y_n + \Delta t\frac{f(Y_n, Y_{n-i_n})}{1+\Delta t|f(Y_n, Y_{n-i_n})|} + g(Y_n, Y_{n-i_n})\Delta W_n + h(Y_n, Y_{n-i_n})\Delta N_n. \tag{5.3}$$

The above methods can be viewed as the extended versions to the corresponding methods in [9] and [27], respectively.

Example 5.1. Consider the linear stiff problem of SDEs with JDPCAs

$$\begin{cases} dx(t) = \left[-100x(t^-) + \frac{1}{4}x(\lfloor t^-\rfloor)\right]dt + \left[0.01x(t^-) + \frac{1}{4}x(\lfloor t^-\rfloor)\right]dW(t) \\ \qquad\qquad + \frac{1}{2}[x(t^-) + x(\lfloor t^-\rfloor)]dN(t), \quad t > 0, \\ x(0) = 1, \end{cases} \tag{5.4}$$

where the intensity λ of $N(t)$ is taken as 2. It can be verified that (5.4) satisfies the conditions $\mathcal{A}_1$ and $\mathcal{A}_2$ with

$$\gamma_1 = -100, \quad \gamma_2 = \frac{1}{4}, \quad \gamma_3 = 0.0002, \quad \gamma_4 = \frac{1}{8}, \quad \gamma_5 = \frac{1}{2}, \quad \gamma_6 = \frac{1}{2}.$$

Hence, by Theorem 2.2, the solution $x(t)$ of (5.4) satisfies the estimation (2.14).

Let $C_n = 3\Delta t + 5|\Delta W_n|$ in the CSSB method (3.2). Then, when applying the CSSB method (3.2) to (5.4), the condition $\mathcal{A}_3$ is fulfilled. Thus, in accordance with Theorem 3.3, the corresponding continuous-time approximation solution converges to the analytical solution of (5.4) in the mean-square sense. In order to give a numerical confirmation, we apply the CSSB method (3.2) with the above C_n and the stepsizes $\Delta t = 2^{-i}$ $(i = 0, 1, \ldots, 7)$ to (5.4) on $[0, 5]$. The error behaviors of the derived numerical solutions are shown in Table 1 and Figure 1 (in log-log scale), where the analytical solutions are approximately taken as the corresponding numerical solutions with the stepsize $\Delta t = 2^{-10}$. These numerical results verify the computational effectiveness of CSSB method (3.2) and Theorem 3.3. From Table 1 and Figure 1, we can find that the CSSB method with $C_n = 3\Delta t + 5|\Delta W_n|$ has the higher accuracy than CSSBE, SSBE and TE methods when the stepsize is not small. Moreover, we can observe from Figure 1 that the mean-square convergence order of CSSB method is approximately 0.5.

Table 1 E(·) of the numerical solutions for (5.4)

Δt	2^0	2^{-1}	2^{-2}	2^{-3}
CSSB method	1.315E−02	1.359E−02	1.174E−02	1.043E−02
CSSBE method	3.619	3.036E−01	1.380E−01	6.354E−02
SSBE method	1.861E+01	7.222E−01	1.937E−01	7.394E−02
TE method	6.800E+04	2.681E+05	2.826E+05	4.200E+05
Δt	2^{-4}	2^{-5}	2^{-6}	2^{-7}
CSSB method	6.496E−03	3.385E−03	1.044E−03	3.074E−04
CSSBE method	2.703E−02	1.003E−02	2.940E−03	6.762E−04
SSBE method	2.833E−02	1.160E−02	2.991E−03	8.278E−04
TE method	1.502E−01	4.480E−02	2.855E−03	3.814E−04

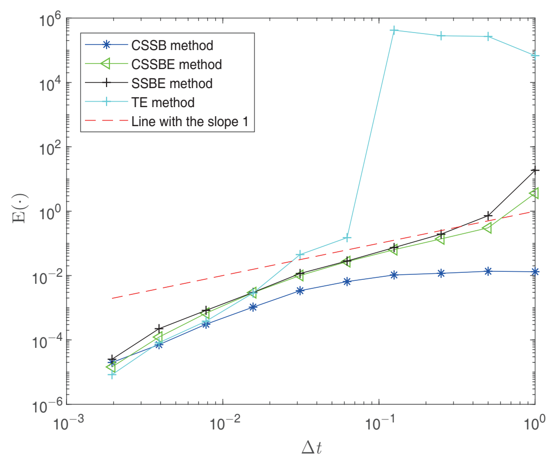

Figure 1 (Color online) A comparison of mean-square errors of the four methods for (5.4)

Next, we take an insight into the mean-square exponential stability of the CSSB method (3.2) for (5.4). Firstly, we note that (5.4) satisfies the conditions $\mathcal{A}_2$ and

$$\widetilde{\beta}_1 + \widetilde{\beta}_2 + \widetilde{\beta}_3 + \widetilde{\beta}_4 < 0$$

with

$$\widetilde{\beta}_1 = -195.5074, \quad \widetilde{\beta}_2 = 1.0002, \quad \widetilde{\beta}_3 = 1.1250, \quad \widetilde{\beta}_4 = 1.6642.$$

Thus, by Theorem 4.3, the analytical solution of problem (5.4) is mean-square exponentially stable. Also, when $C_n = 3\Delta t + 5|\Delta W_n|$, it holds that $0 < K \leqslant 1$, which implies that the condition (4.13) is true. Hence, it follows from Theorem 4.6 that the CSSB method for (5.4) is mean-square exponentially stable whenever $0 < \Delta t < 0.8651$, where the upper bound 0.8651 of Δt is computed by taking $K = 1$ in (4.20). Moreover, when setting $K = 1$, we can obtain by solving (4.14) that

$$\lim_{\Delta t \to 0} \eta_{\Delta t} \approx 4.223.$$

As an example, in Figure 2, we take the stepsize $\Delta t = 0.1$ to plot the curve of $RC(t_n)$, which is under the line $y = -4.223$. This is consistent with the conclusion of Theorem 4.6.

Example 5.2. Consider the nonlinear stiff problem of SDEs with JDPCAs

$$\begin{cases} dx(t) = [-3x(t^-)^3 - 6x(t^-) + x(\lfloor t^- \rfloor)]dt + 0.02[\sin(x(t^-)) + \sin(x(\lfloor t^- \rfloor))]dW(t), \\ \qquad\qquad + \dfrac{1}{2}[x(t^-) + x(\lfloor t^- \rfloor)]dN(t), \quad t > 0, \\ x(0) = 1, \end{cases} \tag{5.5}$$

where the intensity λ of $N(t)$ is taken as 2. It can be confirmed that (5.5) satisfies the conditions $\mathcal{A}_1$ and $\mathcal{A}_2$ with

$$\gamma_1 = -6, \quad \gamma_2 = 1, \quad \gamma_3 = 0.0008, \quad \gamma_4 = 0.0008, \quad \gamma_5 = \frac{1}{2}, \quad \gamma_6 = \frac{1}{2}.$$

Thus, by Theorem 2.2, the solution $x(t)$ of (5.5) satisfies the estimation (2.14).

Let $C_n = \Delta t + 2|\Delta W_n|$ in the CSSB method (3.2). Then, when applying the CSSB method (3.2) to (5.5), the condition $\mathcal{A}_3$ is satisfied. Hence, in terms of Theorem 3.3, the corresponding continuous-time approximation solution converges to the analytical solution of (5.5) in the mean-square sense. For presenting a numerical confirmation, we apply the CSSB method (3.2) with C_n and the stepsizes

$$\Delta t = 2^{-i} \quad (i = 0, 1, \ldots, 7)$$

to (5.5) on $[0, 5]$. The error of the derived numerical solutions are shown in Table 2 and Figure 3 (in log-log scale), where the analytical solutions are taken as the corresponding numerical solutions with the stepsize $\Delta t = 2^{-10}$ approximately. These numerical results verify the computational effectiveness of the CSSB method (3.2) and Theorem 3.3. In Table 2 and Figure 3, we also give the error behaviors of the SSBE method, the CSSBE method, and the TE method for (5.5). From Table 2 and Figure 3 again, we can find that the CSSB method with

$$C_n = \Delta t + 2|\Delta W_n|$$

has the higher accuracy than CSSBE, SSBE and TE methods when the stepsize is not small. Moreover, we can observe from Figure 3 that the mean-square convergence order of the CSSB method is approximately 0.5.

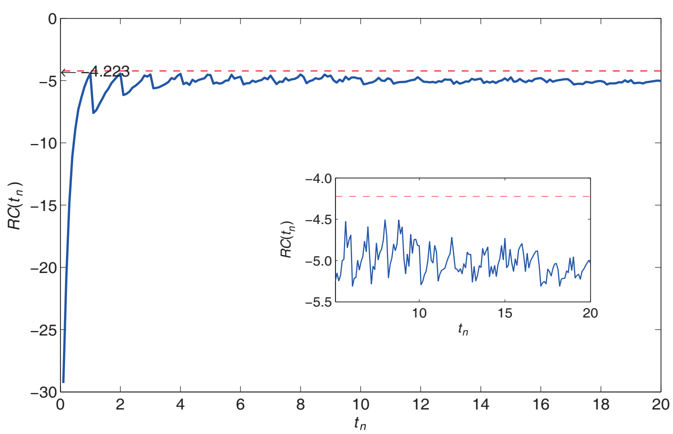

Figure 2 (Color online) 2nd-moment Lyapunov exponent of the CSSB method for (5.4)

Table 2 E(·) of the numerical solutions for (5.5)

Δt	2^{0}	2^{-1}	2^{-2}	2^{-3}
CSSB method	1.243E−01	7.383E−02	3.392E−02	1.496E−02
CSSBE method	3.788	6.976E−01	2.085E−01	5.485E−02
SSBE method	1.530E+02	6.889	7.885E−01	1.436E−01
TE method	7.370E+04	2.676E+05	2.016E+05	3.883E+04
Δt	2^{-4}	2^{-5}	2^{-6}	2^{-7}
CSSB method	5.341E−03	1.943E−03	9.531E−04	4.577E−04
CSSBE method	1.537E−02	5.281E−03	1.229E−03	3.439E−04
SSBE method	3.591E−02	9.243E−03	2.494E−03	4.965E−04
TE method	3.035E−02	4.133E−03	9.714E−04	2.007E−04

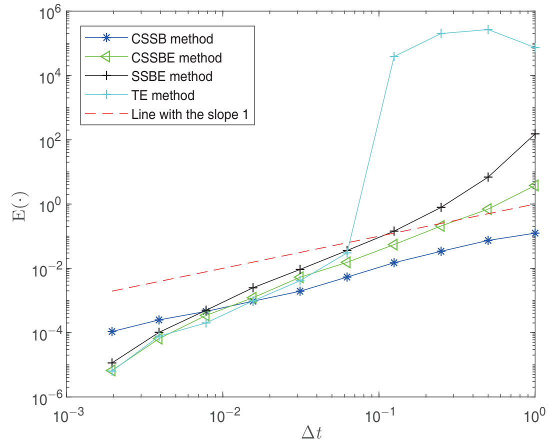

Figure 3 (Color online) A comparison of mean-square errors of the four methods for (5.5)

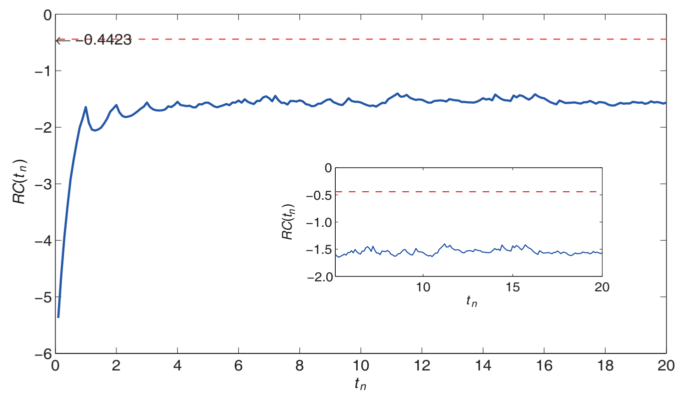

Figure 4 (Color online) 2nd-moment Lyapunov exponent of the CSSB method for (5.5)

Next, we consider the mean-square exponential stability of the CSSB method (3.2) for (5.5). Firstly, we note that (5.5) satisfies the conditions $\mathcal{A}_2$ and

$$\widetilde{\beta}_1+\widetilde{\beta}_2+\widetilde{\beta}_3+\widetilde{\beta}_4<0$$

with

$$\widetilde{\beta}_1=-6.757,\quad \widetilde{\beta}_2=1.0008,\quad \widetilde{\beta}_3=1.0008,\quad \widetilde{\beta}_4=2.414.$$

Hence, it follows from Theorem 4.3 that the analytical solution of (5.5) is mean-square exponentially stable. Also, when $C_n=\Delta t+2|\Delta W_n|$, it holds that $0<K\leqslant 1$, which implies that the condition (4.13) is true. Therefore, by Theorem 4.6 that the CSSB method for (5.5) is mean-square exponentially stable when $0<\Delta t<0.1352$, where the upper bound 0.1352 of Δt is obtained by taking $K=1$ in (4.20). Moreover, when taking $K=1$, we have by solving (4.14) that $\lim_{\Delta t\to 0}\eta_{\Delta t}\approx 0.4423$. As an example, in Figure 4, we take $\Delta t=0.1$ to plot the curve of $RC(t_n)$, which is under the line $y=-0.4423$. This confirms the conclusion of Theorem 4.6.

The above examples testify the computational effectiveness and theoretical results of CSSB methods for stiff SDEs with JDPCAs. In particular, the presented numerical results show that the CSSB method is comparable in computational accuracy with CSSBE, SSBE and TE methods. Nevertheless, we must state that the concerned four methods in Examples 5.1–5.2 are all sensitive on the stepsize to different extent. This is due to the fact that the solved problems are all stiff and the stability of the methods depend on the used stepsize in different degree. In particular, the TE method is an explicit method, which is not suitable for solving stiff SDEs. In the end, we also point out that, for the non-stiff problems of SDEs, the above methods are still quite effective, which can be found in the listed references of this paper.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 11971010) and Scientific Research Project of Education Department of Hubei Province (Grant No. B2019184).

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