SCIENCE CHINA
Mathematics

• ARTICLES •

June 2021 Vol. 64 No. 6: 1305–1320
https://doi.org/10.1007/s11425-019-1627-7

Superconvergence of local discontinuous Galerkin methods with generalized alternating fluxes for 1D linear convection-diffusion equations

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Received June 10, 2019; accepted December 18, 2019; published online January 21, 2020

Abstract This paper investigates superconvergence properties of the local discontinuous Galerkin methods with generalized alternating fluxes for one-dimensional linear convection-diffusion equations. By the technique of constructing some special correction functions, we prove the $(2k+1)$-th-order superconvergence for the cell averages, and the numerical traces in the discrete L^2 norm. In addition, superconvergence of orders $k+2$ and $k+1$ is obtained for the error and its derivative at generalized Radau points. All the theoretical findings are confirmed by numerical experiments.

Keywords local discontinuous Galerkin method, superconvergence, correction function, Radau points

MSC(2020) 65M12, 65M60

Citation: Liu X B, Zhang D Z, Meng X, et al. Superconvergence of local discontinuous Galerkin methods with generalized alternating fluxes for 1D linear convection-diffusion equations. Sci China Math, 2021, 64: 1305–1320, https://doi.org/10.1007/s11425-019-1627-7

1 Introduction

In this paper, we consider the local discontinuous Galerkin (LDG) methods for one-dimensional linear convection-diffusion equations

$$u_t + u_x - u_{xx} = 0, \quad (x,t) \in [0,2\pi] \times (0,T], \tag{1.1a}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.1b}$$

where u_0 is sufficiently smooth. We consider the periodic boundary condition $u(0,t) = u(2\pi,t)$, the mixed boundary condition $u(0,t) = g_1(t)$, $u_x(2\pi,t) = g_2(t)$, and the Dirichlet boundary condition $u(0,t) = g_3(t)$, $u(2\pi,t) = g_4(t)$. We study the superconvergence property concerning Radau points, cell averages, and supercloseness of the LDG method with generalized alternating numerical fluxes, including the case for which the parameters involved in the numerical fluxes for the prime variable regarding the convection part and the diffusion part are independently chosen for solving (1.1).

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As an extension of the discontinuous Galerkin (DG) method for solving first-order hyperbolic equations, the LDG method was proposed by Cockburn and Shu [16] in the framework of solving second-order convection-diffusion equations. The idea of the LDG methods is to rewrite the original equation with high spatial derivatives as a first-order system so that the DG method can be applied. Note that, in addition to the stability issue, the local solvability of auxiliary variables introduced should also be guaranteed when choosing numerical fluxes.

Being a deeper insight of DG methods, superconvergence has been investigated basically measured in the discrete L^2 norm for Radau points as well as cell averages, in the L^2 norm for the error between the numerical solution and a particular projection of the exact solution (supercloseness), and in the weak negative-order norm for enhancing accuracy. For example, by virtue of the duality argument in combination with the standard optimal *a priori* error estimates, Cockburn et al. [15] proved that the post-processed error is of order $2k+1$ superconvergent in the L^2 norm for linear hyperbolic systems and Ji et al. [21] demonstrated that the smoothness-increasing accuracy-conserving (SIAC) filter can be extended to the multi-dimensional linear convection-diffusion equation in order to obtain a $(2k+m)$-th-order superconvergence, where $m=0$, $m=\frac{1}{2}$ or $m=1$. Here and below, k denotes the polynomial degree of the discontinuous finite element space. Later, to efficiently compute multi-dimensional problems, the line filter and the one-dimensional kernel are designed via rotation in [17], and a rigorous proof of the post-processed errors is also given. For arbitrary non-uniform regular meshes, by establishing the relationship of the numerical solution and the auxiliary variable as well as its time derivative, superconvergence of order $k+3/2$ is proved for linear convection-diffusion equations [14]. For supercloseness results concerning high order equations, see, for example, [20, 23]. Note that aforementioned supercloseness results are not sharp. In view of this, Yang and Shu [25] adopted the dual argument to study the sharp superconvergence of the LDG method for one-dimensional linear parabolic equations, and the improved superconvergence results of order $k+2$ were obtained in terms of supercloseness and Radau points.

Recently, motivated by the successful applications of correction function techniques to finite element methods and finite volume methods for elliptic equations [11], Cao et al. [5–8] studied superconvergence properties of DG and LDG methods for linear hyperbolic and parabolic equations. Specifically, they offered a novel proof to derive the $(2k+1)$-th- or $(2k+1/2)$-th-order superconvergence rate for the cell average and numerical fluxes, which will lead to the sharp $(k+2)$-th-order superconvergence for supercloseness as well as the function errors at downwind-biased points. Note that these superconvergent results are based on a supercloseness property of the DG solution to an interpolation function consisting of the difference between a standard Gauss-Radau projection of the exact solution and a carefully designed correction function. It is worth pointing out that a suitable correction is introduced to balance the difference between the projection errors for the inner product term and the DG spatial operator term, and for standard optimal error estimates when a Gauss-Radau projection is used, the projection error involved in the DG operator term is exactly zero. This indicates that the standard Gauss-Radau projection is not the best choice for superconvergence analysis. The superconvergence of the direct DG (DDG) method for the one-dimensional linear convection-diffusion equation was studied in [4]. We would like to remark that all the works mentioned above are focused on purely upwind and alternating numerical fluxes.

In order to obtain flexible numerical dissipation with potential applications to nonlinear systems, the upwind-biased flux was proposed in [24], which is a linear combination of the numerical solution from both sides of interfaces. Stability and optimal error estimates were obtained by constructing and analyzing some suitable *global* projections with emphasis on the analysis to some circulant matrices. Note that the design of *global* projections is similar to those in the work for the Burgers-Poisson equation [22]. Moreover, Cheng et al. [13] studied the LDG methods for the linear convection-diffusion equations when the generalized alternating fluxes were used, and they obtained the optimal L^2 norm error estimate in a unified setting, especially when numerical fluxes with different weights are considered. In [3], Cao et al. investigated the superconvergence of DG methods based on upwind-biased fluxes for one-dimensional linear hyperbolic equations. More recently, Frean and Ryan [18] proved that the use of SIAC filters was still able to extract the superconvergence information and obtained a globally smooth and superconvergent solution of order $2k+1$ for linear hyperbolic equations based on upwind-biased fluxes. Moreover, the

$\alpha\beta$-fluxes, which were introduced as linear combinations of the average and jumps of the solution as well as the auxiliary variables at cell interfaces, have been a hot research topic in recent years [1, 12, 19].

In the current paper, we aim at analyzing the superconvergence properties of LDG methods by using generalized alternating numerical fluxes for the convection-diffusion equations. The contribution of this paper is to consider the more flexible generalized alternating fluxes. The critical step in deriving superconvergence is to construct special interpolation functions for both variables (the exact solution u and the auxiliary variable q) with the aid of some suitable correction functions, essentially following [3]. Taking into account the stability result, we use special projections to eliminate or control the troublesome jump terms involving projection errors (see, e.g., Lemma 3.2 below). To be more precise, we will establish the superconvergence between the LDG solution (u_h, q_h) and special interpolation functions $u_I^\ell = P_\theta u - W_u^\ell$ as well as $q_I^\ell = P_{\tilde{\theta}} q - W_q^\ell$, where W_u^ℓ and W_q^ℓ are correction functions to be specified later, with the main technicality being the construction and analysis of some suitable projections tailored to the very choice of the numerical fluxes. By a rigorous mathematical proof, we prove a superconvergence rate of $2k+1$ for the errors of numerical traces and for the cell averages, and $k+2$ for the DG error at generalized Radau points.

The rest of this paper is organized as follows. In Section 2, we present the LDG method with generalized alternating fluxes. In Section 3, we construct special functions to correct the error between the LDG solution and the standard Gauss-Radau projections of the exact solution. Section 4 is the main body of the paper, in which we show and prove some superconvergence phenomena for cell averages and generalized Radau points for the periodic boundary conditions. Other boundary cases including the mixed boundary condition and Dirichlet boundary condition will be considered in Section 5, and the choice of numerical initial discretization is also given. In Section 6, we present some numerical experiments that confirm the sharpness of our theoretical results. We will end in Section 7 with concluding remarks and some possible future work.

2 The LDG scheme

In this section, we present the LDG scheme with generalized alternating fluxes for the linear convection-diffusion equation (1.1). As usual, we divide the computational domain $\Omega = [0, 2\pi]$ into N cells

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi.$$

For any positive integer r, we define $\mathbb{Z}_r = \{1, \ldots, r\}$ and denote

$$x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}), \quad I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad j \in \mathbb{Z}_N$$

as the cell centers and cells, respectively. Let $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ be the length of the cell I_j for $j \in \mathbb{Z}_N$ and $h = \max_{1\leqslant j\leqslant N} h_j$. We assume that the partition Ω_h is quasi-uniform in the sense that there exists a constant C independent of h such that $Ch \leqslant h_j \leqslant h$, as h goes to zero. Define the finite element space

$$V_h^k = \{v \in L^2(\Omega) : v|_{I_j} \in P^k(I_j), \forall\, j \in \mathbb{Z}_N\},$$

where $P^k(I_j)$ is the space of polynomials on I_j of degree at most $k \geqslant 0$. We use

$$\bar{u}_{j+\frac{1}{2}} = \frac{1}{2}(u^+_{j+\frac{1}{2}} + u^-_{j+\frac{1}{2}}), \quad [u]_{j+\frac{1}{2}} = u^+_{j+\frac{1}{2}} - u^-_{j+\frac{1}{2}}$$

to denote the mean and jump of the function u at each element boundary point $x_{j+\frac{1}{2}}$, and the weighted average is denoted by $u^{(\theta)}_{j+\frac{1}{2}} = \theta u^-_{j+\frac{1}{2}} + \tilde{\theta} u^+_{j+\frac{1}{2}}$, $\tilde{\theta} = 1 - \theta$, where $u^+_{j+\frac{1}{2}}$ and $u^-_{j+\frac{1}{2}}$ are the traces from the right and left cells, respectively.

Throughout this paper, we employ $W^{\ell,p}(D)$ to denote the standard Sobolev space on D equipped with the norm $\|\cdot\|_{W^{\ell,p}(D)}$ with $\ell \geqslant 0, p = 2$ and $p = \infty$. For simplicity, we set $\|\cdot\|_{W^{\ell,p}(D)} = \|\cdot\|_{\ell,p,D}$ with D

equal to Ω or I_j. The subscript D will be omitted when $D=\Omega$, and $W^{\ell,p}(D)$ can be written as $H^\ell(D)$ when $p=2$.

In order to construct the LDG scheme, we first introduce an auxiliary variable $q=u_x$; then the problem (1.1) can be written into a first-order system

$$u_t+(u-q)_x=0,\quad q-u_x=0, \tag{2.1}$$

where $(u-q,u)$ is the physical flux and u is the so-called prime variable. The LDG scheme is thus to find $u_h,q_h\in V_h^k$ such that for all test functions $v,\psi\in V_h^k$,

$$(u_{ht},v)_j-(u_h-q_h,v_x)_j+(\tilde{u}_h-\hat{q}_h)v^-\,|_{j+\frac12}-(\tilde{u}_h-\hat{q}_h)v^+\,|_{j-\frac12}=0, \tag{2.2a}$$

$$(q_h,\psi)_j+(u_h,\psi_x)_j-\hat{u}_h\psi^-\,|_{j+\frac12}+\hat{u}_h\psi^+\,|_{j-\frac12}=0. \tag{2.2b}$$

Here, $(u,v)_j=\int_{I_j}uvdx$, and $\tilde{u}_h,\hat{q}_h$ and $\hat{u}_h$ are numerical fluxes. We use the generalized alternating numerical fluxes related to arbitrary parameters λ and θ as in [13], i.e.,

$$(\tilde{u}_h-\hat{q}_h,\hat{u}_h)=(u_h^{(\lambda)}-q_h^{(\tilde{\theta})},u_h^{(\theta)}). \tag{2.3}$$

Note that the parameters in the numerical flux regarding the convection part and diffusion part can be chosen independently, and to ensure stability the weight λ should satisfy $\lambda\geqslant\frac12$.

For simplicity, we introduce the notation pertaining to the DG discretization operator

$$\mathcal{H}^1(u,q;v)=\sum_{j=1}^N\mathcal{H}_j^1(u,q;v),\quad \mathcal{H}^2(u;\psi)=\sum_{j=1}^N\mathcal{H}_j^2(u;\psi),$$

where

$$\mathcal{H}_j^1(u,q;v)=(q-u,v_x)_j-(\hat{q}-\tilde{u})v^-\,|_{j+\frac12}+(\hat{q}-\tilde{u})v^+\,|_{j-\frac12},$$
$$\mathcal{H}_j^2(u;\psi)=(u,\psi_x)_j-\hat{u}\psi^-\,|_{j+\frac12}+\hat{u}\psi^+\,|_{j-\frac12}.$$

Thus, by Galerkin orthogonality, the cell error equation can be written as

$$(e_{ut},v)_j+(e_q,\psi)_j+\mathcal{H}_j^1(e_u,e_q;v)+\mathcal{H}_j^2(e_u;\psi)=0,\quad \forall\,v,\psi\in V_h^k, \tag{2.4}$$

where $e_u=u-u_h,e_q=q-q_h$.

For optimal error estimates of the LDG scheme using the generalized numerical fluxes (2.3) solving convection-diffusion equations with the periodic boundary conditions, a globally defined projection P_θ together with $P_{\tilde{\theta}}$ is usually needed. For $z\in H^1(\Omega_h)=\bigcup_{j\in\mathbb{Z}_N}H^1(I_j)$, the generalized Gauss-Radau projection $P_\theta z$ is defined as the element of V_h^k that satisfies

$$\int_{I_j}(P_\theta z-z)v_hdx=0,\quad \forall\,v_h\in P^{k-1}(I_j), \tag{2.5a}$$

$$(P_\theta z)_{j+\frac12}^{(\theta)}=(z^{(\theta)})_{j+\frac12},\quad \forall\,j\in\mathbb{Z}_N. \tag{2.5b}$$

It has been shown in [22,24] that the projection $P_\theta z$ is well defined for $\theta\neq\frac12$, and for $\theta=1/2$ some restrictions on the mesh as well as the polynomial degree are needed to guarantee the existence and optimal approximation property of the projection [2]. Note that when the parameter θ is taken as 0 or 1, the projection P_θ reduces to the standard local Gauss-Radau projection P_h^+ or P_h^- as defined in [10]. Besides, the projection P_θ satisfies the following optimal approximation property [22,24]:

$$\|z-P_\theta z\|_{I_j}+h^{\frac12}\|z-P_\theta z\|_{\infty,I_j}\leqslant Ch^{k+\frac32}\|z\|_{k+1,\infty}, \tag{2.6}$$

where $C>0$ is independent of h and z.

To obtain the superconvergence results, the following lemma is useful in describing correction functions.

Lemma 2.1 (See [3]). *Suppose A is an $N\times N$ circulant matrix with the first row $(\theta,(-1)^k(1-\theta),0,\ldots,0)$ and the last row $((-1)^k(1-\theta),0,0,\ldots,\theta)$, where $\theta>1/2$. Then, for any vectors $X=(x_1,\ldots,x_N)^{\mathrm{T}}$ and $b=(b_1,\ldots,b_N)^{\mathrm{T}}$ satisfying $AX=b$, it holds that $|x_j|\lesssim\|b\|_\infty$, $\forall\,j\in\mathbb{Z}_N$.*

3 Correction functions

In what follows, we present the construction of correction functions. The cases for the weights of the prime variable u_h in (2.3) being the same or different are discussed in the following two subsections.

3.1 The case with $\lambda = \theta$ in (2.3)

When $\lambda = \theta$ in (2.3), to construct special interpolation functions (u_I^ℓ, q_I^ℓ) by modifying generalized Gauss-Radau projections with correction functions so that they are superclose to the LDG solution (u_h, q_h), we start by denoting by $L_{j,k}$ the standard Legendre polynomial of degree k on the interval I_j, and assume that the function $v(x,t)$ has the following Legendre expansion, i.e., on each $I_j, j \in \mathbb{Z}_N$,

$$v(x,t) = \sum_{m=0}^{\infty} v_{j,m}(t) L_{j,m}(x), \quad v_{j,m} = \frac{2m+1}{h_j}(v, L_{j,m})_j.$$

By the definition of P_θ in (2.5a), we can rewrite $P_\theta v$ into the following form:

$$P_\theta v = \sum_{m=0}^{k} v_{j,m}(t) L_{j,m}(x) + \bar{v}_{j,k}(t) L_{j,k}(x),$$

where $\bar{v}_{j,k}$ can be determined by $(v - P_\theta v)^{(\theta)}_{j+1/2} = 0$ with

$$v - P_\theta v = -\bar{v}_{j,k}(t) L_{j,k}(x) + \sum_{m=k+1}^{\infty} v_{j,m}(t) L_{j,m}(x). \tag{3.1}$$

It follows from the orthogonality of Legendre polynomials and (2.6) that

$$|\bar{v}_{j,k}| \lesssim \frac{2k+1}{h_j} |(v - P_\theta v, L_{j,k})_j| \lesssim h^{k+1} \|v\|_{k+1,\infty}.$$

Following [3], to balance projection errors for the inner product term and the DG operator term, we define an integral operator D_x^{-1} by

$$D_x^{-1} u(x) = \frac{1}{\bar{h}_j} \int_{x_{j-\frac{1}{2}}}^{x} u(\tau) d\tau, \quad \tau \in I_j,$$

where $\bar{h}_j = h_j/2$. Obviously, $u(x) = \bar{h}_j (D_x^{-1} u(x))_x$. Moreover, by the properties of Legendre polynomials, we have

$$D_x^{-1} L_{j,k}(x) = \frac{1}{2k+1}(L_{j,k+1} - L_{j,k-1})(x). \tag{3.2}$$

To clearly see how to cancel terms involving projection errors with the goal of obtaining superconvergence, we split the errors e_u and e_q into two parts:

$$e_u = u - u_h = u - u_I^\ell + u_I^\ell - u_h \triangleq \eta_u + \xi_u, \quad e_q = q - q_h = q - q_I^\ell + q_I^\ell - q_h \triangleq \eta_q + \xi_q.$$

Then the error equation (2.4) becomes

$$(\xi_{ut}, v)_j + (\xi_q, \psi)_j + \mathcal{H}_j^1(\xi_u, \xi_q; v) + \mathcal{H}_j^2(\xi_u; \psi) = -(\eta_{ut}, v)_j - (\eta_q, \psi)_j - \mathcal{H}_j^1(\eta_u, \eta_q; v) - \mathcal{H}_j^2(\eta_u; \psi).$$

For the periodic boundary conditions, by choosing $v = \xi_u, \psi = \xi_q$ and summing over all j, we have

$$\frac{1}{2}\frac{d}{dt}\|\xi_u\|^2 + \|\xi_q\|^2 + \left(\lambda - \frac{1}{2}\right) \sum_{j=1}^{N} [\xi_u]^2_{j+\frac{1}{2}} = -(\eta_{ut}, \xi_u) - (\eta_q, \xi_q) - \mathcal{H}^1(\eta_u, \eta_q; \xi_u) - \mathcal{H}^2(\eta_u; \xi_q). \tag{3.3}$$

From the equation (3.3), we can see that in order to obtain the supercloseness properties between the numerical solution u_h and interpolation function u_I^ℓ, we need to obtain a sharp superconvergent bound for the right-hand term, essentially using the switch of the time derivative and spatial derivative through the integral operator D_x^{-1} in combination with integration by parts (see Lemma 3.2 below). Next, we show how to construct interpolation functions and estimate the right-hand side of (3.3).

To construct the interpolation functions (u_I^ℓ, q_I^ℓ), we define a series of functions $w_{u,i}, w_{q,i} \in V_h^k, i \in \mathbb{Z}_k$ as follows:

$$(w_{u,i} - \bar{h}_j D_x^{-1} w_{q,i-1}, v)_j = 0, \quad (w_{u,i}^{(\theta)})_{j+\frac{1}{2}} = 0, \tag{3.4a}$$

$$(w_{q,i} - w_{u,i} - \bar{h}_j D_x^{-1} \partial_t w_{u,i-1}, v)_j = 0, \quad (w_{q,i}^{(\tilde{\theta})})_{j+\frac{1}{2}} = 0, \tag{3.4b}$$

where $v \in P^{k-1}(I_j)$ and $w_{u,0} = u - P_\theta u$, $w_{q,0} = q - P_{\tilde{\theta}} q$.

Lemma 3.1. *The functions $w_{u,i}, w_{q,i}, i \in \mathbb{Z}_k$ defined in* (3.4) *have the following properties*:

$$\|\partial_t w_{u,i}\|_\infty \lesssim h^{k+i+1} \|u\|_{k+i+3,\infty}, \quad \|w_{q,i}\|_\infty \lesssim h^{k+i+1} \|u\|_{k+i+2,\infty}, \tag{3.5a}$$

$$(w_{u,i}, v)_j = 0, \quad (w_{q,i}, v)_j = 0, \quad \forall\, v \in P^{k-i-1}(I_j). \tag{3.5b}$$

Proof. The proof of this lemma is based on deriving the following expression of $w_{u,i}$ and $w_{q,i}$ in each element I_j, which can be obtained by induction. It holds that

$$w_{u,i}\,|_{I_j} = \sum_{m=k-i}^{k} \beta_{i,m}^j L_{j,m}(x), \quad w_{q,i}\,|_{I_j} = \sum_{m=k-i}^{k} \gamma_{i,m}^j L_{j,m}(x), \quad i \in \mathbb{Z}_k. \tag{3.6}$$

Step 1. When $i = 1$, by taking $v = L_{j,m}$ with $m \leqslant k-1$ in (3.4a) and using (3.2) together with the orthogonality property of Legendre polynomials, we obtain

$$(w_{u,1} - \bar{h}_j D_x^{-1} w_{q,0}, v) = \left(\beta_{1,k-1}^j L_{j,k-1} - \frac{\bar{q}_{j,k}}{2k+1} \bar{h}_j L_{j,k-1}, v\right) = 0.$$

Obviously, $\beta_{1,k-1}^j = \frac{\bar{q}_{j,k}}{2k+1} \bar{h}_j$, where $\bar{q}_{j,k}$ is the coefficient of the Legendre expansion for q; see (3.1) with v replaced by q and P_θ replaced by $P_{\tilde{\theta}}$. Using the fact that $(w_{u,1}^{(\theta)})_{j+\frac{1}{2}} = 0$ we have

$$\theta \beta_{1,k}^j + (-1)^k (1-\theta) \beta_{1,k}^{j+1} = (-1)^k (1-\theta) \beta_{1,k-1}^{j+1} - \theta \beta_{1,k-1}^j. \tag{3.7}$$

Then the linear system (3.7) can be written in the matrix-vector form $A\beta_{1,k} = b$, where $A = \operatorname{circ}(\theta, (-1)^k (1-\theta), 0, \ldots, 0)$ is an $N \times N$ circulant matrix and

$$\beta_{1,k} = \begin{pmatrix} \beta_{1,k}^1 \\ \beta_{1,k}^2 \\ \vdots \\ \beta_{1,k}^N \end{pmatrix}, \quad b = \begin{pmatrix} -\theta \beta_{1,k-1}^1 + (-1)^k (1-\theta) \beta_{1,k-1}^2 \\ -\theta \beta_{1,k-1}^2 + (-1)^k (1-\theta) \beta_{1,k-1}^3 \\ \vdots \\ -\theta \beta_{1,k-1}^N + (-1)^k (1-\theta) \beta_{1,k-1}^1 \end{pmatrix}.$$

It is easy to compute the determinant of the matrix A in the form

$$|A| = \theta^N (1 - p^N), \quad p = \frac{(-1)^k (\theta - 1)}{\theta},$$

and for $\theta \neq \frac{1}{2}$ the matrix A is always invertible. Therefore, the linear system (3.7) has the unique solution. Moreover, by Lemma 2.1, we have

$$|\beta_{1,k}^j| \lesssim \max_{1 \leqslant \ell \leqslant N} |b_\ell| \lesssim h^{k+2} \|u\|_{k+2,\infty}, \quad \forall\, j \in \mathbb{Z}_N.$$

Thus,

$$\|\partial_t w_{u,1}\|_{\infty,I_j} = \|\partial_t (\beta_{1,k-1}^j L_{j,k-1} + \beta_{1,k}^j L_{j,k})\|_{\infty,I_j} \lesssim h_j |\partial_t \bar{q}_{j,k}| \lesssim h^{k+2} \|u\|_{k+4,\infty}.$$

Similarly, when choosing $v = L_{j,m}, m \leqslant k-1$ in (3.4b), we obtain

$$w_{q,1}|_{I_j} = \sum_{m=k-1}^{k} \gamma_{1,m}^j L_{j,m},$$

where $\gamma_{1,k-1}^j = \beta_{1,k-1}^j + \frac{\partial_t \bar{u}_{j,k}}{2k+1}\bar{h}_j$, and $\gamma_{1,k}^j$ is the solution of the linear system $\tilde{A}\gamma_{1,k} = \tilde{b}$, with $\tilde{A} = \mathrm{circ}(\tilde{\theta}, (-1)^k(1-\tilde{\theta}), 0, \ldots, 0)$ being an $N \times N$ circulant matrix and

$$\gamma_{1,k} = \begin{pmatrix} \gamma_{1,k}^1 \\ \gamma_{1,k}^2 \\ \vdots \\ \gamma_{1,k}^N \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} -\tilde{\theta}\gamma_{1,k-1}^1 + (-1)^k(1-\tilde{\theta})\gamma_{1,k-1}^2 \\ -\tilde{\theta}\gamma_{1,k-1}^2 + (-1)^k(1-\tilde{\theta})\gamma_{1,k-1}^3 \\ \vdots \\ -\tilde{\theta}\gamma_{1,k-1}^N + (-1)^k(1-\tilde{\theta})\gamma_{1,k-1}^1 \end{pmatrix}.$$

Consequently, the estimate of $\|w_{q,1}\|_\infty$ in (3.5a) follows by using Lemma 2.1 and the optimal approximation property (2.6). Moreover, (3.5b) is a trivial consequence of the expression (3.6) when the orthogonality property of Legendre polynomials is taken into account.

Step 2. Suppose that (3.5a) and (3.6) are valid for all $i \leqslant k-1$ and we want to prove that it still holds for $i+1$. From (3.4a), we can get

$$\left(w_{u,i+1} - \bar{h}_j D_x^{-1}\left(\sum_{m=k-i}^{k} \gamma_{i,m}^j L_{j,m}\right), v\right)_j = 0, \quad \forall\, v \in P^{k-1}(I_j).$$

In order to get the superconvergent bounds of $w_{u,i+1}$, we need to find out the expression of coefficient $\beta_{i+1,m}$. After a direct calculation, we have

$$\beta_{i+1,k-i-1}^j = -\frac{\gamma_{i,k-i}^j \bar{h}_j}{2(k-i)+1}, \quad \beta_{i+1,k-i}^j = -\frac{\gamma_{i,k-i+1}^j \bar{h}_j}{2(k-i)+3},$$
$$\beta_{i+1,m}^j = \bar{h}_j\left(\frac{\gamma_{i,m-1}^j}{2m-1} - \frac{\gamma_{i,m+1}^j}{2m+3}\right), \quad m = k-i+1, \ldots, k-1.$$

Moreover, by the fact that $w_{u,i+1}^{(\theta)} = 0$, we get

$$\theta(\beta_{i+1,k-i-1}^j + \cdots + \beta_{i+1,k}^j) + (1-\theta)(-1)^{k-i-1}\beta_{i+1,k-i-1}^{j+1} + \cdots + (1-\theta)(-1)^k\beta_{i+1,k}^{j+1} = 0.$$

Again, we can write the above linear system into the matrix-vector form $A\beta_{i+1,k} = c$, and when $\theta \neq \frac{1}{2}$, we arrive at the unique existence of the system. Consequently, it follows from Lemma 2.1 that

$$\begin{aligned}\|\partial_t w_{u,i+1}\|_{\infty,I_j} &\lesssim \sum_{m=k-i-1}^{k} |\partial_t \beta_{i+1,m}^j| \lesssim h \sum_{m=k-i}^{k} |\partial_t \gamma_{i,m}^j| \\ &\lesssim h\|\partial_t w_{q,i}\|_\infty \lesssim h^{k+i+2}\|\partial_t q\|_{k+i+1,\infty} \lesssim h^{k+i+2}\|u\|_{k+i+4,\infty}.\end{aligned}$$

Analogously, the other estimate of (3.5a) can be obtained, and the orthogonality property in (3.5b) is a trivial consequence of expressions of $w_{u,i}$ and $w_{q,i}$ in (3.6) with i replaced by $i+1$. This finishes the proof of Lemma 3.1. □

We are now ready to define the correction functions as follows. For any positive integer $\ell \in \mathbb{Z}_k$, we define in each element I_j,

$$W_u^\ell = \sum_{i=1}^{\ell} w_{u,i}, \quad W_q^\ell = \sum_{i=1}^{\ell} w_{q,i}, \tag{3.8}$$

and the special interpolation functions are

$$u_I^\ell = P_\theta u - W_u^\ell, \quad q_I^\ell = P_{\tilde{\theta}} q - W_q^\ell. \tag{3.9}$$

Lemma 3.2. *Suppose $u \in W^{k+\ell+3,\infty}(\Omega), \ell \in \mathbb{Z}_k$ is the solution of (1.1), and u_I^ℓ and q_I^ℓ are defined by (3.9). Then $\forall\, v, \psi \in V_h^k$, we have*

$$|((u-u_I^\ell)_t, v)_j - (W_u^\ell, v_x)_j + (W_q^\ell, v_x)_j| \lesssim h^{k+\ell+1}\|u\|_{k+\ell+3,\infty}\|v\|_{1,I_j}, \tag{3.10a}$$

$$|(q-q_I^\ell, \psi)_j + (W_u^\ell, \psi_x)_j| \lesssim h^{k+\ell+1}\|u\|_{k+\ell+2,\infty}\|\psi\|_{1,I_j}. \tag{3.10b}$$

Proof. By the orthogonality property of $w_{u,i}$ and $w_{q,i}, i \in \mathbb{Z}_{k-1}$, we have

$$D_x^{-1} w_{u,i}(x^-_{j+\frac{1}{2}}) = \frac{1}{\bar{h}_j}(w_{u,i}, 1)_j = 0 = D_x^{-1} w_{u,i}(x^+_{j-\frac{1}{2}}),$$
$$D_x^{-1} w_{q,i}(x^-_{j+\frac{1}{2}}) = \frac{1}{\bar{h}_j}(w_{q,i}, 1)_j = 0 = D_x^{-1} w_{q,i}(x^+_{j-\frac{1}{2}}).$$

It follows from integration by parts that

$$(\partial_t w_{u,i}, v)_j = -\bar{h}_j(D_x^{-1}\partial_t w_{u,i}, v_x)_j = -(w_{q,i+1} - w_{u,i+1}, v_x), \quad v \in V_h^k, \quad i \in \mathbb{Z}_{k-1},$$
$$(w_{q,i}, v)_j = -\bar{h}_j(D_x^{-1} w_{q,i}, v_x)_j = -(w_{u,i+1}, v_x), \quad v \in V_h^k, \quad i \in \mathbb{Z}_{k-1}.$$

Then

$$\begin{aligned}((u-u_I^\ell)_t, v)_j - (W_u^\ell, v_x)_j + (W_q^\ell, v_x)_j &= ((u-P_\theta u)_t, v)_j + \sum_{i=1}^{\ell}[(\partial_t w_{u,i}, v)_j + (w_{q,i} - w_{u,i}, v_x)_j] \\ &= (\partial_t w_{u,\ell}, v)_j.\end{aligned}$$

Similarly, it holds that

$$(q - q_I^\ell, \psi)_j + (W_u^\ell, \psi_x)_j = (w_{q,\ell}, \psi)_j.$$

By (3.5a), we can get the desired result (3.10). □

3.2 The case with $\lambda \neq \theta$ in (2.3)

When parameters λ and θ in (2.3) pertaining to convection and diffusion terms are chosen differently, a pair of suitable interpolation functions in possession of the supercloseness property should be constructed, which are based on a combination of modified projections and new correction functions. To do that, let us first recall a new modified projection [13], i.e., $\Pi_h(u,q) = (P_\theta u, P_{\tilde{\theta}}^* q)$, in which $P_\theta u \in V_h^k$ has been given in (2.5a), and $P_{\tilde{\theta}}^* q \in V_h^k$ depends on both u and q satisfying

$$\int_{I_j} (P_{\tilde{\theta}}^* q) v_h dx = \int_{I_j} q v_h dx, \quad \forall\, v_h \in P^{k-1}(I_j),$$
$$(P_{\tilde{\theta}}^* q)^{(\tilde{\theta})}_{j+\frac{1}{2}} = (q^{(\tilde{\theta})})_{j+\frac{1}{2}} + (\lambda - \theta)[u - P_\theta u]_{j+\frac{1}{2}}$$

for any $j = 1, \ldots, N$. Moreover, this projection have the following approximation property:

$$\|q - P_{\tilde{\theta}}^* q\|_{I_j} \leqslant C h^{k+\frac{3}{2}}(\|q\|_{k+1,\infty} + |\lambda - \theta| \cdot \|u\|_{k+1,\infty}).$$

From the above estimate of $q - P_{\tilde{\theta}}^* q$, it is easy to see that the coefficient $\bar{q}_{j,k}$ can be controled by the prime and auxiliary variables. It holds that

$$|\bar{q}_{j,k}| \lesssim \frac{2k+1}{h_j}|(q - P_{\tilde{\theta}}^* q, L_{j,k})| \lesssim h^{k+1}(\|q\|_{k+1,\infty} + |\lambda - \theta| \cdot \|u\|_{k+1,\infty}) \lesssim h^{k+1}\|u\|_{k+2,\infty}.$$

Next, the corresponding correction functions pertaining to two different weights λ and θ can be easily defined. Specifically, we define the functions $w_{u,i}, w_{q,i}, i \in \mathbb{Z}_k$ satisfying

$$(w_{u,i} - \bar{h}_j D_x^{-1} w_{q,i-1}, z)_j = 0, \quad (w_{u,i}^{(\theta)})_{j+\frac{1}{2}} = 0, \tag{3.11a}$$

$$(w_{q,i} - w_{u,i} - \bar{h}_j D_x^{-1}\partial_t w_{u,i-1}, z)_j = 0, \quad (w_{q,i}^{(\tilde{\theta})})_{j+\frac{1}{2}} = (w_{u,i}^{(\lambda)})_{j+\frac{1}{2}}, \tag{3.11b}$$

where $z \in P^{k-1}(I_j)$, and $w_{u,0} = u - P_\theta u, w_{q,0} = q - P_{\tilde{\theta}}^* q$.

Let us finish this section by providing the following theorem.

Theorem 3.3. *When $\lambda \neq \theta$ in (2.3), the functions $w_{u,i}, w_{q,i}, i \in \mathbb{Z}_k$ defined in (3.11) still have the following properties*:

$$\|\partial_t w_{u,i}\|_\infty \lesssim h^{k+i+1}\|u\|_{k+i+3,\infty}, \quad \|w_{q,i}\|_\infty \lesssim h^{k+i+1}\|u\|_{k+i+2,\infty},$$
$$(w_{u,i}, v)_j = 0, \quad (w_{q,i}, v)_j = 0, \quad \forall\, v \in P^{k-i-1}(I_j).$$

Moreover, when $u \in W^{k+\ell+3,\infty}(\Omega), \ell \in \mathbb{Z}_k$, the special interpolation functions

$$u_I^\ell = P_\theta u - W_u^\ell, \quad q_I^\ell = P_{\tilde{\theta}}^* q - W_q^\ell$$

with (3.8) *satisfy*

$$|((u - u_I^\ell)_t, v)_j - (W_u^\ell, v_x)_j + (W_q^\ell, v_x)_j| \lesssim h^{k+\ell+1}\|u\|_{k+\ell+3,\infty}\|v\|_{1,I_j},$$
$$|(q - q_I^\ell, \psi)_j + (W_u^\ell, \psi_x)_j| \lesssim h^{k+\ell+1}\|u\|_{k+\ell+2,\infty}\|\psi\|_{1,I_j}.$$

Proof. Since there is only slight difference between (3.4) and (3.11) in terms of different boundary collocations, Theorem 3.3 can thus be proved by an argument similar to that in Subsection 3.1 with different vectors b, $\tilde{b}$ and c, etc. The detailed proof is omitted. □

4 Superconvergence

In this section, we show the superconvergence properties for the LDG solution at some special points as well as cell averages, which are mainly based on the supercloseness result for the error between the LDG solution (u_h, q_h) and the newly designed interpolation functions (u_I^ℓ, q_I^ℓ).

Theorem 4.1. *Let $u \in W^{k+\ell+3,\infty}(\Omega)$, $\ell \in \mathbb{Z}_k$ be the exact solution of* (1.1), *and u_h and q_h be the numerical solutions of LDG scheme* (2.2). *Then for the periodic boundary conditions, we have*

$$\|u_I^\ell - u_h\| + \left(\int_0^t \|q_I^\ell - q_h\|^2 d\tau\right)^{\frac{1}{2}} \leqslant C(1+t)h^{k+\ell+1},$$

where C depends on $\|u\|_{k+\ell+3,\infty}$.

Proof. Using Lemma 3.2, we obtain

$$\begin{aligned}
&|(\eta_{ut}, v)_j + (\eta_q, \psi)_j + \mathcal{H}_j^1(\eta_u, \eta_q; v) + \mathcal{H}_j^2(\eta_u; \psi)| \\
&\quad = |((u - u_I^\ell)_t, v)_j - (W_u^\ell, v_x)_j + (W_q^\ell, v_x)_j + (q - q_I^\ell, \psi)_j + (W_u^\ell, \psi_x)_j| \\
&\quad = |(\partial_t w_{u,\ell}, v)_j + (w_{q,\ell}, \psi)_j| \\
&\quad \lesssim h^{k+\ell+1}\|u\|_{k+\ell+3,\infty}(\|v\|_{1,I_j} + \|\psi\|_{1,I_j}).
\end{aligned}$$

Inserting the above estimate into (3.3) and summing over all j, one has

$$\frac{1}{2}\frac{d}{dt}\|\xi_u\|^2 + \|\xi_q\|^2 \lesssim h^{k+\ell+1}\|u\|_{k+\ell+3,\infty}(\|\xi_u\| + \|\xi_q\|).$$

If we choose a suitable initial condition such that $\|\xi_u(0)\| = 0$, then Theorem 4.1 follows by using Young's inequality and Gronwall's inequality. □

4.1 Superconvergence of numerical fluxes

In this subsection, we present the superconvergence results of the numerical fluxes.

Theorem 4.2. *Assume that $u \in W^{2k+3,\infty}(\Omega), k \geqslant 1$ is the solution of* (1.1), *and u_h and q_h are the numerical solutions of the LDG scheme* (2.2) *with the initial solution $u_h(\cdot, 0) = u_I^k(\cdot, 0)$. Then for the periodic boundary conditions, we have*

$$\|e_{u,n}\| \lesssim C(1+t)h^{2k+1}, \quad \left(\int_0^t \|e_{q,n}\|^2 d\tau\right)^{\frac{1}{2}} \lesssim C(1+t)h^{2k+1},$$

where

$$\|e_{v,n}\| = \left(\frac{1}{N}\sum_{j=1}^{N}|(v-\hat{v}_h)(x_{j+\frac{1}{2}},t)|^2\right)^{\frac{1}{2}}, \quad v=u \quad or \quad v=q.$$

Proof. It follows from the inverse inequality and the supercloseness result in Theorem 4.1 that

$$\begin{aligned}\|e_{u,n}\| &= \left(\frac{1}{N}\sum_{j=1}^{N}|(\hat{u}_I-\hat{u}_h)(x_{j+\frac{1}{2}},t)|^2\right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{1}{N}\sum_{j=1}^{N}h_j^{-1}\|u_I-u_h\|^2_{I_j\cup I_{j+1}}\right)^{\frac{1}{2}} \\ &\lesssim \|u_I^k-u_h\| \lesssim C(1+t)h^{2k+1}.\end{aligned}$$

By using the superclosenece result in Theorem 4.1 again, superconvergence of the auxiliary variable q can be derived analogously. This finishes the proof of Theorem 4.2. □

4.2 Superconvergence for cell averages

Theorem 4.3. *Assume that the conditions of Theorem* 4.1 *are satisfied. Then for the periodic boundary conditions, we have*

$$\|e_u\|_c \lesssim (1+t)h^{2k+1}\|u\|_{2k+3,\infty}, \quad \left(\int_0^t \|e_q\|_c^2 d\tau\right)^{\frac{1}{2}} \lesssim (1+t)h^{2k+1}\|u\|_{2k+3,\infty}, \tag{4.1}$$

where $\|e_v\|_c = (\frac{1}{N}\sum_{j=1}^{N}(\frac{1}{h_j}(e_v,1)_j)^2)^{\frac{1}{2}}$, $v=u$ *or* $v=q$.

Proof. Taking $\|e_u\|_c$ as an example, by the properties of P_θ and the definition of u_I^k, we obtain

$$(e_u,1)_j = (u_I^k-u_h,1)_j + (W_u^k,1)_j. \tag{4.2}$$

The superconvergence result can thus be proved by using the orthogonality property in (3.5b), the Cauchy-Schwarz inequality and Theorem 4.1. □

4.3 Superconvergence at generalized Radau points

As a natural extension of Radau points for $\theta=1$, the roots of generalized Radau polynomials for the weight θ are introduced in [18]. To be more specific, the generalized Radau polynomials are defined as

$$R_{k+1}^* = \begin{cases} L_{k+1}-(2\theta-1)L_k, & \text{when } k \text{ is even,} \\ (2\theta-1)L_{k+1}-L_k, & \text{when } k \text{ is odd.}\end{cases} \tag{4.3}$$

For superconvergence analysis, instead of using the global projection $P_\theta u$, a much simpler local projection $P_h u$ is introduced [3]:

$$\begin{aligned}&\int_{I_j}(P_hu-u)v=0, \quad \forall\, v\in P^{k-1}(I_j), \\ &\theta P_hu(x^-_{j+\frac{1}{2}})+(1-\theta)P_hu(x^+_{j-\frac{1}{2}}) = \theta u(x^-_{j+\frac{1}{2}})+(1-\theta)u(x^+_{j-\frac{1}{2}}).\end{aligned}$$

Lemma 4.4 (See [3])**.** *Suppose* $u\in W^{k+2,\infty}(\Omega)$ *and* P_hu *is the local projection of* u *defined above with* $\theta\neq\frac{1}{2}$. *Then*

$$\begin{aligned}&|(u-P_hu)(R^r_{j,m})| \lesssim h^{k+2}\|u\|_{k+2,\infty}, \\ &|\partial_x(u-P_hu)(R^{r*}_{j,m})| \lesssim h^{k+1}\|u\|_{k+2,\infty}, \\ &\|P_hu-P_\theta u\|_\infty \lesssim h^{k+2}\|u\|_{k+2,\infty}.\end{aligned}$$

Here, $R^r_{j,m}$ *and* $R^{r*}_{j,m}$ *are the roots of rescaled Radau polynomials* $R^*_{j,m+1}$ *and* $\partial_x R^*_{j,m+1}$.

We are now ready to show the superconvergence result at generalized Radau points.

Theorem 4.5. *Let $u \in W^{k+5,\infty}(\Omega)$ and u_h be the numerical solution of (1.1). Suppose $u_I^\ell, \ell \geqslant 2$ is the special interpolation function defined in (3.9). Then for the periodic boundary conditions, we have*

$$\|e_{u,r}\| \lesssim (1+t)h^{k+2}\|u\|_{k+5,\infty}, \quad \|e_{u,rx}\| \lesssim (1+t)h^{k+1}\|u\|_{k+5,\infty},$$
$$\left(\int_0^t \|e_{q,l}\|^2 d\tau\right)^{\frac{1}{2}} \lesssim (1+t)h^{k+2}\|u\|_{k+5,\infty}, \quad \left(\int_0^t \|e_{q,lx}\|^2 d\tau\right)^{\frac{1}{2}} \lesssim (1+t)h^{k+1}\|u\|_{k+5,\infty},$$

where

$$\|e_{u,r}\| = \max_{j\in\mathbb{Z}_N} |(u-u_h)(R^r_{j,m})|, \quad \|e_{u,rx}\| = \max_{j\in\mathbb{Z}_N} |(u-u_h)_x(R^{r*}_{j,m})|,$$
$$\|e_{q,l}\| = \max_{j\in\mathbb{Z}_N} |(q-q_h)(R^l_{j,m})|, \quad \|e_{q,lx}\| = \max_{j\in\mathbb{Z}_N} |(q-q_h)_x(R^{l*}_{j,m})|.$$

Here, $R^l_{j,m}$ and $R^{l*}_{j,m}$ are the roots of the rescaled Radau polynomials $R^*_{j,m+1}$ and $\partial_x R^*_{j,m+1}$ in (4.3) with θ replaced by $\tilde\theta$.

Proof. By choosing $\ell = 2$ in Theorem 4.1, we obtain $\|u_h - u_I^2\| \lesssim (1+t)h^{k+3}\|u\|_{k+5,\infty}$. From the inverse inequality, we can get $\|u_h - u_I^2\|_\infty \lesssim h^{-\frac{1}{2}}\|u_h - u_I^2\| \lesssim (1+t)h^{k+\frac{5}{2}}\|u\|_{k+5,\infty}$. By the triangle inequality,

$$|(u-u_h)(R^r_{j,m})| \lesssim \|u_h - u_I^2\|_\infty + |(u-P_hu)(R^r_{j,m})| + \|P_hu - P_\theta u\|_\infty + \|W_u^2\|_\infty \lesssim h^{k+2}\|u\|_{k+2,\infty}.$$

The superconvergence results for the derivative of errors and the auxiliary variable q can be obtained by the same arguments. This completes the proof of Theorem 4.5. □

Remark 4.6. The analysis of superconvergence is mainly based on the supercloseness between the LDG solution (u_h, q_h) and the interpolation function (u_I^ℓ, q_I^ℓ) by asking for (W_u^ℓ, W_q^ℓ) satisfying

$$(W_u^\ell)^{(\theta)}_{j+\frac{1}{2}} = 0, \quad (W_q^\ell)^{(\tilde\theta)}_{j+\frac{1}{2}} = 0, \quad j \in \mathbb{Z}_N.$$

Therefore, when $\lambda \neq \theta$, the superconvergence results for auxiliary variable q are no longer valid, since (3.11b) is needed indicating that $(W_q^\ell)^{(\tilde\theta)}_{j+\frac{1}{2}} \neq 0$. Another reason is that the superconvergent result of order $k+2$ for the difference between the local Gauss-Radau projection and the modified global projection no longer holds. In addition, when $\lambda = \theta$, superconvergence of q can be proved in the $L^2([0,T]; L^2[0,2\pi])$ norm while superconvergence can be observed numerically in the $L^\infty([0,T]; L^2[0,2\pi])$ norm.

5 Other boundary conditions

5.1 Mixed boundary conditions

For mixed boundary conditions

$$u(0,t) = g_1(t), \quad u_x(2\pi,t) = g_2(t), \tag{5.1}$$

the numerical fluxes are chosen as

$$(\tilde u_h - \hat q_h, \hat u_h)_{j+\frac{1}{2}} = \begin{cases} (g_1 - q_h^+, g_1), & j = 0, \\ (u_h^\theta - q_h^{\tilde\theta}, u_h^\theta), & j = 1,\ldots,N-1, \\ (u_h^- - g_2, u_h^-), & j = N. \end{cases} \tag{5.2}$$

The corresponding global projections P_θ and $P_{\tilde\theta}$ are modified to be in the following piecewise global version, i.e.,

$$\begin{cases} (\tilde P_\theta u, v)_j = (u,v)_j, & \forall\, v \in P^{k-1}(I_j), \\ (\tilde P_\theta u)^{(\theta)}_{j+\frac{1}{2}} = u^{(\theta)}_{j+\frac{1}{2}}, & j = 1,\ldots,N-1, \\ (\tilde P_\theta u)^-_{N+\frac{1}{2}} = u^-_{N+\frac{1}{2}}, & j = N \end{cases}$$

and

$$\begin{cases}(\tilde{P}_{\tilde{\theta}}q,\eta)_j=(q,\eta)_j, & \forall\,\eta\in P^{k-1}(I_j),\\ (\tilde{P}_{\tilde{\theta}}q)^{(\tilde{\theta})}_{j-\frac12}=q^{(\tilde{\theta})}_{j-\frac12}, & j=2,\dots,N,\\ (\tilde{P}_{\tilde{\theta}}q)^{+}_{\frac12}=q^{+}_{\frac12}, & j=1.\end{cases} \tag{5.3}$$

Obviously, the projection $\tilde{P}_\theta$ can be decoupled starting from the cell I_N and $\tilde{P}_{\tilde{\theta}}$ can be computed from the cell I_1. Moreover, we have the following optimal approximation properties.

Lemma 5.1 (See [24]). *Assume $z\in W^{k+1,\infty}(I_j)$ with $\theta\neq\frac12$. Then projection $P=\tilde{P}_\theta$ or $P=\tilde{P}_{\tilde{\theta}}$ defined above satisfies the following approximation property*:

$$\|z-Pz\|_{I_j}+h^{\frac12}\|z-Pz\|_{\infty,I_j}\leqslant Ch^{k+\frac32}\|z\|_{k+1,\infty},$$

where C is independent of h and z.

Replacing P_θ ($P_{\tilde{\theta}}$) by $\tilde{P}_\theta$ ($\tilde{P}_{\tilde{\theta}}$), we are able to construct the following correction functions in possession of supercloseness properties, i.e., for $z\in P^{k-1}(I_j)$,

$$\begin{aligned}&(w_{u,i}-\bar{h}_jD_x^{-1}w_{q,i-1},z)_j=0,\quad (w^{(\theta)}_{u,i})_{j+\frac12}=0,\quad \forall\,j\in\mathbb{Z}_{N-1},\\ &(w_{q,i}-w_{u,i}-\bar{h}_jD_x^{-1}\partial_tw_{u,i-1},z)_j=0,\quad (w^{(\tilde{\theta})}_{q,i})_{j+\frac12}=0,\quad \forall\,j\in\mathbb{Z}_{N-1},\\ &(w^-_{u,i})_{N+\frac12}=0,\quad (w^+_{q,i})_{\frac12}=0.\end{aligned}$$

The superconvergence results can thus be obtained if we follow the same argument as that in Sections 3 and 4.

5.2 Dirichlet boundary conditions

For Dirichlet boundary conditions

$$u(0,t)=g_3(t),\quad u(2\pi,t)=g_4(t) \tag{5.4}$$

following [9], we choose the numerical fluxes as

$$(\tilde{u}_h-\hat{q}_h,\hat{u}_h)_{j+\frac12}=\begin{cases}(g_3-q^+_h,g_3), & j=0,\\ (u^\theta_h-q^{\tilde{\theta}}_h,u^\theta_h), & j=1,\dots,N-1,\\ (u^-_h-q^-_h,g_4), & j=N.\end{cases}$$

Similarly, we still need to make slight changes to the projection. For the projection $\tilde{P}_{\tilde{\theta}}$, we still adopt the definition in (5.3), while the projection $\tilde{P}_\theta$ is modified as follows:

$$\begin{cases}(\tilde{P}_\theta u,v)_j=(u,v)_j, & \forall\,v\in P^{k-1}(I_j),\\ (\tilde{P}_\theta u)^{(\theta)}_{j+\frac12}=u^{(\theta)}_{j+\frac12}, & j\in\mathbb{Z}_{N-1},\\ (\tilde{P}_\theta u)^-_{N+\frac12}=u^-_{N+\frac12}+(\tilde{P}_{\tilde{\theta}}q-q)^-_{N+\frac12}.\end{cases}$$

From the last equation we can see that, compared with the mixed boundary condition, the left limit of the projection at point $x_{N+\frac12}$ consists of two parts. One is the left limit of the exact solution u at point $x_{N+\frac12}$, and the other is the left limit of the projection error of the auxiliary variable q at point $x_{N+\frac12}$. Since we do not have any information about the auxiliary variable q at the boundary, we need to use the prime variable u to eliminate the boundary term introduced by $\tilde{P}_{\tilde{\theta}}q-q$ at point $x_{N+\frac12}$.

The superconvergence results can be obtained if we define the following correction functions: for $z\in P^{k-1}(I_j)$,

$$\begin{aligned}&(w_{u,i}-\bar{h}_jD_x^{-1}w_{q,i-1},z)_j=0,\quad (w^{(\theta)}_{u,i})_{j+\frac12}=0,\quad \forall\,j\in\mathbb{Z}_{N-1},\\ &(w_{q,i}-w_{u,i}-\bar{h}_jD_x^{-1}\partial_tw_{u,i-1},z)_j=0,\quad (w^{(\tilde{\theta})}_{q,i})_{j+\frac12}=0,\quad \forall\,j\in\mathbb{Z}_{N-1},\\ &(w^+_{q,i})_{\frac12}=0,\quad (w^-_{u,i})_{N+\frac12}=(w^-_{q,i})_{N+\frac12}.\end{aligned}$$

5.3 Initial discretization

In this section, we consider how to discretize the initial datum. Initial value discretization is very important for the study of superconvergence, which can be obtained by using the same technique as that in [3]. Specifically, for the periodic boundary conditions,

(1) according to the definition of projections P_θ and $P_{\tilde{\theta}}$, calculate $w_{u,0}$ and $w_{q,0}$;
(2) calculate $w_{u,i}$ and $w_{q,i}$ by the equations (3.4);
(3) calculate $W_u^\ell = \sum_{i=1}^{\ell} w_{u,i}$ and $u_I^\ell = P_\theta u - W_u^\ell$;
(4) let $u_h(\cdot, 0) = u_I^\ell(\cdot, 0)$.

6 Numerical results

In this section, we provide numerical examples to illustrate our theoretical findings. For the time discretization, we use the explicit third-order total variation diminishing method and take $\Delta t = \mathrm{CFL} * h^2$.

Example 6.1. We consider the following problem:

$$u_t + u_x - u_{xx} = 0, \quad (x,t) \in [0, 2\pi] \times (0, T],$$
$$u(x,0) = \sin(x) - x, \quad x \in \mathbb{R}$$

with the periodic boundary condition, where the exact solution is $u(x,t) = \mathrm{e}^{-t}\sin(x-t)$.

Table 1 lists the results for u with $\lambda = \theta$, from which we observe the $(2k+1)$-th-order superconvergence for numerical traces as well as cell averages, and that the convergence orders of the error and its derivative are $k+2$ and $k+1$, respectively. Table 2 shows errors and orders for q, demonstrating that our results hold true for the auxiliary variable when $\lambda = \theta$. Moreover, the results with different weights for λ and θ are given in Table 3, and similar conclusions can be observed for u, indicating that choosing different parameters for the convection term and the diffusion term does not affect the superconvergence results as far as the prime variable u is concerned.

Example 6.2. We consider the following problem:

$$u_t + u_x - u_{xx} = 0, \quad (x,t) \in [0, 2\pi] \times (0, T],$$
$$u(x,0) = \sin(x) - x, \quad x \in \mathbb{R}$$

with the mixed boundary conditions $u(0,t) = \mathrm{e}^{-t}\sin(t) - t$ and $u_x(2\pi, t) = \mathrm{e}^{-t}\cos(t) + 1$; the exact solution is $u(x,t) = \mathrm{e}^{-t}\sin(x-t) + x - t$.

Table 1 Errors and orders for u ($\lambda = \theta, T = 1.0$, $k = 2, 3, 4$)

	N	$\|e_{un}\|$	Order	$\|e_u\|_c$	Order	$\|e_{u,r}\|$	Order	$\|e_{u,rx}\|$	Order
$k=2$	20	5.20E−08	–	1.91E−07	–	4.53E−06	–	6.95E−05	–
CFL = 0.01	40	1.83E−09	4.83	6.23E−09	4.94	2.80E−07	4.01	8.74E−06	2.99
$\lambda = 0.8$	80	6.09E−11	4.91	1.99E−10	4.96	1.74E−08	4.01	1.10E−06	2.99
$\theta = 0.8$	160	1.96E−12	4.96	6.32E−12	4.98	1.08E−09	4.00	1.38E−07	2.99
$k=3$	15	5.35E−10	–	6.62E−10	–	1.90E−07	–	1.27E−05	–
CFL = 0.005	30	3.82E−12	7.13	5.69E−12	6.86	5.53E−09	5.10	7.86E−07	4.02
$\lambda = 0.9$	45	2.07E−13	7.19	3.50E−13	6.88	7.12E−10	5.05	1.55E−07	4.00
$\theta = 0.9$	60	2.66E−14	7.13	4.80E−14	6.91	1.67E−10	5.01	4.89E−08	4.01
$k=4$	10	1.60E−11	–	5.08E−11	–	8.34E−08	–	7.88E−06	–
CFL = 0.001	15	2.04E−13	10.76	1.23E−12	9.17	7.35E−09	5.98	1.06E−06	4.94
$\lambda = 1.2$	20	1.07E−14	10.23	7.91E−14	9.54	1.31E−09	6.01	2.54E−07	4.98
$\theta = 1.2$	25	6.74E−15	2.10	9.02E−15	9.73	3.41E−10	6.01	8.33E−08	4.99

Table 2 Errors and orders for q ($\lambda = \theta, T = 1.0,\ k = 2, 3, 4$)

	N	$\|e_{qn}\|$	Order	$\|e_q\|_c$	Order	$\|e_{q,l}\|$	Order	$\|e_{q,lx}\|$	Order
$k=2$	20	1.28E−07	–	2.50E−08	–	5.10E−06	–	8.50E−05	–
CFL = 0.01	40	4.15E−09	4.94	1.04E−09	4.59	3.19E−07	3.99	1.06E−05	3.00
$\lambda = 0.7$	80	1.33E−10	4.96	3.76E−11	4.79	2.00E−08	4.00	1.33E−06	2.99
$\theta = 0.7$	160	4.21E−12	4.98	1.26E−12	4.90	1.25E−09	4.00	1.67E−07	3.00
$k=3$	15	1.55E−09	–	5.31E−10	–	4.14E−07	–	1.44E−05	–
CFL = 0.005	30	1.26E−11	6.94	3.81E−12	7.12	1.25E−08	5.05	8.92E−07	4.01
$\lambda = 0.9$	45	7.52E−13	6.96	2.07E−13	7.19	1.63E−09	5.02	1.76E−07	4.01
$\theta = 0.9$	60	1.02E−13	6.93	2.66E−14	7.12	3.83E−10	5.03	5.54E−08	4.01
$k=4$	10	5.72E−11	–	1.58E−11	–	1.02E−07	–	8.12E−06	–
CFL = 0.001	15	1.60E−12	8.82	2.03E−13	10.74	8.79E−09	6.05	1.11E−06	4.91
$\lambda = 1.2$	20	1.32E−13	8.68	1.08E−14	10.21	1.55E−09	6.03	2.63E−07	5.00
$\theta = 1.2$	25	2.00E−14	8.45	6.80E−15	2.06	4.04E−10	6.03	8.70E−08	4.95

Table 3 Errors and orders for u ($\lambda \neq \theta, T = 1.0,\ k = 2, 3, 4$)

	N	$\|e_{un}\|$	Order	$\|e_u\|_c$	Order	$\|e_{u,r}\|$	Order	$\|e_{u,rx}\|$	Order
$k=2$	20	1.41E−07	–	3.09E−07	–	4.75E−06	–	6.71E−05	–
CFL = 0.01	40	4.60E−09	4.93	9.89E−09	4.97	2.91E−07	4.03	8.58E−06	2.97
$\lambda = 1.2$	80	1.47E−10	4.96	3.13E−10	4.98	1.80E−08	4.01	1.09E−06	2.97
$\theta = 0.8$	160	4.66E−12	4.98	9.88E−12	4.99	1.12E−09	4.01	1.37E−07	2.99
$k=3$	15	1.85E−10	–	7.44E−10	–	2.59E−07	–	4.80E−06	–
CFL = 0.002	30	1.60E−12	6.85	5.46E−12	7.09	8.03E−09	5.01	3.01E−07	3.99
$\lambda = 0.9$	45	9.64E−14	6.93	3.13E−13	7.04	1.05E−09	5.01	5.96E−08	3.99
$\theta = 1.1$	60	1.28E−14	7.02	4.16E−14	7.02	2.50E−10	5.00	1.88E−08	4.00
$k=4$	10	1.87E−10	–	1.69E−10	–	8.13E−08	–	7.88E−06	–
CFL = 0.001	15	4.93E−12	8.97	4.63E−12	8.88	7.20E−09	5.98	1.06E−06	4.95
$\lambda = 0.8$	20	3.54E−13	9.15	3.35E−13	9.12	1.28E−09	6.00	2.53E−07	4.98
$\theta = 1.2$	25	4.33E−14	9.41	4.09E−14	9.43	3.38E−10	5.97	8.31E−08	4.99

Table 4 Errors and rates for the mixed boundary condition (5.1)

	N	$\lambda = \theta = 0.8$				$\lambda = \theta = 1.2$			
		$\|e_{un}\|$	Order	$\|e_u\|_c$	Order	$\|e_{un}\|$	Order	$\|e_u\|_c$	Order
P^1	40	2.30E−05	–	3.48E−05	–	8.25E−06	–	1.35E−05	–
	80	2.72E−06	3.08	4.30E−06	3.02	1.06E−06	2.95	1.75E−06	2.94
	160	3.31E−07	3.04	5.35E−07	3.01	1.35E−07	2.97	2.24E−07	2.97
	320	4.07E−08	3.02	6.67E−08	3.00	1.71E−08	2.99	2.82E−08	2.98
P^2	20	7.36E−08	–	1.83E−07	–	4.49E−07	–	7.09E−07	–
	40	2.05E−09	5.16	5.64E−09	5.02	1.37E−08	5.04	2.24E−08	4.98
	80	6.10E−11	5.07	1.76E−10	5.00	4.16E−10	5.04	6.99E−10	5.00
	160	1.87E−12	5.04	5.10E−12	5.10	1.27E−11	5.03	2.15E−11	5.02
P^3	20	9.64E−11	–	1.56E−10	–	3.90E−11	–	8.65E−11	–
	30	5.53E−12	7.05	9.63E−12	6.88	2.26E−12	7.02	4.93E−12	7.07
	40	7.04E−13	7.16	1.32E−12	6.90	3.12E−13	6.88	6.45E−13	7.07
	50	1.65E−13	6.50	2.93E−13	6.77	9.82E−14	5.17	1.82E−13	5.66

The problem is solved by the LDG scheme (2.2) with $k = 1$, $k = 2$ and $k = 3$, respectively, and the numerical fluxes are chosen as (5.2). We list various errors and corresponding convergence rates when $\lambda = \theta = 0.8$, $\lambda = \theta = 1.2$ in Table 4. The superconvergence results of order $2k+1$ at numerical traces and

Table 5 Errors and rates for the Dirichlet boundary condition (5.4)

	N	$\lambda=\theta=0.7$				$\lambda=\theta=0.9$			
		$\|e_{un}\|$	Order	$\|e_u\|_c$	Order	$\|e_{un}\|$	Order	$\|e_u\|_c$	Order
P^1	40	3.44E−05	–	5.27E−05	–	1.54E−05	–	2.51E−05	–
	80	4.15E−06	3.05	6.62E−06	2.99	1.95E−06	2.98	3.21E−06	2.96
	160	5.09E−07	3.03	8.27E−07	3.00	2.46E−07	2.99	4.07E−07	2.98
	320	6.28E−08	3.02	1.03E−07	3.00	3.09E−08	2.99	5.12E−08	2.99
P^2	20	2.03E−08	–	1.13E−07	–	7.07E−08	–	2.47E−07	–
	40	7.83E−10	4.69	3.69E−09	4.93	2.34E−09	4.92	7.89E−09	4.97
	80	3.13E−11	4.65	1.20E−10	4.94	7.56E−11	4.95	2.50E−10	4.98
	160	1.06E−12	4.88	3.81E−12	4.98	2.38E−12	4.99	7.83E−12	5.00
P^3	20	1.26E−10	–	2.05E−10	–	6.23E−11	–	1.25E−10	–
	30	7.77E−12	6.87	1.17E−11	7.06	3.78E−12	6.91	6.87E−12	7.15
	40	1.05E−12	6.94	1.65E−12	6.81	5.36E−13	6.79	1.02E−12	6.62
	50	2.57E−13	6.33	3.26E−13	7.26	1.98E−13	4.47	1.93E−13	7.48

cell averages demonstrate that the superconvergence also holds for mixed boundary conditions. In addition, to verify theoretical results for Dirichlet boundary conditions, we consider Example 6.2 with the following Dirichlet boundary conditions:

$$u(0,t)=\mathrm{e}^{-t}\sin(t)+t,\quad u(2\pi,t)=\mathrm{e}^{-t}\cos(t)-1.$$

The results are shown in Table 5, which confirms that the conclusion still holds for Dirichlet boundary conditions.

7 Concluding remarks

In this paper, we obtain the superconvergence of the LDG methods with generalized alternating numerical fluxes for solving the convection-diffusion equations. The main techniques are the construction of correction functions and analysis of the generalized Gauss-Radau projections and their modified counterparts, with the purpose of obtaining a superconvergent $(2k+1)$-th-order for the error between a special interpolation function and the LDG solution. Different boundary conditions including periodic, mixed, and Dirichlet boundary conditions are considered. The sharpness of the theoretical results is confirmed by numerical experiments. In our further work, we will consider the degenerate diffusion problems and multi-dimensional equations.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 11971132, U1637208, 71773024, 51605114 and 11501149) and the National Key Research and Development Program of China (Grant No. 2017YFB1401801).

References

1 Ainsworth M, Monk P, Muniz W. Dispersive and dissipative properties of discontinuous Galerkin finite element methods for the second-order wave equation. J Sci Comput, 2006, 27: 5–40

2 Bona J L, Chen H, Karakashian O, et al. Conservative, discontinuous Galerkin-methods for the generalized Korteweg-de Vries equation. Math Comp, 2013, 82: 1401–1432

3 Cao W, Li D, Yang Y, et al. Superconvergence of discontinuous Galerkin methods based on upwind-biased fluxes for 1D linear hyperbolic equations. M2AN Math Model Numer Anal, 2017, 51: 467–486

4 Cao W, Liu H, Zhang Z. Superconvergence of the direct discontinuous Galerkin method for convection-diffusion equations. Numer Methods Partial Differential Equations, 2017, 33: 290–317

5 Cao W, Shu C W, Yang Y, et al. Superconvergence of discontinuous Galerkin method for 2-D hyperbolic equations. SIAM J Numer Anal, 2015, 53: 1651–1671

6 Cao W, Shu C W, Zhang Z. Superconvergence of discontinuous Galerkin methods for 1-D linear hyperbolic equations with degenerate variable coefficients. M2AN Math Model Numer Anal, 2017, 51: 2213–2235

7 Cao W, Zhang Z. Superconvergence of local discontinuous Galerkin method for one-dimensional linear parabolic equations. Math Comp, 2016, 85: 63–84

8 Cao W, Zhang Z, Zou Q. Superconvergence of discontinuous Galerkin method for linear hyperbolic equations. SIAM J Numer Anal, 2014, 52: 2555–2573

9 Castillo P. An optimal error estimate for the local discontinuous Galerkin method. In: Discontinuous Galerkin Methods. Lecture Notes in Computational Science and Engineering, vol. 11. Berlin-Heidelberg: Springer, 2000, 285–290

10 Castillo P, Cockburn B, Schötzau D, et al. Optimal a priori error estimates for the *hp*-version of the local discontinuous Galerkin method for convection-diffusion problems. Math Comp, 2002, 71: 455–478

11 Chen C, Hu S. The highest order superconvergence for *bi-k* degree rectangular elements at nodes a proof of $2k$-conjecture. Math Comp, 2013, 82: 1337–1355

12 Cheng Y, Chou C S, Li F, et al. L^2 stable discontinuous Galerkin methods for one-dimensional two-way wave equations. Math Comp, 2017, 86: 121–155

13 Cheng Y, Meng X, Zhang Q. Application of generalized Gauss-Radau projections for the local discontinuous Galerkin method for linear convection-diffusion equations. Math Comp, 2017, 86: 1233–1267

14 Cheng Y, Shu C W. Superconvergence of discontinuous Galerkin and local discontinuous Galerkin schemes for linear hyperbolic and convection-diffusion equations in one space dimension. SIAM J Numer Anal, 2010, 47: 4044–4072

15 Cockburn B, Luskin M, Shu C W, et al. Enhanced accuracy by post-processing for finite element methods for hyperbolic equations. Math Comp, 2003, 72: 577–606

16 Cockburn B, Shu C W. The local discontinuous Galerkin method for time-dependent convection-diffusion systems. SIAM J Numer Anal, 1997, 35: 2440–2463

17 Docampo-S J, Ryan J K, Mirzargar M, et al. Multi-dimensional filtering: Reducing the dimension through rotation. SIAM J Sci Comput, 2017, 39: 2179–2200

18 Frean D, Ryan J. Superconvergence and the numerical flux: A study using the upwind-biased flux in discontinuous Galerkin methods. Commun Appl Math Comput, 2020, 2: 461–486

19 Fu P, Cheng Y, Li F, et al. Discontinuous Galerkin methods with optimal L^2 accuracy for one dimensional linear PDEs with high order spatial derivatives. J Sci Comput, 2019, 78: 816–863

20 Hufford C, Xing Y. Superconvergence of the local discontinuous Galerkin method for the linearized Korteweg-de Vries equation. J Comput Appl Math, 2014, 255: 441–455

21 Ji L, Xu Y, Ryan J. Accuracy-enhancement of discontinuous Galerkin solutions for convection-diffusion equations in multiple-dimensions. Math Comp, 2012, 81: 1929–1950

22 Liu H, Ploymaklam N. A local discontinuous Galerkin method for the Burgers-Poisson equation. Numer Math, 2015, 129: 321–351

23 Meng X, Shu C W, Wu B. Superconvergence of the local discontinuous Galerkin method for linear fourth-order time-dependent problems in one space dimension. IMA J Numer Anal, 2012, 32: 1294–1328

24 Meng X, Shu C W, Wu B. Optimal error estimates for discontinuous Galerkin methods based on upwind-biased fluxes for linear hyperbolic equations. Math Comp, 2016, 85: 1225–1261

25 Yang Y, Shu C W. Analysis of optimal superconvergence of local discontinuous Galerkin method for one-dimensional linear parabolic equations. J Comput Math, 2015, 33: 323–340