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Twisted toroidal Lie algebras and Moody-Rao-Yokonuma presentation

Fulin Chen[1], Naihuan Jing[2], Fei Kong[3,*] & Shaobin Tan[1]

[1]*School of Mathematical Sciences, Xiamen University, Xiamen* 361005, *China;*
[2]*Department of Mathematics, North Carolina State University, Raleigh, NC* 27695, *USA;*
[3]*School of Mathematics and Statistics, Hunan Normal University, Changsha* 410006, *China*

Email: chenf@xmu.edu.cn, jing@math.ncsu.edu, kongmath@hunnu.edu.cn, tans@xmu.edu.cn

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Abstract Let $\mathfrak{g}$ be a (twisted or untwisted) affine Kac-Moody algebra, and μ be a diagram automorphism of $\mathfrak{g}$. In this paper, we give an explicit realization for the universal central extension $\widehat{\mathfrak{g}}[\mu]$ of the twisted loop algebra of $\mathfrak{g}$ with respect to μ, which provides a Moody-Rao-Yokonuma presentation for the algebra $\widehat{\mathfrak{g}}[\mu]$ when μ is non-transitive, and the presentation is indeed related to the quantization of twisted toroidal Lie algebras.

Keywords Moody-Rao-Yokonuma presentation, loop algebra, universal central extension, extended affine Lie algebra

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1 Introduction

Let $\mathfrak{g}$ be a (twisted or untwisted) affine Kac-Moody algebra (without derivation), and $\bar{\mathfrak{g}}$ be the quotient algebra of $\mathfrak{g}$ modulo its center. When $\mathfrak{g}$ is of untwisted type, the universal central extension $\widehat{\mathfrak{g}}$ of the loop algebra $\mathbb{C}[t_1,t_1^{-1}]\otimes\bar{\mathfrak{g}}$ is called a *toroidal Lie algebra*. This algebra was first introduced by Moody et al. [19], where the authors introduced the famous Moody-Rao-Yokonuma (MRY) presentation. The presentation makes it more effective to study representations of toroidal Lie algebras in a manner similar to that of untwisted affine Lie algebras [7, 8, 14–16, 19, 25, 26]. Moreover, it turns out that the classical limit of the quantum toroidal algebra is just the MRY presentation of the toroidal Lie algebra [11, 13].

Let μ be a diagram automorphism of $\mathfrak{g}$ of order N, and $\bar{\mu}$ be the automorphism on $\bar{\mathfrak{g}}$ induced from μ. The twisted loop algebra $\mathcal{L}(\bar{\mathfrak{g}},\bar{\mu})$ of $\bar{\mathfrak{g}}$ is defined as follows:

$$\mathcal{L}(\bar{\mathfrak{g}},\bar{\mu})=\bigoplus_{n\in\mathbb{Z}}\mathbb{C}t_1^n\otimes\bar{\mathfrak{g}}_{(n)},$$

where $\bar{\mathfrak{g}}_{(n)}=\{x\in\bar{\mathfrak{g}}\mid\bar{\mu}(x)=\xi^n x\}$ and $\xi=\mathrm{e}^{2\pi\sqrt{-1}/N}$. In this paper, we study the universal central extension $\widehat{\mathfrak{g}}[\mu]$ of $\mathcal{L}(\bar{\mathfrak{g}},\bar{\mu})$, and give the Moody-Rao-Yokonuma presentation for $\widehat{\mathfrak{g}}[\mu]$ when μ is non-transitive.

* Corresponding author

Just as the untwisted case, one may expect that the MRY presentation could be used to study the representation and quantization for the twisted toroidal Lie algebras [11, 13, 19].

An *extended affine Lie algebra* (EALA) is a complex Lie algebra, together with a non-zero finite-dimensional Cartan subalgebra and a non-degenerate invariant symmetric bilinear form, which satisfies a list of natural axioms [1, 12, 21]. The root system of an EALA is a disjoint union of isotropic and non-isotropic root systems, and the rank of the free abelian group generated by the isotropic root system is defined to be the *nullity* of the EALA [1]. It is known that the nullity 0 EALAs are finite-dimensional simple Lie algebras over the complex number field, and the nullity 1 EALAs are precisely the affine Kac-Moody algebras [3]. We remark that the nullity 2 EALAs are the most important class of EALAs other than the finite-dimensional simple Lie algebras and affine Kac-Moody algebras, which are closely related to the singularity theory studied by Saito [22] and Slodowy [23]. In addition, the nullity 2 EALAs are classified in [5] (also see [10]).

For a given EALA $\mathfrak{L}$, the subalgebra of $\mathfrak{L}$ generated by the set of non-isotropic root vectors is called the *core* of $\mathfrak{L}$ [1]. We denote by $\mathbb{E}_2$ the class of all Lie algebras that are isomorphic to the *centerless cores* (cores modulo their centers) of EALAs with nullity 2. Let $\mathfrak{sl}_n(\mathbb{C}_q)$ $(n \geqslant 2)$ be the special linear Lie algebra over the quantum torus $\mathbb{C}_q$ in two variables [6]. It is proved in [5] that any Lie algebra in $\mathbb{E}_2$ is either isomorphic to $\mathfrak{sl}_n(\mathbb{C}_q)$ with $q \in \mathbb{C}^\times$ not a root of unity, or isomorphic to a Lie algebra of the form $\mathcal{L}(\bar{\mathfrak{g}}, \bar{\mu})$ with μ non-transitive. The universal central extension $\widehat{\mathfrak{sl}}_n(\mathbb{C}_q)$ of $\mathfrak{sl}_n(\mathbb{C}_q)$ is given in [6], and its MRY presentation is obtained in [27] for the purpose of determining the classical limit of the two-parameter quantum toroidal algebras. The purpose of this paper is to study the universal central extension $\widehat{\mathfrak{g}}[\mu]$ of $\mathcal{L}(\bar{\mathfrak{g}}, \bar{\mu})$, and the MRY presentation for $\widehat{\mathfrak{g}}[\mu]$ with μ non-transitive.

The rest of this paper is organized as follows. In Section 2, we recall some facts for the affine Kac-Moody algebras which will be used later on. In Section 3, we show that any diagram automorphism μ of an affine Kac-Moody algebra $\mathfrak{g}$ can be lifted to an automorphism $\widehat{\mu}$ for the universal central extension $\widehat{\mathfrak{g}}$ of $\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id})$. The Lie subalgebra of $\widehat{\mathfrak{g}}$ fixed by $\widehat{\mu}$ is denoted by $\widehat{\mathfrak{g}}[\mu]$. We prove that $\widehat{\mathfrak{g}}[\mu]$ is the universal central extension of $\mathcal{L}(\bar{\mathfrak{g}}, \bar{\mu})$ (see Theorem 3.3), and give the MRY presentation for $\widehat{\mathfrak{g}}[\mu]$ with μ non-transitive (see Theorem 3.6). Sections 4 and 5 are devoted to the proofs of Theorems 3.3 and 3.6.

We denote the sets of non-zero complex numbers, non-zero integers, and positive integers, respectively by $\mathbb{C}^\times$, $\mathbb{Z}^\times$ and $\mathbb{Z}_+$. For $M \in \mathbb{Z}_+$, we set $\xi_M = \mathrm{e}^{2\pi\sqrt{-1}/M}$ and $\mathbb{Z}_M = \mathbb{Z}/M\mathbb{Z}$.

2 Diagram automorphisms of affine Kac-Moody algebras

2.1 Affine Kac-Moody algebras

In this subsection, we collect some basics about affine Kac-Moody algebras that will be used later on.

Let $A = (a_{ij})_{i,j=0}^{\ell}$ be a *generalized Cartan matrix* (GCM) of affine type, and $\mathfrak{g}$ be the affine Kac-Moody algebra (without derivation) associated to the GCM A. We denote the set $\{0, 1, \ldots, \ell\}$ by I. By definition, the Lie algebra $\mathfrak{g}$ is generated by the Chevalley generators

$$\alpha_i^\vee, \quad e_i^\pm, \quad i \in I$$

with the defining relations $(i, j \in I)$

$$[\alpha_i^\vee, \alpha_j^\vee] = 0, \quad [\alpha_i^\vee, e_j^\pm] = \pm a_{ij}\, e_j^\pm, \quad [e_i^+, e_j^-] = \delta_{ij}\alpha_i^\vee, \quad \mathrm{ad}(e_i^\pm)^{1-a_{ij}}(e_j^\pm) = 0, \quad i \neq j.$$

Let Δ be the root system (including 0) of $\mathfrak{g}$, $\Delta^\times$ be the set of real roots in Δ, and $\Delta^0 = \Delta\backslash\Delta^\times = \mathbb{Z}\delta_2$ be the set of imaginary roots in Δ. Then $\mathfrak{g}$ has a root space decomposition $\mathfrak{g} = \bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$. Let $\Pi = \{\alpha_i, i \in I\}$ be the simple root system of $\mathfrak{g}$ such that $e_i^\pm \in \mathfrak{g}_{\pm\alpha_i}$ for $i \in I$, and $Q = \bigoplus_{i\in I}\mathbb{Z}\alpha_i$ be the root lattice of $\mathfrak{g}$. Then the root space decomposition naturally induces a Q-grading on $\mathfrak{g}$. In addition, let $\bar{\mathfrak{g}}$ be the quotient algebra of $\mathfrak{g}$ modulo its center. Then the Q-grading on $\mathfrak{g}$ naturally induces a Q-grading $\bar{\mathfrak{g}} = \bigoplus_{\alpha\in Q}\bar{\mathfrak{g}}_\alpha$ on $\bar{\mathfrak{g}}$.

Now we recall the twisted loop realization of the affine Kac-Moody algebra $\mathfrak{g}$ (see [17, Chapters 7 and 8]). Using the notations given in [17, Chapter 4, Tables Aff 1–3], we assume that the GCM A is of type $X_n^{(r)}$.

We start with a finite-dimensional simple Lie algebra $\dot{\mathfrak{g}}$ of type X_n. Let

$$\dot{\alpha}_i^\vee, \quad \dot{E}_i^\pm, \quad i = 1, 2, \ldots, n$$

be the Chevalley generators of $\dot{\mathfrak{g}}$, and $\dot{\mathfrak{h}} = \bigoplus_{i=1}^n \mathbb{C}\dot{\alpha}_i^\vee$ be a Cartan subalgebra of $\dot{\mathfrak{g}}$. We denote by $\dot{\Delta}$ the root system (containing 0) of $\dot{\mathfrak{g}}$ with respect to $\dot{\mathfrak{h}}$. Then $\dot{\mathfrak{g}}$ has a root space decomposition $\dot{\mathfrak{g}} = \bigoplus_{\dot{\alpha}\in\dot{\Delta}} \dot{\mathfrak{g}}_{\dot{\alpha}}$ such that $\dot{\mathfrak{g}}_0 = \dot{\mathfrak{h}}$. Let $\dot{\Pi}$ be a fixed simple root system of $\dot{\Delta}$, and $\dot{\Delta}_+$ be the set of positive roots with respect to $\dot{\Pi}$. In addition, for each $\dot{\alpha} \in \dot{\Delta}_+$, there exist $\dot{E}_{\dot{\alpha}}^\pm \in \dot{\mathfrak{g}}_{\pm\dot{\alpha}}$ and $\dot{\alpha}^\vee \in \dot{\mathfrak{h}}$, such that $\{\dot{E}_{\dot{\alpha}}^+, \dot{\alpha}^\vee, \dot{E}_{\dot{\alpha}}^-\}$ form an $\mathfrak{sl}_2$ triple. Moreover, for a simple root $\dot{\alpha}_i \in \dot{\Pi}$, we assume that $\dot{E}_{\dot{\alpha}_i}^\pm = \dot{E}_i^\pm$.

Let $\dot{\nu}$ be a diagram automorphism of $\dot{\mathfrak{g}}$ of order r. By definition, there exists a permutation $\dot{\nu}$ on the set $\{1, 2, \ldots, n\}$, such that

$$\dot{\nu}(\dot{E}_i^\pm) = \dot{E}_{\dot{\nu}(i)}^\pm \quad \text{and} \quad \dot{\nu}(\dot{\alpha}_i^\vee) = \dot{\alpha}_{\dot{\nu}(i)}^\vee \quad \text{for} \quad i = 1, 2, \ldots, n.$$

For each $x \in \dot{\mathfrak{g}}$ and $m \in \mathbb{Z}$, we set

$$x_{[m]} = r^{-1} \sum_{p\in\mathbb{Z}_r} \xi_r^{-mp} \dot{\nu}^p(x) \quad \text{and} \quad \dot{\mathfrak{g}}_{[m]} = \{x_{[m]} \mid x \in \dot{\mathfrak{g}}\}.$$

In addition, define the Lie algebra

$$\mathrm{Aff}(\dot{\mathfrak{g}}, \dot{\nu}) = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}t_2^m \otimes \dot{\mathfrak{g}}_{[m]} \oplus \mathbb{C}\mathrm{k}_2$$

with Lie bracket given by

$$[t_2^{m_1} \otimes x + a_1\mathrm{k}_2, t_2^{m_2} \otimes y + a_2\mathrm{k}_2] = t_2^{m_1+m_2} \otimes [x, y] + \langle x, y\rangle \delta_{m_1+m_2,0} m_1 \mathrm{k}_2,$$

where $m_1, m_2 \in \mathbb{Z}$, $x \in \dot{\mathfrak{g}}_{[m_1]}, y \in \dot{\mathfrak{g}}_{[m_2]}$, $a_1, a_2 \in \mathbb{C}$ and $\langle \cdot, \cdot \rangle$ is the normalized symmetric invariant bilinear form on $\dot{\mathfrak{g}}$.

We denote

$$\dot{\theta} = \begin{cases} \text{the highest root of } \dot{\mathfrak{g}}, & \text{if } r = 1 \text{ or } X_n = A_{2\ell}, \quad r = 2, \\ \dot{\alpha}_1 + \cdots + \dot{\alpha}_\ell, & \text{if } X_n = D_{\ell+1}, \quad r = 2, 3, \\ \dot{\alpha}_1 + \cdots + \dot{\alpha}_{2\ell-2}, & \text{if } X_n = A_{2\ell-1}, \quad r = 2, \\ \dot{\alpha}_1 + 2\dot{\alpha}_2 + 2\dot{\alpha}_3 + \dot{\alpha}_4 + \dot{\alpha}_5 + \dot{\alpha}_6, & \text{if } X_n = E_6, \quad r = 2. \end{cases}$$

In addition, for each $i = 1, 2, \ldots, n$, we let r_i be the cardinality of the set $\{\dot{\nu}^k(i) \,|\, k \in \mathbb{Z}_r\}$. If the GCM A is of type $A_{2\ell}^{(2)}$, we set

$$E_i^\pm = r_i \dot{E}_{i[0]}^\pm, \quad E_\ell^\pm = \dot{E}_{\dot{\theta}[1]}^\mp, \quad E_0^\pm = 2\sqrt{2}\dot{E}_{\ell[0]}^\pm, \quad H_i = r_i \dot{\alpha}_{i[0]}^\vee, \quad H_\ell = -\dot{\theta}^\vee, \quad H_0 = 4\dot{\alpha}_{\ell[0]}^\vee,$$

where $i = 1, \ldots, \ell - 1$. Otherwise, we set

$$E_i^\pm = r_i \dot{E}_{i[0]}^\pm, \quad H_i = r_i \dot{\alpha}_{i[0]}^\vee, \quad E_0^\pm = r\dot{E}_{\dot{\theta}[1]}^\mp, \quad H_0 = -r\dot{\theta}_{[0]}^\vee, \quad i = 1, \ldots, \ell. \tag{2.1}$$

It is proved in [17, Theorem 8.3] that $\mathfrak{g}$ is isomorphic to $\mathrm{Aff}(\dot{\mathfrak{g}}, \dot{\nu})$ with

$$\alpha_\epsilon^\vee = ra_0^{-1}\mathrm{k}_2 + 1 \otimes H_0, \quad e_\epsilon^\pm = t^{\pm 1} \otimes E_\epsilon^\pm, \quad \alpha_i^\vee = 1 \otimes H_i, \quad e_i^\pm = 1 \otimes E_i^\pm, \quad i \neq \epsilon, \tag{2.2}$$

where $\epsilon = 0, a_0 = 1$ except that the GCM A is of type $A_{2\ell}^{(2)}$, in which case $\epsilon = \ell, a_0 = 2$. From now on, we will often use the following identifications:

$$\mathfrak{g} = \mathrm{Aff}(\dot{\mathfrak{g}}, \dot{\nu}) \quad \text{and} \quad \bar{\mathfrak{g}} = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}t_2^m \otimes \dot{\mathfrak{g}}_{[m]}$$

without further explanation.

Let $\dot{Q}=\bigoplus_{i=1}^{n}\dot{\alpha}_i$ be the root lattice of $\dot{\mathfrak{g}}$. Note that $\dot{\nu}$ induces an automorphism of $\dot{Q}$ such that $\dot{\nu}(\dot{\alpha}_i)=\dot{\alpha}_{\dot{\nu}(i)}$ for $i=1,2,\ldots,n$. For $\dot{\alpha}\in\dot{Q}$, set

$$\dot{\alpha}_{[0]}=r^{-1}\sum_{p\in\mathbb{Z}_r}\dot{\nu}^p(\dot{\alpha})$$

and also set

$$\dot{Q}_{[0]}=\{\dot{\alpha}_{[0]}\mid\dot{\alpha}\in\dot{Q}\}\subset\dot{\mathfrak{h}}^*.$$

Then the root lattice Q of $\mathfrak{g}$ is equivalent to $\dot{Q}_{[0]}\oplus\mathbb{Z}\delta_2$ and the simple root system Π of $\mathfrak{g}$ is equivalent to

$$\{\alpha_\epsilon=-\dot{\theta}_{[0]}+\delta_2,\,\alpha_{\ell-\epsilon}=\dot{\alpha}_{\ell[0]},\,\alpha_i=\dot{\alpha}_{i[0]},\,i\neq\epsilon,\ell-\epsilon\}.$$

We extend the normalized bilinear form $\langle\cdot,\cdot\rangle$ on $\dot{\mathfrak{g}}$ to a symmetric invariant bilinear form on $\mathfrak{g}$ by letting

$$\langle t_2^{m_1}\otimes x+a_1\mathrm{k}_2,t_2^{m_2}\otimes y+a_2\mathrm{k}_2\rangle=\delta_{m_1+m_2,0}\,\langle x,y\rangle,$$

where $m_1,m_2\in\mathbb{Z}$, $x\in\dot{\mathfrak{g}}_{[m_1]}$, $y\in\dot{\mathfrak{g}}_{[m_2]}$ and $a_1,a_2\in\mathbb{C}$. Since the restriction of $\langle\cdot,\cdot\rangle$ on $\dot{\mathfrak{h}}$ is non-degenerate, we get a non-degenerate bilinear form $(\cdot,\cdot)$ on $\dot{\mathfrak{h}}^*$ by duality. In addition, the bilinear form $(\cdot,\cdot)$ can be extended to a symmetric bilinear form on Q by letting

$$(\alpha+m\delta_2,\beta+n\delta_2)=(\alpha,\beta),\tag{2.3}$$

where $\alpha,\beta\in\dot{Q}_{[0]}$ and $m,n\in\mathbb{Z}$.

2.2 Diagram automorphisms

Throughout this paper, we let μ be a permutation of I with order N such that $a_{ij}=a_{\mu(i)\mu(j)}$ for $i,j\in I$. It is known that μ induces a *diagram automorphism* μ of $\mathfrak{g}$ such that

$$\mu(\alpha_i^\vee)=\alpha_{\mu(i)}^\vee,\quad\mu(e_i^\pm)=e_{\mu(i)}^\pm,\quad i\in I.\tag{2.4}$$

This subsection is devoted to an explicit description of the action of μ on $\mathfrak{g}$.

It is immediate to see that the permutation μ induces an automorphism of Q such that $\mu(\delta_2)=\delta_2$. Recall from [17, Proposition 8.3] that the finite-dimensional simple Lie algebra $\dot{\mathfrak{g}}$ can be generated by the elements E_i^+, $i\in I$ defined in (2.1). Then we have the following lemma.

Lemma 2.1. (a) *The action*

$$E_i^+\mapsto E_{\mu(i)}^+,\quad i\in I\tag{2.5}$$

defines (uniquely) an automorphism $\dot{\mu}$ of $\dot{\mathfrak{g}}$.

(b) *The Cartan subalgebra $\dot{\mathfrak{h}}$ of $\dot{\mathfrak{g}}$ is stable under $\dot{\mu}$, and*

$$\dot{\mu}(\dot{\nu}(h))=\dot{\nu}(\dot{\mu}(h)),\quad\forall\,h\in\dot{\mathfrak{h}}.\tag{2.6}$$

(c) *There is a homomorphism $\rho_\mu:\dot{Q}\to\mathbb{Z}$ of abelian groups such that*

$$\rho_\mu(\dot{\nu}(\dot{\alpha}))=\rho_\mu(\dot{\alpha}),\quad\mu(\dot{\alpha}_{[0]})=\dot{\mu}(\dot{\alpha})_{[0]}+\rho_\mu(\dot{\alpha})\delta_2,\quad\dot{\alpha}\in\dot{Q}.\tag{2.7}$$

(d) *For $\dot{\alpha}\in\dot{\Delta}$, $x\in\dot{\mathfrak{g}}_{\dot{\alpha}}$ and $m\in\mathbb{Z}$, we have*

$$\dot{\mu}(x_{[m]})=\dot{\mu}(x)_{[m+\rho_\mu(\dot{\alpha})]}.\tag{2.8}$$

Proof. We first consider the case where $\dot{\nu} = \mathrm{id}$. For each $\dot{\alpha} \in \dot{Q}$, write

$$\mu(\dot{\alpha}) = \dot{\mu}(\dot{\alpha}) + \rho_\mu(\dot{\alpha})\delta_2 \quad \text{with} \quad \dot{\mu}(\dot{\alpha}) \in \dot{Q} \quad \text{and} \quad \rho_\mu(\alpha) \in \mathbb{Z}.$$

Then the map

$$\dot{\mu} : \dot{Q} \to \dot{Q}, \quad \dot{\alpha} \mapsto \dot{\mu}(\dot{\alpha})$$

is an automorphism of $\dot{Q}$ (with order N) and the map

$$\rho_\mu : \dot{Q} \to \mathbb{Z}, \quad \dot{\alpha} \mapsto \rho_\mu(\dot{\alpha})$$

is a homomorphism of abelian groups. We define a linear map $\dot{\mu}$ on $\dot{\mathfrak{g}}$ as follows:

$$\dot{\mu} : \dot{\mathfrak{g}} \to \dot{\mathfrak{g}}, \quad \dot{E}^{\pm}_{\dot{\alpha}} \mapsto \dot{\mu}(\dot{E}^{\pm}_{\dot{\alpha}}), \quad \dot{\alpha}^\vee \mapsto \dot{\mu}(\dot{\alpha}^\vee), \quad \text{for} \quad \dot{\alpha} \in \dot{\Delta}_+,$$

where $\dot{\mu}(\dot{E}^{\pm}_{\dot{\alpha}})$ are the elements in $\dot{\mathfrak{g}}_{\pm\dot{\mu}(\dot{\alpha})}$ determined by the following equation:

$$\mu(1 \otimes \dot{E}^{\pm}_{\dot{\alpha}}) = t_2^{\rho_\mu(\pm\dot{\alpha})} \otimes \dot{\mu}(\dot{E}^{\pm}_{\dot{\alpha}}).$$

It is easy to see that $\dot{\mu}$ is an automorphism of $\dot{\mathfrak{g}}$ (with order N). Moreover, one can check that the automorphism $\dot{\mu}$ and the homomorphism ρ_μ defined above satisfy all the assertions in the lemma.

Next, we consider the case where $\dot{\nu} \neq \mathrm{id}$. If $\mu = \mathrm{id}$, then we only need to take $\dot{\mu} = \mathrm{id}$ and $\rho_\mu = 0$. So we assume further that μ is nontrivial. Then either

$$X_n^{(r)} = A_{2\ell-1}^{(2)} \quad \text{and} \quad \mu = (0, 1)$$

or

$$X_n^{(r)} = D_{\ell+1}^{(2)} \quad \text{and} \quad \mu = \prod_{0 \leqslant i \leqslant \lfloor \frac{\ell-1}{2} \rfloor} (i, l - i).$$

Observe that, if $X_n^{(r)} = A_{2\ell-1}^{(2)}$ ($D_{\ell+1}^{(2)}$, respectively), then the set $\{-\dot{\nu}(\dot{\theta}), \dot{\alpha}_2, \ldots, \dot{\alpha}_{2\ell-2}, -\dot{\theta}\}$ ($\{\alpha_{\ell-1}, \dot{\alpha}_{\ell-2}, \ldots, \dot{\alpha}_1, -\dot{\theta}, -\dot{\nu}(\dot{\theta})\}$, respectively) is another simple root system of $\dot{\mathfrak{g}}$. Thus, if $X_n^{(r)} = A_{2\ell-1}^{(2)}$, then there is an automorphism $\dot{\mu}$ on $\dot{\mathfrak{g}}$ given by

$$\dot{E}_1^{\pm} \mapsto -\dot{E}^{\mp}_{\dot{\nu}(\dot{\theta})}, \quad \dot{E}_i^{\pm} \mapsto \dot{E}_i^{\pm}, \quad 2 \leqslant i \leqslant 2\ell - 2, \quad \dot{E}^{\pm}_{2\ell-1} \mapsto \dot{E}^{\mp}_{\dot{\theta}}.$$

In addition, if $X_n^{(r)} = D_{\ell+1}^{(2)}$, then there is an automorphism $\dot{\mu}$ on $\dot{\mathfrak{g}}$ given by

$$\dot{E}_i^{\pm} \mapsto \dot{E}^{\pm}_{\ell-i}, \quad 1 \leqslant i \leqslant \ell - 1, \quad \dot{E}^{\pm}_\ell \mapsto \dot{E}^{\mp}_{\dot{\theta}}, \quad \dot{E}^{\pm}_{\ell+1} \mapsto -\dot{E}^{\mp}_{\dot{\nu}(\dot{\theta})}.$$

It is straightforward to check that in both cases the automorphism $\dot{\mu}$ defined above satisfies the properties (2.5) and (2.6). This proves the assertions (a) and (b).

For the assertion (c), we define a homomorphism $\rho_\mu : \dot{Q} \to \mathbb{Z}$ by letting

$$\begin{aligned} &\rho_\mu(\dot{\alpha}_1) = 1 = \rho_\mu(\dot{\alpha}_{2\ell-1}), \quad \rho_\mu(\dot{\alpha}_i) = 0, \quad 2 \leqslant i \leqslant 2\ell - 2, \quad \text{if} \quad X_n^{(r)} = A^{(2)}_{2\ell-1}, \\ &\rho_\mu(\dot{\alpha}_1) = 0, \quad 1 \leqslant i \leqslant \ell - 1, \quad \rho_\mu(\dot{\alpha}_\ell) = 1 = \rho_\mu(\dot{\alpha}_{\ell+1}), \quad \text{if} \quad X_n^{(r)} = D^{(2)}_{\ell+1}. \end{aligned}$$

It is obvious that the property (2.7) holds true for all $\dot{\alpha}_i \in \dot{\Pi}$ and hence for all $\dot{\alpha} \in \dot{Q}$. Finally, it can be checked case by case that, the property (2.8) holds true for every $x = \dot{E}_i^{\pm}$, $i = 1, 2, \ldots, n$. For the general case, we may assume that $\dot{\alpha} = \dot{\alpha}_{i_1} + \cdots + \dot{\alpha}_{i_s}$ and $x = [\dot{E}^+_{i_1}, \ldots, [\dot{E}^+_{i_{s-1}}, \dot{E}^+_{i_s}]]$ for some $i_1, \ldots, i_s \in \dot{I}$. Then

$$\begin{aligned} \dot{\mu}(x) &= \dot{\mu}\bigg(\sum_{k_1, \ldots, k_s \in \mathbb{Z}_r} [\dot{E}^+_{i_1[k_1]}, \ldots, [\dot{E}^+_{i_{s-1}[k_{s-1}]}, \dot{E}^+_{i_s[k_s]}]] \bigg) \\ &= \sum_{k_1, \ldots, k_s \in \mathbb{Z}_r} [\dot{\mu}(\dot{E}_{i_1})_{[k_1 + \rho_\mu(\dot{\alpha}_{i_1})]}, \ldots, [\dot{\mu}(\dot{E}_{i_{s-1}})_{[k_{s-1} + \rho_\mu(\dot{\alpha}_{i_{s-1}})]}, \dot{\mu}(\dot{E}_{i_s})_{[k_s + \rho_\mu(\dot{\alpha}_{i_s})]}]]. \end{aligned}$$

It implies that

$$\begin{aligned}\dot{\mu}(x)_{[m+\rho_\mu(\dot{\alpha})]} &= \sum_{k_1+\cdots+k_s=m} [\dot{\mu}(\dot{E}_{i_1})_{[k_1+\rho_\mu(\dot{\alpha}_{i_1})]}, \ldots, [\dot{\mu}(\dot{E}_{i_{s-1}})_{(k_{s-1}+\rho_\mu(\dot{\alpha}_{i_{s-1}}))}, \dot{\mu}(\dot{E}_{i_s})_{(k_s+\rho_\mu(\dot{\alpha}_{i_s}))}]] \\ &= \dot{\mu}\Bigg(\sum_{k_1+\cdots+k_s=m} [\dot{E}^+_{i_1[k_1]}, \ldots, [\dot{E}^+_{i_{s-1}[k_{s-1}]}, \dot{E}^+_{i_s[k_s]}]]\Bigg) = \dot{\mu}(x_{[m]})\end{aligned}$$

holds true for every $m \in \mathbb{Z}_r$. This completes the proof of the assertion (d). □

Let $\dot{\mu}$ and ρ_μ be as in Lemma 2.1. Since the bilinear form $\langle \cdot, \cdot \rangle$ is non-degenerate on $\dot{\mathfrak{h}}$, we may and do identify $\dot{\mathfrak{h}}$ with its dual space $\dot{\mathfrak{h}}^*$, and extend ρ_μ to a linear functional on $\dot{\mathfrak{h}}$ by $\mathbb{C}$-linearity. The following result is an explicit description of the action of the diagram automorphism μ.

Proposition 2.2. *For each $m \in \mathbb{Z}$, $\dot{\alpha} \in \dot{\Delta} \setminus \{0\}$, $x \in \dot{\mathfrak{g}}_{\dot{\alpha}}$ and $h \in \dot{\mathfrak{h}}$, we have*

$$\begin{aligned}&\mu(t_2^m \otimes x_{[m]}) = t_2^{m+\rho_\mu(\dot{\alpha})} \otimes \dot{\mu}(x_{[m]}), \quad \mu(\mathrm{k}_2) = \mathrm{k}_2, \\ &\mu(t_2^m \otimes h_{[m]}) = t_2^m \otimes \dot{\mu}(h_{[m]}) + \delta_{m,0}\, \rho_\mu(h)\, \mathrm{k}_2.\end{aligned} \tag{2.9}$$

Proof. Using Lemma 2.1 and the identification (2.2), one can check that the action given in (2.9) defines an automorphism of $\mathfrak{g}$ such that the equation (2.4) holds, as desired. □

3 The Lie algebra $\widehat{\mathfrak{g}}[\mu]$ and its MRY presentation

In this section, we define the twisted toroidal Lie algebra $\widehat{\mathfrak{g}}[\mu]$ and state its Moody-Rao-Yokonuma presentation.

3.1 The Lie algebra $\widehat{\mathfrak{g}}[\mu]$

In this subsection, we introduce the definition of the Lie algebra $\widehat{\mathfrak{g}}[\mu]$.

For $M_1, M_2 \in \mathbb{Z}_+$, let $\mathcal{K}_{M_1,M_2}$ be the $\mathbb{C}$-vector space spanned by the symbols

$$t_1^{m_1} t_2^{m_2} \mathrm{k}_1, \quad t_1^{m_1} t_2^{m_2} \mathrm{k}_2, \quad m_1 \in M_1\mathbb{Z}, \quad m_2 \in M_2\mathbb{Z}$$

subject to the relation

$$m_1 t_1^{m_1} t_2^{m_2} \mathrm{k}_1 + m_2 t_1^{m_1} t_2^{m_2} \mathrm{k}_2 = 0.$$

We define

$$\widehat{\mathfrak{g}} = \bigoplus_{m,n \in \mathbb{Z}} \mathbb{C} t_1^m t_2^n \otimes \dot{\mathfrak{g}}_{[n]} \oplus \mathcal{K}_{1,r} \subset (\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}] \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K}_{1,r}$$

to be a Lie algebra with Lie bracket given by

$$[t_1^{m_1} t_2^{m_2} \otimes x, t_1^{n_1} t_2^{n_2} \otimes y] = t_1^{m_1+n_1} t_2^{m_2+n_2} \otimes [x, y] + \langle x, y \rangle \Bigg(\sum_{i=1}^{2} m_i t_1^{m_1+n_1} t_2^{m_2+n_2} \mathrm{k}_i\Bigg), \tag{3.1}$$

where $x \in \dot{\mathfrak{g}}_{[m_2]}$, $y \in \dot{\mathfrak{g}}_{[n_2]}$, $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ and $\mathcal{K}_{1,r}$ is the center. It follows from [19, 24] that the projective map

$$\psi : \widehat{\mathfrak{g}} \to \bigoplus_{m,n \in \mathbb{Z}} \mathbb{C} t_1^m t_2^n \otimes \dot{\mathfrak{g}}_{[n]} = \mathbb{C}[t_1, t_1^{-1}] \otimes \bar{\mathfrak{g}}$$

is the universal central extension of the loop algebra $\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id})$ of $\bar{\mathfrak{g}}$.

For convenience, we view $\mathbb{C}[t_1, t_1^{-1}] \otimes \mathfrak{g}$ as a subspace of $\widehat{\mathfrak{g}}$ in the following way:

$$t_1^{m_1} \otimes x = t_1^{m_1} t_2^{m_2} \otimes \dot{x} + a t_1^{m_1} \mathrm{k}_2$$

for $x = t_2^{m_2} \otimes \dot{x} + a\mathrm{k}_2 \in \mathfrak{g}$, $m_1 \in \mathbb{Z}$. Then it is easy to see that the Lie algebra $\widehat{\mathfrak{g}}$ is spanned by the elements $t_1^{m_1} \otimes x$, k_1, $t_1^{n_1} t_2^{n_2} \mathrm{k}_1$, $x \in \mathfrak{g}$, $m_1, n_1 \in \mathbb{Z}$ and $n_2 \in r\mathbb{Z}^{\times}$. Moreover, the commutator relations among these elements are as follows.

Lemma 3.1. *Let $\alpha,\beta\in\Delta$, $x\in\mathfrak{g}_\alpha, y\in\mathfrak{g}_\beta$ and $m_1,n_1\in\mathbb{Z}$. If $\alpha+\beta\in\Delta^\times\cup\{0\}$, then*

$$[t_1^{m_1}\otimes x, t_1^{n_1}\otimes y]=t_1^{m_1+n_1}\otimes[x,y]+m_1\delta_{m_1,n_1}\langle x,y\rangle \mathrm{k}_1. \tag{3.2}$$

If $x=t_2^{m_2}\otimes\dot{x}$, $y=t_2^{n_2}\otimes\dot{y}$ and $\alpha+\beta\in\Delta^0\setminus\{0\}$, then

$$[t_1^{m_1}\otimes x, t_1^{n_1}\otimes y]=t_1^{m_1+n_1}\otimes[x,y]+\langle\dot{x},\dot{y}\rangle\frac{m_1n_2-m_2n_1}{m_2+n_2}t_1^{m_1+n_1}t_2^{m_2+n_2}\mathrm{k}_1. \tag{3.3}$$

Observe that the Lie algebra $\widehat{\mathfrak{g}}$ is generated by the elements

$$t_1^m\otimes e_i^\pm,\quad t_1^m\otimes\alpha_i^\vee,\quad \mathrm{k}_1,\quad i\in I,\quad m\in\mathbb{Z}. \tag{3.4}$$

Similar to (2.4), the permutation μ induces an automorphism of $\widehat{\mathfrak{g}}$ as follows.

Lemma 3.2. *The assignment*

$$t_1^m\otimes e_i^\pm\mapsto\xi^{-m}t_1^m\otimes e_{\mu(i)}^\pm,\quad t_1^m\otimes\alpha_i^\vee\mapsto\xi^{-m}t_1^m\otimes\alpha_{\mu(i)}^\vee,\quad \mathrm{k}_1\mapsto\mathrm{k}_1 \tag{3.5}$$

for $i\in I$, $m\in\mathbb{Z}$, defines an automorphism $\widehat{\mu}$ of $\widehat{\mathfrak{g}}$.

Proof. We define a linear transformation $\widehat{\mu}$ on $\widehat{\mathfrak{g}}$ by letting

$$\begin{aligned}&t_1^{m_1}\otimes x\mapsto\xi^{-m_1}t_1^{m_1}\otimes\mu(x),\\&t_1^{m_1}\otimes h\mapsto\xi^{-m_1}\left(t_1^{m_1}\otimes\mu(h)-\frac{m_1}{m_2}\rho_\mu(\dot{h})t_1^{m_1}t_2^{m_2}\mathrm{k}_1\right),\\&\mathrm{k}_1\mapsto\mathrm{k}_1,\quad t_1^{n_1}t_2^{n_2}\mathrm{k}_1\mapsto\xi^{-n_1}t_1^{n_1}t_2^{n_2}\mathrm{k}_1,\end{aligned}$$

where $m_1,n_1\in\mathbb{Z}$, $x\in\mathfrak{g}_\alpha,\alpha\in\Delta^\times\cup\{0\}$, $h=t_2^{m_2}\otimes\dot{h}$, $m_2\in\mathbb{Z}^\times$, $n_2\in r\mathbb{Z}^\times$ and $\dot{h}\in\dot{\mathfrak{h}}_{[m_2]}$. Note that $\rho_\mu(\dot{h})\neq0$ only if $m_2\in r\mathbb{Z}$, and so $\widehat{\mu}$ is well defined.

By using the explicit action of μ given in Proposition 2.2 and the commutator relations of $\widehat{\mathfrak{g}}$ given in Lemma 3.1, one can easily verify that the map $\widehat{\mu}$ is an automorphism of $\widehat{\mathfrak{g}}$. Moreover, it is obvious that the actions of $\widehat{\mu}$ on those generators in (3.4) coincide with that in (3.5). This completes the proof. □

We define $\widehat{\mathfrak{g}}[\mu]$ to be the subalgebra of $\widehat{\mathfrak{g}}$ fixed by $\widehat{\mu}$. Recall from Section 1 that $\bar{\mu}$ is the automorphism of $\bar{\mathfrak{g}}$ induced from μ, and that $\mathcal{L}(\bar{\mathfrak{g}},\bar{\mu})$ is the twisted loop algebra of $\bar{\mathfrak{g}}$ related to $\bar{\mu}$. Note that $\mathcal{L}(\bar{\mathfrak{g}},\bar{\mu})$ is the subalgebra of $\mathcal{L}(\bar{\mathfrak{g}},\mathrm{id})$ fixed by the automorphism

$$\xi^{-\mathrm{d}_1}\otimes\bar{\mu}:\mathcal{L}(\bar{\mathfrak{g}},\mathrm{id})\to\mathcal{L}(\bar{\mathfrak{g}},\mathrm{id}),\quad t_1^m\otimes x\mapsto\xi^{-m}t_1^m\otimes\bar{\mu}(x),\quad m\in\mathbb{Z},\quad x\in\bar{\mathfrak{g}}.$$

It follows from (3.5) that

$$\psi\circ\widehat{\mu}=(\xi^{-\mathrm{d}_1}\otimes\bar{\mu})\circ\psi. \tag{3.6}$$

Thus, by taking the restriction of ψ on $\widehat{\mathfrak{g}}[\mu]$, one gets a Lie algebra homomorphism

$$\psi_\mu=\psi\,|_{\widehat{\mathfrak{g}}[\mu]}:\widehat{\mathfrak{g}}[\mu]\to\mathcal{L}(\bar{\mathfrak{g}},\bar{\mu}).$$

The following theorem is the first main result of this paper, whose proof will be presented in Section 4.

Theorem 3.3. *The Lie algebra homomorphism $\psi_\mu:\widehat{\mathfrak{g}}[\mu]\to\mathcal{L}(\bar{\mathfrak{g}},\bar{\mu})$ is a universal central extension of the twisted loop algebra $\mathcal{L}(\bar{\mathfrak{g}},\bar{\mu})$.*

3.2 The MRY presentation

Here we state an MRY presentation for $\widehat{\mathfrak{g}}[\mu]$. Throughout this subsection, we assume that μ is non-transitive. Observe that a diagram automorphism on $\mathfrak{g}$ is transitive if and only if $\mathfrak{g}$ is of type $A_\ell^{(1)}$ ($\ell\geqslant1$), and the diagram automorphism is an order $\ell+1$ rotation of the Dynkin diagram.

We first introduce some notations. Set $V=\mathbb{R}\otimes_{\mathbb{Z}}Q$ and extend $(\cdot,\cdot)$ (see (2.3)) to a bilinear form on V by $\mathbb{R}$-linearity. For $i,j\in I$, we set

$$\check{\alpha}_i=\frac{1}{N}\sum_{k\in\mathbb{Z}_N}\alpha_{\mu^k(i)}\quad\text{and}\quad\check{a}_{ij}=2\frac{(\check{\alpha}_i,\check{\alpha}_j)}{(\check{\alpha}_i,\check{\alpha}_i)}.$$

We fix a representative subset of I as follows:

$$\check{I}=\{i\in I\,|\,\mu^k(i)\geqslant i\text{ for }k\in\mathbb{Z}_N\}.$$

It was proved in [5, Proposition 12.1.10] (see also [9]) that the folded matrix

$$\check{A}=(\check{a}_{ij})_{i,j\in\check{I}}$$

of the GCM A associated with μ is also a GCM of affine type.

For $i\in I$, we denote by $\mathcal{O}(i)\subset I$ the orbit containing i under the action of the group $\langle\mu\rangle$. The following result was proved in [5, Lemma 12.1.5].

Lemma 3.4. *For each $i\in I$, exactly one of the following holds*:

(a) *The elements $\alpha_p,p\in\mathcal{O}(i)$ are pairwise orthogonal*;

(b) $\mathcal{O}(i)=\{i,\mu(i)\}$ *and* $a_{i\mu(i)}=-1=a_{\mu(i)i}$.

As in [5], for $i\in I$, we set

$$s_i=\begin{cases}1, & \text{if (a) holds in Lemma 3.4,}\\ 2, & \text{if (b) holds in Lemma 3.4.}\end{cases}$$

Now we introduce the following definition.

Definition 3.5. Define $\mathcal{M}(\mathfrak{g},\mu)$ to be the Lie algebra generated by the elements

$$h_{i,m},\quad x^{\pm}_{i,m},\quad c,\quad i\in I,\quad m\in\mathbb{Z} \tag{3.7}$$

subject to the relations

(T0) $h_{\mu(i),m}=\xi^m h_{i,m},\quad x^{\pm}_{\mu(i),m}=\xi^m x^{\pm}_{i,m}$,

(T1) $[c,h_{i,n}]=0=[c,x^{\pm}_{i,n}]$,

(T2) $[h_{i,m},h_{j,n}]=\sum_{k\in\mathbb{Z}_N}mN\langle\alpha_i^\vee,\alpha^\vee_{\mu^k(j)}\rangle\delta_{m+n,0}m\xi^{km}c$,

(T3) $[h_{i,m},x^{\pm}_{j,n}]=\pm\sum_{k\in\mathbb{Z}_N}a_{i\mu^k(j)}x^{\pm}_{j,m+n}\xi^{km}$,

(T4) $[x^+_{i,m},x^-_{j,n}]=\sum_{k\in\mathbb{Z}_N}\delta_{i,\mu^k(j)}\left(h_{j,m+n}+\frac{mN\langle\alpha_i^\vee,\alpha_i^\vee\rangle}{2}\delta_{m+n,0}c\right)\xi^{km}$,

(T5) $(\mathrm{ad}\,x^{\pm}_{i,0})^{1-\check{a}_{ij}}\left(x^{\pm}_{j,m}\right)=0,\quad\text{if }\ \check{a}_{ij}\leqslant 0$,

(T6) $[x^{\pm}_{i,m_1},\dots,[x^{\pm}_{i,m_{s_i}},x^{\pm}_{i,m_{s_i+1}}]]=0$.

In view of (3.4) and (3.5), we know that the Lie algebra $\widehat{\mathfrak{g}}[\mu]$ is generated by the following elements:

$$t_1^m\otimes e^{\pm}_{i(m)},\quad t_1^m\otimes\alpha^\vee_{i(m)},\quad \mathrm{k}_1,\quad i\in I,\quad m\in\mathbb{Z}, \tag{3.8}$$

where $x_{(m)}=\sum_{p\in\mathbb{Z}_N}\xi^{-pm}\mu^p(x)$ for $x\in\mathfrak{g}$. The following theorem is the second main result of this paper, whose proof will be presented in Section 5.

Theorem 3.6. *The assignment*

$$c\mapsto\mathrm{k}_1,\quad h_{i,m}\mapsto t_1^m\otimes\alpha^\vee_{i(m)},\quad x^{\pm}_{i,m}\mapsto t_1^m\otimes e^{\pm}_{i(m)},\quad i\in I,\quad m\in\mathbb{Z}$$

determines a Lie algebra isomorphism from $\mathcal{M}(\mathfrak{g},\mu)$ to $\widehat{\mathfrak{g}}[\mu]$.

When $\mathfrak{g}$ is of untwisted type and $\mu=\mathrm{id}$, Theorem 3.6 is proved in [19].

4 Proof of Theorem 3.3

4.1 Multiloop algebras

We start by recalling the definition of multiloop algebras (see [2]). Let $\mathfrak{k}$ be an arbitrary Lie algebra, and let σ_1, $\sigma_2, \ldots, \sigma_s$ be pairwise commuting automorphisms on $\mathfrak{k}$. From now on, we denote by $\mathfrak{k}^{\sigma_1,\sigma_2,\ldots,\sigma_s}$ the fixed point subalgebra of $\mathfrak{k}$ under the automorphisms $\sigma_1, \sigma_2, \ldots, \sigma_s$. Suppose further that each automorphism σ_i has a finite period M_i, i.e., $\sigma^{M_i} = 1$, $i = 1, \ldots, s$. The *multiloop algebra* associated with $\mathfrak{k}$, $\sigma_1, \sigma_2, \ldots, \sigma_s$ is by definition the following subalgebra of $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_s^{\pm 1}] \otimes \mathfrak{k}$:

$$\mathcal{L}_{M_1,M_2,\ldots,M_s}(\mathfrak{k}, \sigma_1, \sigma_2, \ldots, \sigma_s) = \bigoplus_{m_1,m_2,\ldots,m_s \in \mathbb{Z}} \mathbb{C}t_1^{m_1} t_2^{m_2} \cdots t_s^{m_s} \otimes \mathfrak{k}_{(m_1,m_2,\ldots,m_s)},$$

where

$$\mathfrak{k}_{(m_1,m_2,\ldots,m_s)} = \{x \in \mathfrak{k} \mid \sigma_i(x) = \xi_{M_i}^{m_i} x,\ i = 1, 2, \ldots, s\},$$

and when each M_i is the order of σ_i we often write

$$\mathcal{L}(\mathfrak{k}, \sigma_1, \sigma_2, \ldots, \sigma_s) = \mathcal{L}_{M_1,M_2,\ldots,M_r}(\mathfrak{k}, \sigma_1, \sigma_2, \ldots, \sigma_s).$$

Let σ be an automorphism of $\mathfrak{k}$, and $(c_1, c_2, \ldots, c_s)$ be an s-tuple in $(\mathbb{C}^\times)^s$. Let

$$c_1^{-\mathrm{d}_1} \otimes c_2^{-\mathrm{d}_2} \otimes \cdots \otimes c_s^{-\mathrm{d}_s} \otimes \sigma$$

be the automorphism of $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_s^{\pm 1}] \otimes \mathfrak{k}$ defined by

$$t_1^{m_1} t_2^{m_2} \cdots t_s^{m_s} \otimes x \mapsto c_1^{-m_1} c_2^{-m_2} \cdots c_s^{-m_s} t_1^{m_1} t_2^{m_2} \cdots t_s^{m_s} \otimes \sigma(x),$$

where $x \in \mathfrak{k}$ and $m_i \in \mathbb{Z}$. It is obvious that the multiloop algebra $\mathcal{L}_{M_1,\ldots,M_s}(\mathfrak{k}, \sigma_1, \ldots, \sigma_s)$ is the subalgebra of $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_s^{\pm 1}] \otimes \mathfrak{k}$ fixed by the following commuting automorphisms:

$$\xi_{M_1}^{-\mathrm{d}_1} \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma_1, \quad 1 \otimes \xi_{M_2}^{-\mathrm{d}_2} \otimes \cdots \otimes 1 \otimes \sigma_2, \quad \ldots, \quad 1 \otimes \cdots \otimes 1 \otimes \xi_{M_s}^{-\mathrm{d}_s} \otimes \sigma_s.$$

4.2 The functor $\mathfrak{uce}$

In this subsection, we recall the endofunctor $\mathfrak{uce}$ on the category of Lie algebras introduced in [20]. Let $\mathfrak{k}$ be an arbitrary Lie algebra, and B be the subspace of $\mathfrak{k} \otimes \mathfrak{k}$ spanned by all elements of the form

$$x \otimes y + y \otimes x \quad \text{and} \quad x \otimes [y, z] + y \otimes [z, x] + z \otimes [x, y], \quad x, y, z \in \mathfrak{k}.$$

We define $\mathfrak{uce}(\mathfrak{k}) = \mathfrak{k} \otimes \mathfrak{k}/B$ to be a Lie algebra with Lie bracket

$$[x \otimes x', y \otimes y']_{\mathfrak{uce}(\mathfrak{k})} = [x, x'] \otimes [y, y'] + B.$$

Then we have the following well-defined Lie algebra homomorphism:

$$\mathfrak{u}_{\mathfrak{k}} : \mathfrak{uce}(\mathfrak{k}) \to [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad x \otimes y \mapsto [x, y],$$

which is in fact a central extension of $[\mathfrak{k}, \mathfrak{k}]$.

Let $f : \mathfrak{k} \to \mathfrak{k}_0$ be a homomorphism of Lie algebras. Then the map

$$\begin{aligned} \mathfrak{uce}(f) : \mathfrak{uce}(\mathfrak{k}) &\to \mathfrak{uce}(\mathfrak{k}_0), \\ x \otimes y &\mapsto f(x) \otimes f(y) \end{aligned}$$

is also a Lie algebra homomorphism. Note that $\mathfrak{uce}$ is a covariant functor. Therefore, if f is an isomorphism, then so is $\mathfrak{uce}(f)$.

We say that a homomorphism $\hat{f} : \mathfrak{uce}(\mathfrak{k}) \to \mathfrak{uce}(\mathfrak{k}_0)$ *covers* $f : \mathfrak{k} \to \mathfrak{k}_0$ if

$$\mathfrak{u}_{\mathfrak{k}_0} \circ \hat{f} = f \circ \mathfrak{u}_{\mathfrak{k}}.$$

The following results were proved in [20].

Proposition 4.1. *Let $\mathfrak{k}$ be a perfect Lie algebra. Then*

(a) *the map $\mathfrak{u}_{\mathfrak{k}} : \mathfrak{uce}(\mathfrak{k}) \to \mathfrak{k}$ is the universal central extension of $\mathfrak{k}$, and* $\ker(\mathfrak{u}_{\mathfrak{k}})$ *is the center of* $\mathfrak{uce}(\mathfrak{k})$ *when $\mathfrak{k}$ is centerless*;

(b) *for any homomorphism $f : \mathfrak{k} \to \mathfrak{k}_0$ of Lie algebras, the map $\mathfrak{uce}(f)$ is the unique homomorphism from $\mathfrak{uce}(\mathfrak{k})$ to $\mathfrak{uce}(\mathfrak{k}_0)$ that covers f.*

We also record the following straightforward result as a lemma that will be used later on.

Lemma 4.2. *Let $\sigma_1, \ldots, \sigma_s$ and $\tau_1, \ldots, \tau_s$ be pairwise commuting automorphisms of Lie algebras $\mathfrak{k}$ and $\mathfrak{k}_0$, respectively. Assume that there is a homomorphism $\gamma : \mathfrak{k} \to \mathfrak{k}_0$ such that $\gamma \circ \sigma_i = \tau_i \circ \gamma$ for each $i = 1, \ldots, s$. Then one has*

(a) *if the map $\mathfrak{uce}(\gamma)$ is injective, then*

$$\mathfrak{uce}(\gamma)(\mathfrak{uce}(\mathfrak{k})^{\mathfrak{uce}(\sigma_1),\ldots,\mathfrak{uce}(\sigma_s)}) = \mathfrak{uce}(\mathfrak{k}_0)^{\mathfrak{uce}(\tau_1),\ldots,\mathfrak{uce}(\tau_s)} \cap \mathrm{im}(\mathfrak{uce}(\gamma));$$

(b) *if the map γ is an isomorphism, then*

$$\mathfrak{uce}(\gamma) : \mathfrak{uce}(\mathfrak{k})^{\mathfrak{uce}(\sigma_1),\ldots,\mathfrak{uce}(\sigma_s)} \cong \mathfrak{uce}(\mathfrak{k}_0)^{\mathfrak{uce}(\tau_1),\ldots,\mathfrak{uce}(\tau_s)}.$$

Suppose now that σ_1 and σ_2 are two commuting automorphisms of $\dot{\mathfrak{g}}$ with periods M_1 and M_2, respectively. We define

$$\widehat{\mathcal{L}}_{M_1,M_2}(\dot{\mathfrak{g}}, \sigma_1, \sigma_2) = \mathcal{L}_{M_1,M_2}(\dot{\mathfrak{g}}, \sigma_1, \sigma_2) \oplus \mathcal{K}_{M_1,M_2}$$

to be the Lie algebra with Lie bracket as in (3.1). In particular, we have $\widehat{\mathfrak{g}} = \widehat{\mathcal{L}}_{1,r}(\dot{\mathfrak{g}}, \mathrm{id}, \dot{\nu})$. It was proved in [24] that $\widehat{\mathcal{L}}_{M_1,M_2}(\dot{\mathfrak{g}}, \sigma_1, \sigma_2)$ is the universal central extension of $\mathcal{L}_{M_1,M_2}(\dot{\mathfrak{g}}, \sigma_1, \sigma_2)$. For convenience, when M_i is the order of σ_i for $i = 1, 2$, we also write $\widehat{\mathcal{L}}(\dot{\mathfrak{g}}, \sigma_1, \sigma_2) = \widehat{\mathcal{L}}_{M_1,M_2}(\dot{\mathfrak{g}}, \sigma_1, \sigma_2)$.

For $\sigma \in \mathrm{Aut}(\dot{\mathfrak{g}})$ and $c_1, c_2 \in \mathbb{C}^\times$, one can easily verify that the assignment

$$\begin{aligned} &t_1^{m_1} t_2^{m_2} \otimes x \mapsto c_1^{-m_1} c_2^{-m_2} t_1^{m_1} t_2^{m_2} \otimes \sigma(x), \quad x \in \dot{\mathfrak{g}}, \quad m_1, m_s \in \mathbb{Z}, \\ &t_1^{m_1} t_2^{m_2} \mathrm{k}_i \mapsto c_1^{-m_1} c_2^{-m_2} t_1^{m_1} t_2^{m_2} \mathrm{k}_i, \quad i = 1, 2 \end{aligned}$$

determines an automorphism on $\widehat{\mathcal{L}}(\dot{\mathfrak{g}}, \mathrm{id}, \mathrm{id}) = \mathfrak{uce}(\mathcal{L}(\dot{\mathfrak{g}}, \mathrm{id}, \mathrm{id}))$. Note that this automorphism covers $c_1^{-\mathrm{d}_1} \otimes c_2^{-\mathrm{d}_2} \otimes \sigma$, and hence coincides with $\mathfrak{uce}(c_1^{-\mathrm{d}_1} \otimes c_2^{-\mathrm{d}_2} \otimes \sigma)$ (see Proposition 4.1(b)). By using this, it is easy to see that

$$\widehat{\mathcal{L}}_{M_1,M_2}(\dot{\mathfrak{g}}, \sigma_1, \sigma_2) = (\widehat{\mathcal{L}}(\dot{\mathfrak{g}}, \mathrm{id}, \mathrm{id}))^{\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1} \otimes 1^{-\mathrm{d}_2} \otimes \sigma_1), \mathfrak{uce}(1^{-\mathrm{d}_1} \otimes \xi_{M_2}^{-\mathrm{d}_2} \otimes \sigma_2)}.$$

In other words, we have the following isomorphism:

$$\mathfrak{uce}(\mathcal{L}(\dot{\mathfrak{g}}, \mathrm{id}, \mathrm{id})^{\xi_{M_1}^{-\mathrm{d}_1} \otimes 1^{-\mathrm{d}_2} \otimes \sigma_1, 1^{-\mathrm{d}_1} \otimes \xi_{M_2}^{-\mathrm{d}_2} \otimes \sigma_2}) \cong \mathfrak{uce}(\mathcal{L}(\dot{\mathfrak{g}}, \mathrm{id}, \mathrm{id}))^{\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1} \otimes 1^{-\mathrm{d}_2} \otimes \sigma_1), \mathfrak{uce}(1^{-\mathrm{d}_1} \otimes \xi_{M_2}^{-\mathrm{d}_2} \otimes \sigma_2)}. \quad (4.1)$$

4.3 Automorphism groups

In this subsection we collect some basics on the automorphism group of $\mathfrak{g}$, one may consult [5, Section 6] for details. Let $\mathrm{Aut}(A)$ be the group of diagram automorphisms of $\mathfrak{g}$. Define the outer automorphism group of $\mathfrak{g}$ to be

$$\mathrm{Out}(A) = \langle \omega \rangle \times \mathrm{Aut}(A),$$

where ω is the Chevalley involution of $\mathfrak{g}$.

Let $\mathrm{Hom}(Q, \mathbb{C}^\times)$ denote the set of group homomorphisms from Q to $\mathbb{C}^\times$, which is viewed as a group under pointwise multiplication. The group $\mathrm{Hom}(Q, \mathbb{C}^\times)$ can be identified as a subgroup of $\mathrm{Aut}(\mathfrak{g})$ in the following way:

$$\mathrm{Hom}(Q, \mathbb{C}^\times) \hookrightarrow \mathrm{Aut}(\mathfrak{g}), \quad \rho \mapsto (x \mapsto \rho(\alpha)x), \quad x \in \mathfrak{g}_\alpha, \quad \alpha \in \Delta. \quad (4.2)$$

Define the inner automorphism group of $\mathfrak{g}$ to be

$$\mathrm{Aut}^0(\mathfrak{g}) = \langle \exp(\mathrm{ad} x_\alpha) \mid \alpha \in \Delta^\times \rangle \cdot \mathrm{Hom}(Q, \mathbb{C}^\times).$$

Consider now the group homomorphism

$$\bar{\chi} : \mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(\bar{\mathfrak{g}}),$$

where $\bar{\chi}(\tau) = \bar{\tau}$ is the automorphism of $\bar{\mathfrak{g}}$ induced from τ. Note that the restriction of $\bar{\chi}$ on $\mathrm{Out}(A)$ and $\mathrm{Hom}(Q, \mathbb{C}^\times)$ are both injective. Thus we may view them as subgroups of $\mathrm{Aut}(\bar{\mathfrak{g}})$. The following statements were proved in [5, Propositions 6.1.5 and 6.1.8].

Proposition 4.3. *The homomorphism $\bar{\chi}$ is an isomorphism. Furthermore,*

$$\mathrm{Aut}(\mathfrak{g}) = \mathrm{Aut}^0(\mathfrak{g}) \rtimes \mathrm{Out}(A), \quad \mathrm{Aut}(\bar{\mathfrak{g}}) = \mathrm{Aut}^0(\bar{\mathfrak{g}}) \rtimes \mathrm{Out}(A),$$

where $\mathrm{Aut}^0(\bar{\mathfrak{g}}) = \bar{\chi}(\mathrm{Aut}^0(\mathfrak{g}))$.

By Proposition 4.3, we have the following projections:

$$p : \mathrm{Aut}(\mathfrak{g}) \to \mathrm{Out}(A) \quad \text{and} \quad \bar{p} : \mathrm{Aut}(\bar{\mathfrak{g}}) \to \mathrm{Out}(A)$$

such that $\bar{p} \circ \bar{\chi} = p$. An automorphism σ of $\mathfrak{g}$ (resp. $\bar{\mathfrak{g}}$) is said to be of the *first kind* if $p(\sigma)$ (resp. $\bar{p}(\sigma)$) lies in $\mathrm{Aut}(A)$. Otherwise, we say that σ is of the *second kind*.

4.4 Universal central extensions

This subsection is devoted to a proof of the following theorem.

Theorem 4.4. *Let $\bar{\eta}$ be an automorphism of $\bar{\mathfrak{g}}$ of the first kind with period M. Then the Lie algebra $\mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}))^{\mathfrak{uce}(\xi_M^{-\mathrm{d}_1} \otimes \bar{\eta})}$ is the universal central extension of the loop algebra $\mathcal{L}_M(\bar{\mathfrak{g}}, \bar{\eta}) = \mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id})^{\xi_M^{-\mathrm{d}_1} \otimes \bar{\eta}}$.*

Recall that the automorphism $\widehat{\mu}$ of $\widehat{\mathfrak{g}} = \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}))$ covers the automorphism $\xi^{-\mathrm{d}_1} \otimes \bar{\mu}$ of $\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id})$ (see (3.6)), and so coincides with $\mathfrak{uce}(\xi^{-\mathrm{d}_1} \otimes \bar{\mu})$ (see Proposition 4.1(b)). Thus, Theorem 3.3 is just a special case of Theorem 4.4.

We first establish some technical results. Let $\bar{\sigma}$ be an automorphism of $\bar{\mathfrak{g}}$ with period M. It is known that the twisted loop algebra of $\bar{\mathfrak{g}}$ related to $\bar{\sigma}$ is independent from the choice of its periods [4, Lemma 2.3]. In the following, we extend this result to their universal central extensions.

Lemma 4.5. *Let $\bar{\sigma}$ be an automorphism of $\bar{\mathfrak{g}}$ of finite period, and M and M' two periods of $\bar{\sigma}$. Then*

$$\mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}))^{\mathfrak{uce}(\xi_M^{-\mathrm{d}_1} \otimes \bar{\sigma})} \cong \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}))^{\mathfrak{uce}(\xi_{M'}^{-\mathrm{d}_1} \otimes \bar{\sigma})}. \tag{4.3}$$

Proof. We may (and do) assume that $M' = bM$ for some $b \in \mathbb{Z}_+$. Consider the natural imbedding

$$i_b : \mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}) \to \mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}), \quad t_1^m \otimes x \mapsto t_1^{bm} \otimes x,$$

where $m \in \mathbb{Z}$ and $x \in \bar{\mathfrak{g}}$. It is clear that the image of i_b is the Lie algebra $\mathcal{L}_b(\bar{\mathfrak{g}}, \mathrm{id}) = \mathcal{L}_{b,r}(\dot{\mathfrak{g}}, \mathrm{id}, \dot{\nu})$ and that

$$(\xi_{M'}^{-\mathrm{d}_1} \otimes \bar{\sigma}) \circ i_b = i_b \circ (\xi_M^{-\mathrm{d}_1} \otimes \bar{\sigma}). \tag{4.4}$$

By using Proposition 4.1(b), it is easy to see that the action of $\mathfrak{uce}(i_b)$ on the center of $\mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}))$ $= \mathfrak{uce}(\mathcal{L}_{1,r}(\dot{\mathfrak{g}}, \mathrm{id}, \dot{\nu}))$ is given by

$$t_1^{m_1} t_2^{m_2} \mathrm{k}_i \mapsto t_1^{bm_1} t_2^{m_2} \mathrm{k}_i, \quad i = 1, 2, \quad m_1 \in \mathbb{Z}, \quad m_2 \in r\mathbb{Z}.$$

This implies

$$\text{the map } \mathfrak{uce}(i_b) \text{ is injective} \tag{4.5}$$

and

$$\mathrm{im}(\mathfrak{uce}(i_b)) = \mathfrak{uce}(\mathcal{L}_{b,r}(\dot{\mathfrak{g}}, \mathrm{id}, \dot{\nu}))$$

$$
\begin{aligned}
&= \mathfrak{uce}(\mathcal{L}(\dot{\mathfrak{g}}, \mathrm{id}, \mathrm{id}))^{\mathfrak{uce}(\xi_b^{-\mathrm{d}_1} \otimes 1^{-\mathrm{d}_2} \otimes 1), \mathfrak{uce}(1^{-\mathrm{d}_1} \otimes \xi_r^{-\mathrm{d}_2} \otimes \dot{\nu})} \\
&= (\mathfrak{uce}(\mathcal{L}(\dot{\mathfrak{g}}, \mathrm{id}, \mathrm{id}))^{\mathfrak{uce}(1^{-\mathrm{d}_1} \otimes \xi_r^{-\mathrm{d}_2} \otimes \dot{\nu})})^{\mathfrak{uce}(\xi_b^{-\mathrm{d}_1} \otimes 1^{-\mathrm{d}_2} \otimes 1)} \\
&= \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}))^{\mathfrak{uce}(\xi_b^{-\mathrm{d}_1} \otimes 1)}.
\end{aligned} \tag{4.6}
$$

Note that we also have

$$
\begin{aligned}
\mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}))^{\mathfrak{uce}(\xi_{M'}^{-\mathrm{d}_1} \otimes \bar{\sigma})} &\subset \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}))^{(\mathfrak{uce}(\xi_{M'}^{-\mathrm{d}_1} \otimes \bar{\sigma}))^M} \\
&= \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}))^{\mathfrak{uce}((\xi_{M'}^{-\mathrm{d}_1} \otimes \bar{\sigma})^M)} \\
&= \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}))^{\mathfrak{uce}(\xi_b^{-\mathrm{d}_1} \otimes 1)}.
\end{aligned}
$$

This together with (4.6) gives

$$
\mathrm{im}(\mathfrak{uce}(i_b)) \cap \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}))^{\mathfrak{uce}(\xi_{M'}^{-\mathrm{d}_1} \otimes \bar{\sigma})} = \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}))^{\mathfrak{uce}(\xi_{M'}^{-\mathrm{d}_1} \otimes \bar{\sigma})}. \tag{4.7}
$$

Now the assertion is implied by (4.4), (4.5), (4.7) and Lemma 4.2(a). □

Let $\bar{\sigma}$ be an automorphism of $\bar{\mathfrak{g}}$ with the period M. Now $\bar{\mathfrak{g}} = \mathcal{L}(\dot{\mathfrak{g}}, \dot{\nu})$ itself is a twisted loop algebra and so is independent from the choice of the period of $\dot{\nu}$. Namely, if M' is another period of $\dot{\nu}$, then one has the natural isomorphism $\bar{\mathfrak{g}} \cong \mathcal{L}_{M'}(\dot{\mathfrak{g}}, \dot{\nu})$. Via this isomorphism, $\bar{\sigma}$ induces an automorphism, say $\bar{\sigma}'$, of $\mathcal{L}_{M'}(\dot{\mathfrak{g}}, \dot{\nu})$ with the period M. Similar to Lemma 4.5, we have the following lemma.

Lemma 4.6. *Let $\bar{\sigma}, M, M'$ and $\bar{\sigma}'$ be as above. Then one has*

$$
\mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}))^{\mathfrak{uce}(\xi_M^{-\mathrm{d}_1} \otimes \bar{\sigma})} \cong \mathfrak{uce}(\mathcal{L}(\mathcal{L}_{M'}(\dot{\mathfrak{g}}, \dot{\nu}), \mathrm{id}))^{\mathfrak{uce}(\xi_M^{-\mathrm{d}_1} \otimes \bar{\sigma}')}. \tag{4.8}
$$

Proof. Set $b = M'/r$ and define the embedding

$$
j_b : \bar{\mathfrak{g}} = \mathcal{L}(\dot{\mathfrak{g}}, \dot{\nu}) \to \mathcal{L}(\dot{\mathfrak{g}}, \mathrm{id}), \quad t_2^{m_2} \otimes x \mapsto t_2^{bm_2} \otimes x, \quad m_2 \in \mathbb{Z}, \quad x \in \dot{\mathfrak{g}}.
$$

Then the image of j_b is the Lie algebra $\mathcal{L}_{M'}(\dot{\mathfrak{g}}, \dot{\nu})$ and

$$
j_b \circ \bar{\sigma} = \bar{\sigma}' \circ j_b. \tag{4.9}
$$

Moreover, the action of $\mathfrak{uce}(1^{-\mathrm{d}_1} \otimes j_b)$ on the center of $\mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}))$ is given by

$$
t_1^{m_1} t_2^{m_2} \mathrm{k}_i \mapsto t_1^{m_1} t_2^{bm_2} \mathrm{k}_i, \quad i = 1, 2, \quad m_1 \in \mathbb{Z}, \quad m_2 \in r\mathbb{Z}.
$$

This implies

$$
\text{the map } \mathfrak{uce}(1^{-\mathrm{d}_1} \otimes j_b) \text{ is injective} \tag{4.10}
$$

and

$$
\mathrm{im}(\mathfrak{uce}(1^{-\mathrm{d}_1} \otimes j_b)) = \mathfrak{uce}(\mathcal{L}_{1,M'}(\dot{\mathfrak{g}}, \mathrm{id}, \dot{\nu})) = \mathfrak{uce}(\mathcal{L}(\mathcal{L}_{M'}(\dot{\mathfrak{g}}, \dot{\nu}), \mathrm{id})). \tag{4.11}
$$

Then the lemma follows from (4.9)–(4.11) and Lemma 4.2(a). □

Using Lemma 4.5, we have the following result.

Lemma 4.7. *Let $\bar{\sigma}$ be an automorphism of $\bar{\mathfrak{g}}$ with period M. Then*

$$
\mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}))^{\mathfrak{uce}(\xi_M^{-\mathrm{d}_1} \otimes \bar{\sigma})} \cong \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}}, \mathrm{id}))^{\mathfrak{uce}(\xi_M^{-\mathrm{d}_1} \otimes \bar{p}(\bar{\sigma}))}. \tag{4.12}
$$

Proof. Recall the isomorphism $\bar{\chi} : \mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(\bar{\mathfrak{g}})$ given in Proposition 4.3. Then we may choose an automorphism σ of $\mathfrak{g}$ such that $\sigma^M = \mathrm{id}$ and $\bar{\chi}(\sigma) = \bar{\sigma}$. This together with [18, Lemma 4.31] gives that there exists a $\rho \in \mathrm{Hom}(Q, \mathbb{C}^\times)$ such that

$$
\rho\, \bar{p}(\bar{\sigma}) = \bar{p}(\bar{\sigma})\, \rho, \quad \rho^M = \mathrm{id} \quad \text{and} \quad \bar{\sigma} \text{ is conjugate to } \bar{p}(\bar{\sigma})\rho.
$$

Note that the automorphisms ρ and $\bar{p}(\bar{\sigma})$ of $\bar{\mathfrak{g}}$ satisfy all the assumptions stated in [4, Theorem 5.1]. Then it follows from [4, (5.3)] that the automorphism $\xi_M^{-\mathrm{d}_1}\otimes\rho\bar{p}(\bar{\sigma})$ is conjugate to $\xi_{M^2}^{-\mathrm{d}_1}\otimes\bar{p}(\bar{\sigma})$. This together with Lemmas 4.2(b) and 4.5 gives

$$\begin{aligned}\mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}},\mathrm{id}))^{\mathfrak{uce}(\xi_M^{-\mathrm{d}_1}\otimes\bar{\sigma})} &\cong \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}},\mathrm{id}))^{\mathfrak{uce}(\xi_M^{-\mathrm{d}_1}\otimes\rho\bar{p}(\bar{\sigma}))}\\ &\cong \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}},\mathrm{id}))^{\mathfrak{uce}(\xi_{M^2}^{-\mathrm{d}_1}\otimes\bar{p}(\bar{\sigma}))}\\ &\cong \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}},\mathrm{id}))^{\mathfrak{uce}(\xi_M^{-\mathrm{d}_1}\otimes\bar{p}(\bar{\sigma}))}.\end{aligned}$$

Therefore, we complete the proof. □

Let $\mathrm{Hom}(\dot{Q},\mathbb{C}^\times)$ be the set of group homomorphisms from $\dot{Q}$ to $\mathbb{C}^\times$. Similar to (4.2), we may (and do) view $\mathrm{Hom}(\dot{Q},\mathbb{C}^\times)$ as a subgroup of $\mathrm{Aut}(\dot{\mathfrak{g}})$. From now on, let $\bar{\eta}$ be as in Theorem 4.4. The following characterization of $\mathcal{L}_M(\bar{\mathfrak{g}},\bar{\eta})$ plays a key role in the proof of Theorem 4.4.

Lemma 4.8. *There exist finite order automorphisms $\dot{\rho}$ and $\dot{\tau}$ of $\dot{\mathfrak{g}}$ such that*

$$\dot{\rho}\in\mathrm{Hom}(\dot{Q},\mathbb{C}^\times),\quad \dot{\nu}\dot{\rho}=\dot{\rho}\dot{\nu},\quad (\dot{\nu}\dot{\rho})\dot{\tau}=\dot{\tau}(\dot{\nu}\dot{\rho})\quad \textit{and}\quad \mathcal{L}_{M_1,M_2}(\dot{\mathfrak{g}},\dot{\tau},\dot{\nu}\dot{\rho})\cong\mathcal{L}_M(\bar{\mathfrak{g}},\bar{\eta}),$$

where M_1 and M_2 are some periods of $\dot{\tau}$ and $\dot{\nu}\dot{\rho}$, respectively.

Proof. By [5, Theorem 10.1.1], there exist finite order automorphisms $\dot{\tau}$ and $\dot{\sigma}$ such that $\mathcal{L}(\bar{\mathfrak{g}},\bar{\eta})\cong\mathcal{L}(\dot{\mathfrak{g}},\dot{\tau},\dot{\sigma})$. Up to conjugation, we may assume that $\dot{\sigma}$ is of the form $\dot{\rho}\,\dot{\vartheta}$, where $\dot{\rho}\in\mathrm{Hom}(\dot{Q},\mathbb{C}^\times)$ and $\dot{\vartheta}$ is a diagram automorphism of $\dot{\mathfrak{g}}$ such that $\dot{\rho}\,\dot{\vartheta}=\dot{\vartheta}\,\dot{\rho}$. If $\mathfrak{g}$ is of untwisted type, then it follows from the proof of [5, Theorem 10.1.1] that one may take $\dot{\vartheta}=\mathrm{id}=\dot{\nu}$. If $\mathfrak{g}$ is of twisted type, then by comparing the classification results (the relative and absolute types) given in [5, Table 3] and [10, Table 9.2.4], we find out that the diagram automorphism $\dot{\vartheta}$ can also be taken to be $\dot{\nu}$. □

Notice that the automorphisms $\dot{\rho}$ and $\dot{\nu}$ satisfy the assumptions given in [4, Theorem 5.1]. Thus, there is an automorphism φ of $\mathcal{L}(\dot{\mathfrak{g}},\mathrm{id})$ such that

$$\varphi\circ(\xi_{M_2}^{-\mathrm{d}_2}\otimes\dot{\nu}\dot{\rho})\circ\varphi^{-1}=\xi_{M_2^2}^{-\mathrm{d}_2}\otimes\dot{\nu}. \tag{4.13}$$

Denote by τ' the automorphism

$$\varphi\circ(1^{-\mathrm{d}_2}\otimes\dot{\tau})\circ\varphi^{-1}$$

of $\mathcal{L}(\dot{\mathfrak{g}},\mathrm{id})$. Then τ' commutes with the automorphism $\xi_{M_2^2}^{-\mathrm{d}_2}\otimes\dot{\nu}$, and hence preserves the Lie algebra $\mathcal{L}_{M_2^2}(\dot{\mathfrak{g}},\dot{\nu})$. Write τ'' for the restriction of τ' on $\mathcal{L}_{M_2^2}(\dot{\mathfrak{g}},\dot{\nu})$, and $\bar{\tau}$ for the automorphism of $\bar{\mathfrak{g}}$ induced from τ'' via the isomorphism $\bar{\mathfrak{g}}\cong\mathcal{L}_{M_2^2}(\dot{\mathfrak{g}},\dot{\nu})$. So by definition we have

$$\begin{aligned}\mathcal{L}(\dot{\mathfrak{g}},\mathrm{id},\mathrm{id})^{\xi_{M_1}^{-\mathrm{d}_1}\otimes1^{-\mathrm{d}_2}\otimes\dot{\tau},1^{-\mathrm{d}_1}\otimes\xi_{M_2}^{-\mathrm{d}_2}\otimes\dot{\rho}\dot{\nu}} &\cong \mathcal{L}(\mathcal{L}(\dot{\mathfrak{g}},\mathrm{id}),\mathrm{id})^{\xi_{M_1}^{-\mathrm{d}_1}\otimes\tau',1^{-\mathrm{d}_1}\otimes(\xi_{M_2^2}^{-\mathrm{d}_2}\otimes\dot{\nu})}\\ &\cong \mathcal{L}(\mathcal{L}_{M_2^2}(\dot{\mathfrak{g}},\dot{\nu}),\mathrm{id})^{\xi_{M_1}^{-\mathrm{d}_1}\otimes\tau''}\cong\mathcal{L}_{M_1}(\bar{\mathfrak{g}},\bar{\tau}).\end{aligned} \tag{4.14}$$

Lemma 4.9. *One has*

$$\mathfrak{uce}(\mathcal{L}(\mathcal{L}(\dot{\mathfrak{g}},\mathrm{id}),\mathrm{id}))^{\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1}\otimes\tau'),\mathfrak{uce}(1^{-\mathrm{d}_1}\otimes(\xi_{M_2^2}^{-\mathrm{d}_2}\otimes\dot{\nu}))}\cong\mathfrak{uce}(\mathcal{L}(\mathcal{L}_{M_2^2}(\dot{\mathfrak{g}},\dot{\nu}),\mathrm{id}))^{\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1}\otimes\tau'')}.$$

Proof. Due to the isomorphisms

$$\mathfrak{uce}(\mathcal{L}(\dot{\mathfrak{g}},\mathrm{id},\mathrm{id}))^{\mathfrak{uce}(1^{-\mathrm{d}_1}\otimes\xi_{M_2^2}^{-\mathrm{d}_2}\otimes\dot{\nu})}\cong\mathfrak{uce}(\mathcal{L}_{1,M_2^2}(\dot{\mathfrak{g}},\mathrm{id},\dot{\nu}))\cong\mathfrak{uce}(\mathcal{L}(\mathcal{L}_{M_2^2}(\dot{\mathfrak{g}},\dot{\nu}),\mathrm{id})),$$

it suffices to show that the restriction of $\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1}\otimes\tau')$ on $\mathfrak{uce}(\mathcal{L}(\mathcal{L}_{M_2^2}(\dot{\mathfrak{g}},\dot{\nu}),\mathrm{id}))$ coincides with $\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1}\otimes\tau'')$. Set $\mathfrak{k}=\mathcal{L}(\dot{\mathfrak{g}},\mathrm{id},\mathrm{id})$ and $\mathfrak{k}_0=\mathcal{L}_{1,M_2^2}(\dot{\mathfrak{g}},\mathrm{id},\dot{\nu})=\mathcal{L}(\mathcal{L}_{M_2^2}(\dot{\mathfrak{g}},\dot{\nu}),\mathrm{id})$. Then by definition one has

$$\mathfrak{u}_{\mathfrak{k}}\circ\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1}\otimes\tau')=(\xi_{M_1}^{-\mathrm{d}_1}\otimes\tau')\circ\mathfrak{u}_{\mathfrak{k}},\quad \mathfrak{u}_{\mathfrak{k}_0}\circ\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1}\otimes\tau'')=(\xi_{M_1}^{-\mathrm{d}_1}\otimes\tau'')\circ\mathfrak{u}_{\mathfrak{k}_0},$$

$$\mathfrak{u}_{\mathfrak{k}_0} = \mathfrak{u}_{\mathfrak{k}}\,|_{\mathfrak{uce}(\mathfrak{k}_0)=\widehat{\mathcal{L}}_{1,M_2^2}(\dot{\mathfrak{g}},\mathrm{id},\dot{\nu})} \quad \text{and} \quad \xi_{M_1}^{-\mathrm{d}_1} \otimes \tau'' = \xi_{M_1}^{-\mathrm{d}_1} \otimes \tau'\,|_{\mathfrak{k}_0}.$$

This implies that the restriction of $\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1} \otimes \tau')$ on $\mathfrak{uce}(\mathfrak{k}_0)$ covers $\xi_{M_1}^{-\mathrm{d}_1} \otimes \tau''$. Combining this with Proposition 4.1(b), we obtain the desired result. □

Now, by using Lemmas 4.9, 4.2(b) and 4.6, we can extend the isomorphisms given in (4.14) to their universal central extensions as follows:

$$\begin{aligned} &\mathfrak{uce}(\mathcal{L}(\dot{\mathfrak{g}},\mathrm{id},\mathrm{id}))^{\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1}\otimes 1^{-\mathrm{d}_2}\otimes\dot{\tau}),\mathfrak{uce}(1^{-\mathrm{d}_1}\otimes\xi_{M_2}^{-\mathrm{d}_2}\otimes\dot{\rho}\dot{\nu})} \\ &\quad\cong \mathfrak{uce}(\mathcal{L}(\mathcal{L}(\dot{\mathfrak{g}},\mathrm{id}),\mathrm{id}))^{\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1}\otimes\tau'),\mathfrak{uce}(1^{-\mathrm{d}_1}\otimes(\xi_{M_2^2}^{-\mathrm{d}_2}\otimes\dot{\nu}))} \\ &\quad\cong \mathfrak{uce}(\mathcal{L}(\mathcal{L}_{M_2^2}(\dot{\mathfrak{g}},\dot{\nu}),\mathrm{id}))^{\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1}\otimes\tau'')} \\ &\quad\cong \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}},\mathrm{id}))^{\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1}\otimes\bar{\tau})}. \end{aligned} \tag{4.15}$$

Combining Lemma 4.8 with (4.14), we get the isomorphism

$$\mathcal{L}_{M_1}(\bar{\mathfrak{g}},\bar{\tau}) \cong \mathcal{L}_M(\bar{\mathfrak{g}},\bar{\eta}).$$

By using [5, Theorem 10.1.1 and Corollary 10.1.5], we get that $\bar{\tau}$ is of the first kind. Moreover, it follows from [5, Theorem 13.2.3] that the diagram automorphism $\bar{p}(\bar{\tau})$ is conjugate to $\bar{p}(\bar{\eta})$. Thus, one can conclude from Lemmas 4.5 and 4.7 that

$$\begin{aligned} \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}},\mathrm{id}))^{\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1}\otimes\bar{\tau})} &\cong \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}},\mathrm{id}))^{\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1}\otimes\bar{p}(\bar{\tau}))} \\ &\cong \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}},\mathrm{id}))^{\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1}\otimes\bar{p}(\bar{\eta}))} \\ &\cong \mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}},\mathrm{id}))^{\mathfrak{uce}(\xi_{M}^{-\mathrm{d}_1}\otimes\bar{\eta})}. \end{aligned}$$

Combining this with (4.15), we get that

$$\mathfrak{uce}(\mathcal{L}(\bar{\mathfrak{g}},\mathrm{id}))^{\mathfrak{uce}(\xi_{M}^{-\mathrm{d}_1}\otimes\bar{\eta})} \cong \mathfrak{uce}(\mathcal{L}(\dot{\mathfrak{g}},\mathrm{id},\mathrm{id}))^{\mathfrak{uce}(\xi_{M_1}^{-\mathrm{d}_1}\otimes 1^{-\mathrm{d}_2}\otimes\dot{\tau}),\mathfrak{uce}(1^{-\mathrm{d}_1}\otimes\xi_{M_2}^{-\mathrm{d}_2}\otimes\dot{\rho}\dot{\nu})}$$

is centrally closed. This completes the proof of Theorem 4.4.

5 Proof of Theorem 3.6

Throughout this section, we assume that the diagram automorphism μ is non-transitive.

5.1 The root system of $\widehat{\mathfrak{g}}[\mu]$

In this subsection, we determine the non-isotropic roots in $\widehat{\mathfrak{g}}[\mu]$. As indicated in [5, Section 14], this affords an explicit realization of all nullity 2 reduced extended affine root systems given by Saito [22].

Recall that $V = \mathbb{R}\otimes_{\mathbb{Z}} Q$, and we extend μ to a linear automorphism on V by $\mathbb{R}$-linearity. We denote by V_μ the fixed point subspace of V under the isometry μ, $\pi_\mu : V \to V_\mu$ the canonical projection of V onto V_μ, and $\widehat{Q}_\mu$ the abelian group $\pi_\mu(Q)\times\mathbb{Z}$.

Define a $Q\times\mathbb{Z}$-grading on $\widehat{\mathfrak{g}} = \bigoplus_{(\alpha,n)\in Q\times\mathbb{Z}} \widehat{\mathfrak{g}}_{\alpha,n}$ by letting

$$t_1^{n_1}\otimes x \in \widehat{\mathfrak{g}}_{\alpha,n_1}, \quad \mathrm{k}_1 \in \widehat{\mathfrak{g}}_{0,0}, \quad t_1^{n_1}t_2^{n_2}\mathrm{k}_1 \in \widehat{\mathfrak{g}}_{n_2\delta_2,n_1},$$

where $x\in\mathfrak{g}_\alpha$, $\alpha\in\Delta$, $n_1\in\mathbb{Z}$ and $n_2\in r\mathbb{Z}^\times$. The above grading induces a $\widehat{Q}_\mu$-grading

$$\widehat{\mathfrak{g}}[\mu] = \bigoplus_{(\alpha,n)\in\widehat{Q}_\mu} \widehat{\mathfrak{g}}[\mu]_{\alpha,n}$$

on $\widehat{\mathfrak{g}}[\mu]$ such that for any $(\alpha,n)\in\widehat{Q}_\mu$,

$$\widehat{\mathfrak{g}}[\mu]_{\alpha,n} = \{x\in\widehat{\mathfrak{g}}[\mu]\cap\widehat{\mathfrak{g}}_{\beta,n} \mid \beta\in Q, \pi_\mu(\beta)=\alpha\}.$$

Notice that this is the unique $\widehat{Q}_\mu$-grading on $\widehat{\mathfrak{g}}[\mu]$ such that

$$t_1^n \otimes e_{i(n)}^{\pm} \in \widehat{\mathfrak{g}}[\mu]_{\pm\check{\alpha}_i,n}, \quad t_1^n \otimes \alpha_{i(n)}^{\vee} \in \widehat{\mathfrak{g}}[\mu]_{0,n}, \quad \mathrm{k}_1 \in \widehat{\mathfrak{g}}[\mu]_{0,0} \tag{5.1}$$

for $i \in I$ and $n \in \mathbb{Z}$.

Consider now the following subsets of $\widehat{Q}_\mu$:

$$\begin{aligned}\Phi_\mu &= \{(\alpha, n) \in \widehat{Q}_\mu \mid \widehat{\mathfrak{g}}[\mu]_{\alpha,n} \neq 0\},\\ \widehat{Q}_\mu^\times &= \{(\alpha, n) \in \widehat{Q}_\mu \mid (\alpha, \alpha) \neq 0\},\\ \Phi_\mu^\times &= \Phi_\mu \cap \widehat{Q}_\mu^\times = \{(\alpha, n) \in \Phi_\mu \mid (\alpha, \alpha) \neq 0\}.\end{aligned}$$

It is obvious that $\Phi_\mu \subset \pi_\mu(\Delta) \times \mathbb{Z}$ and so we have

$$\Phi_\mu^\times \subset \pi_\mu(\Delta)^\times \times \mathbb{Z}, \tag{5.2}$$

where $\pi_\mu(\Delta)^\times = \{\alpha \in \pi_\mu(\Delta) \,|\, (\alpha, \alpha) \neq 0\}$. By definition, for each $i \in I$ we have $\check{\alpha}_i = \pi_\mu(\alpha_i)$. In addition, for $i \in I$ with $s_i = 2$, we have $2\check{\alpha}_i = \pi_\mu(\alpha_i + \alpha_{\mu(i)})$. This shows

$$k_i \check{\alpha}_i \in \pi_\mu(\Delta)^\times, \quad 1 \leqslant k_i \leqslant s_i, \quad i \in \check{I}. \tag{5.3}$$

For $i \in I$, we let N_i be the cardinality of the orbit $\mathcal{O}(i)$ in I and set $d_i = \frac{N}{N_i}$. Denote by $\check{W}$ the Weyl group of the folded GCM $\check{A}$. Then we have the following description of the set $\Phi_\mu^\times$.

Proposition 5.1. *One has*

$$\Phi_\mu^\times = \{(\check{w}(k_i \check{\alpha}_i), p) \mid \check{w} \in \check{W}, i \in \check{I}, 1 \leqslant k_i \leqslant s_i, p \in (k_i - 1)d_i + k_i d_i \mathbb{Z}\} \tag{5.4}$$

and that

$$\dim \widehat{\mathfrak{g}}[\mu]_{\alpha,p} = 1, \quad \forall\, (\alpha, p) \in \Phi_\mu^\times. \tag{5.5}$$

Before proving Proposition 5.1, we first give a characterization of the set $\pi_\mu(\Delta)^\times$. This result is a slight generalization of [5, Proposition 12.1.16].

Lemma 5.2. *One has*

$$\pi_\mu(\Delta)^\times = \{\check{w}(k_i \check{\alpha}_i) \mid \check{w} \in \check{W}, i \in \check{I}, 1 \leqslant k_i \leqslant s_i\}. \tag{5.6}$$

Proof. For convenience, we set

$$\check{\Delta}^{\mathrm{en}} = \{\check{w}(k_i \check{\alpha}_i) \mid \check{w} \in \check{W}, i \in \check{I}, 1 \leqslant k_i \leqslant s_i\}.$$

We first show that

$$\check{W}(\pi_\mu(\Delta)) \subset \pi_\mu(\Delta). \tag{5.7}$$

Let $r_{\check{\alpha}_i}$, $i \in \check{I}$ denote the reflections associated to $\check{\alpha}_i$. Note that the Weyl group $\check{W}$ is generated by these reflections. Thus we only need to show that

$$r_{\check{\alpha}_i}(\pi_\mu(\Delta)) \subset \pi_\mu(\Delta), \quad i \in \check{I}. \tag{5.8}$$

If $s_i = 1$, it is shown in the proof of [5, Proposition 12.1.16] that for each $\alpha \in \Delta$, the following relation holds true:

$$r_{\check{\alpha}_i}(\pi_\mu(\alpha)) = \pi_\mu\bigg(\bigg(\prod_{p \in \mathcal{O}(i)} r_{\alpha_p}\bigg)(\alpha)\bigg) \in \pi_\mu(\Delta).$$

If $s_i = 2$, then $2\check{\alpha}_i = \alpha_i + \alpha_{\mu(i)} \in \Delta$ as $a_{i,\mu(i)} = -1$. Note that $\pi_\mu(\check{\alpha}_i) = \check{\alpha}_i$ and hence $(\check{\alpha}_i, \pi_\mu(\alpha)) = (\check{\alpha}_i, \alpha)$ for all $\alpha \in \Delta$. This implies that

$$\begin{aligned} r_{\check{\alpha}_i}(\pi_\mu(\alpha)) &= \pi_\mu(\alpha) - 2\frac{(\check{\alpha}_i, \pi_\mu(\alpha))}{(\check{\alpha}_i, \check{\alpha}_i)}\check{\alpha}_i = \pi_\mu\left(\alpha - 2\frac{(\check{\alpha}_i, \alpha)}{(\check{\alpha}_i, \check{\alpha}_i)}\check{\alpha}_i\right) \\ &= \pi_\mu\left(\alpha - 2\frac{(\check{\alpha}_i, \alpha)}{(2\check{\alpha}_i, 2\check{\alpha}_i)}2\check{\alpha}_i\right) = \pi_\mu(r_{2\check{\alpha}_i}(\alpha)) \in \pi_\mu(\Delta). \end{aligned}$$

Thus we complete the proof of the assertion (5.8) and hence the assertion (5.7). Now, as the reflections preserve the bilinear form $(\cdot,\cdot)$, we have

$$\check{W}(\pi_\mu(\Delta)^\times) \subset \pi_\mu(\Delta)^\times.$$

This together with (5.3) gives

$$\check{\Delta}^{\mathrm{en}} \subset \check{W}(\pi_\mu(\Delta)^\times) \subset \pi_\mu(\Delta)^\times.$$

For the reverse inclusion, observe first that any non-zero element $\beta \in \pi_\mu(\Delta)$ can be written uniquely in the form $\beta = \sum_{i\in\check{I}} n_i\check{\alpha}_i$, where n_i's are either all non-negative integers or all non-positive integers. Set $\mathrm{ht}\beta = \sum_{i\in\check{I}} n_i$. Assume that $\beta \in \pi_\mu(\Delta)^\times$. We then show that $\beta \in \check{\Delta}^{\mathrm{en}}$ by using induction on $\mathrm{ht}\beta$. Without loss of generality, we may assume that $\mathrm{ht}\beta > 0$. Since $(\beta,\beta) > 0$, there are some $i \in \check{I}$ such that $(\beta, \check{\alpha}_i) > 0$ and that $n_i > 0$. If $r_{\check{\alpha}_i}(\beta)$ is positive, then we are done by the induction hypothesis. If $r_{\check{\alpha}_i}(\beta)$ is negative, then $\beta = q\check{\alpha}_i$ for some positive integer q. This implies that $\beta = \pi_\mu(\alpha)$ for some

$$\alpha = \sum_{p\in\mathcal{O}(i)} m_p\alpha_p \in \Delta \quad \text{with} \quad \sum_{p\in\mathcal{O}(i)} m_p = q.$$

If $s_i = 1$, then q must equal 1 as all α_p, $p \in \mathcal{O}(i)$ are pairwise orthogonal. If $s_i = 2$, then q can be 1 or 2, as $|\mathcal{O}(i)| = 2$ and $a_{i\mu(i)} = -1$. This completes the proof. □

As a by-product of Lemma 5.2, we have the following corollary.

Corollary 5.3. *Let $i, j \in I$ with $\check{a}_{ij} \leqslant 0$. Then for every $p \in \mathbb{Z}$, the elements $((1-\check{a}_{ij})\check{\alpha}_i + \check{\alpha}_j, p)$ and $((s_i+1)\check{\alpha}_i, p)$ are contained in $\widehat{Q}_\mu^\times$ but not contained in $\Phi_\mu^\times$.*

Proof. By Lemma 5.2, it suffices to show that if $\check{a}_{ij} \leqslant 0$, then $(1-\check{a}_{ij})\check{\alpha}_i + \check{\alpha}_j$ is non-isotropic. Otherwise,

$$0 = 2\frac{(\check{\alpha}_i, (1-\check{a}_{ij})\check{\alpha}_i + \check{\alpha}_j)}{(\check{\alpha}_i, \check{\alpha}_i)} = 2(1-\check{a}_{ij}) + \check{a}_{ij} = 2 - \check{a}_{ij},$$

which leads to a contradiction. □

Let $\check{\mathfrak{g}}$ be the subalgebra of $\widehat{\mathfrak{g}}[\mu]$ generated by the elements $\alpha_{i(0)}^\vee, e_{i(0)}^\pm$, $i \in \check{I}$. Then by applying Corollary 5.3 we have the following corollary.

Corollary 5.4. *The Lie algebra $\check{\mathfrak{g}}$ is isomorphic to the derived subalgebra of the Kac-Moody algebra associated with $\check{A}$.*

Proof. It suffices to check that the elements $\alpha_{i(0)}^\vee, e_{i(0)}^\pm$, $i \in \check{I}$ satisfy the defining relations of the derived subalgebra of the Kac-Moody algebra associated with $\check{A}$. Only the Serre relations

$$(\mathrm{ad} e_{i(0)}^\pm)^{1-\check{a}_{ij}}(e_{i(0)}^\pm) = 0, \quad i \neq j \in \check{I}$$

are non-trivial. But such relations are immediate from Corollary 5.3. □

Now we are ready to complete the proof of Proposition 5.1. Using (5.2) and Lemma 5.2, we know that any element in $\Phi_\mu^\times$ has the form

$$(\check{w}(k_i\check{\alpha}_i), p), \quad \check{w} \in \check{W}, \quad 1 \leqslant k_i \leqslant s_i, \quad i \in \check{I}, \quad p \in \mathbb{Z}. \tag{5.9}$$

Regard $\widehat{\mathfrak{g}}[\mu]$ as a module of the affine Kac-Moody algebra $\check{\mathfrak{g}}$ (see Corollary 5.4) via the adjoint action. Then it is integrable, and for each $p\in\mathbb{Z}$, the graded subspace $\widehat{\mathfrak{g}}[\mu]_p$ of $\widehat{\mathfrak{g}}[\mu]$ is a $\check{\mathfrak{g}}$-submodule, where

$$\widehat{\mathfrak{g}}[\mu]_p=\bigoplus_{\check{\alpha}\in\pi_\mu(\Delta)}\widehat{\mathfrak{g}}[\mu]_{\check{\alpha},p}.$$

Using this and the standard $\mathfrak{sl}_2$-theory, we obtain that $(\check{w}(k_i\check{\alpha}_i),p)\in\Phi_\mu^\times$ if and only if $(k_i\check{\alpha}_i,p)\in\Phi_\mu^\times$. Moreover, we have $\dim\widehat{\mathfrak{g}}[\mu]_{\check{w}(k_i\check{\alpha}_i),p}=\dim\widehat{\mathfrak{g}}[\mu]_{k_i\check{\alpha}_i,p}$. So we only need to treat the case where $\check{w}=1$.

We first consider the case where $k_i=1$. Note that for each $i\in\check{I}$,

$$\widehat{\mathfrak{g}}[\mu]_{\check{\alpha}_i,p}=\mathbb{C}t_1^p\otimes e_{i(p)}^+=\mathbb{C}\sum_{s\in\mathbb{Z}_{N_i}}\Bigg(\sum_{k\in\mathbb{Z}_{d_i}}\xi_{d_i}^{-kr}\Bigg)\xi^{-ps}t_1^p\otimes e_{\mu^s(i)}^+.$$

This together with the fact

$$\sum_{k\in\mathbb{Z}_{d_i}}\xi_{d_i}^{-pk}\neq 0\Leftrightarrow p\in d_i\mathbb{Z}$$

gives that $(\check{\alpha}_i,p)\in\Phi_\mu^\times$ if and only if $p\in d_i\mathbb{Z}$. Next, for the case where $k_i=2$ (and hence $s_i=2$), we have

$$\widehat{\mathfrak{g}}[\mu]_{2\check{\alpha}_i,p}=\mathbb{C}t_1^p\otimes[e_i^+,e_{\mu(i)}^+]_{(p)}.$$

This together with the fact

$$\mu([e_i^+,e_{\mu(i)}^+])=[e_{\mu(i)}^+,e_i^+]=-[e_i^+,e_{\mu(i)}^+]$$

gives that $(2\check{\alpha}_i,p)\in\Phi_\mu^\times$ if and only if $p\in d_i+N\mathbb{Z}$. Therefore, we complete the proof of Proposition 5.1.

5.2 Proof of Theorem 3.6

We start with the following lemma.

Lemma 5.5. *The assignment*

$$c\mapsto \mathrm{k}_1,\quad h_{i,m}\mapsto t_1^m\otimes\alpha_{i(m)}^\vee,\quad x_{i,m}^\pm\mapsto t_1^m\otimes e_{i(m)}^\pm,\quad i\in I,\quad m\in\mathbb{Z}$$

determines (*uniquely*) *a surjective Lie homomorphism from* $\mathcal{M}(\mathfrak{g},\mu)$ *to* $\widehat{\mathfrak{g}}[\mu]$.

Proof. One needs to check that the generators $\alpha_{i(m)}^\vee,e_{i(m)}^\pm,\mathrm{k}_1$, $i\in I,m\in\mathbb{Z}$ of $\widehat{\mathfrak{g}}[\mu]$ satisfy the defining relations (T0)–(T6) of $\mathcal{M}(\mathfrak{g},\mu)$. The relations (T0)–(T4) follow from a direct verification by using (3.1), and the relations (T5)–(T6) are immediate from Proposition 5.1. □

Denote by $\phi_\mu:\mathcal{M}(\mathfrak{g},\mu)\to\widehat{\mathfrak{g}}$ the Lie homomorphism given in Lemma 5.5, and

$$\bar{\phi}_\mu=\psi_\mu\circ\phi_\mu:\mathcal{M}(\mathfrak{g},\mu)\to\mathcal{L}(\bar{\mathfrak{g}},\bar{\mu})$$

the composition of the map ϕ_μ and the universal central extension $\psi_\mu:\widehat{\mathfrak{g}}[\mu]\to\mathcal{L}(\bar{\mathfrak{g}},\bar{\mu})$. By the universal property of ψ_μ, we see that Theorem 3.6 follows from the following result.

Proposition 5.6. *The Lie homomorphism* $\bar{\phi}_\mu:\mathcal{M}(\mathfrak{g},\mu)\to\mathcal{L}(\bar{\mathfrak{g}},\bar{\mu})$ *is a central extension.*

The rest part of this subsection is devoted to a proof of Proposition 5.6. Notice that there is a (unique) $\widehat{Q}_\mu$-grading $\mathcal{M}(\mathfrak{g},\mu)=\bigoplus_{(\alpha,n)\in\widehat{Q}_\mu}\mathcal{M}(\mathfrak{g},\mu)_{\alpha,n}$ on $\mathcal{M}(\mathfrak{g},\mu)$ such that

$$\deg c=(0,0),\quad \deg h_{i,m}=(0,m)\quad\text{and}\quad \deg x_{i,m}^\pm=(\pm\check{\alpha}_i,m),\quad i\in I,\quad m\in\mathbb{Z}.$$

We also introduce a $\widehat{Q}_\mu$-grading structure $\mathcal{L}(\bar{\mathfrak{g}},\bar{\mu})=\bigoplus_{(\alpha,n)\in\widehat{Q}_\mu}\mathcal{L}(\bar{\mathfrak{g}},\bar{\mu})_{\alpha,n}$ so that the quotient map $\psi_\mu:\widehat{\mathfrak{g}}[\mu]\to\mathcal{L}(\bar{\mathfrak{g}},\bar{\mu})$ is graded. It is obvious that the homomorphism ϕ_μ is $\widehat{Q}_\mu$-graded (see (5.1)) and so is the homomorphism $\bar{\phi}_\mu$.

Let $\mathcal{M}(\mathfrak{g},\mu)^\pm$ be the subalgebra of $\mathcal{M}(\mathfrak{g},\mu)$ generated by $\{x_{i,m}^\pm\mid i\in I,m\in\mathbb{Z}\}$, and $\mathcal{M}(\mathfrak{g},\mu)^0$ the subalgebra of $\mathcal{M}(\mathfrak{g},\mu)$ generated by $\{h_{i,m}\mid i\in I,m\in\mathbb{Z}\}$. Then we have the following triangular decomposition of $\mathcal{M}(\mathfrak{g},\mu)$, whose proof is straightforward and omitted.

Lemma 5.7. *One has* $\mathcal{M}(\mathfrak{g},\mu)=\mathcal{M}(\mathfrak{g},\mu)^+\oplus\mathcal{M}(\mathfrak{g},\mu)^0\oplus\mathcal{M}(\mathfrak{g},\mu)^-$.

Recall from Lemma 5.2 that

$$\pi_\mu(\Delta)^\times=\{\check{w}(k_i\check{\alpha}_i)\mid \check{w}\in\check{W}, i\in\check{I}, 1\leqslant k_i\leqslant s_i\}.$$

Lemma 5.8. *Let* $(\alpha,p)\in\widehat{Q}_\mu^\times$. *Then the following results hold true*:

(1) *if* $\mathcal{M}(\mathfrak{g},\mu)_{\alpha,p}\neq 0$, *then* $\alpha\in\pi_\mu(\Delta)^\times$;

(2) *if* $\alpha=\check{w}(\check{\alpha}_i)$ *for some* $i\in\check{I}$ *and* $\check{w}\in\check{W}$, *then the dimension of the graded subspace* $\mathcal{M}(\mathfrak{g},\mu)_{\alpha,p}$ *is* 1 *if* $p\in d_i\mathbb{Z}$, *and is* 0 *otherwise*;

(3) *if* $\alpha=\check{w}(2\check{\alpha}_i)$ *for some* $i\in\check{I}$ *with* $s_i=2$ *and* $\check{w}\in\check{W}$, *then the dimension of the graded subspace* $\mathcal{M}(\mathfrak{g},\mu)_{\alpha,p}$ *is* 1 *if* $p\in d_i+N\mathbb{Z}$, *and is* 0 *otherwise.*

Proof. Denote by $\mathcal{M}_0(\mathfrak{g},\mu)$ the subalgebra of $\mathcal{M}(\mathfrak{g},\mu)$ generated by the elements $h_{i,0}$, $x_{i,0}^{\pm}$ and $i\in\check{I}$. Then one concludes from the relations (T2)–(T5) that $\mathcal{M}_0(\mathfrak{g},\mu)$ is the derived subalgebra of the Kac-Moody algebra associated with $\check{A}$. Viewing $\mathcal{M}(\mathfrak{g},\mu)$ as an $\mathcal{M}_0(\mathfrak{g},\mu)$-module by the adjoint action, we see from (T3)–(T6) that the $\mathcal{M}_0(\mathfrak{g},\mu)$-module $\mathcal{M}(\mathfrak{g},\mu)$ is integrable. Moreover, for each $p\in\mathbb{Z}$, the subspace

$$\mathcal{M}(\mathfrak{g},\mu)_p=\bigoplus_{(\alpha,p)\in\widehat{Q}_\mu}\mathcal{M}(\mathfrak{g},\mu)_{\alpha,p}$$

of $\mathcal{M}(\mathfrak{g},\mu)$ is an $\mathcal{M}_0(\mathfrak{g},\mu)$-submodule. A standard $\mathfrak{sl}_2$-theory argument gives that

$$\dim\mathcal{M}(\mathfrak{g},\mu)_{\alpha,p}=\dim\mathcal{M}(\mathfrak{g},\mu)_{\check{w}(\alpha),p},\quad \check{w}\in\check{W}.$$

Assume now that $\mathcal{M}(\mathfrak{g},\mu)_{\alpha,p}\neq 0$ for some $(\alpha,p)\in\widehat{Q}_\mu$. We now prove that $\alpha\in\pi_\mu(\Delta)^\times$ by using induction on $\operatorname{ht}\alpha$. Here and as before, $\operatorname{ht}\alpha=\sum_{i\in\check{I}}n_i$ if $\alpha=\sum_{i\in\check{I}}n_i\check{\alpha}_i$. By Lemma 5.7, the integers $n_i, i\in\check{I}$ are either all non-negative or all non-positive. We assume that $\operatorname{ht}\alpha>0$, so that all n_i are non-negative. Then there exist some $i\in\check{I}$ such that $(\check{\alpha}_i,\alpha)>0$ and $n_i>0$. If $\operatorname{ht}r_{\check{\alpha}_i}(\alpha)>0$, then we are done by the induction hypothesis. Otherwise $\operatorname{ht}r_{\check{\alpha}_i}(\alpha)<0$ and so $\alpha=k\check{\alpha}_i$ for some positive integer k. But the relation (T6) forces that $1\leqslant k\leqslant s_i$. This proves the assertion (1).

The assertion (2) is implied by (T0) as $\mathcal{M}(\mathfrak{g},\mu)_{\check{\alpha}_i,p}=\mathbb{C}x_{i,p}^+$. As for the assertion (3), we have $N_i=2$ and $\alpha_{i\mu(i)}=-1$ in this case. Then by the assertion (2) and Lemma 5.7, we get that

$$\mathcal{M}(\mathfrak{g},\mu)_{2\check{\alpha}_i,p}=\sum_{\substack{m+n=p\\ m,n\in(N/2)\mathbb{Z}}}[\mathcal{M}(\mathfrak{g},\mu)_{\check{\alpha}_i,m},\mathcal{M}(\mathfrak{g},\mu)_{\check{\alpha}_i,n}]. \tag{5.10}$$

So the proof of the assertion (3) can be reduced to the proof of the following facts: $\mathcal{M}(\mathfrak{g},\mu)_{2\check{\alpha}_i,p}=0$ if $p\in N\mathbb{Z}$, and $\dim\mathcal{M}(\mathfrak{g},\mu)_{2\check{\alpha}_i,p}=1$ if $p\in N/2+N\mathbb{Z}$. We first show that $\mathcal{M}(\mathfrak{g},\mu)_{2\check{\alpha}_i,p}=0$ if $p\in N\mathbb{Z}$. By (5.10), this is implied by

$$[x_{i,mN/2}^+,x_{i,nN/2}^+]=0\quad\text{if}\quad m\equiv n \pmod 2. \tag{5.11}$$

Using (T4), we have

$$[x_{i,mN/2}^+,x_{i,nN/2}^-]=\frac{N}{2}h_{i,(m+n)N/2}+ac$$

for some $a\in\mathbb{C}$. In addition, by (T3), we have

$$[h_{i,mN/2},x_{i,nN/2}^+]=\frac{(2-(-1)^m)N}{2}x_{i,(m+n)N/2}^+.$$

Thus, if $m\equiv n \pmod 2$, then

$$[[x_{i,0}^+,x_{i,0}^-],[x_{i,mN/2}^+,x_{i,nN/2}^+]]=\frac{N^2}{2}[x_{i,mN/2}^+,x_{i,nN/2}^+],$$
$$[[x_{i,mN/2}^+,x_{i,nN/2}^+],x_{i,0}^-]=0.$$

Combining these with (T6), we get

$$\begin{aligned}\frac{N^2}{2}[x^+_{i,mN/2}, x^+_{i,nN/2}] &= [[x^+_{i,0}, x^-_{i,0}], [x^+_{i,mN/2}, x^+_{i,nN/2}]] \\ &= [[x^+_{i,0}, [x^+_{i,mN/2}, x^+_{i,nN/2}]], x^-_{i,0}] + [x^+_{i,0}, [[x^+_{i,mN/2}, x^+_{i,nN/2}], x^-_{i,0}]] \\ &= [[x^+_{i,0}, [x^+_{i,mN/2}, x^+_{i,nN/2}]], x^-_{i,0}] = 0.\end{aligned}$$

This completes the verification of (5.11).

We now prove that $\dim \mathcal{M}(\mathfrak{g},\mu)_{2\check{\alpha}_i,p} = 1$ if $p \in N/2 + N\mathbb{Z}$. It follows from (T3) and (T4) that

$$[x^-_{i,0}, [x^-_{i,0}, \mathcal{M}(\mathfrak{g},\mu)_{2\check{\alpha}_i,p}]] \subset \mathbb{C}h_{i,p}. \tag{5.12}$$

It is immediate from the (T2)–(T4) that $\mathbb{C}x^+_{i,0} + \mathbb{C}x^-_{i,0} + \mathbb{C}h_{i,0} \cong \mathfrak{sl}_2$. Then by (5.10), (5.12) and the assertion (1), we find that the space spanned by $\mathcal{M}(\mathfrak{g},\mu)_{k\check{\alpha}_i,p}$, $h_{i,p}$, $k=\pm1$ and $k=\pm2$ is an irreducible $\mathfrak{sl}_2$-module. This gives that $\dim \mathcal{M}(\mathfrak{g},\mu)_{2\check{\alpha}_i,p} \leqslant 1$. But one can conclude from Proposition 5.1 that

$$\dim \mathcal{M}(\mathfrak{g},\mu)_{2\check{\alpha}_i,p} \geqslant \dim \widehat{\mathfrak{g}}[\mu]_{2\check{\alpha}_i,p} = 1,$$

as ϕ_μ is a graded surjective homomorphism. Thus we complete the proof of the assertion (3). □

Now we are in a position to complete the proof of Proposition 5.6. It follows from Proposition 5.1 and Lemma 5.8 that

$$\ker \bar{\phi}_\mu \subset \mathcal{M}(\mathfrak{g},\mu)^{\mathrm{iso}} = \bigoplus_{(\alpha,p)\in\widehat{Q}^0_\mu} \mathcal{M}(\mathfrak{g},\mu)_{\alpha,p}, \tag{5.13}$$

where

$$\widehat{Q}^0_\mu = \{(\alpha,p) \in \widehat{Q}_\mu \mid (\alpha,\alpha) = 0\}.$$

Note that $\widehat{Q}^0_\mu + \widehat{Q}^\times_\mu \subset \widehat{Q}^\times_\mu$, which in particular shows that

$$[x^\pm_{i,m}, \mathcal{M}(\mathfrak{g},\mu)^{\mathrm{iso}}] \cap \mathcal{M}(\mathfrak{g},\mu)^{\mathrm{iso}} = \{0\}, \quad \text{for } \ i \in I, \quad m \in \mathbb{Z}. \tag{5.14}$$

Finally, Proposition 5.6 follows from (5.13) and (5.14), as the Lie algebra $\mathcal{M}(\mathfrak{g},\mu)$ is generated by the elements $x^\pm_{i,m}, i \in I, m \in \mathbb{Z}$.

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