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Lindeberg's central limit theorems for martingale-like sequences under sub-linear expectations

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Abstract The central limit theorem of martingales is the fundamental tool for studying the convergence of stochastic processes, especially stochastic integrals and differential equations. In this paper, the central limit theorem and the functional central limit theorem are obtained for martingale-like random variables under the sub-linear expectation. As applications, the Lindeberg's central limit theorem is obtained for independent but not necessarily identically distributed random variables, and a new proof of the Lévy characterization of a G-Brownian motion without using stochastic calculus is given. For proving the results, Rosenthal's inequality and the exponential inequality for the martingale-like random variables are established.

Keywords capacity, central limit theorem, functional central limit theorem, martingale difference, sub-linear expectation

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1 Introduction and notations

Non-additive probabilities and non-additive expectations are useful tools for studying uncertainties in the statistics, measures of risk, superhedging in finance and non-linear stochastic calculus (see Denis and Martini [4], Gilboa [5], Marinacci [13] and Peng [14–16, 18, 19], etc.). Peng [16] introduced the notion of the sub-linear expectation. Under the sub-linear expectation, Peng [16–21] gave the notions of the G-normal distributions, G-Brownian motions, G-martingales, independence of random variables, identical distribution of random variables and so on, and developed the weak law of large numbers and central limit theorem for independent and identically distributed (i.i.d.) random variables. Furthermore, Peng established the stochastic calculus with respect to the G-Brownian motion. As a result, Peng's framework of nonlinear expectation gives a generalization of Kolmogorov's probability theory. Recently, Bayraktar and Munk [1] proved an α-stable central limit theorem for independent and identically distributed random variables. This paper considers the general central limit theorem for random variables which are not necessarily i.i.d. under the sub-linear expectation. We establish a central limit theorem and a functional central limit theorem under the conditional Lindeberg's condition for a kind of martingale-difference-like random variables. As applications, the central limit theorem for independent but not necessary identically distributed random variables under the popular Lindeberg's condition is obtained. The tool for proving the central limit theorem is a promotion of Peng's [20] and gives also a new normal approximation method for classical martingale differences instead of the characteristic function. For proving the functional central limit theorem, we also establish Rosenthal's inequalities for the martingale-like random variables. As the central limit theorem of classical martingales which is the fundamental tool for studying the convergence of stochastic processes under the framework of the probability and linear expectation, especially stochastic integrals and differential equations (see Jacod and Shiryaev [9]), the (functional) central limit theorem of martingale-difference-like random variables under the sub-linear expectation will provide a way to study the weak convergence of stochastic integrals and difference equations with respect to the G-Brownian motion.

In the rest of this section, we state some notations about sub-linear expectations. The main results on the central limit theorem and functional central limit theorem are stated in Sections 2 and 3 with the proofs given in the last section. In Section 4, we establish the Rosenthal-type inequalities and an exponential inequality for the maximal sums of the martingale-difference-like random variables. In Section 5, we consider the Lévy characterization of a G-Brownian motion in a general sub-linear expectation space. The Lévy characterization of a G-Brownian motion under G-expectation in a Wiener space is established by Xu and Zhang [25, 26] and extended by Lin [11] by the method of the stochastic calculus. We give an elementary proof without using stochastic calculus. We find that the functional central limit theorem gives a new way to show the Lévy characterization.

We use the framework and notations of Peng [20]. Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathscr{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \ldots, X_n \in \mathscr{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathscr{H}$ for each $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}_n)$, where $C_{l,\mathrm{Lip}}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \leqslant C(1 + |\boldsymbol{x}|^m + |\boldsymbol{y}|^m)|\boldsymbol{x} - \boldsymbol{y}|, \quad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_n,$$
$$\text{for some } C > 0, \quad m \in \mathbb{N} \text{ depending on } \varphi.$$

$\mathscr{H}$ is considered as a space of "random variables". In this case, we denote $X \in \mathscr{H}$. We also denote the space of bounded Lipschitz functions and the space of bounded continuous functions on $\mathbb{R}_n$ by $C_{b,\mathrm{Lip}}(\mathbb{R}_n)$ and $C_b(\mathbb{R}_n)$, respectively.

Definition 1.1. A sub-linear expectation $\widehat{\mathrm{E}}$ on $\mathscr{H}$ is a function $\widehat{\mathrm{E}} : \mathscr{H} \to \overline{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathscr{H}$,

(1) monotonicity: if $X \geqslant Y$ then $\widehat{\mathrm{E}}[X] \geqslant \widehat{\mathrm{E}}[Y]$;

(2) constant preserving: $\widehat{\mathrm{E}}[c] = c$;

(3) sub-additivity: $\widehat{\mathrm{E}}[X + Y] \leqslant \widehat{\mathrm{E}}[X] + \widehat{\mathrm{E}}[Y]$ whenever $\widehat{\mathrm{E}}[X] + \widehat{\mathrm{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(4) positive homogeneity: $\widehat{\mathrm{E}}[\lambda X] = \lambda \widehat{\mathrm{E}}[X]$, $\lambda \geqslant 0$.

Here, $\overline{\mathbb{R}} = [-\infty, \infty]$. The triple $(\Omega, \mathscr{H}, \widehat{\mathrm{E}})$ is called a sub-linear expectation space. Given a sub-linear expectation $\widehat{\mathrm{E}}$, let us denote the conjugate expectation $\widehat{\mathcal{E}}$ of $\widehat{\mathrm{E}}$ by $\widehat{\mathcal{E}}[X] := -\widehat{\mathrm{E}}[-X]$, $\forall\, X \in \mathscr{H}$.

A sub-linear expectation $\widehat{\mathrm{E}}$ is countably sub-additive, if

$$\widehat{\mathrm{E}}\left[\sum_{i=1}^{\infty} X_i\right] \leqslant \sum_{i=1}^{\infty} \widehat{\mathrm{E}}[X_i] \quad \text{for all random variables } X_i \geqslant 0.$$

If X is not in $\mathscr{H}$, we define its sub-linear expectation by $\widehat{\mathrm{E}}^*[X] = \inf\{\widehat{\mathrm{E}}[Y] : X \leqslant Y \in \mathscr{H}\}$. When there is no ambiguity, we also denote it by $\widehat{\mathrm{E}}$. From the definition, it is easily shown that $\widehat{\mathcal{E}}[X] \leqslant \widehat{\mathrm{E}}[X]$, $\widehat{\mathrm{E}}[X + c] = \widehat{\mathrm{E}}[X] + c$ and $\widehat{\mathrm{E}}[X - Y] \geqslant \widehat{\mathrm{E}}[X] - \widehat{\mathrm{E}}[Y]$ for all $X, Y \in \mathscr{H}$ with $\widehat{\mathrm{E}}[Y]$ being finite. Furthermore, if $\widehat{\mathrm{E}}[|X|]$ is finite, then $\widehat{\mathcal{E}}[X]$ and $\widehat{\mathrm{E}}[X]$ are both finite.

Definition 1.2 (See [19, 20])**.** (i) (*Identical distribution*) Let $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ be two n-dimensional random vectors defined, respectively, in the sub-linear expectation spaces $(\Omega_1, \mathscr{H}_1, \widehat{\mathrm{E}}_1)$ and $(\Omega_2, \mathscr{H}_2, \widehat{\mathrm{E}}_2)$. They are called identically distributed, denoted by $\boldsymbol{X}_1 \overset{d}{=} \boldsymbol{X}_2$, if

$$\widehat{\mathrm{E}}_1[\varphi(\boldsymbol{X}_1)] = \widehat{\mathrm{E}}_2[\varphi(\boldsymbol{X}_2)], \quad \forall\, \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}_n),$$

whenever the sub-expectations are finite. A sequence $\{X_n; n \geqslant 1\}$ of random variables is said to be identically distributed if $X_i \overset{d}{=} X_1$ for each $i \geqslant 1$.

(ii) (*Independence*) In a sub-linear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$, a random vector $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$, $Y_i \in \mathscr{H}$ is said to be independent to another random vector $\boldsymbol{X} = (X_1, \ldots, X_m)$, $X_i \in \mathscr{H}$ under $\widehat{\mathbb{E}}$, if for each test function $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}_m \times \mathbb{R}_n)$ we have $\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}, \boldsymbol{Y})] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(\boldsymbol{x}, \boldsymbol{Y})]\,|_{\boldsymbol{x}=\boldsymbol{X}}]$, whenever $\overline{\varphi}(\boldsymbol{x}) := \widehat{\mathbb{E}}[|\varphi(\boldsymbol{x}, \boldsymbol{Y})|] < \infty$ for all $\boldsymbol{x}$ and $\widehat{\mathbb{E}}[|\overline{\varphi}(\boldsymbol{X})|] < \infty$.

Random variables $X_1, \ldots, X_n$ are said to be independent if for each $2 \leqslant k \leqslant n$, X_k is independent of $(X_1, \ldots, X_{k-1})$. A sequence of random variables is said to be independent if for each n, $X_1, \ldots, X_n$ are independent.

Next, we introduce the capacities corresponding to the sub-linear expectation. We denote the pair $(\mathbb{V}, \mathcal{V})$ of capacities on $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ by setting

$$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I_A \leqslant \xi, \, \xi \in \mathscr{H}\}, \quad \mathcal{V}(A) := 1 - \mathbb{V}(A^c), \quad \forall\, A \in \mathcal{F},$$

where A^c is the complement set of A. Then it is obvious that $\mathbb{V}$ is sub-additive, i.e., $\mathbb{V}(A \cup B) \leqslant \mathbb{V}(A) + \mathbb{V}(B)$. But $\mathcal{V}$ and $\widehat{\mathcal{E}}$ are not. However, we have

$$\mathcal{V}(A \cup B) \leqslant \mathcal{V}(A) + \mathbb{V}(B) \quad \text{and} \quad \widehat{\mathcal{E}}[X + Y] \leqslant \widehat{\mathcal{E}}[X] + \widehat{\mathbb{E}}[Y]$$

due to the fact that $\mathbb{V}(A^c \cap B^c) = \mathbb{V}(A^c \backslash B) \geqslant \mathbb{V}(A^c) - \mathbb{V}(B)$ and $\widehat{\mathbb{E}}[-X - Y] \geqslant \widehat{\mathbb{E}}[-X] - \widehat{\mathbb{E}}[Y]$.

The Choquet integrals/expectations of $(C_{\mathbb{V}}, C_{\mathcal{V}})$ are defined by

$$C_V[X] = \int_0^\infty V(X \geqslant t)dt + \int_{-\infty}^0 [V(X \geqslant t) - 1]dt$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$, respectively.

Finally, we recall the notations of G-normal distribution and G-Brownian motion which are introduced by Peng [20, 22].

Definition 1.3 (G-normal random variable)**.** For $0 \leqslant \underline{\sigma}^2 \leqslant \overline{\sigma}^2 < \infty$, a random variable ξ in a sub-linear expectation space $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$ is called a normal $N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$ distributed random variable (written as $\xi \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$ under $\widetilde{\mathbb{E}}$), if for any $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R})$, the function $u(x, t) = \widetilde{\mathbb{E}}[\varphi(x + \sqrt{t}\xi)]$ $(x \in \mathbb{R}, t \geqslant 0)$ is the unique viscosity solution of the following heat equation:

$$\partial_t u - G(\partial_{xx}^2 u) = 0, \quad u(0, x) = \varphi(x),$$

where $G(\alpha) = \frac{1}{2}(\overline{\sigma}^2 \alpha^+ - \underline{\sigma}^2 \alpha^-)$.

That ξ is a normal distributed random variable is equivalent to that, if ξ' is an independent copy of ξ, then

$$\widetilde{\mathbb{E}}[\varphi(\alpha\xi + \beta\xi')] = \widetilde{\mathbb{E}}[\varphi(\sqrt{\alpha^2 + \beta^2} X)], \quad \forall\, \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}) \quad \text{and} \quad \forall\, \alpha, \beta \geqslant 0$$

(see Peng [22, Definition II.1.4 and Example II.1.13]).

Definition 1.4 (G-Brownian motion)**.** A random process $(W_t)_{t \geqslant 0}$ in the sub-linear expectation space $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$ is called a G-Brownian motion (see Peng [22, Definition III.1.2]) if

(i) $W_0 = 0$;

(ii) for each $0 \leqslant t_1 \leqslant \cdots \leqslant t_d \leqslant t \leqslant s$,

$$\begin{aligned} &\widetilde{\mathbb{E}}[\varphi(W_{t_1}, \ldots, W_{t_d}, W_s - W_t)] \\ &\quad = \widetilde{\mathbb{E}}[\widetilde{\mathbb{E}}[\varphi(x_1, \ldots, x_d, \sqrt{t - s})\xi]\,|_{x_1 = W_{t_1}, \ldots, x_d = W_{t_d}}], \quad \forall\, \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}_{d+1}), \end{aligned} \tag{1.1}$$

where $\xi \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$.

In some papers, for example, [25, 26] by Xu and Zhang, the test functions φ are only required to be elements in $C_{b,\mathrm{Lip}}(\mathbb{R}_{d+1})$. It can be shown that if $\widetilde{\mathbb{E}}[|W_t|^p] < \infty$ for all $p > 0$ and t, then that (1.1) holds for all $\varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}_{d+1})$ is equivalent to that it holds for all $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}_{d+1})$. Furthermore, if the sub-linear expectation $\widetilde{\mathrm{E}}$ is countably sub-additive, then these two kinds of definitions are equivalent because, if X is a random variable in $(\Omega, \mathscr{H}, \widehat{\mathrm{E}})$ such that

$$\widehat{\mathrm{E}}[\varphi(X)] = \widetilde{\mathrm{E}}[\varphi(\xi)], \quad \forall\, \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}), \tag{1.2}$$

then $\widehat{\mathrm{E}}[|X|^p] < \infty$ for all $p > 0$. In fact, if $\xi \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$ under $\widetilde{\mathrm{E}}$, then (see Peng [22, p. 22])

$$\widetilde{\mathrm{E}}[|\xi|^p] = \overline{\sigma}^p \int_{-\infty}^{\infty} |x|^p \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} dx = c_p \overline{\sigma}^p, \quad \forall\, p \geqslant 1.$$

Now, for any $z > 0$, one can choose a function $\varphi \in C_{b,\mathrm{Lip}}(\mathbb{R})$ such that $I\{x > z\} \leqslant \varphi(x) \leqslant I\{x > z - \epsilon\}$. From (1.2), it follows that

$$\mathbb{V}(|X| > z) \leqslant \widehat{\mathrm{E}}[\varphi(X)] = \widetilde{\mathrm{E}}[\varphi(\xi)] \leqslant \widetilde{\mathbb{V}}(|\xi| > z - \epsilon).$$

Hence

$$\mathbb{V}(|X| > z) \leqslant \widetilde{\mathbb{V}}\bigg(|\xi| \geqslant \frac{z}{2}\bigg) \leqslant \frac{2^{2p}\widetilde{\mathrm{E}}[|\xi|^{2p}]}{z^{2p}} = \frac{\overline{\sigma}^{2p} c_{2p}}{z^{2p}}.$$

It follows that

$$C_{\mathbb{V}}(|X|^p) = \int_0^{\infty} \mathbb{V}(|X|^p > z) dz \leqslant 1 + \int_1^{\infty} \frac{\overline{\sigma}^{2p} c_{2p}}{z^2} dz \leqslant 1 + \overline{\sigma}^{2p} c_{2p} < \infty, \quad \forall\, p \geqslant 2.$$

So, if $\widehat{\mathrm{E}}$ is countably sub-additive or $\widehat{\mathrm{E}}[|X|^p] = \lim_{c\to\infty} \widehat{\mathrm{E}}[(|X| \wedge c)^p]$, then $\widehat{\mathrm{E}}[|X|^p] \leqslant C_{\mathbb{V}}(|X|^p) < \infty$ for all $p > 0$ by Zhang [28, Lemma 3.9].

Let $C_{[0,1]}$ be a function space of continuous real functions on $[0,1]$ equipped with the supremum norm $\|x\| = \sup_{0\leqslant t\leqslant 1} |x(t)|$ and $C_b(C_{[0,1]})$ is the set of bounded continuous functions $h(x) : C_{[0,1]} \to \mathbb{R}$. As showed in [3,19,22], there is a sub-linear expectation space $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathrm{E}})$ with $\widetilde{\Omega} = C_{[0,1]}$ and $C_b(C_{[0,1]}) \subset \widetilde{\mathscr{H}}$ such that $(\widetilde{\mathscr{H}}, \widetilde{\mathrm{E}}[\|\cdot\|])$ is a Banach space, and the canonical process $W(t)(\omega) = \omega_t (\omega \in \widetilde{\Omega})$ is a G-Brownian motion. In the sequel of this paper, the G-normal random variables and G-Brownian motions are considered in $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathrm{E}})$.

2 Lindeberg's central limit theorem for independent random variables

We write $\eta_n \xrightarrow{\mathbb{V}} \eta$ if $\mathbb{V}(|\eta_n - \eta| \geqslant \epsilon) \to 0$ for any $\epsilon > 0$, and write $\eta_n \xrightarrow{d} \eta$ if $\widehat{\mathrm{E}}[\varphi(\eta_n)] \to \widehat{\mathrm{E}}[\varphi(\eta)]$ holds for all bounded and continuous functions φ. In this section, we consider the independent random variables $\{X_{n,k}; k = 1, \ldots, k_n\}$. Denote $\overline{\sigma}_{n,k}^2 = \widehat{\mathrm{E}}[X_{n,k}^2]$, $\underline{\sigma}_{n,k}^2 = \widehat{\mathcal{E}}[X_{n,k}^2]$ and $B_n^2 = \sum_{k=1}^{k_n} \overline{\sigma}_{n,k}^2$. We have the following Lindeberg's central limit theorem.

Theorem 2.1. *Suppose that the following Lindeberg's condition is satisfied*:

$$\frac{1}{B_n^2} \sum_{k=1}^{k_n} \widehat{\mathrm{E}}[(X_{n,k}^2 - \epsilon B_n^2)^+] \to 0, \quad \forall\, \epsilon > 0. \tag{2.1}$$

Furthermore, there is a constant $r \in [0,1]$ such that

$$\frac{\sum_{k=1}^{k_n} |r\overline{\sigma}_{n,k}^2 - \underline{\sigma}_{n,k}^2|}{B_n^2} \to 0, \tag{2.2}$$

and also,

$$\frac{\sum_{k=1}^{k_n} \{|\widehat{\mathrm{E}}[X_{n,k}]| + |\widehat{\mathcal{E}}[X_{n,k}]|\}}{B_n} \to 0. \tag{2.3}$$

Then for any bounded continuous function φ,

$$\lim_{n\to\infty} \widehat{\mathrm{E}}\bigg[\varphi\bigg(\frac{\sum_{k=1}^{k_n} X_{n,k}}{B_n}\bigg)\bigg] = \widetilde{\mathrm{E}}[\varphi(\xi)], \tag{2.4}$$

where $\xi \sim N(0, [r, 1])$ under $\widetilde{\mathrm{E}}$.

Theorem 2.1 will be a direct corollary of Theorem 3.1 on the central limit theorem for the martingale-like sequence. The central limit theorem for independent and identically distributed random variables under the sub-linear expectation was obtained by Peng [20]. Li and Shi [10] generalized Peng's result to a central limit theorem for independent random variables $\{X_n; n \geqslant 1\}$ satisfying $\widehat{\mathbb{E}}[X_i] = \widehat{\mathcal{E}}[X_i] = 0$, $\widehat{\mathbb{E}}[|X_i|^3] \leqslant M < \infty$, $i = 1, 2, \ldots$, and

$$\frac{1}{n}\sum_{i=1}^{n}|\widehat{\mathbb{E}}[X_i^2] - \overline{\sigma}^2| \to 0, \quad \frac{1}{n}\sum_{i=1}^{n}|\widehat{\mathcal{E}}[X_i^2] - \underline{\sigma}^2| \to 0.$$

It is easily seen that the array $\{\frac{1}{\sqrt{n}}X_k; k = 1, \ldots, n\}$ satisfies the condition (2.2) with $r = \underline{\sigma}^2/\overline{\sigma}^2$, (2.3) and (2.1).

When $\widehat{\mathbb{E}}$ is a classical linear expectation, (2.2) is automatically satisfied with $r = 1$. It is easily seen that (2.2) implies

$$\frac{\sum_{k=1}^{k_n}\underline{\sigma}_{n,k}^2}{\sum_{k=1}^{k_n}\overline{\sigma}_{n,k}^2} \to r. \tag{2.5}$$

One may conjecture that (2.2) can be weakened to (2.5). The following example tells us that it is not the truth.

Example 2.2. Let $0 < \tau_1, \tau_2 < 1$, and $\{X_{n,k}; k = 1, \ldots, 2n\}$ be a sequence of independent normal random variables such that

$$X_{n,k} \overset{d}{\sim} N(0, [\tau_1, 1]), \quad k = 1, \ldots, n \quad \text{and} \quad X_{n,k} \overset{d}{\sim} N(0, [\tau_2, 1]), \quad k = n+1, \ldots, 2n.$$

It is easily seen that $\{X_{n,k}; k = 1, \ldots, 2n\}$ satisfies the conditions (2.1), (2.3) and (2.5) with $r = (\tau_1+\tau_2)/2$, and $B_n^2 = 2n$. It is obvious that

$$\frac{\sum_{k=1}^{2n}X_{n,k}}{\sqrt{n}} = \frac{\sum_{k=1}^{n}X_{n,k}}{\sqrt{n}} + \frac{\sum_{k=n+1}^{2n}X_{n,k}}{\sqrt{n}} \overset{d}{\sim} \xi + \eta,$$

where ξ and η are independent normal random variables with $\xi \overset{d}{\sim} N(0, [\tau_1, 1])$, $\eta \overset{d}{\sim} N(0, [\tau_2, 1])$. Song [24] showed that $\xi + \eta$ is not G-normal distributed if $\tau_1 \neq \tau_2$, and hence (2.4) fails.

3 The central limit theorem for the martingale-like sequence

In this section, we consider a general martingale. First, we recall the definition of the conditional expectation under the sub-linear expectation. Let $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ be a sub-linear expectation space. We write $X \leqslant Y$ in L_p if $\widehat{\mathbb{E}}[((X-Y)^+)^p] = 0$, $X = Y$ in L_p if both $X \leqslant Y$ and $Y \leqslant X$ hold in L_p.

Let $\mathscr{H}_{n,0} \subset \cdots \subset \mathscr{H}_{n,k_n}$ be subspaces of $\mathscr{H}$ such that

(1) any constant $c \in \mathscr{H}_{n,k}$, and

(2) if $X_1, \ldots, X_d \in \mathscr{H}_{n,k}$, then $\varphi(X_1, \ldots, X_d) \in \mathscr{H}_{n,k}$ for any $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}_d)$, $k = 0, \ldots, k_n$.

Denote $\mathscr{L}(\mathscr{H}) = \{X : \widehat{\mathbb{E}}[|X|] < \infty, X \in \mathscr{H}\}$. We consider a system of operators in $\mathscr{L}(\mathscr{H})$,

$$\widehat{\mathbb{E}}_{n,k} : \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H}_{n,k})$$

and denote $\widehat{\mathbb{E}}[X \mid \mathscr{H}_{n,k}] = \widehat{\mathbb{E}}_{n,k}[X]$, $\widehat{\mathcal{E}}[X \mid \mathscr{H}_{n,k}] = -\widehat{\mathbb{E}}_{n,k}[-X]$. $\widehat{\mathbb{E}}[X \mid \mathscr{H}_{n,k}]$ is called the conditional sub-linear expectation of X given $\mathscr{H}_{n,k}$, and $\widehat{\mathbb{E}}_{n,k}$ is called the conditional expectation operator. Suppose that the operators $\widehat{\mathbb{E}}_{n,k}$ satisfy the following properties: for all $X, Y \in \mathscr{L}(\mathscr{H})$,

(a) $\widehat{\mathbb{E}}_{n,k}[X+Y] = X + \widehat{\mathbb{E}}_{n,k}[Y]$ in L_1 if $X \in \mathscr{H}_{n,k}$, and $\widehat{\mathbb{E}}_{n,k}[XY] = X^+\widehat{\mathbb{E}}_{n,k}[Y] + X^-\widehat{\mathbb{E}}_{n,k}[-Y]$ in L_1 if $X \in \mathscr{H}_{n,k}$ and $XY \in \mathscr{L}(\mathscr{H})$;

(b) $\widehat{\mathbb{E}}[\widehat{\mathbb{E}}_{n,k}[X]] = \widehat{\mathbb{E}}[X]$.

It is easily seen that (a) implies that $\widehat{\mathbb{E}}_{n,k}[c] = c$ in L_1 and $\widehat{\mathbb{E}}_{n,k}[\lambda X] = \lambda\widehat{\mathbb{E}}_{n,k}[X]$ in L_1 if $\lambda \geqslant 0$. The definition of the conditional sub-linear expectation can be found in Peng [22], and Xu and Zhang [25,26] with the operators satisfying (a), (b) and $\widehat{\mathbb{E}}_{n,k}[X] \leqslant \widehat{\mathbb{E}}_{n,k}[Y]$ if $X \leqslant Y$, $\widehat{\mathbb{E}}_{n,k}[X] - \widehat{\mathbb{E}}_{n,k}[Y] \leqslant \widehat{\mathbb{E}}_{n,k}[X-Y]$, $\widehat{\mathbb{E}}_{n,k}[[\widehat{\mathbb{E}}_{n,l}[X]]] = \widehat{\mathbb{E}}_{n,l\wedge k}[X]$. It can be showed that these properties can be implied by (a) and (b) (see Lemma 4.3).

Now, we assume that $\{Z_{n,k}; k = 1,\dots,k_n\}$ is an array of random variables such that $Z_{n,k} \in \mathscr{H}_{n,k}$ and $\widehat{\mathbb{E}}[Z_{n,k}^2] < \infty$, $k = 1,\dots,k_n$. The following is the central limit theorem.

Theorem 3.1. *Suppose that the operators $\widehat{\mathbb{E}}_{n,k}$ satisfy* (a) *and* (b)*. Assume that the following Lindeberg's condition is satisfied*:

$$\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[(Z_{n,k}^2 - \epsilon)^+ \mid \mathscr{H}_{n,k-1}] \xrightarrow{\mathbb{V}} 0, \quad \forall\, \epsilon > 0. \tag{3.1}$$

Furthermore, there are constants $\rho \geqslant 0$ and $r \in [0,1]$ such that

$$\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Z_{n,k}^2 \mid \mathscr{H}_{n,k-1}] \xrightarrow{\mathbb{V}} \rho, \tag{3.2}$$

$$\sum_{k=1}^{k_n} |r\widehat{\mathbb{E}}[Z_{n,k}^2 \mid \mathscr{H}_{n,k-1}] - \widehat{\mathcal{E}}[Z_{n,k}^2 \mid \mathscr{H}_{n,k-1}]| \xrightarrow{\mathbb{V}} 0, \tag{3.3}$$

$$\sum_{k=1}^{k_n} \{|\widehat{\mathbb{E}}[Z_{n,k} \mid \mathscr{H}_{n,k-1}]| + |\widehat{\mathcal{E}}[Z_{n,k} \mid \mathscr{H}_{n,k-1}]|\} \xrightarrow{\mathbb{V}} 0. \tag{3.4}$$

Then for any bounded continuous function φ,

$$\lim_{n\to\infty} \widehat{\mathbb{E}}\bigg[\varphi\bigg(\sum_{k=1}^{k_n} Z_{n,k}\bigg)\bigg] = \widetilde{\mathbb{E}}[\varphi(\sqrt{\rho}\xi)], \tag{3.5}$$

i.e., $\sum_{k=1}^{k_n} Z_{n,k} \xrightarrow{d} \sqrt{\rho}\xi$, where $\xi \sim N(0,[r,1])$ under $\widetilde{\mathbb{E}}$.

Remark 3.2. When $\widehat{\mathbb{E}}[Z_{n,k} \mid \mathscr{H}_{n,k-1}] = 0$ and $\widehat{\mathcal{E}}[Z_{n,k} \mid \mathscr{H}_{n,k-1}] = 0$, then $\{Z_{n,k}; k = 1,\dots,k_n\}$ is an array of symmetric martingale differences (see Xu and Zhang [25]). If $\widehat{\mathbb{E}}[\cdot] = \mathsf{E}_P[\cdot]$ is a classical linear expectation, then (3.3) is satisfied with $r = 1$, and the conclusion coincides with Hall and Heyde [6, Corollary 3.1].

The following is a direct corollary of Theorem 3.1.

Corollary 3.3. *Let $\{\eta_n\}$ be a sequence of independent random variables on $(\Omega,\mathscr{H},\widehat{\mathbb{E}})$ with $\widehat{\mathbb{E}}[\eta_n] = \widehat{\mathcal{E}}[\eta_n] = 0$, $\widehat{\mathbb{E}}[\eta_n^2] =: \overline{\sigma}_n^2 \to \overline{\sigma}^2$, $\widehat{\mathcal{E}}[\eta_n^2] := \underline{\sigma}_n^2 \to \underline{\sigma}^2$ and $\sup_n \widehat{\mathbb{E}}[(\eta_n^2 - c)^+] \to 0$ as $c \to \infty$. Suppose that $\{a_{n,i}; i = 1,\dots,k_n\}$ is an array of real random variables in $\mathscr{H}$ with $a_{n,i}$ being a function of $\eta_1,\dots,\eta_{i-1}$,*

$$\max_i |a_{n,i}| \xrightarrow{\mathbb{V}} 0 \quad \textit{and} \quad \sum_{i=1}^{k_n} a_{n,i}^2 \xrightarrow{\mathbb{V}} \rho,$$

where $\rho \geqslant 0$ is a constant. Then

$$\lim_{n\to\infty} \widehat{\mathbb{E}}\bigg[\varphi\bigg(\sum_{i=1}^{k_n} a_{n,i}\eta_i\bigg)\bigg] = \widetilde{\mathbb{E}}[\varphi(\xi)] \tag{3.6}$$

for any bounded continuous function φ, where $\xi \sim N(0,[\rho\underline{\sigma}^2,\rho\overline{\sigma}^2])$ under $\widetilde{\mathbb{E}}$.

The following corollary is a central limit theorem for moving average processes which include the autoregressive moving average (ARMA) model.

Corollary 3.4. *Let $\{\eta_n\}$ be a sequence of independent and identically distributed random variables in $(\Omega,\mathscr{H},\widehat{\mathbb{E}})$ with $\widehat{\mathbb{E}}[\eta_1] = \widehat{\mathcal{E}}[\eta_1] = 0$, $\widehat{\mathbb{E}}[\eta_1^2] = \overline{\sigma}^2$ and $\widehat{\mathcal{E}}[\eta_1^2] = \underline{\sigma}^2$, and $\{a_n; n \geqslant 0\}$ be a sequence of real numbers with $\sum_{n=0}^{\infty} |a_n| < \infty$. Let $X_k = \sum_{i=0}^{\infty} a_i\eta_{i+k}$. Then*

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k \xrightarrow{d} N(0,[a^2\underline{\sigma}^2, a^2\overline{\sigma}^2]), \tag{3.7}$$

where $a=\sum_{j=0}^{\infty}a_j$.

Proof. Let $a_n=0$ if $n<0$. Then $X_k=\sum_{i=1}^{\infty}a_{i-k}\eta_i$ and

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}X_k=\sum_{i=1}^{\infty}\bigg(\frac{\sum_{k=1}^{n}a_{i-k}}{\sqrt{n}}\bigg)\eta_i.$$

Let $a_{n,i}=\frac{\sum_{k=1}^{n}a_{i-k}}{\sqrt{n}}$. Then $\max_i|a_{n,i}|\leqslant n^{-1/2}\sum_{i=-\infty}^{\infty}|a_i|\to 0$ and $\sum_{i=1}^{\infty}a_{n,i}^2\to a^2$. The result follows from Corollary 3.3. □

Finally, we give the functional central limit theorems.

Let $D_{[0,1]}$ be the space of right continuous functions having finite left limits which is endowed with the Skorohod topology, and $\tau_n(t)$ be a non-decreasing function in $D_{[0,1]}$ which takes integer values with $\tau_n(0)=0$, $\tau_n(1)=k_n$. Define $S_{n,i}=\sum_{k=1}^{i}Z_{n,k}$,

$$W_n(t)=S_{n,\tau_n(t)}. \tag{3.8}$$

Theorem 3.5. *Suppose that the operators* $\widehat{\mathbb{E}}_{n,k}$ *satisfy* (a) *and* (b). *Assume that the conditions* (3.1), (3.3) *and* (3.4) *in Theorem* 3.1 *are satisfied. Furthermore, there is a continuous non-decreasing non-random function* $\rho(t)$ *such that*

$$\sum_{k\leqslant\tau_n(t)}\widehat{\mathbb{E}}[Z_{n,k}^2\,|\,\mathscr{H}_{n,k-1}]\xrightarrow{\mathbb{V}}\rho(t),\quad t\in[0,1]. \tag{3.9}$$

Then for any $0=t_0<\cdots<t_d\leqslant 1$,

$$(W_n(t_1),\dots,W_n(t_d))\xrightarrow{d}(W(\rho(t_1)),\dots,W(\rho(t_d))), \tag{3.10}$$

and for any bounded continuous function $\varphi:D_{[0,1]}\to\mathbb{R}$,

$$\lim_{n\to\infty}\widehat{\mathbb{E}}[\varphi(W_n)]=\widetilde{\mathbb{E}}[\varphi(W\circ\rho)], \tag{3.11}$$

where W *is G-Brownian motion on* $[0,1]$ *with* $W(1)\sim N(0,[r,1])$ *under* $\widetilde{\mathbb{E}}$, *and* $W\circ\rho(t)=W(\rho(t))$.

Because the proofs of Theorems 3.1 and 3.5 are a little long and need some preparation, we will give them in the last section.

4 Moment inequalities and exponential inequalities

To prove the central limit theorems and functional central limit theorems, we need some inequalities on the sums of martingale-difference-like random variables as basic tools. Before we give the inequalities, we state some properties of the sub-linear expectations $\widehat{\mathbb{E}}$ and $\widehat{\mathbb{E}}_{n,k}$. The first is Hölder's inequality which is Proposition 16 of Denis et al. [3].

Lemma 4.1 (Hölder's inequality). *Let* $p,q>1$ *be two real numbers satisfying* $\frac{1}{p}+\frac{1}{q}=1$. *Then, for two random variables* X,Y *in* $(\Omega,\mathscr{H},\widehat{\mathbb{E}})$ *we have* $\widehat{\mathbb{E}}[|XY|]\leqslant(\widehat{\mathbb{E}}[|X|^p])^{\frac{1}{p}}(\widehat{\mathbb{E}}[|Y|^q])^{\frac{1}{q}}$ *whenever* $\widehat{\mathbb{E}}[|X|^p]<\infty$, $\widehat{\mathbb{E}}[|Y|^q]<\infty$.

The next two lemmas are on the properties of the sub-linear expectation, the capacity and the operators $\widehat{\mathbb{E}}_{n,k}$. The proofs will be given in Appendix A. We write $X\leqslant Y$ in capacity $\mathbb{V}$ if $\mathbb{V}(X-Y\geqslant\epsilon)=0$ for all $\epsilon>0$, and $X=Y$ in capacity $\mathbb{V}$ if both $X\leqslant Y$ and $Y\leqslant X$ holds in $\mathbb{V}$.

Lemma 4.2. *We have*

(1) *if* $X\leqslant Y$ *in* L_p, *then* $X\leqslant Y$ *in* $\mathbb{V}$;

(2) *if* $X\leqslant Y$ *in* $\mathbb{V}$ *and* $\widehat{\mathbb{E}}[((X-Y)^+)^p]<\infty$, *then* $X\leqslant Y$ *in* L_q *for* $0<q<p$;

(3) *if* $X\leqslant Y$ *in* $\mathbb{V}$, $f(x)$ *is non-decreasing continuous function and* $\mathbb{V}(|Y|\geqslant M)\to 0$ *as* $M\to\infty$, *then* $f(X)\leqslant f(Y)$ *in* $\mathbb{V}$;

(4) *if* $p\geqslant 1$, $X,Y\geqslant 0$ *in* L_p, $X\leqslant Y$ *in* L_p, *then* $\widehat{\mathbb{E}}[X^p]\leqslant\widehat{\mathbb{E}}[Y^p]$;

(5) *if* $\widehat{\mathbb{E}}$ *is countably additive, then* $X\leqslant Y$ *in* $\mathbb{V}$ *is equivalent to* $X\leqslant Y$ *in* L_p *for any* $p>0$.

Lemma 4.3. *Suppose that the operators $\widehat{\mathrm{E}}_{n,k}$ satisfy* (a) *and* (b). *For $X, Y \in \mathscr{L}(\mathscr{H})$, we have*

(1) *if $X \leqslant Y$ in L_1, then $\widehat{\mathrm{E}}_{n,k}[X] \leqslant \widehat{\mathrm{E}}_{n,k}[Y]$ in L_1;*

(2) *$\widehat{\mathrm{E}}_{n,k}[X] - \widehat{\mathrm{E}}_{n,k}[Y] \leqslant \widehat{\mathrm{E}}_{n,k}[X-Y] \leqslant \widehat{\mathrm{E}}_{n,k}[|X-Y|]$ in L_1;*

(3) *$\widehat{\mathrm{E}}_{n,k}[[\widehat{\mathrm{E}}_{n,l}[X]]] = \widehat{\mathrm{E}}_{n,l\wedge k}[X]$ in L_1;*

(4) *if $|X| \leqslant M$ in L_p for all $p \geqslant 1$, then $|\widehat{\mathrm{E}}_{n,k}[X]| \leqslant M$ in L_p for all $p \geqslant 1$.*

For the martingale-difference-like random variables, we have the following theorem on the Rosenthal-type inequalities.

Theorem 4.4. *Set $S_0 = 0$ and $S_k = \sum_{i=1}^k Z_{n,i}$. Suppose that $\{Z_{n,i}\}$ are a set of bounded random variables. Then,*

$$\widehat{\mathrm{E}}\Big[\Big(\max_{k\leqslant k_n}(S_{k_n} - S_k)\Big)^2\Big] \leqslant \widehat{\mathrm{E}}\Bigg[\sum_{k=1}^{k_n}\widehat{\mathrm{E}}[Z_{n,k}^2 \mid \mathscr{H}_{n,k-1}]\Bigg] \tag{4.1}$$

when $\widehat{\mathrm{E}}[Z_{n,k}|\mathscr{H}_{n,k-1}] \leqslant 0$, $k = 1, \ldots, k_n$, and in general,

$$\begin{aligned}\widehat{\mathrm{E}}\Big[\max_{k\leqslant k_n}|S_k|^2\Big] \leqslant 256\Bigg\{&\widehat{\mathrm{E}}\Bigg[\sum_{k=1}^{k_n}\widehat{\mathrm{E}}[Z_{n,k}^2 \mid \mathscr{H}_{n,k-1}]\Bigg]\\ &+ \widehat{\mathrm{E}}\Bigg[\Bigg\{\sum_{k=1}^{k_n}((\widehat{\mathrm{E}}[Z_{n,k} \mid \mathscr{H}_{n,k-1}])^+ + (\widehat{\mathcal{E}}[Z_{n,k} \mid \mathscr{H}_{n,k-1}])^-)\Bigg\}^2\Bigg]\Bigg\}.\end{aligned} \tag{4.2}$$

Moreover, for $p \geqslant 2$ there is a constant C_p such that

$$\begin{aligned}\widehat{\mathrm{E}}\Big[\max_{k\leqslant k_n}|S_k|^p\Big] \leqslant C_p\Bigg\{&\widehat{\mathrm{E}}\Bigg[\sum_{k=1}^{k_n}\widehat{\mathrm{E}}[|Z_{n,k}|^p \mid \mathscr{H}_{n,k-1}]\Bigg] + \widehat{\mathrm{E}}\Bigg[\Bigg(\sum_{k=1}^{k_n}\widehat{\mathrm{E}}[Z_{n,k}^2 \mid \mathscr{H}_{n,k}]\Bigg)^{p/2}\Bigg]\\ &+ \widehat{\mathrm{E}}\Bigg[\Bigg\{\sum_{k=1}^{k_n}((\widehat{\mathrm{E}}[Z_{n,k} \mid \mathscr{H}_{n,k}])^+ + (\widehat{\mathcal{E}}[Z_{n,k} \mid \mathscr{H}_{n,k}])^-)\Bigg\}^p\Bigg]\Bigg\}.\end{aligned} \tag{4.3}$$

Proof. Let $Q_k = \max\{Z_{n,k}, Z_{n,k} + Z_{n,k-1}, \ldots, Z_{n,k} + \cdots + Z_{n,1}\}$, $M_k = \max_{i\leqslant k}|S_i|$. Then, $Q_k = Z_{n,k} + Q_{k-1}^+$, $Q_k^2 = Z_{n,k}^2 + 2Z_{n,k}Q_{k-1}^+ + (Q_{k-1}^+)^2$, $|Q_k| \leqslant 2M_{k_n}$. It follows that

$$\begin{aligned}\Big(\max_{k\leqslant k_n}(S_{k_n} - S_k)\Big)^2 = (Q_{k_n}^+)^2 &\leqslant \sum_{k=1}^{k_n} Z_{n,k}^2 + 2\sum_{k=1}^{k_n} Z_{n,k}Q_{k-1}^+\\ &\leqslant \sum_{k=1}^{k_n}\widehat{\mathrm{E}}[Z_{n,k}^2 \mid \mathscr{H}_{n,k-1}] + \sum_{k=1}^{k_n}(Z_{n,k}^2 - \widehat{\mathrm{E}}[Z_{n,k}^2 \mid \mathscr{H}_{n,k-1}])\\ &\quad + 2\sum_{k=1}^{k_n}\widehat{\mathrm{E}}[Z_{n,k} \mid \mathscr{H}_{n,k-1}]Q_{k-1}^+ + 2\sum_{k=1}^{k_n}(Z_{n,k} - \widehat{\mathrm{E}}[Z_{n,k} \mid \mathscr{H}_{n,k-1}])Q_{k-1}^+\\ &\leqslant \sum_{k=1}^{k_n}\widehat{\mathrm{E}}[Z_{n,k}^2 \mid \mathscr{H}_{n,k-1}] + 4\sum_{k=1}^{k_n}(\widehat{\mathrm{E}}[Z_{n,k} \mid \mathscr{H}_{n,k-1}])^+ M_{k_n}\\ &\quad + \sum_{k=1}^{k_n}(Z_{n,k}^2 - \widehat{\mathrm{E}}[Z_{n,k}^2 \mid \mathscr{H}_{n,k-1}]) + 2\sum_{k=1}^{k_n}(Z_{n,k} - \widehat{\mathrm{E}}[Z_{n,k} \mid \mathscr{H}_{n,k-1}])Q_{k-1}^+.\end{aligned}$$

By the fact that $Z_{n,i}$s are bounded, Lemma 4.3(4) and Hölder's inequality, the random variables considered above and in the sequel have finite moments of any order. So, the properties of the conditional expectation operator can be applied freely. The sub-linear expectations of the last two sums above are non-positive, and the sub-linear expectation of the second sum is also zero when $\widehat{\mathrm{E}}[Z_{n,k} \mid \mathscr{H}_{n,k}] \leqslant 0$, $k = 1, \ldots, k_n$. Taking the sub-linear expectation yields (4.1). By considering $\{-Z_{n,k}\}$, for $\max_{k\leqslant k_n}(-S_{k_n}$

$+ S_k)$ we have a similar estimate. Note $M_{k_n} \leqslant 2\max_{k\leqslant k_n}|S_n - S_k|$. It follows that

$$\begin{aligned}\widehat{\mathrm{E}}[M_{k_n}^2] &\leqslant 8\widehat{\mathrm{E}}\bigg[\sum_{k=1}^{k_n}\widehat{\mathrm{E}}[Z_{n,k}^2 \,|\, \mathscr{H}_{n,k-1}]\bigg] \\ &\quad + 16\widehat{\mathrm{E}}\bigg[\sum_{k=1}^{k_n}\{(\widehat{\mathrm{E}}[Z_{n,k} \,|\, \mathscr{H}_{n,k-1}])^+ + (\widehat{\mathcal{E}}[Z_{n,k} \,|\, \mathscr{H}_{n,k-1}])^-\}M_{k_n}\bigg] \\ &\leqslant 8\widehat{\mathrm{E}}\bigg[\sum_{k=1}^{k_n}\widehat{\mathrm{E}}[Z_{n,k}^2 \,|\, \mathscr{H}_{n,k-1}]\bigg] + \frac{1}{2}\widehat{\mathrm{E}}[M_{k_n}^2] \\ &\quad + 128\widehat{\mathrm{E}}\bigg[\bigg(\sum_{k=1}^{k_n}\{(\widehat{\mathrm{E}}[Z_{n,k} \,|\, \mathscr{H}_{n,k-1}])^+ + (\widehat{\mathcal{E}}[Z_{n,k} \,|\, \mathscr{H}_{n,k-1}])^-\}\bigg)^2\bigg],\end{aligned}$$

where the last inequality is due to $ab \leqslant \frac{a^2+b^2}{2}$.

For (4.3), we apply the elementary inequality

$$|x+y|^p \leqslant 2^p p^2 |x|^p + |y|^p + px|y|^{p-1}\mathrm{sgn}y + 2^p p^2 x^2 |y|^{p-2}, \quad p \geqslant 2,$$

and yield

$$|Q_k|^p \leqslant 2^p p^2 |Z_{n,k}|^p + |Q_{k-1}|^p + pZ_{n,k}(Q_{k-1}^+)^{p-1} + 2^p p^2 Z_{n,k}^2 (Q_{k-1}^+)^{p-2}.$$

It follows that

$$\begin{aligned}\Big(\max_{k\leqslant k_n}(S_{k_n} - S_k)\Big)^p &\leqslant |Q_{k_n}|^p \\ &\leqslant 2^p p^2 \sum_{k=1}^{k_n}|Z_{n,k}|^p + p\sum_{k=1}^{k_n} Z_{n,k}(Q_{k-1}^+)^{p-1} + 2^p p^2 \sum_{k=1}^{k_n} Z_{n,k}^2 (Q_{k-1}^+)^{p-2} \\ &\leqslant 2^p p^2 \sum_{k=1}^{k_n}\widehat{\mathrm{E}}[|Z_{n,k}|^p \,|\, \mathscr{H}_{n,k-1}] + p\sum_{k=1}^{k_n}(\widehat{\mathrm{E}}[Z_{n,k} \,|\, \mathscr{H}_{n,k-1}])^+(Q_{k-1}^+)^{p-1} \\ &\quad + 2^p p^2 \sum_{k=1}^{k_n}\widehat{\mathrm{E}}[Z_{n,k}^2 \,|\, \mathscr{H}_{n,k-1}](Q_{k-1}^+)^{p-2} + 2^p p^2 \sum_{k=1}^{k_n}(|Z_{n,k}|^p - \widehat{\mathrm{E}}[|Z_{n,k}|^p \,|\, \mathscr{H}_{n,k-1}]) \\ &\quad + p\sum_{k=1}^{k_n}(Z_{n,k} - \widehat{\mathrm{E}}[Z_{n,k} \,|\, \mathscr{H}_{n,k-1}])(Q_{k-1}^+)^{p-1} \\ &\quad + 2^p p^2 \sum_{k=1}^{k_n}(Z_{n,k}^2 - \widehat{\mathrm{E}}[Z_{n,k}^2 \,|\, \mathscr{H}_{n,k-1}])(Q_{k-1}^+)^{p-2}.\end{aligned}$$

The sub-linear expectations of the last three sums are non-positive. Note $Q_k \leqslant 2M_{k_n}$ and for

$$\Big(\max_{k\leqslant k_n}(-S_{k_n} + S_k)\Big)^p,$$

we have a similar estimate. It follows that

$$\begin{aligned}\widehat{\mathrm{E}}[M_{k_n}^p] &\leqslant C_p\bigg\{\widehat{\mathrm{E}}\bigg[\sum_{k=1}^{k_n}\widehat{\mathrm{E}}[|Z_{n,k}|^p \,|\, \mathscr{H}_{n,k-1}]\bigg] + \widehat{\mathrm{E}}\bigg[\sum_{k=1}^{k_n}\widehat{\mathrm{E}}[Z_{n,k}^2 \,|\, \mathscr{H}_{n,k-1}]M_{k_n}^{p-2}\bigg] \\ &\quad + \widehat{\mathrm{E}}\bigg[\sum_{k=1}^{k_n}\{(\widehat{\mathrm{E}}[Z_{n,k} \,|\, \mathscr{H}_{n,k-1}])^+ + (\widehat{\mathcal{E}}[Z_{n,k} \,|\, \mathscr{H}_{n,k-1}])^-\}M_{k_n}^{p-1}\bigg]\bigg\} \\ &\leqslant C_p\bigg\{\widehat{\mathrm{E}}\bigg[\sum_{k=1}^{k_n}\widehat{\mathrm{E}}[|Z_{n,k}|^p \,|\, \mathscr{H}_{n,k-1}]\bigg] + \widehat{\mathrm{E}}\bigg[\bigg(\sum_{k=1}^{k_n}\widehat{\mathrm{E}}[Z_{n,k}^2 \,|\, \mathscr{H}_{n,k-1}]\bigg)^{p/2}\bigg]\end{aligned}$$

$$+\widehat{\mathbb{E}}\bigg[\bigg(\sum_{k=1}^{k_n}\{(\widehat{\mathbb{E}}[Z_{n,k}\,|\,\mathscr{H}_{n,k-1}])^+ + (\widehat{\mathcal{E}}[Z_{n,k}\,|\,\mathscr{H}_{n,k-1}])^-\}\bigg)^p\bigg]\bigg\} + \frac{1}{2}\widehat{\mathbb{E}}[M_{k_n}^p],$$

where the last inequality is due to $ab \leqslant \frac{2}{p}|a|^{p/2} + (1-\frac{2}{p})|b|^{p/(p-2)}$ and $ab \leqslant \frac{1}{p}|a|^p + (1-\frac{1}{p})|b|^{p/(p-1)}$. The proof is completed. □

The next theorem gives the exponential inequality of the martingale-like sequences.

Theorem 4.5. *Suppose that the operators $\widehat{\mathbb{E}}_{n,k}$ satisfy* (a) *and* (b), *$\{Z_{n,k}; k=1,\dots,k_n\}$ is an array of random variables such that $Z_{n,k}\in\mathscr{H}_{n,k}$ and $\widehat{\mathbb{E}}[Z_{n,k}^2]<\infty$, $k=1,\dots,k_n$. Assume that $\widehat{\mathbb{E}}[Z_{n,k}\,|\,\mathscr{H}_{n,k-1}] \leqslant 0$ in L_1, $k=1,\dots,k_n$. Then for all $x,y,A>0$,*

$$\mathbb{V}\bigg(\max_{m\leqslant k_n}\sum_{k=1}^m Z_{n,k}\geqslant x\bigg) \leqslant \mathbb{V}\bigg(\max_{k\leqslant k_n} Z_{n,k}\geqslant y \text{ or } \sum_{k=1}^{k_n}\widehat{\mathbb{E}}[Z_{n,k}^2\,|\,\mathscr{H}_{n,k-1}]\geqslant A\bigg) + \exp\bigg\{-\frac{x^2}{2(xy+A)}\bigg(1+\frac{2}{3}\ln\bigg(1+\frac{xy}{A}\bigg)\bigg)\bigg\}. \tag{4.4}$$

Proof. Let $X_k = Z_{n,k}\wedge y$. Then $Z_{n,k}-X_k = (Z_{n,k}-y)^+ \geqslant 0$. Denote $\sigma_{n,k}^2 = \widehat{\mathbb{E}}[Z_{n,k}^2\,|\,\mathscr{H}_{n,k-1}]$, $\delta_k = \sum_{i=1}^k \sigma_{n,i}^2$, $k=1,\dots,k_n$. Let $f(x)$ be a function with bounded derivative such that $I\{x\leqslant A\}\leqslant f(x)\leqslant I\{x\leqslant A+\epsilon\}$. Let $Y_k = X_k f(\delta_k)$, $T_k=\sum_{i=1}^k Y_k$. Then $\widehat{\mathbb{E}}[Y_k\,|\,\mathscr{H}_{n,k-1}]\leqslant f(\delta_k)\widehat{\mathbb{E}}[Z_{n,k}\,|\,\mathscr{H}_{n,k-1}]\leqslant 0$ in L_1, $\widehat{\mathbb{E}}[Y_k^2\,|\,\mathscr{H}_{n,k-1}]\leqslant f^2(\delta_k)\widehat{\mathbb{E}}[Z_{n,k}^2\,|\,\mathscr{H}_{n,k-1}] = f^2(\delta_k)\sigma_{n,k}^2$ in L_1. Denote $\delta_k^* = \sum_{i=1}^k f^2(\delta_k)\sigma_{n,k}^2$. It follows that for any $x,y,A>0$,

$$\mathbb{V}\bigg(\max_{m\leqslant k_n}\sum_{k=1}^m Z_{n,k}\geqslant x\bigg)\leqslant \mathbb{V}\Big(\max_{k\leqslant k_n} Z_{n,k}\geqslant y \text{ or } \delta_{k_n}>A\Big) + \mathbb{V}\Big(\max_{k\leqslant k_n} T_k\geqslant x\Big).$$

For any $t>0$, by noting $Y_k\leqslant y$, $0\leqslant f^2(\delta_k)\sigma_{n,k}^2\leqslant \delta_k^*\leqslant A+\epsilon$ and

$$\mathrm{e}^{tY_k} = 1+tY_k+\frac{\mathrm{e}^{tY_k}-1-tY_k}{Y_k^2}Y_k^2 \leqslant 1+tY_k+\frac{\mathrm{e}^{ty}-1-ty}{y^2}Y_k^2,$$

we have

$$\begin{aligned}&\exp\bigg\{-\frac{\mathrm{e}^{ty}-1-ty}{y^2}f^2(\delta_k)\sigma_{n,k}^2\bigg\}\widehat{\mathbb{E}}[\mathrm{e}^{tY_k}\,|\,\mathscr{H}_{n,k-1}]\\ &\quad\leqslant \exp\bigg\{-\frac{\mathrm{e}^{ty}-1-ty}{y^2}f^2(\delta_k)\sigma_{n,k}^2\bigg\}\bigg\{1+\frac{\mathrm{e}^{ty}-1-ty}{y^2}\widehat{\mathbb{E}}[Y_k^2\,|\,\mathscr{H}_{n,k-1}]\bigg\}\\ &\quad\leqslant 1 \quad \text{in} \quad L_1.\end{aligned}$$

Write

$$U_0=1,\quad U_k=\exp\bigg\{-\frac{\mathrm{e}^{ty}-1-ty}{y^2}\delta_k^*\bigg\}\mathrm{e}^{tT_k},\quad k=1,\dots,k_n.$$

Then

$$\widehat{\mathbb{E}}[U_k\,|\,\mathscr{H}_{n,k-1}]\leqslant U_{k-1}\quad\text{in}\quad L_1,\quad k=1,\dots,k_n. \tag{4.5}$$

Next, we show that for any $\alpha>0$,

$$\mathbb{V}\Big(\max_{k\leqslant k_n} U_k\geqslant\alpha\Big)\leqslant\frac{\widehat{\mathbb{E}}[U_0]}{\alpha}. \tag{4.6}$$

For a given $\beta\in(0,\alpha)$, let $f(x)$ be a continuous function with bounded derivation such that $I\{x\leqslant\alpha-\beta\}\leqslant f(x)\leqslant I\{x\leqslant\alpha\}$. Define $f_0=1$, $f_k=f(U_1)\cdots f(U_k)$. Then $f_k\in\mathscr{H}_k$, $0\leqslant f_k\leqslant 1$ and

$$f_0U_0+\sum_{k=1}^n f_{k-1}(U_k-U_{k-1}) = f_nU_n+\sum_{k=1}^n f_{k-1}(1-f(U_k))U_k$$

$$\begin{aligned}&\geqslant f_nU_n+\sum_{k=1}^n f_{k-1}(1-f(U_k))(\alpha-\beta)\\&=(\alpha-\beta)(1-f_n)+f_nU_n\\&\geqslant(\alpha-\beta)I\Big\{\max_{k\leqslant k_n}U_k\geqslant\alpha\Big\}.\end{aligned}$$

By (4.5),

$$\begin{aligned}\widehat{\mathrm{E}}[f_{k-1}(U_k-U_{k-1})]&=\widehat{\mathrm{E}}[\widehat{\mathrm{E}}[f_{k-1}(U_k-U_{k-1})\,|\,\mathscr{H}_{k-1}]]\\&=\widehat{\mathrm{E}}[f_{k-1}(\widehat{\mathrm{E}}[U_k\,|\,\mathscr{H}_{k-1}]-U_{k-1})]\leqslant 0.\end{aligned}$$

It follows that

$$(\alpha-\beta)\mathbb{V}\Big(\max_{k\leqslant k_n}U_k\geqslant\alpha\Big)\leqslant\widehat{\mathrm{E}}[f_0U_0]=\widehat{\mathrm{E}}[U_0].$$

(4.6) is proved. Now, note $\delta_k^*\leqslant A+\epsilon$. We have for any $t>0$,

$$\exp\Big\{t\max_{k\leqslant k_n}T_k\Big\}\leqslant\max_{k\leqslant k_n}U_k\exp\Big\{\frac{\mathrm{e}^{ty}-1-ty}{y^2}(A+\epsilon)\Big\}.$$

Hence by (4.6),

$$\begin{aligned}\mathbb{V}\Big(\max_{k\leqslant k_n}T_k\geqslant x\Big)&\leqslant\mathbb{V}\Big(\max_{k\leqslant k_n}U_k\geqslant\exp\Big\{tx-\frac{\mathrm{e}^{ty}-1-ty}{y^2}(A+\epsilon)\Big\}\Big)\\&\leqslant\exp\Big\{-tx+\frac{\mathrm{e}^{ty}-1-ty}{y^2}(A+\epsilon)\Big\}.\end{aligned}$$

Choosing $t=\frac{1}{y}\ln(1+\frac{xy}{A+\epsilon})$, we have

$$\mathbb{V}\Big(\max_{k\leqslant k_n}T_k\geqslant x\Big)\leqslant\exp\Big\{\frac{x}{y}-\frac{x}{y}\Big(\frac{A+\epsilon}{xy}+1\Big)\ln\Big(1+\frac{xy}{A+\epsilon}\Big)\Big\}.$$

Applying the elementary inequality

$$\ln(1+t)\geqslant\frac{t}{1+t}+\frac{t^2}{2(1+t)^2}\Big(1+\frac{2}{3}\ln(1+t)\Big),$$

we have

$$\Big(\frac{A+\epsilon}{xy}+1\Big)\ln\Big(1+\frac{xy}{A+\epsilon}\Big)\geqslant 1+\frac{xy}{2(xy+A+\epsilon)}\Big(1+\frac{2}{3}\ln\Big(1+\frac{xy}{A+\epsilon}\Big)\Big).$$

(4.4) is proved by letting $\epsilon\to 0$. □

5 Lévy's characterization of a G-Brownian motion

In this section, we give a Lévy characterization of a G-Brownian motion as an application of Theorem 3.5. Let $\{\mathscr{H}_t;t\geqslant 0\}$ be a non-decreasing family of subspaces of $\mathscr{H}$ such that (1) a constant $c\in\mathscr{H}_t$, and (2) if $X_1,\ldots,X_d\in\mathscr{H}_t$, then $\varphi(X_1,\ldots,X_d)\in\mathscr{H}_t$ for any $\varphi\in C_{l,\mathrm{Lip}}$. We consider a system of operators in $\mathscr{L}(\mathscr{H})$, $\widehat{\mathrm{E}}_t:\mathscr{L}(\mathscr{H})\to\mathscr{L}(\mathscr{H}_t)$ and denote $\widehat{\mathrm{E}}[X\,|\,\mathscr{H}_t]=\widehat{\mathrm{E}}_t[X]$, $\widehat{\mathcal{E}}[X\,|\,\mathscr{H}_t]=-\widehat{\mathrm{E}}_t[-X]$. Suppose that the operators $\widehat{\mathrm{E}}_t$ satisfy the following properties: for all $X,Y\in\mathscr{L}(\mathscr{H})$,

(i) $\widehat{\mathrm{E}}_t[X+Y]=X+\widehat{\mathrm{E}}_t[Y]$ in L_1 if $X\in\mathscr{H}_t$, and $\widehat{\mathrm{E}}_t[XY]=X^+\widehat{\mathrm{E}}_t[Y]+X^-\widehat{\mathrm{E}}_t[-Y]$ in L_1 if $X\in\mathscr{H}_t$ and $XY\in\mathscr{L}(\mathscr{H})$;

(ii) $\widehat{\mathrm{E}}[\widehat{\mathrm{E}}_t[X]]=\widehat{\mathrm{E}}[X]$.

Example 5.1. Let W_t be a G-Brownian motion in a sub-linear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$, and

$$\widetilde{\mathscr{H}} = \{X = \varphi(W_{t_1}, \dots, W_{t_d}) : 0 \leqslant t_1 \leqslant \cdots \leqslant t_d, \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}_d), d \geqslant 1\},$$
$$\mathscr{H}_t = \{X = \varphi(W_{t_1}, \dots, W_{t_d}) : 0 \leqslant t_1 \leqslant \cdots \leqslant t_d \leqslant t, \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}_d), d \geqslant 1\}.$$

For $X = \varphi(W_{t_1}, \dots, W_{t_d}) \in \widetilde{\mathscr{H}}$, assume $0 \leqslant t_1 \leqslant t_i \leqslant t \leqslant t_{i+1} \leqslant \cdots \leqslant t_d$, and define

$$\widehat{\mathbb{E}}_t[X] = \widehat{\mathbb{E}}[\varphi(w_{t_1}, \dots, w_{t_i}, W_{t_{i+1}} - W_t + w_t, \dots, W_{t_d} - W_t + w_t)]\,|_{w_{t_1}=W_{t_1}, \dots, w_{t_i}=W_{t_i}, w_t=W_t}.$$

Then, in the sub-linear expectation space $(\Omega, \widetilde{\mathscr{H}}, \widehat{\mathbb{E}})$, the family $\{\mathscr{H}_t, \widehat{\mathbb{E}}_t\}_{t\geqslant 0}$ satisfies the properties (i)–(iii).

Definition 5.2. A process M_t is called a martingale, if $M_t \in \mathscr{L}(\mathscr{H})$, $M_t \in \mathscr{H}_t$ and

$$\widehat{\mathbb{E}}[M_t \mid \mathscr{H}_s] = M_s, \quad s \leqslant t.$$

Denote

$$w_T(M, \delta) = \sup_{|t-s|<\delta, t,s\in[0,T]} |M(t) - M(s)|$$

and

$$W_T(M, \delta) = \sup_{t_i} \widehat{\mathbb{E}}\Big[\max_{1\leqslant i\leqslant n} |M(t_i) - M(t_{i-1})| \wedge 1\Big],$$

where the supremum $\sup_{t_i}$ is taken over all t_is with

$$0 = t_0 < t_1 < \cdots < t_n = T, \quad \frac{\delta}{2} < t_i - t_{i-1} < \delta, \quad i = 1, \dots, n.$$

The following theorem gives a Lévy characterization of a G-Brownian motion.

Theorem 5.3. *Let M_t be a random process in $(\Omega, \mathscr{H}, \mathscr{H}_t, \widehat{\mathbb{E}})$ with $M_0 = 0$,*

$$\text{for all } p > 0 \text{ and } t \geqslant 0, \quad C_{\mathbb{V}}(|M_t|^p) < \infty \Rightarrow \widehat{\mathbb{E}}[|M_t|^p] < \infty. \tag{5.1}$$

Suppose that M_t satisfies

(I) *both M_t and $-M_t$ are martingales*;

(II) *for a constant $\overline{\sigma}^2 > 0$, $M_t^2 - \overline{\sigma}^2 t$ is a martingale*;

(III) *for a constant $0 < \underline{\sigma}^2 \leqslant \overline{\sigma}^2$, $-(M_t^2 - \underline{\sigma}^2 t)$ is a martingale*;

(IV) *for any $T > 0$, $\lim_{\delta\to 0} W_T(M, \delta) = 0$.*

Then, M_t satisfies Property (ii) *as in Definition* 1.4 *with $M_1 \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$.*

Remark 5.4. The assumption (I) implies that $\widehat{\mathbb{E}}[M_t - M_s \mid \mathscr{H}_s] = \widehat{\mathcal{E}}[M_t - M_s \mid \mathscr{H}_s] = 0$ for all $t > s$. Also, under the assumption (I), the assumption (II) is equivalent to that $\widehat{\mathbb{E}}[(M_t - M_s)^2 \mid \mathscr{H}_s] = \overline{\sigma}^2(t - s)$ for all $t > s$, and (III) is equivalent to that $\widehat{\mathcal{E}}[(M_t - M_s)^2 \mid \mathscr{H}_s] = \underline{\sigma}^2(t - s)$ for all $t > s$.

The assumption (IV) means that M_t is continuous. Note $W_T(M, \delta) \leqslant \epsilon + \mathbb{V}(w_T(M, \delta) > \epsilon)$. It is satisfied if

(IV′) for any $T, \epsilon > 0$, $\lim_{\delta\to 0} \mathbb{V}(w_T(M, \delta) > \epsilon) = 0$.

The condition (IV′) means that M_t is continuous in capacity $\mathbb{V}$ uniformly in t on each finite interval. Also, $W_T(M, \delta) \leqslant \sup_{t_i}(\sum_i \widehat{\mathbb{E}}[|M(t_i) - M(t_{i-1})|^{2+\alpha}])^{\frac{1}{2+\alpha}}$. (IV) is also satisfied if

(IV′′) there is a constant $\alpha > 0$ such that for any $t > s > 0$, $\widehat{\mathbb{E}}[|M_t - M_s|^{2+\alpha}] = o(t - s)$ as $t - s \to 0$.

Remark 5.5. Lévy characterization of a G-Brownian motion is first established under G-expectation in a Wiener space by Xu and Zhang [25, 26] by using the stochastic calculus. We give an elementary proof by using the functional central limit theorem.

Remark 5.6. If $\widehat{\mathbb{E}}$ is countably sub-additive, then the condition (5.1) is automatically satisfied. The G-expectation space considered in Xu and Zhang [25, 26] is complete and so the sub-linear expectation is countably additive, and (5.1) is satisfied.

In [25, 26], the operators $\widehat{\mathbb{E}}_t$ are also supposed to have the following assumptions:

(iii) if $X \leqslant Y$, then $\widehat{\mathrm{E}}_t[X] \leqslant \widehat{\mathrm{E}}_t[Y]$;
(iv) $\widehat{\mathrm{E}}_t[X] - \widehat{\mathrm{E}}_t[Y] \leqslant \widehat{\mathrm{E}}_t[X-Y]$;
(v) $\widehat{\mathrm{E}}_t[[\widehat{\mathrm{E}}_s[X]]] = \widehat{\mathrm{E}}_{t\wedge s}[X]$.
As in Lemma 4.3, (iii)–(v) hold in L_1 if the operators satisfy (i) and (ii).

For proving Theorem 5.3 we need a more lemma.

Lemma 5.7. *Suppose that the operators* $\widehat{\mathrm{E}}_t$ *satisfy* (i) *and* (ii), M_t *is a martingale in* $(\Omega, \mathscr{H}, \mathscr{H}_t, \widehat{\mathrm{E}})$ *such that Theorem* 5.3(IV) *is satisfied and* $\widehat{\mathrm{E}}[(M_t - M_s)^2 \,|\, \mathscr{H}_s] \leqslant (t-s)\sigma^2$ *for all* $t > s \geqslant 0$, *where* σ *is a positive constant. Then,*

$$\mathbb{V}(M_t - M_s \geqslant x) \leqslant \exp\left\{-\frac{x^2}{2(t-s)\sigma^2}\right\}, \quad \textit{for all} \ \ t > s \geqslant 0, \quad x \geqslant 0. \tag{5.2}$$

In particular, for any $p > 0$, $C_{\mathbb{V}}([(M_t - M_s)^+]^p) \leqslant c_p (t-s)^{p/2}\sigma^p$.

Proof. Let $s = t_0 < t_1 < \cdots < t_k = t$ be a partition of $[s,t]$ with $\delta/2 < t_i - t_{i-1} < \delta$. Note $\widehat{\mathrm{E}}[M_{t_i} - M_{t_{i-1}} \,|\, \mathscr{H}_{t_{i-1}}] = 0$ and $\widehat{\mathrm{E}}[(M_{t_i} - M_{t_{i-1}})^2 \,|\, \mathscr{H}_{t_{i-1}}] \leqslant (t_i - t_{i-1})\sigma^2$. So, $\sum_{i=1}^k \widehat{\mathrm{E}}[(M_{t_i} - M_{t_{i-1}})^2 \,|\, \mathscr{H}_{t_{i-1}}] \leqslant (t-s)\sigma^2$. By Theorem 4.5, for $0 < y < 1$ and $x > 0$,

$$\begin{aligned}\mathbb{V}(M_t - M_s \geqslant x) &\leqslant \mathbb{V}\Big(\max_i (M_{t_i} - M_{t_{i-1}}) \geqslant y\Big) \\ &\quad + \exp\left\{-\frac{x^2}{2(xy + (t-s)\sigma^2)}\left(1 + \frac{2}{3}\ln\left(1 + \frac{xy}{(t-s)\sigma^2}\right)\right)\right\} \\ &\leqslant \frac{W_T(M,\delta)}{y} + \exp\left\{-\frac{x^2}{2(xy + (t-s)\sigma^2)}\left(1 + \frac{2}{3}\ln\left(1 + \frac{xy}{(t-s)\sigma^2}\right)\right)\right\}.\end{aligned}$$

By letting $\delta \to 0$ and then $y \to 0$, we conclude (5.2). Finally, for $p > 0$,

$$\begin{aligned}C_{\mathbb{V}}([(M_t - M_s)^+]^p) &\leqslant \int_0^\infty \mathbb{V}(M_t - M_s \geqslant x^{1/p})dx \\ &\leqslant (t-s)^{p/2}\sigma^p \int_0^\infty \exp\left\{-\frac{x^{2/p}}{2}\right\}dx \leqslant c_p (t-s)^{p/2}\sigma^p.\end{aligned}$$

The proof is completed. □

Proof of Theorem 5.3. Suppose that (I)–(IV) are satisfied. Note that both M_t and $-M_t$ are martingales, and $\widehat{\mathrm{E}}[(M_t - M_s)^2 \,|\, \mathscr{H}_{t_{i-1}}] = (t-s)\overline{\sigma}^2$. By Lemma 5.7,

$$C_{\mathbb{V}}(|M_t - M_s|^p) \leqslant c_p (t-s)^{p/2}\overline{\sigma}^p.$$

By the assumption (5.1), $\widehat{\mathrm{E}}[|M_t - M_s|^p] < \infty$ for any $p > 0$ and t, s. Let W_t be a G-Brownian motion in a sub-linear expectation $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathrm{E}})$ with $W_1 \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$. It is sufficient to show that for any $0 < t_1 < \cdots < t_d$ and $\varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}_d)$,

$$\widehat{\mathrm{E}}[\varphi(M_{t_1}, \ldots, M_{t_d})] = \widetilde{\mathrm{E}}[\varphi(W_{t_1}, \ldots, W_{t_d})]. \tag{5.3}$$

Actually, by noting $\widehat{\mathrm{E}}[|M_t|^p] < \infty$ for any $p > 0$, we can extend φ from $C_{b,\mathrm{Lip}}(\mathbb{R}_d)$ to $C_{l,\mathrm{Lip}}(\mathbb{R}_d)$ by an elementary argument.

Now, without loss of generality, we assume $0 < t_1 < \cdots < t_d \leqslant 1$. Note $\widehat{\mathrm{E}}[(|M_t - M_s|^3 - c^3)^+] \leqslant \widehat{\mathrm{E}}[|M_t - M_s|^4]/c \to 0$ as $c \to \infty$. Then $\widehat{\mathrm{E}}[|M_t - M_s|^3] \leqslant C_{\mathbb{V}}(|M_t - M_s|^3) = o(t-s)$ as $t - s \to 0$. Let

$$k_n = 2^n, \quad Z_{n,k} = M_{k/2^n} - M_{(k-1)/2^n}, \quad \mathscr{H}_{n,k} = \mathscr{H}_{k/2^n}, \quad k = 1, \ldots, k_n$$

and $\tau_n(t) = [t2^n]$. Then $\widehat{\mathrm{E}}[Z_{n,k} \,|\, \mathscr{H}_{n,k-1}] = \widehat{\mathcal{E}}[Z_{n,k} \,|\, \mathscr{H}_{n,k-1}] = 0$,

$$\widehat{\mathrm{E}}[Z_{n,k}^2 \,|\, \mathscr{H}_{n,k-1}] = \frac{\overline{\sigma}^2}{2^n}, \quad \widehat{\mathcal{E}}[Z_{n,k}^2 \,|\, \mathscr{H}_{n,k-1}] = \frac{\underline{\sigma}^2}{2^n}.$$

Hence it is easily seen that $\{Z_{n,k}, \mathscr{H}_{n,k}\}$ satisfy the conditions (3.3), (3.4) and (3.9) with $\rho(t) = t\overline{\sigma}^2$, $r = \underline{\sigma}^2/\overline{\sigma}^2$. Furthermore,

$$\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[|Z_{n,k}|^3] = \sum_{k=1}^{2^n} o\left(\frac{1}{2^n}\right) \to 0.$$

So, the Lindeberg's condition (3.1) is satisfied. Let $W_n(\cdot)$ be defined as in (3.8). By Theorem 3.5, $(W_n(t_1), \ldots, W_n(t_d)) \xrightarrow{d} (W_{t_1}, \ldots, W_{t_d})$. On the other hand,

$$|W_n(t) - M_t| = |M_t - M_{[2^n t]/2^n}| \xrightarrow{\mathbb{V}} 0.$$

So, (5.3) holds for all $\varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}_d)$. The proof is now completed. $\square$

6 Proofs of the central limit theorems for martingales

6.1 Proof of the central limit theorem

We give the proof of Theorem 3.1. By (3.1), there exists a sequence of positive numbers $1/2 > \epsilon_n \searrow 0$ such that

$$\epsilon_n^{-2} \sum_{k=1}^{k_n} \widehat{\mathbb{E}}[(Z_{n,k}^2 - \epsilon_n^2)^+ \mid \mathscr{H}_{n,k-1}] \xrightarrow{\mathbb{V}} 0.$$

Let $Z_{n,k}^* = (-2\epsilon_n) \vee Z_{n,k} \wedge (2\epsilon_n)$. Then

$$\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[(Z_{n,k} - Z_{n,k}^*)^2 \mid \mathscr{H}_{n,k-1}] \leqslant \sum_{k=1}^{k_n} \widehat{\mathbb{E}}[(Z_{n,k}^2 - \epsilon_n^2)^+ \mid \mathscr{H}_{n,k-1}] \xrightarrow{\mathbb{V}} 0$$

and

$$\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[|Z_{n,k} - Z_{n,k}^*| \,|\mathscr{H}_{n,k-1}] \leqslant \epsilon_n^{-1} \sum_{k=1}^{k_n} \widehat{\mathbb{E}}[(Z_{n,k}^2 - \epsilon_n^2)^+ \mid \mathscr{H}_{n,k-1}] \xrightarrow{\mathbb{V}} 0.$$

Hence, $\{Z_{n,k}^*; k = 1, \ldots, k_n\}$ satisfy the conditions (3.2)–(3.4). Furthermore, let

$$h_k = \epsilon_n^{-2} \sum_{i=1}^{k} \widehat{\mathbb{E}}[(Z_{n,k}^2 - \epsilon_n^2)^+ \mid \mathscr{H}_{n,k-1}]$$

and f be a bounded Lipschitz function such that $I\{x \leqslant \epsilon\} \leqslant f(x) \leqslant I\{x \leqslant 2\epsilon\}$. Then,

$$\begin{aligned}
&\mathbb{V}(Z_{n,k} \neq Z_{n,k}^* \text{ for some } k)\\
&= \mathbb{V}\Big(\max_{k\leqslant k_n} |Z_{n,k}| \geqslant 2\epsilon_n\Big) \leqslant \mathbb{V}\bigg(\sum_{k=1}^{k_n}[1 \wedge (Z_{n,k}^2 - \epsilon_n^2)^+] \geqslant \epsilon_n^2\bigg)\\
&\leqslant \mathbb{V}\bigg(\sum_{k=1}^{k_n}[1 \wedge (Z_{n,k}^2 - \epsilon_n^2)^+] \geqslant \epsilon_n^2, h_{k_n} \leqslant \epsilon\bigg) + \mathbb{V}(h_{k_n} \geqslant \epsilon)\\
&= \mathbb{V}\bigg(\sum_{k=1}^{k_n}[1 \wedge (Z_{n,k}^2 - \epsilon_n^2)^+] f(h_k) \geqslant \epsilon_n^2, h_{k_n} \leqslant \epsilon\bigg) + \mathbb{V}(h_{k_n} \geqslant \epsilon)\\
&\leqslant \widehat{\mathbb{E}}\bigg[\epsilon_n^{-2} \sum_{k=1}^{k_n}[1 \wedge (Z_{n,k}^2 - \epsilon_n^2)^+] f(h_k)\bigg] + \mathbb{V}(h_{k_n} \geqslant \epsilon)\\
&\leqslant \widehat{\mathbb{E}}\bigg[\epsilon_n^{-2} \sum_{k=1}^{k_n} f(h_k) \widehat{\mathbb{E}}[[1 \wedge (Z_{n,k}^2 - \epsilon_n^2)^+] \mid \mathscr{H}_{n,k-1}]\bigg] + \mathbb{V}(h_{k_n} \geqslant \epsilon)\\
&\leqslant 2\epsilon + \mathbb{V}(h_{k_n} \geqslant \epsilon) \to 0 \quad \text{as} \quad n \to \infty \quad \text{and then} \quad \epsilon \to 0.
\end{aligned}$$

It follows that for any bounded function φ,

$$\widehat{\mathrm{E}}\left[\left|\varphi\left(\sum_{k=1}^{k_n} Z_{n,k}\right)-\varphi\left(\sum_{k=1}^{k_n} Z_{n,k}^*\right)\right|\right] \leqslant 2\sup_x|\varphi(x)|\mathbb{V}(Z_{n,k}\neq Z_{n,k}^* \text{ for some } k)\to 0.$$

So, without loss of generality we can assume that there is a positive sequence $1\geqslant \epsilon_n\searrow 0$ such that $|Z_{n,k}|\leqslant \epsilon_n$, $k=1,\ldots,k_n$.

Denote $S_0=0$, $\delta_0=0$, $S_k=\sum_{i=1}^k Z_{n,i}$, $a_{n,k}^2=\widehat{\mathrm{E}}[Z_{n,k}^2\,|\,\mathscr{H}_{n,k-1}]$, $\delta_k=\sum_{i=1}^k a_{n,i}^2$, $k=1,\ldots,k_n$. Let $f(x)$ be a function with bounded derivative such that $I\{x\leqslant \rho+\epsilon/2\}\leqslant f(x)\leqslant I\{x\leqslant\rho+\epsilon\}$. Let $Z_{n,k}^*=Z_{n,k}f(\delta_k)$. Then $\{Z_{n,k}^*;k=1,\ldots,k_n\}$ satisfy the conditions (3.2)–(3.4), and

$$\sum_{k=1}^{k_n}\widehat{\mathrm{E}}[(Z_{n,k}^*)^2\,|\,\mathscr{H}_{n,k-1}]=\delta_{k_n}^*, \tag{6.1}$$

where $\delta_{k_n}^*=\sum_{k=1}^{k_n} f(\delta_k)\widehat{\mathrm{E}}[Z_{n,k}^2\,|\,\mathscr{H}_{n,k-1}]\leqslant \rho+\epsilon$. The above equalities hold in L_1 by the property (a) of the operators $\widehat{\mathrm{E}}_{n,k}$ and then hold in any L_q by Lemma 4.2(2) since $\delta_{k_n}^*$ is bounded in L_q by Lemma 4.3(4). Furthermore,

$$\{Z_{n,k}\neq Z_{n,k}^* \text{ for some } k\}\subset\left\{\sum_{k=1}^{k_n}a_{n,k}^2>\rho+\frac{\epsilon}{2}\right\}.$$

So, without loss of generality we can further assume that

$$\delta_{k_n}=\sum_{k=1}^{k_n}\widehat{\mathrm{E}}[Z_{n,k}^2\,|\,\mathscr{H}_{n,k-1}]\leqslant\rho+\epsilon$$

in L_1. Similarly, we can assume

$$\chi_{k_n}:=\sum_{k=1}^{k_n}\{|\widehat{\mathrm{E}}[Z_{n,k}\,|\,\mathscr{H}_{n,k-1}]|+|\widehat{\mathcal{E}}[Z_{n,k}\,|\,\mathscr{H}_{n,k-1}]|\}<\epsilon<1$$

in L_1. Lemma 4.3(4) implies that all the random variables considered above and in the sequel are bounded in L_p for all $p>0$.

Now, by Theorem 4.4,

$$\widehat{\mathrm{E}}\left[\max_{k\leqslant k_n}\left(\sum_{i=1}^k Z_{n,i}\right)^2\right]\leqslant 256\widehat{\mathrm{E}}[\delta_{k_n}]+256\widehat{\mathrm{E}}[\chi_{k_n}^2]. \tag{6.2}$$

If $\rho=0$, then $\delta_{k_n}\xrightarrow{\mathbb{V}}0$. Note $\chi_{k_n}\xrightarrow{\mathbb{V}}0$. So, $\widehat{\mathrm{E}}[(\sum_{i=1}^{k_n}Z_{n,i})^2]\to 0$, and then the result is obvious. In the sequel, we suppose $\rho\neq 0$. Let φ be a bounded continuous function with bounded derivation. Without loss of generality, we assume $|\varphi(x)|\leqslant 1$. We want to show that

$$\widehat{\mathrm{E}}[\varphi(S_{k_n})]\to\widetilde{\mathrm{E}}[\varphi(\sqrt{\rho}\xi)]. \tag{6.3}$$

In the classical probability space, the above convergence is usually shown by verifying the convergence of the related characteristic functions (see Hall and Heyde [6, pp. 60–63] and Pollard [23, pp. 171–174]). As shown by Hu and Li [8], the characteristic function cannot determine the distribution of random variables in the sub-linear expectation space. Peng [18, 20] developed a method to show the above convergence for independent random variables. Here we promote Peng's argument such that it is also valid for martingale differences which give also a new normal approximation method for classical martingale differences instead of the characteristic function.

Now, for a small but fixed $h>0$, let $V(t,x)$ be the unique viscosity solution of the following equation:

$$\partial_t V+G(\partial_{xx}^2V)=0,\quad (t,x)\in[0,\rho+h]\times\mathbb{R},\quad V\,|_{t=\rho+h}=\varphi(x), \tag{6.4}$$

where $G(\alpha)=\frac{1}{2}(\alpha^+-r\alpha^-)$. Then by the interior regularity of V,

$$\|V\|_{C^{1+\alpha/2,2+\alpha}([0,\rho+h/2]\times R)}<\infty,\quad \text{for some}\ \ \alpha\in(0,1). \tag{6.5}$$

According to the definition of G-normal distribution, we have $V(t,x)=\widetilde{\mathrm{E}}[\varphi(x+\sqrt{\rho+h-t}\xi)]$ where $\xi\sim N(0,[r,1])$ under $\widetilde{\mathrm{E}}$. In particular,

$$V(h,0)=\widetilde{\mathrm{E}}[\varphi(\sqrt{\rho}\xi)],\quad V(\rho+h,x)=\varphi(x).$$

It is obvious that, if $\varphi(\cdot)$ is a global Lipschitz function, i.e., $|\varphi(x)-\varphi(y)|\leqslant C|x-y|$, then $|V(t,x)-V(t,y)|\leqslant C|x-y|$ and

$$|V(t,x)-V(s,x)|\leqslant C\widetilde{\mathrm{E}}[|\xi|]|\sqrt{\rho+h-t}-\sqrt{\rho+h-s}|\leqslant C\widetilde{\mathrm{E}}[|\xi|]|t-s|^{1/2}.$$

So, $|\partial_x V(t,x)|\leqslant C$, $|\partial_t V(t,x)|\leqslant C\widetilde{\mathrm{E}}[|\xi|]/\sqrt{\rho+h-t}$, $|V(\rho+h,x)-V(\rho,x)|\leqslant C\widetilde{\mathrm{E}}[|\xi|]\sqrt{h}$ and $|V(h,0)-V(0,0)|\leqslant C\widetilde{\mathrm{E}}[|\xi|]\sqrt{h}$. Following the proof of Lemma 5.4 of [20], it is sufficient to show that

$$\lim_{n\to\infty}\widehat{\mathrm{E}}[V(\rho,S_{k_n})]=V(0,0). \tag{6.6}$$

As we have shown, we can assume that $\delta_{k_n}\leqslant\rho+h/4=:h_0<2\rho$ in L_1. It is obvious that $|V(t,x)|\leqslant 1$ and

$$\widehat{\mathrm{E}}[|V(\rho,S_{k_n})-V(\delta_{k_n}\wedge h_0,S_{k_n})|]\leqslant C\widehat{\mathrm{E}}[|\delta_{k_n}\wedge h_0-\rho|^{1/2}]\to 0.$$

Hence, it is sufficient to show that

$$\lim_{n\to\infty}\widehat{\mathrm{E}}[V(\delta_{k_n}\wedge h_0,S_{k_n})]=V(0,0). \tag{6.7}$$

Let $\widetilde{\delta}_i=\delta_i\wedge h_0$. Then $\widetilde{\delta}_{i+1}-\widetilde{\delta}_i\leqslant a_{n,i+1}^2$, $|\widetilde{\delta}_i|\leqslant h_0=\rho+h/4$. It follows that

$$|\partial_x V(\widetilde{\delta}_i,S_i)|\leqslant C,\quad |\partial_t V(\widetilde{\delta}_i,S_i)|\leqslant\frac{C}{\sqrt{h}}\leqslant C.$$

Also, by the fact that $\partial_{xx}V$ is uniformly α-Hölder continuous in x and $\alpha/2$-Hölder continuous in t on $[0,\rho+h/2]\times\mathbb{R}$, it follows that

$$|\partial_{xx}^2 V(\widetilde{\delta}_i,S_i)|\leqslant|\partial_{xx}^2 V(0,0)|+C|\widetilde{\delta}_i|^{\alpha/2}+C|S_i|^{\alpha}\leqslant C+C|S_i|^{\alpha}.$$

Now, applying Taylor's expansion, we have

$$\begin{aligned}&V(\widetilde{\delta}_{k_n},S_{k_n})-V(0,0)\\&\quad=\sum_{i=0}^{k_n-1}\{[V(\widetilde{\delta}_{i+1},S_{i+1})-V(\widetilde{\delta}_i,S_{i+1})]+[V(\widetilde{\delta}_i,S_{i+1})-V(\widetilde{\delta}_i,S_i)]\}=:\sum_{i=0}^{k_n-1}\{I_n^i+J_n^i\}\end{aligned}$$

with

$$\begin{aligned}J_n^i&=\partial_t V(\widetilde{\delta}_i,S_i)(\widetilde{\delta}_{i+1}-\widetilde{\delta}_i)+\frac{1}{2}\partial_{xx}^2 V(\widetilde{\delta}_i,S_i)Z_{n,i+1}^2+\partial_x V(\widetilde{\delta}_i,S_i)Z_{n,i+1}\\&=\left\{a_{n,i+1}^2\partial_t V(\widetilde{\delta}_i,S_i)+\frac{1}{2}\partial_{xx}^2 V(\widetilde{\delta}_i,S_i)Z_{n,i+1}^2-\frac{1}{2}(\partial_{xx}^2 V(\widetilde{\delta}_i,S_i))^-(ra_{n,i+1}^2-\widehat{\mathcal{E}}[Z_{n,i+1}^2\,|\,\mathscr{H}_{n,i}])\right\}\\&\quad+\{\partial_x V(\widetilde{\delta}_i,S_i)Z_{n,i+1}\}+\left\{\frac{1}{2}(\partial_{xx}^2 V(\widetilde{\delta}_i,S_i))^-(ra_{n,i+1}^2-\widehat{\mathcal{E}}[Z_{n,i+1}^2\,|\,\mathscr{H}_{n,i}])\right\}\\&\quad+\{\partial_t V(\widetilde{\delta}_i,S_i)(\widetilde{\delta}_{i+1}-\widetilde{\delta}_i-a_{n,i+1}^2)\}\\&=:J_{n,1}^i+J_{n,2}^i+J_{n,3}^i+J_{n,4}^i\end{aligned}$$

and

$$I_n^i=(\widetilde{\delta}_{i+1}-\widetilde{\delta}_i)[(\partial_t V(\widetilde{\delta}_i+\gamma(\widetilde{\delta}_{i+1}-\widetilde{\delta}_i),S_{i+1})-\partial_t V(\widetilde{\delta}_i,S_{i+1}))$$

$$+ (\partial_t V(\widetilde{\delta}_i, S_{i+1}) - \partial_t V(\widetilde{\delta}_i, S_i))]$$
$$+ \frac{1}{2}[\partial^2_{xx} V(\widetilde{\delta}_i, S_i + \beta Z_{n,i+1}) - \partial^2_{xx} V(\widetilde{\delta}_i, S_i)] Z^2_{n,i+1},$$

where γ and β are between 0 and 1. Thus

$$\begin{aligned}
&\bigg|\widehat{\mathrm{E}}[V(\widetilde{\delta}_{k_n}, S_{k_n})] - V(0,0) - \widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n-1}(J^i_{n,1} + J^i_{n,2})\bigg]\bigg| \\
&\quad \leqslant \widehat{\mathrm{E}}\bigg[\bigg|V(\widetilde{\delta}_{k_n}, S_{k_n}) - V(0,0) - \sum_{i=0}^{k_n-1}(J^i_{n,1} + J^i_{n,2})\bigg|\bigg] \\
&\quad \leqslant \widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n-1}(|I^i_n| + |J^i_{n,3}| + |J^i_{n,4}|)\bigg]. \qquad (6.8)
\end{aligned}$$

For $J^i_{n,1}$, it follows that

$$\widehat{\mathrm{E}}[J^i_{n,1} \mid \mathscr{H}_{n,i}] = [\partial_t V(\widetilde{\delta}_i, S_i) + G(\partial^2_{xx} V(\widetilde{\delta}_i, S_i))] a^2_{n,i+1} = 0 \quad \text{in} \quad L_1.$$

It follows that

$$\widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n-1} J^i_{n,1}\bigg] = \widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n-2} J^i_{n,1} + \widehat{\mathrm{E}}[J^{k_n-1}_{n,1} \mid \mathscr{H}_{n,k_n-1}]\bigg] = \widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n-2} J^i_{n,1}\bigg] = \cdots = 0. \qquad (6.9)$$

For $J^i_{n,2}$, we denote $\widetilde{J}^i_{n,2} = |\partial_x V(\widetilde{\delta}_i, S_i)|(|\widehat{\mathrm{E}}[Z_{n,i+1} \mid \mathscr{H}_{n,i}]| + |\widehat{\mathcal{E}}[Z_{n,i+1} \mid \mathscr{H}_{n,i}]|)$. Then

$$\begin{aligned}
&\widehat{\mathrm{E}}[J^i_{n,2} - \widetilde{J}^i_{n,2} \mid \mathscr{H}_{n,i}] \\
&\quad = \widehat{\mathrm{E}}[J^i_{n,2} \mid \mathscr{H}_{n,i}] - \widetilde{J}^i_{n,2} \\
&\quad \leqslant (\partial_x V(\widetilde{\delta}_i, S_i))^+ \widehat{\mathrm{E}}[Z_{n,i+1} \mid \mathscr{H}_{n,i}] - (\partial_x V(\widetilde{\delta}_i, S_i))^- \widehat{\mathcal{E}}[Z_{n,i+1} \mid \mathscr{H}_{n,i}] - \widetilde{J}^i_{n,2} \leqslant 0 \quad \text{in} \quad L_1.
\end{aligned}$$

Similarly $\widehat{\mathrm{E}}[-J^i_{n,2} - \widetilde{J}^i_{n,2} \mid \mathscr{H}_{n,i}] \leqslant 0$ in L_1. It follows that

$$\begin{aligned}
\widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n-1}(\pm J^i_{n,2} - \widetilde{J}^i_{n,2})\bigg] &= \widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n-2}(\pm J^i_{n,2} - \widetilde{J}^i_{n,2}) + \widehat{\mathrm{E}}[\pm J^{k_n-1}_{n,1} - \widetilde{J}^{k_n-1}_{n,2} \mid \mathscr{H}_{n,k_n-1}]\bigg] \\
&\leqslant \widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n-2}(\pm J^i_{n,2} - \widetilde{J}^i_{n,2})\bigg] \leqslant \cdots \leqslant 0. \qquad (6.10)
\end{aligned}$$

Hence

$$\widehat{\mathrm{E}}\bigg[\pm \sum_{i=0}^{k_n-1} J^i_{n,2}\bigg] \leqslant \widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n-1}(\pm J^i_{n,2} - \widetilde{J}^i_{n,2})\bigg] + \widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n-1} \widetilde{J}^i_{n,2}\bigg] \leqslant \widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n-1} \widetilde{J}^i_{n,2}\bigg]. \qquad (6.11)$$

Note $|\partial_x V(\widetilde{\delta}_i, S_i)| \leqslant C$, $\chi_{k_n} \xrightarrow{\mathbb{V}} 0$ and $\chi_{k_n} \leqslant 1$ in any L_p. Combining (6.9) and (6.11) we have

$$\bigg|\widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n-1}(J^i_{n,1} + J^i_{n,2})\bigg]\bigg| \leqslant \widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n-1} \widetilde{J}^i_{n,2}\bigg] \leqslant C\widehat{\mathrm{E}}[\chi_{k_n}] \to 0.$$

For $J^i_{n,3}$, it is easily seen that

$$\sum_{i=0}^{k_n-1} |J^i_{n,3}| \leqslant C\Big(1 + \max_{i \leqslant k_n} |S_i|^\alpha\Big) \sum_{i=1}^{k_n} |r a^2_{n,i} - \widehat{\mathcal{E}}[Z^2_{n,i} \mid \mathscr{H}_{n,i-1}]|. \qquad (6.12)$$

Write $\beta_{k_n} = \sum_{i=1}^{k_n} |r a^2_{n,i} - \widehat{\mathcal{E}}[Z^2_{n,i} \mid \mathscr{H}_{n,i-1}]|$. Note that

$$\beta_{k_n} \xrightarrow{\mathbb{V}} 0 \quad \text{and} \quad \beta_{k_n} \leqslant 2\delta_{k_n} \leqslant 2h_0 \quad \text{in any} \quad L_p$$

and $\widehat{\mathbb{E}}[\max_{i\leqslant k_n}|S_i|^2]\leqslant 256\{\widehat{\mathbb{E}}[\delta_{k_n}]+\widehat{\mathbb{E}}[\chi^2_{k_n}]\}\leqslant 256(h_0+1)$ by (6.2). So

$$\widehat{\mathbb{E}}\bigg[\sum_{i=0}^{k_n-1}|J_{n,3}^i|\bigg]\leqslant C\Big(\widehat{\mathbb{E}}\Big[\Big(1+\max_{i\leqslant k_n}|S_i|^\alpha\Big)^2\Big]\Big)^{1/2}(\widehat{\mathbb{E}}[\beta^2_{k_n}])^{1/2}\to 0.$$

For $J_{n,4}^i$, note that $|\widetilde{\delta}_{i+1}-\widetilde{\delta}_i-a^2_{n,i+1}|\leqslant a^2_{n,i+1}$, and $\widetilde{\delta}_{i+1}-\widetilde{\delta}_i-a^2_{n,i+1}=\delta_{i+1}-\delta_i-a^2_{n,i+1}=0$ when $\delta_{k_n}\leqslant h_0$. It follows that

$$\widehat{\mathbb{E}}\bigg[\sum_{i=0}^{k_n-1}|J_{n,4}^i|\bigg]\leqslant C\widehat{\mathbb{E}}[\delta_{k_n}I\{\delta_{k_n}>h_0\}]\leqslant C(\widehat{\mathbb{E}}[\delta^2_{k_n}])^{1/2}(\mathbb{V}(\delta_{k_n}>h_0))^{1/2}=0.$$

For I_n^i, note both $\partial_t V$ and $\partial_{xx}V$ are uniformly α-Hölder continuous in x and $\alpha/2$-Hölder continuous in t on $[0,\rho+h/2]\times\mathbb{R}$. Without loss of generality, we assume $\alpha<\tau$. Also, $\widetilde{\delta}_{i+1}-\widetilde{\delta}_i\leqslant a_{n,i+1}$. We then have

$$\begin{aligned}|I_n^i|&\leqslant C\big|a_{n,i+1}\big|^{2+\alpha}+Ca^2_{n,i+1}|Z_{n,i+1}|^\alpha+|Z_{n,i+1}|^{2+\alpha}\\&\leqslant C\epsilon_n^\alpha a^2_{n,i+1}+C\epsilon_n^\alpha Z^2_{n,i+1}=C\epsilon_n^\alpha a^2_{n,i+1}+C\epsilon_n^\alpha(Z^2_{n,i+1}-a^2_{n,i+1})\end{aligned}$$

in any L_q by Lemma 4.2. So

$$\sum_{i=0}^{k_n-1}|I_n^i|\leqslant 2C\epsilon_n^\alpha+C\epsilon_n^\alpha\sum_{i=1}^{k_n}(Z^2_{n,i}-a^2_{n,i})\quad\text{in}\quad L_1,\tag{6.13}$$

by noting $\sum_{i=1}^{k_n}a^2_{n,i}\leqslant 2\rho$ in L_1, where the sub-linear expectation under $\widehat{\mathbb{E}}$ of the last term is zero. It follows that

$$\widehat{\mathbb{E}}\bigg[\sum_{i=0}^{k_n-1}|I_n^i|\bigg]\leqslant 2C\epsilon_n^\alpha\to 0.$$

(6.7) is proved. Hence, (6.3) holds for any bounded function φ with bounded derivative.

If φ is a bounded and uniformly continuous function, we define a function φ_δ as a convolution of φ and the density of a normal distribution $N(0,\delta)$, i.e.,

$$\varphi_\delta=\varphi*\psi_\delta\quad\text{with}\quad\psi_\delta(x)=\frac{1}{\sqrt{2\pi\delta}}\exp\bigg\{-\frac{x^2}{2\delta}\bigg\},$$

where $\varphi*\psi_\delta$ denotes the convolution of φ and ψ_δ. Then

$$|\varphi_\delta'(x)|\leqslant\sup_x|\varphi(x)|\delta^{-1/2}\quad\text{and}\quad\sup_x|\varphi_\delta(x)-\varphi(x)|\to 0$$

as $\delta\to 0$. Hence, (6.3) holds for any bounded and uniformly continuous function φ.

Now, for a bounded continuous function φ and given a number $N>1$, we define

$$\varphi_1(x)=\varphi((-N)\vee(x\wedge N)).$$

Then, φ_1 is a bounded and uniformly continuous function, and $|\varphi(x)-\varphi_1(x)|\leqslant CI\{|x|>N\}$. So

$$\begin{aligned}&\sup_n\widehat{\mathbb{E}}\bigg[\bigg|\varphi\bigg(\sum_{k=1}^{k_n}Z_{n,k}\bigg)-\varphi_1\bigg(\sum_{k=1}^{k_n}Z_{n,k}\bigg)\bigg|\bigg]\\&\quad\leqslant C\mathbb{V}\bigg(\bigg|\sum_{k=1}^{k_n}Z_{n,k}\bigg|>N\bigg)\leqslant CN^{-2}\sup_n\widehat{\mathbb{E}}\bigg[\bigg(\sum_{k=1}^{k_n}Z_{n,k}\bigg)^2\bigg]\\&\quad\leqslant CN^{-2}\sup_n(\widehat{\mathbb{E}}[\delta_{k_n}]+\widehat{\mathbb{E}}[\chi^2_{k_n}])\leqslant 3CN^{-2}\to 0\quad\text{as}\quad N\to\infty\end{aligned}$$

by (6.2). The proof of Theorem 3.1 is now completed.

6.2 Proof of the functional central limit theorem

For proving the functional central limit theorem, we need a more lemma.

Lemma 6.1. *Suppose that the operators* $\widehat{\mathbb{E}}_{n,k}$ *satisfy* (a) *and* (b)*,* $\boldsymbol{X}_n \in \mathscr{H}_{n,k_n'} \subset \mathscr{H}$ *is a* d_1*-dimensional random vector, and* $\boldsymbol{Y}_n \in \mathscr{H}$ *is a* d_2*-dimensional random vector. Write* $\mathscr{H}_n = \mathscr{H}_{n,k_n'}$*. Assume that* $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X}$*, and for any bounded Lipschitz function* $\varphi(\boldsymbol{x},\boldsymbol{y}) : \mathbb{R}_{d_1} \bigotimes \mathbb{R}_{d_2} \to \mathbb{R}$*,*

$$\widehat{\mathbb{E}}[|\widehat{\mathbb{E}}[\varphi(\boldsymbol{x},\boldsymbol{Y}_n)\,|\,\mathscr{H}_n] - \widetilde{\mathbb{E}}[\varphi(\boldsymbol{x},\boldsymbol{Y})]|] \to 0, \quad \forall\, \boldsymbol{x}, \tag{6.14}$$

where $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are two random vectors in a sub-linear expectation space* $(\Omega, \mathscr{H}, \widetilde{\mathbb{E}})$ *with* $\widetilde{\mathbb{V}}(\|\boldsymbol{X}\| > \lambda) \to 0$ *and* $\widetilde{\mathbb{V}}(\|\boldsymbol{Y}\| > \lambda) \to 0$ *as* $\lambda \to \infty$*. Then*

$$(\boldsymbol{X}_n, \boldsymbol{Y}_n) \xrightarrow{d} (\widetilde{\boldsymbol{X}}, \widetilde{\boldsymbol{Y}}), \tag{6.15}$$

where $\widetilde{\boldsymbol{Y}}$ *is independent to* $\widetilde{\boldsymbol{X}}$*,* $\widetilde{\boldsymbol{X}} \overset{d}{=} \boldsymbol{X}$ *and* $\widetilde{\boldsymbol{Y}} \overset{d}{=} \boldsymbol{Y}$*.*

Proof. Suppose $\varphi(\boldsymbol{x},\boldsymbol{y}) : \mathbb{R}_{d_1} \bigotimes \mathbb{R}_{d_2} \to \mathbb{R}$ is a bounded continuous function. We want to show that

$$\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}_n,\boldsymbol{Y}_n)] \to \widetilde{\mathbb{E}}[\varphi(\widetilde{\boldsymbol{X}},\widetilde{\boldsymbol{Y}})]. \tag{6.16}$$

First we assume that $\varphi(\boldsymbol{x},\boldsymbol{y})$ is a bounded Lipschitz function. Without loss of generality, we assume $0 \leqslant \varphi(\boldsymbol{x},\boldsymbol{y}) \leqslant 1$ and $|\varphi(\boldsymbol{x}_1,\boldsymbol{y}_1) - \varphi(\boldsymbol{x}_2,\boldsymbol{y}_2)| \leqslant \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| + \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|$. Let $g_n(\boldsymbol{x}) = \widehat{\mathbb{E}}[\varphi(\boldsymbol{x},\boldsymbol{Y}_n)\,|\,\mathscr{H}_n]$ and $g(\boldsymbol{x}) = \widetilde{\mathbb{E}}[\varphi(\boldsymbol{x},\widetilde{\boldsymbol{Y}})]$. Then

$$|g(\boldsymbol{x}_1) - g(\boldsymbol{x}_2)| \leqslant \widetilde{\mathbb{E}}[|\varphi(\boldsymbol{x}_1,\widetilde{\boldsymbol{Y}}) - \varphi(\boldsymbol{x}_2,\widetilde{\boldsymbol{Y}})|] \leqslant \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$$

and

$$|\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}_n,\boldsymbol{Y}_n)\,|\,\mathscr{H}_n] - g_n(\boldsymbol{x})| \leqslant \widehat{\mathbb{E}}[|\varphi(\boldsymbol{X}_n,\boldsymbol{Y}_n) - \varphi(\boldsymbol{x},\boldsymbol{Y}_n)|\,|\,\mathscr{H}_n] \leqslant \|\boldsymbol{X}_n - \boldsymbol{x}\| \quad \text{in} \quad L_1$$

by Lemma 4.3. We use an argument of Hu et al. [7] (see Proposition 3.4) to approximate the function $\varphi(\boldsymbol{x},\boldsymbol{y})$. For fixed $N \geqslant 1$, denote $B_N(0) = \{\boldsymbol{x} : \|x\| \leqslant N\}$. By partition of the unity theorem, there exist $h_i \in C_{b,\mathrm{Lip}}(\mathbb{R}^{d_1})$, $i = 1, \ldots, k_N$, such that $0 \leqslant h_i(\boldsymbol{x}) \leqslant 1$, $I_{B_N(0)} \leqslant \sum_{i=1}^{k_n} h_i(\boldsymbol{x}) \leqslant 1$, and the diameter of support $\lambda(\mathrm{supp}(h_i)) \leqslant 1/N$. Choose $\boldsymbol{x}_i$ such that $h_i(\boldsymbol{x}_i) > 0$. Then

$$\begin{aligned}
&\left|\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}_n,\boldsymbol{Y}_n)\,|\,\mathscr{H}_n] - \sum_{i=1}^{k_N} h_i(\boldsymbol{X}_n) g_n(\boldsymbol{x}_i)\right| \\
&\quad \leqslant \sum_{i=1}^{k_N} h_i(\boldsymbol{X}_n) |\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}_n,\boldsymbol{Y}_n)\,|\,\mathscr{H}_n] - g_n(x_i)| + \left(1 - \sum_{i=1}^{k_N} h_i(\boldsymbol{X}_n)\right) |\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}_n,\boldsymbol{Y}_n)\,|\,\mathscr{H}_n]| \\
&\quad \leqslant \sum_{i=1}^{k_N} h_i(\boldsymbol{X}_n) \|\boldsymbol{X}_n - \boldsymbol{x}_i\| + \left(1 - \sum_{i=1}^{k_N} h_i(\boldsymbol{X}_n)\right) \leqslant \frac{1}{N} + \left(1 - \sum_{i=1}^{k_N} h_i(\boldsymbol{X}_n)\right) \quad \text{in} \quad L_1.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left|\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}_n,\boldsymbol{Y}_n)] - \widehat{\mathbb{E}}\left[\sum_{i=1}^{k_N} h_i(\boldsymbol{X}_n) g_n(x_i)\right]\right| \\
&\quad = \left|\widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}_n,\boldsymbol{Y}_n)\,|\,\mathscr{H}_n]] - \widehat{\mathbb{E}}\left[\sum_{i=1}^{k_N} h_i(\boldsymbol{X}_n) g_n(x_i)\right]\right| \\
&\quad \leqslant \widehat{\mathbb{E}}\left[\left|\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}_n,\boldsymbol{Y}_n)\,|\,\mathscr{H}_n] - \sum_{i=1}^{k_N} h_i(\boldsymbol{X}_n) g_n(\boldsymbol{x}_i)\right|\right] \\
&\quad \leqslant \frac{1}{N} + \widehat{\mathbb{E}}\left[1 - \sum_{i=1}^{k_N} h_i(\boldsymbol{X}_n)\right].
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\bigg|\widetilde{\mathrm{E}}[\varphi(\widetilde{\boldsymbol{X}},\widetilde{\boldsymbol{Y}})]-\widetilde{\mathrm{E}}\bigg[\sum_{i=1}^{k_N}h_i(\widetilde{\boldsymbol{X}})g(\boldsymbol{x}_i)\bigg]\bigg|\\
&=\bigg|\widetilde{\mathrm{E}}[g(\widetilde{\boldsymbol{X}})]-\widetilde{\mathrm{E}}\bigg[\sum_{i=1}^{k_N}h_i(\widetilde{\boldsymbol{X}})g(\boldsymbol{x}_i)\bigg]\bigg|\\
&\leqslant\widetilde{\mathrm{E}}\bigg[\bigg|g(\widetilde{\boldsymbol{X}})-\sum_{i=1}^{k_N}h_i(\widetilde{\boldsymbol{X}})g(\boldsymbol{x}_i)\bigg|\bigg]\leqslant\frac{1}{N}+\widetilde{\mathrm{E}}\bigg[1-\sum_{i=1}^{k_N}h_i(\widetilde{\boldsymbol{X}})\bigg].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\bigg|\widehat{\mathrm{E}}\bigg[\sum_{i=1}^{k_N}h_i(\boldsymbol{X}_n)g_n(\boldsymbol{x}_i)\bigg]-\widehat{\mathrm{E}}\bigg[\sum_{i=1}^{k_N}h_i(\boldsymbol{X}_n)g(\boldsymbol{x}_i)\bigg]\bigg|\\
&\leqslant\sum_{i=1}^{k_N}\widehat{\mathrm{E}}[|g_n(\boldsymbol{x}_i)-g(\boldsymbol{x}_i)|]\quad\text{as}\quad n\to\infty
\end{aligned}$$

by (6.14), and

$$\begin{aligned}
&\widehat{\mathrm{E}}\bigg[\sum_{i=1}^{k_N}h_i(\boldsymbol{X}_n)g(\boldsymbol{x}_i)\bigg]\to\widetilde{\mathrm{E}}\bigg[\sum_{i=1}^{k_N}h_i(\widetilde{\boldsymbol{X}})g(\boldsymbol{x}_i)\bigg],\\
&\widehat{\mathrm{E}}\bigg[1-\sum_{i=1}^{k_N}h_i(\boldsymbol{X}_n)\bigg]\to\widetilde{\mathrm{E}}\bigg[1-\sum_{i=1}^{k_N}h_i(\widetilde{\boldsymbol{X}})\bigg]
\end{aligned}$$

as $n\to\infty$, by the fact that $\boldsymbol{X}_n\xrightarrow{d}\widetilde{\boldsymbol{X}}$. Combining the above arguments, we have

$$\begin{aligned}
&\limsup_{n\to\infty}|\widehat{\mathrm{E}}[\varphi(\boldsymbol{X}_n,\boldsymbol{Y}_n)]-\widetilde{\mathrm{E}}[\varphi(\widetilde{\boldsymbol{X}},\widetilde{\boldsymbol{Y}})]|\\
&\leqslant\frac{2}{N}+2\widetilde{\mathrm{E}}\bigg[1-\sum_{i=1}^{k_N}h_i(\widetilde{\boldsymbol{X}})\bigg]\leqslant\frac{2}{N}+2\widetilde{\mathbb{V}}(\|\boldsymbol{X}\|>N)\to0\quad\text{as}\quad N\to\infty.
\end{aligned}$$

Hence (6.16) is proved for any bounded Lipschitz function φ. For a bounded and uniformly continuous function φ, we define

$$\varphi_\delta=\varphi*\psi_\delta\quad\text{with}\quad\psi_\delta(\boldsymbol{x},\boldsymbol{y})=\frac{1}{(2\pi\delta)^{(d_1+d_2)/2}}\exp\bigg\{-\frac{\sum_{i=1}^{d_1}x_i^2+\sum_{j=1}^{d_2}y_j^2}{2\delta}\bigg\}.$$

Then φ_δ is a bounded Lipschitz function with $\sup_{\boldsymbol{x},\boldsymbol{y}}|\varphi_\delta(\boldsymbol{x},\boldsymbol{y})-\varphi(\boldsymbol{x},\boldsymbol{y})|\to0$ as $\delta\to0$. Hence, (6.16) holds for any bounded and uniformly continuous function φ. Finally, let $\varphi(\boldsymbol{x},\boldsymbol{y})$ be a bounded continuous function with $|\varphi(\boldsymbol{x},\boldsymbol{y})|\leqslant M$. Let $\lambda>0$. For $\boldsymbol{x}=(x_1,\ldots,x_d)$, denote $\boldsymbol{x}_\lambda=((-\lambda)\vee(x_1\wedge\lambda)\lambda,\ldots,(-\lambda)\vee(x_{d_1}\wedge\lambda))$ and define $\boldsymbol{y}_\lambda$ similarly. Let $\varphi_\lambda(\boldsymbol{x},\boldsymbol{y})=\varphi(\boldsymbol{x}_\lambda,\boldsymbol{y}_\lambda)$. Then φ_λ is a bounded uniformly continuous function with

$$|\varphi_\lambda(\boldsymbol{x},\boldsymbol{y})-\varphi(\boldsymbol{x},\boldsymbol{y})|\leqslant2MI\{\|\boldsymbol{x}\|>\lambda\}+2MI\{\|\boldsymbol{y}\|>\lambda\}.$$

It follows that

$$\begin{aligned}
&\limsup_{n\to\infty}|\widehat{\mathrm{E}}[\varphi(\boldsymbol{X}_n,\boldsymbol{Y}_n)]-\widetilde{\mathrm{E}}[\varphi(\widetilde{\boldsymbol{X}},\widetilde{\boldsymbol{Y}})]|\\
&\leqslant\limsup_{n\to\infty}|\widehat{\mathrm{E}}[\varphi_\lambda(\boldsymbol{X}_n,\boldsymbol{Y}_n)]-\widetilde{\mathrm{E}}[\varphi_\lambda(\widetilde{\boldsymbol{X}},\widetilde{\boldsymbol{Y}})]|\\
&\quad+2M\limsup_{n\to\infty}\{\mathbb{V}(\|\boldsymbol{X}_n\|>\lambda)+\mathbb{V}(\|\boldsymbol{Y}_n\|>\lambda)\}\\
&\quad+2M\{\widetilde{\mathbb{V}}(\|\boldsymbol{X}\|>\lambda)+\widetilde{\mathbb{V}}(\|\boldsymbol{Y}\|>\lambda)\}
\end{aligned}$$

$$\leqslant 4M\left\{\widetilde{\mathbb{V}}\left(\|\boldsymbol{X}\|>\frac{\lambda}{2}\right)+\widetilde{\mathbb{V}}\left(\|\boldsymbol{Y}\|>\frac{\lambda}{2}\right)\right\}\to 0 \quad \text{as} \quad \lambda\to\infty.$$

The proof is completed. □

Remark 6.2. In the original proofs of Lemma 6.1 and Theorem 3.5, we need an additional assumption on the operators $\widehat{\mathrm{E}}_{n,k}$ as follows:

(a′) If $\boldsymbol{X}=(X_1,\dots,X_d)\in\mathscr{H}_{n,k}$, $Z\in\mathscr{H}$ and $\varphi(\boldsymbol{x},y)$ is a bounded Lipschitz function, then

$$\widehat{\mathrm{E}}[\varphi(\boldsymbol{X},Z)]=\widehat{\mathrm{E}}[\widehat{\mathrm{E}}_{n,k}[\varphi(\boldsymbol{x},Z)]\,|_{\boldsymbol{x}=\boldsymbol{X}}].$$

We thank one of the referees mentioning us Proposition 3.4 of Hu et al. [7] which helps us to remove this condition, though we fail to verify this proposition when the point by point monotonicity of the conditional sub-linear expectation (see Hu et al. [7, Definition 3.1(1)]) is replaced by the L_1-monotonicity (see Lemma 4.3(1)).

Proof of Theorem 3.5. With the same argument as that at the beginning of the proof of Theorem 3.1, we can assume that

$$\chi_{k_n}:=\sum_{k=1}^{k_n}\{|\widehat{\mathrm{E}}[Z_{n,k}\,|\,\mathscr{H}_{n,k-1}]|+|\widehat{\mathcal{E}}[Z_{n,k}\,|\,\mathscr{H}_{n,k-1}]|\}<1$$

in L_1, $\delta_{k_n}=\sum_{k=1}^{k_n}\widehat{\mathrm{E}}[Z_{n,k}^2\,|\,\mathscr{H}_{n,k-1}]\leqslant 2\rho(1)$ in L_1 and $|Z_{n,k}|\leqslant\epsilon_n$, $k=1,\dots,k_n$, with a sequence $0<\epsilon_n\to 0$. Let $0<t_1<t_2\leqslant 1$. Consider $\{Z_{n,k}^*:=Z_{n,\tau_n(t_1)+k};k=1,\dots,k_n^*\}$, $S_i^*=\sum_{k=1}^{i}Z_{n,\tau_n(t_1)+k}$ and $k_n^*=\tau_n(t_2)-\tau_n(t_1)$. Then

$$S_{k_n^*}^*=S_{n,\tau_n(t_2)}-S_{n,\tau_n(t_1)}=\sum_{k=1}^{k_n^*}Z_{n,\tau_n(t_1)+k}$$

and

$$\sum_{k=1}^{k_n^*}\widehat{\mathrm{E}}[Z_{n,\tau_n(t_1)+k}^2\,|\,\mathscr{H}_{n,\tau_n(t_1)+k-1}]\xrightarrow{\mathbb{V}}\rho(t_2)-\rho(t_1).$$

By Theorem 2.1,

$$S_{n,\tau_n(t_2)}-S_{n,\tau_n(t_1)}\xrightarrow{d}W(\rho(t_2))-W(\rho(t_1)).$$

Furthermore, for any bounded Lipschitz function $\varphi(\boldsymbol{u},x)$, let $V^{\boldsymbol{u}}(t,x)$ be the unique viscosity solution of the following equation:

$$\partial_t V^{\boldsymbol{u}}+G(\partial_{xx}^2V^{\boldsymbol{u}})=0,\quad (t,x)\in[0,\varrho+h]\times\mathbb{R},\quad V^{\boldsymbol{u}}\,|_{t=\varrho+h}=\varphi(\boldsymbol{u},x),$$

where $\varrho=\rho(t_2)-\rho(t_1)$. With the same argument for showing (6.3), we can show that

$$\widehat{\mathrm{E}}[|\widehat{\mathrm{E}}[\varphi(\boldsymbol{u},S_{n,\tau_n(t_2)}-S_{n,\tau_n(t_1)})\,|\,\mathscr{H}_{n,\tau_n(t_1)}]-\widetilde{\mathrm{E}}[\varphi(\boldsymbol{u},W(\rho(t_2))-W(\rho(t_1)))]|]\to 0. \tag{6.17}$$

The only difference is that (6.8)–(6.10) are needed to be replaced, respectively, by

$$\begin{aligned}
&\widehat{\mathrm{E}}\left[\left|\widehat{\mathrm{E}}[V^{\boldsymbol{u}}(\delta_{k_n^*}^*\wedge h_0,S_{k_n^*}^*)\,|\,\mathscr{H}_{n,\tau_n(t_1)}]-V^{\boldsymbol{u}}(0,0)-\widehat{\mathrm{E}}\left[\sum_{i=0}^{k_n^*-1}(J_{n,1,*}^i+J_{n,2,*}^i)\,\middle|\,\mathscr{H}_{n,\tau_n(t_1)}\right]\right|\right]\\
&\quad\leqslant\widehat{\mathrm{E}}\left[\left|V^{\boldsymbol{u}}(\delta_{k_n^*}^*\wedge h_0,S_{k_n^*}^*)-V^{\boldsymbol{u}}(0,0)-\sum_{i=0}^{k_n^*-1}(J_{n,1,*}^i+J_{n,2,*}^i)\right|\right],\\
&\widehat{\mathrm{E}}\left[\sum_{i=0}^{k_n^*-1}J_{n,1,*}^i\,\middle|\,\mathscr{H}_{n,\tau_n(t_1)}\right]=\widehat{\mathrm{E}}\left[\widehat{\mathrm{E}}\left[\sum_{i=0}^{k_n^*-1}J_{n,1,*}^i\,\middle|\,\mathscr{H}_{n,\tau_n(t_1)+k_n^*-1}\right]\,\middle|\,\mathscr{H}_{n,\tau_n(t_1)}\right]\\
&\qquad=\widehat{\mathrm{E}}\left[\sum_{i=0}^{k_n^*-2}J_{n,1,*}^i+\widehat{\mathrm{E}}[J_{n,1,*}^{k_n-1}\,|\,\mathscr{H}_{n,\tau_n(t_1)+k_n^*-1}]\,\middle|\,\mathscr{H}_{n,\tau_n(t_1)}\right]
\end{aligned}$$

$$= \widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n^*-2} J_{n,1,*}^i \bigg| \mathscr{H}_{n,\tau_n(t_1)}\bigg] = \cdots = 0 \quad \text{in} \quad L_1$$

and

$$\begin{aligned}
&\widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n-1}(\pm J_{n,2,*}^i - \widetilde{J}_{n,2,*}^i)\bigg| \mathscr{H}_{n,\tau_n(t_1)}\bigg] \\
&\quad = \widehat{\mathrm{E}}\bigg[\widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n-2}(\pm J_{n,2,*}^i - \widetilde{J}_{n,2,*}^i) + \widehat{\mathrm{E}}[\pm J_{n,1}^{k_n-1} - \widetilde{J}_{n,2}^{k_n-1} \,|\, \mathscr{H}_{n,k_n-1}] \bigg| \mathscr{H}_{n,\tau_n(t_1)}\bigg]\bigg] \\
&\quad \leqslant \widehat{\mathrm{E}}\bigg[\sum_{i=0}^{k_n-2}(\pm J_{n,2,*}^i - \widetilde{J}_{n,2,*}^i)\bigg| \mathscr{H}_{n,\tau_n(t_1)}\bigg] \leqslant \cdots \leqslant 0 \quad \text{in} \quad L_1,
\end{aligned}$$

where $J_{n,1,*}^i$, $J_{n,2,*}^i$ and $\widetilde{J}_{n,2,*}^i$ are defined the same as $J_{n,1}^i$, $J_{n,2}^i$ and $\widetilde{J}_{n,2}^i$ with $\{Z_{n,k}^*\}$ taking the place of $\{Z_{n,k}\}$. On the other hand, note $S_{n,\tau_n(t_1)} \xrightarrow{d} W(\rho(t_1))$. Hence,

$$(S_{n,\tau_n(t_1)}, S_{n,\tau_n(t_2)} - S_{n,\tau_n(t_1)}) \xrightarrow{d} (W(\rho(t_1)), W(\rho(t_2)) - W(\rho(t_1)))$$

by (6.17) and Lemma 6.1. By induction, for any $0 = t_0 < \cdots < t_d \leqslant 1$,

$$\begin{aligned}
&(S_{n,\tau_n(t_1)} - S_{n,\tau_n(t_0)}, \ldots, S_{n,\tau_n(t_d)} - S_{n,\tau_n(t_{d-1})}) \\
&\quad \xrightarrow{d} (W(\rho(t_1)) - W(\rho(t_0)), \ldots, W(\rho(t_d)) - W(\rho(t_{d-1}))),
\end{aligned}$$

which implies (3.10). So, we have shown the convergence of finite-dimensional distributions of W_n. By Peng [22, Theorem 9] on the tightness and the argument of Lin and Zhang [12] or Zhang [27], to show that (3.11) holds for bounded continuous function φ, it is sufficient to show that for any $\epsilon' > 0$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \mathbb{V}(w_\delta(W_n) \geqslant 3\epsilon') = 0, \tag{6.18}$$

where $\omega_\delta(x) = \sup_{|t-s|<\delta, t,s\in[0,1]} |x(t) - x(s)|$ (see Proposition B.1 in Appendix B). Assume $0 < \delta < 1/10$. Let $0 = t_0 < t_1 < \cdots < t_K = 1$ such that $t_k - t_{k-1} = \delta$, and let $t_{K+1} = t_{K+2} = 1$. It is easily seen that

$$\mathbb{V}(w_\delta(W_n) \geqslant 3\epsilon') \leqslant 2\sum_{k=0}^{K-1} \mathbb{V}\Big(\max_{s\in[t_k,t_{k+2}]} |S_{n,\tau_n(s)} - S_{n,\tau_n(t_k)}| \geqslant \epsilon'\Big).$$

On the other hand, for $t, \gamma > 0$, by (4.3) we have

$$\begin{aligned}
&\widehat{\mathrm{E}}\Big[\max_{s\leqslant\gamma} |S_{n,\tau_n(t+s)} - S_{n,\tau_n(t)}|^4\Big] \\
&\quad \leqslant C\widehat{\mathrm{E}}\bigg[\sum_{k=\tau_n(t)+1}^{\tau_n(t+\gamma)} \widehat{\mathrm{E}}[Z_{n,k}^4 \,|\, \mathscr{H}_{n,k-1}]\bigg] + C\widehat{\mathrm{E}}\bigg[\bigg(\sum_{k=\tau_n(t)+1}^{\tau_n(t+\gamma)} \widehat{\mathrm{E}}[Z_{n,k}^2 \,|\, \mathscr{H}_{n,k-1}]\bigg)^2\bigg] \\
&\qquad + C\widehat{\mathrm{E}}\bigg[\bigg(\sum_{k=\tau_n(t)+1}^{\tau_n(t+\gamma)} \{|\widehat{\mathrm{E}}[Z_{n,k} \,|\, \mathscr{H}_{n,k-1}]| + |\widehat{\mathcal{E}}[Z_{n,k} \,|\, \mathscr{H}_{n,k-1}]|\}\bigg)^4\bigg] \\
&\quad \leqslant C\widehat{\mathrm{E}}\bigg[\bigg(\sum_{k=\tau_n(t)+1}^{\tau_n(t+\gamma)} \widehat{\mathrm{E}}[Z_{n,k}^2 \,|\, \mathscr{H}_{n,k-1}]\bigg)^2\bigg] + C\epsilon_n^2 \cdot 2\rho + C\widehat{\mathrm{E}}[\chi_{k_n}^4].
\end{aligned}$$

The last two terms above will go to zero by (3.4). For considering the first term, we note

$$2\rho(1) \geqslant \sum_{k=\tau_n(t)+1}^{\tau_n(t+\gamma)} \widehat{\mathrm{E}}[Z_{n,k}^2 \,|\, \mathscr{H}_{n,k-1}] \xrightarrow{\mathbb{V}} \rho(t+\gamma) - \rho(t). \tag{6.19}$$

It follows that

$$\widehat{\mathrm{E}}\Big[\Big(\sum_{k=\tau_n(t)+1}^{\tau_n(t+\gamma)}\widehat{\mathrm{E}}[Z_{n,k}^2\,|\,\mathscr{H}_{n,k-1}]\Big)^2\Big]\to(\rho(t+\gamma)-\rho(t))^2.$$

So, we conclude that

$$\begin{aligned}&\limsup_n 2\sum_{k=0}^{K-1}\mathbb{V}\Big(\max_{s\in[t_k,t_{k+2}]}|S_{n,\tau_n(s)}-S_{n,\tau_n(t_k)}|\geqslant\epsilon'\Big)\\&\leqslant\limsup_n 2\sum_{k=0}^{K-1}\left(\frac{1}{\epsilon^*}\right)^4\widehat{\mathrm{E}}\Big[\max_{s\in[t_k,t_{k+2}]}|S_{n,\tau_n(s)}-S_{n,\tau_n(t_k)}|^4\Big]\\&\leqslant C\sum_{k=0}^{K-1}\frac{1}{(\epsilon^*)^4}(\rho(t_{k+2})-\rho(t_k))^2\leqslant C\frac{\rho(1)}{(\epsilon^*)^4}\sup_{|t-s|\leqslant 2\delta}|\rho(t)-\rho(s)|\to 0\end{aligned}$$

by taking $\delta\to 0$. Hence, (6.18) is verified. The proof is completed. □

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References

1 Bayraktar E, Munk A. An α-stable limit theorem under sublinear expectation. Bernoulli, 2016, 22: 2548–2578
2 Billingsley P. Convergence of Probability Measures. New York: Wiley, 1968
3 Denis L, Hu M, Peng S. Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion pathes. Potential Anal, 2011, 34: 139–161
4 Denis L, Martini C. A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. Ann Appl Probab, 2006, 16: 827–852
5 Gilboa I. Expected utility with purely subjective nonadditive probabilities. J Math Econom, 1987, 16: 65–68
6 Hall P, Heyde C C. Martingale Limit Theory and Its Applications. New York: Academic Press, 1980
7 Hu M S, Ji X J, Liu G M. Lévy's martingale characterization and reflection principle of G-Brownian motion. J Math Anal Appl, 2019, 480: 12346
8 Hu M S, Li X J. Independence under the G-expectation framework. J Theoret Probab, 2014, 27: 1011–1020
9 Jacod J, Shiryaev A N. Limit Theorems for Stochastic Processes, 2nd ed. Berlin-Heidelberg: Springer-Verlag, 2003
10 Li M, Shi Y F. A general central limit theorem under sublinear expectations. Sci China Math, 2010, 53: 1989–1994
11 Lin Q. General martingale characterization of G-Brownian motion. Stoch Anal Appl, 2013, 31: 1024–1048
12 Lin Z Y, Zhang L-X. Convergence to a self-normalized G-Brownian motion. Probab Uncertain Quant Risk, 2017, 2: 4
13 Marinacci M. Limit laws for non-additive probabilities and their frequentist interpretation. J Econom Theory, 1999, 84: 145–195
14 Peng S. Backward SDE and related g-expectation. In: Backward Stochastic Differential Equations. Pitman Research Notes in Mathematics Series, vol. 364. Harlow: Longman, 1997, 141–159
15 Peng S. Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer type. Probab Theory Related Fields, 1999, 113: 473–499
16 Peng S. G-expectation, G-Brownian motion and related stochastic calculus of Ito type. In: Proceedings of the Second Abel Symposium. Stochastic Analysis and Applications. New York: Springer-Verlag, 2007, 541–567
17 Peng S. Law of large numbers and central limit theorem under nonlinear expectations. arXiv:math/0702358, 2007
18 Peng S. G-Brownian motion and dynamic risk measure under volatility uncertainty. arXiv:0711.2834v1, 2007
19 Peng S. Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation. Stochastic Process Appl, 2008, 118: 2223–2253
20 Peng S. A new central limit theorem under sublinear expectations. arXiv:0803.2656v1, 2008
21 Peng S. Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations. Sci China Ser A, 2009, 52: 1391–1411
22 Peng S. Nonlinear expectations and stochastic calculus under uncertainty. arXiv:1002.4546, 2010
23 Pollard D. Convergence of Stochastic Processes. New York: Springer-Verlag, 1984

24 Song Y S. A note on G-normal distributions. Statist Probab Lett, 2015, 106: 142–146
25 Xu J, Zhang B. Martingale characterization of G-Brownian motion. Stochastic Process Appl, 2009, 119: 232–248
26 Xu J, Zhang B. Martingale property and capacity under G-framework. Electron J Probab, 2010, 15: 2041–2068
27 Zhang L-X. Donsker's invariance principle under the sub-linear expectation with an application to Chung's law of the iterated logarithm. Commun Math Stat, 2015, 3: 187–214
28 Zhang L-X. Rosenthal's inequalities for independent and negatively dependent random variables under sub-linear expectations with applications. Sci China Math, 2016, 59: 751–768

Appendix A The properties of the conditional expectations

In this appendix, we give the proofs of Lemmas 4.2 and 4.3 on the properties of the conditional expectation.

Proof of Lemma 4.2. (1) is obvious. For (2), note that

$$\widehat{\mathrm{E}}[((X-Y)^+)^q] \leqslant \epsilon^q + c^q\mathbb{V}(X-Y\geqslant\epsilon) + \widehat{\mathrm{E}}[((X-Y-c)^+)^q]$$

and

$$\widehat{\mathrm{E}}[((X-Y-c)^+)^q] \leqslant \frac{\widehat{\mathrm{E}}[((X-Y)^+)^p]}{c^{p-q}} \to 0 \quad \text{as} \quad c\to\infty.$$

The result follows.

For (3), let $\epsilon>0$ and $M>0$ be given. Let $0<\delta<1$ such that $|x-y|\leqslant\delta$ and $|y|\leqslant M$ implies $|f(x)-f(y)|\leqslant\epsilon$. Then,

$$\mathbb{V}(f(X)-f(Y)\geqslant\epsilon)\leqslant\mathbb{V}(X-Y\geqslant\delta)+\mathbb{V}(|Y|\geqslant M).$$

The result follows.

For (4), note for $y,x\geqslant 0$, $x^p-y^p\leqslant px^{p-1}(x-y)$. So,

$$\widehat{\mathrm{E}}[X^p]-\widehat{\mathrm{E}}[Y^p]\leqslant p\widehat{\mathrm{E}}[X^{p-1}(X-Y)^+]\leqslant p(\widehat{\mathrm{E}}[X^p])^{1/q}(\widehat{\mathrm{E}}[((X-Y)^+)^p])^{1/p}=0.$$

For (5), note that the countable additivity of $\widehat{\mathrm{E}}$ implies

$$\widehat{\mathrm{E}}[((X-Y)^+)^p]\leqslant\int_0^\infty\mathbb{V}(((X-Y)^+)^p>y)dy=\int_0^\infty\mathbb{V}(X-Y>y^{1/p})dy$$

(see Zhang [28, Lemma 3.9]). The result follows. □

Proof of Lemma 4.3. (1) Let $0\leqslant f\in\mathscr{H}_{n,k}$ be a bounded random variable. Then

$$\begin{aligned}\widehat{\mathrm{E}}[f(\widehat{\mathrm{E}}_{n,k}[X]-\widehat{\mathrm{E}}_{n,k}[Y])] &= \widehat{\mathrm{E}}[\widehat{\mathrm{E}}_{n,k}[fX-\widehat{\mathrm{E}}_{n,k}[fY]]]\\ &= \widehat{\mathrm{E}}[fX-\widehat{\mathrm{E}}_{n,k}[fY]]\leqslant\widehat{\mathrm{E}}[fX-fY+fY-\widehat{\mathrm{E}}_{n,k}[fY]]\\ &\leqslant\widehat{\mathrm{E}}[f(X-Y)^+]+\widehat{\mathrm{E}}[fY-\widehat{\mathrm{E}}_{n,k}[fY]]\leqslant\widehat{\mathrm{E}}[fY-\widehat{\mathrm{E}}_{n,k}[fY]]\\ &= \widehat{\mathrm{E}}[\widehat{\mathrm{E}}_{n,k}[fY-\widehat{\mathrm{E}}_{n,k}[fY]]]=\widehat{\mathrm{E}}[\widehat{\mathrm{E}}_{n,k}[fY]-\widehat{\mathrm{E}}_{n,k}[fY]]=0,\end{aligned}$$

which implies $\widehat{\mathrm{E}}[(\widehat{\mathrm{E}}_{n,k}[X]-\widehat{\mathrm{E}}_{n,k}[Y])^+]=0$. In fact, let $Z=\widehat{\mathrm{E}}_{n,k}[X]-\widehat{\mathrm{E}}_{n,k}[Y]$ and choose f to be a bounded Lipschitz function of Z such that $I\{Z\geqslant 2\epsilon\}\leqslant f\leqslant I\{Z\geqslant\epsilon\}$. Then,

$$\widehat{\mathrm{E}}[Z^+]\leqslant 2\epsilon+\widehat{\mathrm{E}}[fZ]\leqslant 2\epsilon.$$

(2) The second inequality is due to (1). For the first one, let $Z=\widehat{\mathrm{E}}_{n,k}[X]-\widehat{\mathrm{E}}_{n,k}[Y]-\widehat{\mathrm{E}}_{n,k}[X-Y]$. With the same argument as in (1), it is sufficient to show that $\widehat{\mathrm{E}}[fZ]\leqslant 0$ for any bounded $0\leqslant f\in\mathscr{H}_{n,k}$. Now,

$$\begin{aligned}\widehat{\mathrm{E}}[fZ] &= \widehat{\mathrm{E}}[\widehat{\mathrm{E}}_{n,k}[fX-\widehat{\mathrm{E}}_{n,k}[fY]-\widehat{\mathrm{E}}_{n,k}[fX-fY]]]\\ &= \widehat{\mathrm{E}}[fX-\widehat{\mathrm{E}}_{n,k}[fY]-\widehat{\mathrm{E}}_{n,k}[fX-fY]]\end{aligned}$$

$$
\begin{aligned}
&= \widehat{\mathrm{E}}[(fY - \widehat{\mathrm{E}}_{n,k}[fY]) + (fX - fY - \widehat{\mathrm{E}}_{n,k}[fX - fY])] \\
&\leqslant \widehat{\mathrm{E}}[fY - \widehat{\mathrm{E}}_{n,k}[fY]] + \widehat{\mathrm{E}}[fX - fY - \widehat{\mathrm{E}}_{n,k}[fX - fY]] = 0.
\end{aligned}
$$

(3) Suppose $k < l$. Let $Z = \widehat{\mathrm{E}}_{n,k}[[\widehat{\mathrm{E}}_{n,l}[X]]] - \widehat{\mathrm{E}}_{n,k}[X]$ and $f \geqslant 0$ be a bounded random variable in $\mathscr{H}_{n,k}$. Then,

$$
\begin{aligned}
\widehat{\mathrm{E}}[fZ] &= \widehat{\mathrm{E}}[\widehat{\mathrm{E}}_{n,k}[[\widehat{\mathrm{E}}_{n,l}[fX]]] - \widehat{\mathrm{E}}_{n,k}[fX]] \\
&= \widehat{\mathrm{E}}[\widehat{\mathrm{E}}_{n,k}[\widehat{\mathrm{E}}_{n,l}[fX - \widehat{\mathrm{E}}_{n,k}[fX]]]] = \widehat{\mathrm{E}}[fX - \widehat{\mathrm{E}}_{n,k}[fX]] = 0,
\end{aligned}
$$

which implies $\widehat{\mathrm{E}}[Z^+] = 0$. On the other hand, note $-Z \leqslant \widehat{\mathrm{E}}_{n,k}[X - \widehat{\mathrm{E}}_{n,l}[X]]$ by Property (2). We have

$$
\widehat{\mathrm{E}}[f(-Z)] \leqslant \widehat{\mathrm{E}}[\widehat{\mathrm{E}}_{n,k}[fX - \widehat{\mathrm{E}}_{n,l}[fX]]] = \widehat{\mathrm{E}}[fX - \widehat{\mathrm{E}}_{n,l}[fX]] = 0,
$$

which implies $\widehat{\mathrm{E}}[(-Z)^+] = 0$. So, $\widehat{\mathrm{E}}[|Z|] = 0$,

(4) Let $Z = \widehat{\mathrm{E}}_{n,k}[X]$ and $0 \leqslant f \in \mathscr{H}_{n,k}$ be a bounded random variable with $fZ^+ = 0$ and $|f| \leqslant 1$. Then $Z \in \mathscr{L}(\mathscr{H})$. We first show that $f|Z|^p \in \mathscr{L}(\mathscr{H})$ for any $p \geqslant 1$. It is obvious that $f|Z| \in \mathscr{L}(\mathscr{H})$. Assume that $k \geqslant 1$ is an integer, and $f|Z|^p \in \mathscr{L}(\mathscr{H})$ for $p \leqslant k$. Let $p' \geqslant k$, $p' \leqslant p < p' + 1$. Note

$$
0 \leqslant f|X||Z|^{p-1} \leqslant \frac{p' + 1 - p}{p'}|X|^{\frac{p'}{p'+1-p}} + \frac{p-1}{p'}f|Z|^{p'}.
$$

Choosing $p' = k$ yields $X, f|Z|^{p-1}, f|X||Z|^{p-1} \in \mathscr{L}(\mathscr{H})$. So by the properties (a), (b) and (2),

$$
\begin{aligned}
\widehat{\mathrm{E}}[f|Z|^p] &= \widehat{\mathrm{E}}[f|Z|^{p-1}(-\widehat{\mathrm{E}}_{n,k}[X])] = \widehat{\mathrm{E}}[-\widehat{\mathrm{E}}_{n,k}[Xf|Z|^{p-1}]] \\
&\leqslant \widehat{\mathrm{E}}[\widehat{\mathrm{E}}_{n,k}[|X| \cdot f|Z|^{p-1}]] = \widehat{\mathrm{E}}[f|X||Z|^{p-1}] < \infty
\end{aligned}
$$

if $k \leqslant p < k + 1$. Choosing $p' = k + 1/2$ and repeating the same argument, we have $\widehat{\mathrm{E}}[f|Z|^p] < \infty$ if $k + 1/2 \leqslant p < k + 3/2$. So, $f|Z|^p \in \mathscr{L}(\mathscr{H})$ for $k \leqslant p \leqslant k + 1$. By induction, for any $p \geqslant 1$, $\widehat{\mathrm{E}}[f|Z|^p] < \infty$ which implies $\widehat{\mathrm{E}}[(Z^-)^p] < \infty$. In addition, similarly by choosing f such that $fZ^- = 0$, we have $\widehat{\mathrm{E}}[(Z^+)^p] < \infty$. So, we have $\widehat{\mathrm{E}}[|Z|^p] < \infty$ for any $p \geqslant 1$. Finally, by (1), $|Z| \leqslant M$ in L_1. Hence, by Lemma 4.2(2), the result follows. The proof is now completed. □

Appendix B The tightness

Proposition B.1. *Let $\{Z_{n,k}; k = 1, \ldots, k_n\}$ be an array of random variables with $\widehat{\mathrm{E}}[|Z_{n,k}|] < \infty$, $k = 1, \ldots, k_n$, and $\tau_n(t)$ be a non-decreasing function in $D_{[0,1]}$ which takes integer values with $\tau_n(0) = 0$ and $\tau_n(1) = k_n$. Define $S_{n,i} = \sum_{k=1}^{i} Z_{n,k}$,*

$$
W_n(t) = S_{n,\tau_n(t)}. \tag{B.1}
$$

Assume that for any $\epsilon > 0$,

$$
\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{V}(w_\delta(W_n) \geqslant \epsilon) = 0, \tag{B.2}
$$

where $\omega_\delta(x) = \sup_{|t-s|<\delta, t,s \in [0,1]} |x(t) - x(s)|$. Then $\{W_n\}$ is tight in $D_{[0,1]}$ endowed the Skorohod topology, i.e., for any $\eta > 0$, there exists a compact set K in $D_{[0,1]}$ such that

$$
\sup_n \mathbb{V}(W_n \notin K) \leqslant \eta. \tag{B.3}
$$

Furthermore, if (3.10) *holds for any $0 < t_1 < \cdots < t_d \leqslant 1$, then* (3.11) *holds.*

Proof. The proof of the tightness is similar to that of the tightness of probability measures (see Billingsley [2]). The only difference we shall note is that $\mathbb{V}$ may be not countably additive and may be not continuous. For $T_0 \subset [0, 1]$, define

$$
w(x, T_0) = \sup_{t,s \in T_0} |x(t) - x(s)|
$$

and

$$w'_\delta(x) = \inf_{t_i} \max_{1\leqslant i\leqslant \nu} w(x, [t_{i-1}, t_i)),$$

where the infimum extends over all sets $\{t_i\}$ with

$$0 = t_0 < t_1 < \cdots < t_{\nu-1} < t_\nu = 1, \quad \min_{1\leqslant i\leqslant \nu}(t_i - t_{i-1}) > \delta.$$

Note $w'_\delta(x) \leqslant w_{2\delta}(x)$,

$$|x(t)| \leqslant |x(0)| + \sum_{i=1}^k \left| x\left(\frac{it}{k}\right) - x\left(\frac{(i-1)t}{k}\right)\right| \leqslant |x(0)| + kw_{1/k}(x),$$

and $W_n(0) = 0$. From (B.2) it follows that

$$\lim_{a\to\infty} \limsup_{n\to\infty} \mathbb{V}\Big(\sup_t |W_n(t)| > a\Big) = 0 \tag{B.4}$$

and

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{V}(w'_\delta(W_n) \geqslant \epsilon) = 0, \quad \forall\, \epsilon > 0. \tag{B.5}$$

For fixed n, let $0 < t_1^n < \cdots < t_{\nu-1}^n \leqslant 1$ be the jump times of the step function $\tau_n(t)$, $t_0^n = 0$, $t_\nu^n = 1$. Then $w(W_n, [t_{i-1}^n, t_i^n)) = 0$, $i = 1, \ldots, \nu$. Let $\delta_0^n = \min_{1\leqslant i\leqslant \nu-1}(t_i^n - t_{i-1}^n)$ if $t_{\nu-1}^n = 1$, and $= \min_{1\leqslant i\leqslant \nu}(t_i^n - t_{i-1}^n)$ if $t_{\nu-1}^n < 1$. Then

$$w'_\delta(W_n) = 0 \quad \text{when} \quad \delta < \delta_0^n. \tag{B.6}$$

On the other hand, it is obvious that

$$\lim_{a\to\infty} \mathbb{V}\Big(\sup_t |W_n(t)| > a\Big) \leqslant \lim_{a\to\infty} \frac{\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[|Z_{n,k}|]}{a} = 0.$$

Hence, (B.4) and (B.5) imply that

$$\lim_{a\to\infty} \sup_n \mathbb{V}\Big(\sup_t |W_n(t)| > a\Big) = 0 \tag{B.7}$$

and

$$\lim_{\delta\to 0} \sup_n \mathbb{V}(w'_\delta(W_n) \geqslant \epsilon) = 0, \quad \forall\, \epsilon > 0. \tag{B.8}$$

Now, for any $\eta > 0$ and a sequence $0 < \epsilon_k \to 0$, choose $a > 0$ and $0 < \delta_k \to 0$ such that

$$\sup_n \mathbb{V}\Big(\sup_t |W_n(t)| > a\Big) < \frac{\eta}{2} \quad \text{and} \quad \sup_n \mathbb{V}(w'_{\delta_k}(W_n) > \epsilon_k) < \frac{\eta}{2^{k+1}}.$$

Now, let $B_0 = \{x \in D_{[0,1]} : \sup_t |x(t)| \leqslant a\}$, $B_k = \{x \in D_{[0,1]} : w'_{\delta_k}(x) \leqslant \epsilon_k\}$ and $A = \bigcap_{k=0}^\infty B_k$. Then $\sup_{x\in A} \sup_t |x(t)| \leqslant a$ and $\lim_{\delta\to 0} \sup_{x\in A} w'_\delta(x) = 0$. By the Arzalá-Ascoli thorem, the closure of A is a compact set in $D_{[0,1]}$. On the other hand, by noting (B.6),

$$\begin{aligned}\{W_n \notin cl(A)\} &\subset \Big\{\sup_t |W_n(t)| > a\Big\} \bigcup_{k=1}^\infty \{w'_{\delta_k}(W_n) > \epsilon_k\} \\ &\subset \Big\{\sup_t |W_n(t)| > a\Big\} \bigcup_{k:\delta_k \geqslant \delta_0^n} \{w'_{\delta_k}(W_n) > \epsilon_k\}.\end{aligned}$$

By the (finite) sub-additivity of $\mathbb{V}$, it follows that

$$\mathbb{V}(W_n \notin cl(A)) \leqslant \mathbb{V}\Big(\sup_t |W_n(t)| > a\Big) + \sum_{k:\delta_k \geqslant \delta_0^n} \mathbb{V}(w'_{\delta_k}(W_n) > \epsilon_k)$$

$$< \eta/2 + \sum_{k=1}^{\infty} \frac{\eta}{2^{k+1}} = \eta. \tag{B.9}$$

The proof of the tightness (B.3) is completed.

Now, consider the G-Brownian motion W. In [27], it is proved that

$$\lim_{\delta \to 0} \widetilde{\mathbb{V}}(w_\delta(W) \geqslant \epsilon) = 0 \quad \text{for any} \quad \epsilon > 0.$$

Note that $\rho(\cdot)$ is a uniformly continuous function on $[0,1]$. It follows that

$$\lim_{\delta \to 0} \widetilde{\mathbb{V}}(w_\delta(W \circ \rho) \geqslant \epsilon) = 0 \quad \text{for any} \quad \epsilon > 0.$$

With the same argument as (B.9) one can show that for any $\eta > 0$, there exists a compact set K in $D_{[0,1]}$ such that $\widetilde{\mathbb{V}}(W \circ \rho \notin K) < \eta$.

For $0 = t_0 < t_1 < t_2 < \cdots < t_{d-1} < t_d = 1$, we define the projection $\pi_{t_1,\ldots,t_d}$ from $D_{[0,1]}$ to $\mathbb{R}^d$ by

$$\pi_{t_1,\ldots,t_d} x = (x(t_1), \ldots, x(t_d)),$$

and define a map $\Pi^{-1}_{t_1,\ldots,t_d}$ from $\mathbb{R}^d$ to $D_{[0,1]}$ by

$$\Pi^{-1}_{t_1,\ldots,t_d}(x_1,\ldots,x_d) = \begin{cases} 0, & \text{if } t \in [t_0, t_1), \\ x_k, & \text{if } t \in [t_k, t_{k+1}) \ (k = 1, \ldots, d), \\ x_d, & \text{if } t = t_d. \end{cases}$$

Then $\Pi^{-1}_{t_1,\ldots,t_d}$ is a continuous map. Denote $\widetilde{\pi}_{t_1,\ldots,t_d} = \Pi^{-1}_{t_1,\ldots,t_d} \circ \pi_{t_1,\ldots,t_d}$. Let $\varphi \in C_b(D_{[0,1]})$. Then $\varphi(\widetilde{\pi}_{t_1,\ldots,t_d} x) = \varphi \circ \Pi^{-1}_{t_1,\ldots,t_d}(x(t_1), \ldots, x(t_d))$ and $\varphi \circ \Pi^{-1}_{t_1,\ldots,t_d} \in C_b(\mathbb{R}^d)$. By (3.10) on the convergence of the finite-dimensional distributions of W_n, it follows that

$$\begin{aligned}\lim_{n\to\infty} \widehat{\mathbb{E}}[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d} W_n)] &= \lim_{n\to\infty} \widehat{\mathbb{E}}[\varphi \circ \Pi^{-1}_{t_1,\ldots,t_d}(W_n(t_1), \ldots, W_n(t_d))] \\ &= \widetilde{\mathbb{E}}[\varphi \circ \Pi^{-1}_{t_1,\ldots,t_d}(W(\rho(t_1)), \ldots, W(\rho(t_d)))] = \widetilde{\mathbb{E}}[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d} W \circ \rho)].\end{aligned}$$

Now, suppose that $t_{i+1} - t_i < \delta$ for $i = 0, \ldots, d-1$. Recall $\omega_\delta(x) = \sup_{|t-s|<\delta} |x(t) - x(s)|$, and let $d_0(\cdot,\cdot)$ be the Skorohod distance in $D_{[0,1]}$ and $\|x\| = \sup_{0 \leqslant t \leqslant 1} |x(t)|$. It is easily seen that

$$d_0(\widetilde{\pi}_{t_1,\ldots,t_d} x, x) \leqslant \|\widetilde{\pi}_{t_1,\ldots,t_d} x - x\| \leqslant \omega_\delta(x).$$

Let $\epsilon > 0$ be given. Since φ is a continuous function, for each x, there is an $\epsilon_x > 0$ such that

$$|\varphi(x) - \varphi(y)| < \frac{\epsilon}{2} \quad \text{whenever} \quad d_0(x,y) < \epsilon_x.$$

Let $K \subset D_{[0,1]}$ be a compact set. Then it can be covered by a union of finite many of the sets $\{y : d_0(x,y) < \epsilon_x/2\}$, $x \in K$. So, there is an $\epsilon_K > 0$ such that $|\varphi(x) - \varphi(y)| < \epsilon$ whenever $d_0(x,y) < \epsilon_K$ and $x \in K$. Denote $M = \sup_x |\varphi(x)|$. It follows that

$$|\varphi(\widetilde{\pi}_{t_1,\ldots,t_d} x) - \varphi(x)| < \epsilon + 2MI\{\omega_\delta(x) \geqslant \epsilon_K\} + 2MI\{x \notin K\}.$$

By the tightness of $\{W_n\}$ and $W \circ \rho$, respectively, we can choose K and δ such that

$$\sup_n \mathbb{V}(\omega_\delta(W_n) \geqslant \epsilon_K) + \sup_n \mathbb{V}(W_n \notin K) \leqslant \frac{\epsilon}{4M}$$

and

$$\widetilde{\mathbb{V}}(\omega_\delta(W \circ \rho) \geqslant \epsilon_K) + \widetilde{\mathbb{V}}(W \circ \rho \notin K) \leqslant \frac{\epsilon}{4M}.$$

Hence

$$|\widehat{\mathbb{E}}[\varphi(W_n)] - \widetilde{\mathbb{E}}[\varphi(W \circ \rho)]|$$

$$
\begin{aligned}
&\leqslant |\widehat{\mathrm{E}}[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d}W_n)] - \widetilde{\mathrm{E}}[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d}W\circ\rho)]| \\
&\quad + |\widehat{\mathrm{E}}[\varphi(W_n)] - \widehat{\mathrm{E}}[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d}W_n)]| + |\widetilde{\mathrm{E}}[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d}W\circ\rho)] - \widetilde{\mathrm{E}}[\varphi(W\circ\rho)]| \\
&\leqslant |\widehat{\mathrm{E}}[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d}W_n)] - \widetilde{\mathrm{E}}[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d}W\circ\rho)]| \\
&\quad + 2\epsilon + 2M\mathbb{V}(\omega_\delta(W_n)\geqslant\epsilon_K) + 2M\mathbb{V}(W_n\notin K) \\
&\quad + 2M\widetilde{\mathbb{V}}(\omega_\delta(W\circ\rho)\geqslant\epsilon_K) + 2M\widetilde{\mathbb{V}}(W\circ\rho\notin K) \\
&\leqslant |\widehat{\mathrm{E}}[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d}W_n)] - \widetilde{\mathrm{E}}[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d}W\circ\rho)]| + 3\epsilon.
\end{aligned}
$$

Letting $n\to\infty$ and then $\epsilon\to 0$ completes the proof of (3.11). □