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Number of edges in inhomogeneous random graphs

Zhishui Hu* & Liang Dong

Department of Statistics and Finance, School of Management,
University of Science and Technology of China, Hefei 230026, *China*

Email: huzs@ustc.edu.cn, dl040@mail.ustc.edu.cn

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Abstract We study the number of edges in the inhomogeneous random graph when vertex weights have an infinite mean and show that the number of edges is $O(n \log n)$. Central limit theorems for the number of edges are also established.

Keywords inhomogeneous random graphs, number of edges, power law, complex network, infinite mean

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1 Introduction

One of the most studied random graphs is the Erdős-Rényi random graph $G(n,p)$, which has vertex set $[n] = \{1, 2, \ldots, n\}$ and where each pair of vertices is connected by an edge with probability p, independently of all other edges. However, $G(n,p)$ is not suitable for real-world complex networks. The power-law degree sequence is one of the most important features of real networks while $G(n,p)$ has an extremely light tailed degree distribution. In this paper, we will study inhomogeneous random graphs that generalize $G(n,p)$ but have power-law degree distributions.

Let $[n] = \{1, 2, \ldots, n\}$ be the set of vertices. We first assign a weight $W_i > 0$ to each vertex i and assume that these weights are i.i.d. random variables. Conditionally given the weights $\{W_i, i = 1, 2, \ldots, n\}$, we connect each pair of vertices $i, j \in [n]$ independently with probability p_{ij}, which depends on the vertex weights. By choosing different edge occupation probabilities, we can obtain many kinds of inhomogeneous random graph models, for example, the Chung-Lu model [6], the generalized random graph model [5], and the Norrs-Reittu model [15].

For the above three models, the number of edges is linear in the number of vertices when the weights have a finite mean. We call it "the sparse case". The sparse case has been studied in detail; for example, Janson and Luczak [10], Bhamidi et al. [2] and van der Hofstad [18] studied the sizes of the largest components, and Chung and Lu [6, 7] and van den Esker et al. [17] studied the graph distance between two random vertices. From these results, we can conclude that in the sparse case, except the degree distributions, many behaviors of inhomogeneous random graphs are similar to those of the Erdős-Rényi random graph $G(n, p/n)$.

* Corresponding author

In this paper, we consider the properties when the weights have an infinite mean. This case has not received much attention. van der Hofstad [19] and Hu et al. [9] discussed the asymptotic degree distributions. Janson et al. [11] studied large cliques in the Norrs-Reittu model. The behaviors of inhomogeneous random graphs are different from those when the weights have a finite mean. In this paper, we focus on the number of edges. It is known that the number of edges is $O(n)$ if the weights have a finite mean, while it is $O(n\log n)$ here.

Assume that $\{W_i, i=1,2,\ldots\}$ are i.i.d. random variables with a power law of the form

$$\mathrm{P}(W>x)\sim cx^{-\alpha},\quad x\to\infty,$$

where $c>0,\alpha\in(0,1)$ are some real constants. It is well known that (see [8])

$$n^{-1/\alpha}\sum_{i=1}^{n}W_i\xrightarrow{d}S, \tag{1.1}$$

where S is a positive stable random variable with exponent α and

$$\mathrm{E}\mathrm{e}^{\mathrm{i}tS}=\exp\bigg\{\int_0^\infty(\mathrm{e}^{\mathrm{i}tx}-1)\nu(dx)\bigg\}=\exp\bigg\{-c_1|t|^\alpha\bigg(1-\mathrm{i}\tan\bigg(\frac{\pi\alpha}{2}\bigg)\bigg)\bigg\}$$

with $\nu(dx)=c\alpha x^{-1-\alpha}dx$ on $(0,\infty)$, $c_1=cL(\alpha)\cos(\pi\alpha/2)$ and $L(\alpha)=\int_0^\infty\frac{1-\mathrm{e}^{-y}}{y^{\alpha+1}}dy>0$.

Let $\{X_t, 0\leqslant t\leqslant 1\}$ be an α-stable subordinator with $X_1\overset{d}{=}S$. Then $\eta:=\sum_{0\leqslant t\leqslant 1}\delta_{\Delta X_t}$ is a Poisson process on $(0,\infty)$ with intensity $\mathrm{E}\eta=\nu$ (we refer to [1,16] for more details).

Let the edge occupation probability be given by

$$p_{ij}=\frac{W_iW_j}{L_n+W_iW_j}, \tag{1.2}$$

where L_n can be a real constant or random variable. Assume that G_n is the inhomogeneous random graph with the edge probability (1.2), and $e(G_n)$ is the number of edges in G_n. At first, we consider the case where $L_n=n^{1/\alpha}$.

Theorem 1.1. *Assume that $L_n=n^{1/\alpha}$. Then under the above conditions, we have*

$$\frac{e(G_n)}{n\log n}\xrightarrow{p}\frac{c^2\alpha\pi}{2\sin(\alpha\pi)}. \tag{1.3}$$

Furthermore,

$$\frac{e(G_n)-\mathrm{E}e(G_n)}{n}\xrightarrow{d}\int_0^\infty h(x)(\eta(dx)-\nu(dx)), \tag{1.4}$$

where

$$h(x)=\mathrm{E}\bigg(\frac{xW}{1+xW}\bigg),\quad x>0.$$

Remark 1.2. In Theorem 1.1, $h(x)$ is a continuous function on $(0,\infty)$ and bounded above by 1. Furthermore, from the proof of Lemma 2.4 in Section 2, we have $h(x)\leqslant C_0(x^\alpha\wedge 1)$ for some $C_0>0$ and the integral $\int_0^\infty h(x)(\eta(dx)-\nu(dx))$ exists. By applying [13, Lemma 12.2], the limit distribution in (1.4) has the characteristic function

$$\mathrm{E}\exp\bigg\{\mathrm{i}t\int_0^\infty h(x)(\eta(dx)-\nu(dx))\bigg\}=\exp\bigg\{c\alpha\int_0^\infty\frac{\mathrm{e}^{\mathrm{i}th(x)}-1-\mathrm{i}th(x)}{x^{\alpha+1}}dx\bigg\}.$$

Remark 1.3. We can get the exact expression of $h(x)$ by using an integral with respect to the distribution function of W. But unfortunately, the calculation of the integral is generally too complicated. We now consider the special case where $\mathrm{P}(W>x)=x^{-\alpha}$ for all $x\geqslant 1$. Then we have $h(x)=\alpha t^\alpha\int_x^\infty\frac{1}{(1+u)u^\alpha}du$. If we further assume that $\alpha=1/2$, then $h(x)=t^\alpha(\pi/2-\arctan\sqrt{x})$.

The sequel is organized as follows. Section 2 is devoted to the proof of Theorem 1.1 stated above. In Section 3, Theorem 1.1 is generalized to the case where L_n is a random variable, especially $L_n=\sum_{k=1}^n W_k$ and $L_n=Ln^{1/\alpha}$.

2 Proof of Theorem 1.1

Before proving Theorem 1.1, we show four preliminary lemmas.

Lemma 2.1. *Under the conditions of Theorem* 1.1*, we have*

$$\mathrm{E}(e(G_n)) \sim \frac{c^2\alpha\pi}{2\sin(\alpha\pi)} n\log n, \quad n\to\infty.$$

Proof. Let $a_n := n^{1/\alpha}$. Since $\mathrm{P}(W>x)\sim cx^{-\alpha}$ as $x\to\infty$, we get that, for any $\varepsilon>0$, there exists some $x_0>0$ such that

$$(c-\varepsilon)x^{-\alpha} < \mathrm{P}(W>x) < (c+\varepsilon)x^{-\alpha}, \quad x\geqslant x_0. \tag{2.1}$$

At first, we calculate the quantity

$$\theta_n := \mathrm{E}\frac{W_1W_2}{a_n+W_1W_2}. \tag{2.2}$$

By a direct calculation, we have

$$\begin{aligned}\theta_n &= \int_0^1 \mathrm{P}\bigg(\frac{W_1W_2}{a_n+W_1W_2}>x\bigg)dx = \int_0^1 \mathrm{P}\bigg(W_1W_2>\frac{a_nx}{1-x}\bigg)dx\\ &= \int_0^\infty \frac{a_n}{(a_n+y)^2}\mathrm{P}(W_1W_2>y)dy = \int_{x_0^2}^\infty \frac{a_n}{(a_n+y)^2}\mathrm{P}(W_1W_2>y)dy + O\bigg(\frac{1}{a_n}\bigg).\end{aligned} \tag{2.3}$$

At first, we estimate $\mathrm{P}(W_1W_2>y)$. By Fubini's theorem, we can get that

$$\mathrm{E}W^\alpha I(W\leqslant y) = \int_0^y \alpha x^{\alpha-1}\mathrm{P}(W>x)dx - y^\alpha \mathrm{P}(W>y), \quad y>0. \tag{2.4}$$

Then, for any $y>x_0$, we have

$$\begin{aligned}\mathrm{E}W^\alpha I(W\leqslant y) &\geqslant \int_{x_0}^y \alpha x^{\alpha-1}\mathrm{P}(W>x)dx - y^\alpha\mathrm{P}(W>y)\\ &\geqslant (c-\varepsilon)\int_{x_0}^y \alpha x^{\alpha-1}x^{-\alpha}dx-(c+\varepsilon) \geqslant (c-\varepsilon)\alpha(\log y-\log x_0)-(c+\varepsilon).\end{aligned}$$

Thus, for any $y>x_0^2$,

$$\begin{aligned}\mathrm{P}(W_1W_2>y) &\geqslant \mathrm{P}\bigg(W_2\leqslant \frac{y}{x_0}, W_1>\frac{y}{W_2}\bigg)\\ &\geqslant (c-\varepsilon)y^{-\alpha}\mathrm{E}W_2^\alpha I\bigg(W_2\leqslant\frac{y}{x_0}\bigg)\\ &\geqslant (c-\varepsilon)^2\alpha y^{-\alpha}(\log y-2\log x_0)-(c^2-\varepsilon^2)y^{-\alpha}.\end{aligned} \tag{2.5}$$

Similarly we can get the upper bound

$$\begin{aligned}\mathrm{E}W^\alpha I(W\leqslant y) &\leqslant \int_{x_0}^y \alpha x^{\alpha-1}\mathrm{P}(W>x)dx + x_0^\alpha\\ &\leqslant (c+\varepsilon)\alpha(\log y-\log x_0)+x_0^\alpha, \quad y>x_0,\end{aligned}$$

and, for any $y>x_0^2$,

$$\begin{aligned}\mathrm{P}(W_1W_2>y) &\leqslant \mathrm{P}\bigg(W_2\leqslant\frac{y}{x_0}, W_1>\frac{y}{W_2}\bigg)+\mathrm{P}\bigg(W_2>\frac{y}{x_0}\bigg)\\ &\leqslant (c+\varepsilon)y^{-\alpha}\mathrm{E}W_2^\alpha I\bigg(W_2\leqslant\frac{y}{x_0}\bigg)+(c+\varepsilon)x_0^\alpha y^{-\alpha}\\ &\leqslant (c+\varepsilon)^2\alpha y^{-\alpha}(\log y-2\log x_0)+2(c+\varepsilon)x_0^\alpha y^{-\alpha}.\end{aligned} \tag{2.6}$$

Note that

$$\begin{aligned}\int_0^\infty \frac{y^{-\alpha}\log y}{(a_n+y)^2}dy &= a_n^{-1-\alpha}\int_0^\infty \frac{u^{-\alpha}(\log a_n+\log u)}{(1+u)^2}du\\ &\sim a_n^{-1-\alpha}\log a_n\int_0^\infty \frac{1}{u^\alpha(1+u)^2}du\\ &= \frac{\alpha\pi}{\sin(\alpha\pi)}a_n^{-1-\alpha}\log a_n \end{aligned}\tag{2.7}$$

and

$$\int_0^\infty \frac{y^{-\alpha}}{(a_n+y)^2}dy = a_n^{-1-\alpha}\int_0^\infty \frac{1}{u^\alpha(1+u)^2}du = \frac{\alpha\pi}{\sin(\alpha\pi)}a_n^{-1-\alpha}.\tag{2.8}$$

Then it follows from (2.3)–(2.8) that

$$\theta_n \sim \frac{c^2\alpha^2\pi}{\sin(\alpha\pi)}a_n^{-\alpha}\log a_n = \frac{c^2\alpha\pi}{\sin(\alpha\pi)}n^{-1}\log n.\tag{2.9}$$

Let I_{ij} denote the indicator when the edge ij is occupied. Then $e(G_n)=\sum_{1\leqslant i<j\leqslant n} I_{ij}$ and

$$\mathrm{E}(e(G_n)) = \mathrm{E}(\mathrm{E}(e(G_n)\mid W_1,\dots,W_n)) = \sum_{1\leqslant i<j\leqslant n}\mathrm{E}\frac{W_iW_j}{a_n+W_iW_j} = \frac{n(n-1)}{2}\theta_n \sim \frac{c^2\alpha\pi}{2\sin(\alpha\pi)}n\log n.$$

This proves Lemma 2.1. □

Lemma 2.2. *Under the conditions of Theorem* 1.1, *we have* $\mathrm{Var}(e(G_n)) = o(\mathrm{E}(e(G_n))^2)$, $n\to\infty$.

Proof. By using the same method as that in the proof of Lemma 2.1, we have

$$\mathrm{E}\bigg(\frac{W_1W_2}{a_n+W_1W_2}\bigg)^2 = \int_0^1 \mathrm{P}\bigg(\frac{W_1W_2}{a_n+W_1W_2}>\sqrt{x}\bigg)dx = \int_{x_0^2}^\infty \frac{2a_ny}{(a_n+y)^3}\mathrm{P}(W_1W_2>y)dy + O\bigg(\frac{1}{a_n^2}\bigg)$$

and

$$\begin{aligned}\int_0^\infty \frac{y^{1-\alpha}\log y}{(a_n+y)^3}dy &= a_n^{-1-\alpha}\int_0^\infty \frac{u^{1-\alpha}(\log a_n+\log u)}{(1+u)^3}du\\ &\sim a_n^{-1-\alpha}\log a_n\int_0^\infty \frac{u^{1-\alpha}}{(1+u)^3}du = \frac{\pi\alpha(1-\alpha)}{2\sin((1-\alpha)\pi)}a_n^{-1-\alpha}\log a_n,\\ \int_0^\infty \frac{y^{1-\alpha}}{(a_n+y)^3}dy &= a_n^{-1-\alpha}\int_0^\infty \frac{u^{1-\alpha}}{(1+u)^3}du = \frac{\pi\alpha(1-\alpha)}{2\sin((1-\alpha)\pi)}a_n^{-1-\alpha}.\end{aligned}$$

Combining the above facts with (2.5) and (2.6), we get that

$$\mathrm{E}\bigg(\frac{W_1W_2}{a_n+W_1W_2}\bigg)^2 = O(1)a_n^{-\alpha}\log a_n = O(1)n^{-1}\log n.\tag{2.10}$$

Then

$$\begin{aligned}\mathrm{E}(e(G_n)(e(G_n)-1)) &= \sum_{\{i,j\}\neq\{i',j'\}}\mathrm{E}I_{ij}I_{i'j'}\\ &= \sum_{\{i,j\}\cap\{i',j'\}=\emptyset}\mathrm{E}I_{ij}I_{i'j'} + \sum_{\substack{\{i,j\}\neq\{i',j'\}\\ \{i,j\}\cap\{i',j'\}\neq\emptyset}}\mathrm{E}I_{ij}I_{i'j'}\\ &= \sum_{\{i,j\}\cap\{i',j'\}=\emptyset}\mathrm{E}I_{ij}\mathrm{E}I_{i'j'} + 3\binom{n}{3}\mathrm{E}\frac{W_1W_2}{a_n+W_1W_2}\frac{W_1W_3}{a_n+W_1W_3}\\ &\leqslant (\mathrm{E}(e(G_n)))^2 + \frac{n^3}{4}\bigg(\mathrm{E}\bigg(\frac{W_1W_2}{a_n+W_1W_2}\bigg)^2 + \mathrm{E}\bigg(\frac{W_1W_3}{a_n+W_1W_3}\bigg)^2\bigg)\\ &= (\mathrm{E}(e(G_n)))^2 + O(1)n^2\log n.\end{aligned}$$

This together with Lemma 2.1 yields Lemma 2.2. □

Lemma 2.3. *Let $\{X, Y_n, X_m, X_{m,n}, n \geqslant 1, m \geqslant 1\}$ be random variables such that as $n \to \infty$,*

$$X_{m,n} \xrightarrow{d} X_m$$

holds for any fixed $m \geqslant 1$, and moreover $X_m \xrightarrow{d} X$ as $m \to \infty$. Assume that for any $\varepsilon > 0$,

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \mathrm{P}(|X_{m,n} - Y_n| \geqslant \varepsilon) = 0.$$

Then $Y_n \xrightarrow{d} X$, as $n \to \infty$.

Proof. See Theorem 4.2 in [3]. □

Lemma 2.4. *Under the conditions of Theorem 1.1, we have*

$$\frac{1}{n} \sum_{1\leqslant i<j\leqslant n} \left(\frac{W_i W_j}{a_n + W_i W_j} - \theta_n \right) \xrightarrow{d} \int_0^\infty h(x)(\eta(dx) - \nu(dx)),$$

where θ_n is defined in (2.2).

Proof. By the Hoeffding representation (see [14]), we have

$$\sum_{1\leqslant i<j\leqslant n} \frac{W_i W_j}{a_n + W_i W_j} - \frac{n(n-1)\theta_n}{2} = (n-1) \sum_{i=1}^n (g_n(W_i) - Eg_n(W_i)) + \sum_{1\leqslant i<j\leqslant n} h_n(W_i, W_j),$$

where

$$h_n(x,y) = \frac{xy}{a_n + xy} - g_n(x) - g_n(y) + \theta_n, \quad x < 0, \quad y > 0$$

and

$$g_n(x) = \mathrm{E}\bigg(\frac{W_1 W_2}{a_n + W_1 W_2} \bigg| W_1 = x \bigg) = \mathrm{E}\bigg(\frac{xW}{a_n + xW} \bigg) = h\bigg(\frac{x}{a_n} \bigg), \quad x \geqslant 0$$

with

$$\begin{aligned} h(x) = \mathrm{E}\bigg(\frac{xW}{1 + xW} \bigg) &= \int_0^1 \mathrm{P}\bigg(\frac{xW}{1+xW} > y \bigg) dy \\ &= \int_0^\infty \frac{x}{(1+xy)^2} \mathrm{P}(W > y) dy, \quad x \geqslant 0. \end{aligned}$$

By the Lebesgue dominated convergence theorem, we can easily get that $h(x)$ is a continuous function on $[0, \infty)$. For any $\varepsilon > 0$, by applying (2.1), we have

$$\begin{aligned} h(x) &\leqslant \int_0^{x_0} \frac{x}{(1+xy)^2} dy + (c+\varepsilon) \int_{x_0}^\infty \frac{x}{(1+xy)^2} y^{-\alpha} dy \\ &\leqslant \frac{x_0 x}{1 + x_0 x} + (c+\varepsilon) x^\alpha \int_{xx_0}^\infty \frac{1}{(1+y)^2 y^\alpha} dy \\ &\leqslant (x_0 x) \wedge 1 + (c+\varepsilon) x^\alpha I(x \leqslant 1) \int_0^\infty \frac{1}{(1+y)^2 y^\alpha} dy \\ &\quad + (c+\varepsilon) x^\alpha I(x > 1) \frac{1}{1+\alpha} (xx_0)^{-1-\alpha}. \end{aligned}$$

Thus, we obtain that $h(x) \leqslant C_0(x^\alpha \wedge 1)$ for some constant $C_0 > 0$. Now in order to prove Lemma 2.4, it is sufficient to show that

$$\sum_{i=1}^n \left(h\bigg(\frac{W_i}{a_n} \bigg) - \mathrm{E}h\bigg(\frac{W_i}{a_n} \bigg) \right) \xrightarrow{d} \int_0^\infty h(x)(\eta(dx) - \nu(dx)) \tag{2.11}$$

and

$$\frac{1}{n}\sum_{1\leqslant i<j\leqslant n} h_n(W_i,W_j)\xrightarrow{p}0. \tag{2.12}$$

Denote random counting measures (point processes) by $\mu_n=\sum_{i=1}^n \delta_{W_i/a_n}, n\geqslant 1$ on $(0,\infty)$. Then by [13, Theorems 15.28 and 15.29] we have

$$\mu_n\xrightarrow{d}\eta,\quad \mathrm{E}\mu_n\xrightarrow{v}\mathrm{E}\eta=\nu,$$

where $\eta=\sum_{0\leqslant t\leqslant 1}\delta_{\Delta X_t}$ is a Poisson process on $(0,\infty)$ with intensity ν, $\xrightarrow{v}$ denotes the vague convergence on $(0,\infty)$ and $\xrightarrow{d}$ means convergence in distribution of random measures with respect to the vague topology. (Please refer to [12] for the details on convergence of random measures.) For any $\delta>0$ and $K>0$, $h_{\delta,K}(x)=h(x)I(\delta\leqslant x\leqslant K)$ is a continuous function with compact support on $(0,\infty)$. Then we have

$$\begin{aligned}\sum_{i=1}^n\left(h_{\delta,K}\left(\frac{W_i}{a_n}\right)-\mathrm{E}h_{\delta,K}\left(\frac{W_i}{a_n}\right)\right)&=\int_0^\infty h_{\delta,K}(x)(\mu_n(dx)-\mathrm{E}\mu_n(dx))\\&\xrightarrow{d}\int_0^\infty h_{\delta,K}(x)(\eta(dx)-\nu(dx)).\end{aligned} \tag{2.13}$$

Note that η is a Poisson process with intensity $\nu(dx)=c\alpha x^{-\alpha-1}dx$ on $(0,\infty)$. Then, by the fact that $h(x)\leqslant C_0(x^\alpha\wedge 1)$ is a continuous function, we have $\int_0^\infty h^2(x)\nu(dx)<\infty$. Hence, by [13, Theorem 12.13], the integral $\int_0^\infty h(x)(\eta(dx)-\nu(dx))$ exists and

$$\int_0^\infty h_{\delta,K}(x)(\eta(dx)-\nu(dx))\xrightarrow{p}\int_0^\infty h(x)(\eta(dx)-\nu(dx)) \tag{2.14}$$

as $\delta\to 0$ and $K\to\infty$.

Define $h^c_{\delta,K}(x):=h(x)-h(x)I(\delta\leqslant x\leqslant K)$. Then, by noting that $h(x)\leqslant C_0(x^\alpha\wedge 1)$, we have that, for any $0<\delta<1<K$,

$$\begin{aligned}&\mathrm{E}\left(\sum_{i=1}^n\left(h^c_{\delta,K}\left(\frac{W_i}{a_n}\right)-\mathrm{E}h^c_{\delta,K}\left(\frac{W_i}{a_n}\right)\right)\right)^2\\&\quad=n\mathrm{E}\left(h^c_{\delta,K}\left(\frac{W}{a_n}\right)-\mathrm{E}h^c_{\delta,K}\left(\frac{W}{a_n}\right)\right)^2\leqslant n\mathrm{E}\left(h^c_{\delta,K}\left(\frac{W}{a_n}\right)\right)^2\\&\quad\leqslant 2C_0^2n\mathrm{E}\left(\frac{W}{a_n}\right)^{2\alpha}I(W<\delta a_n)+2C_0^2n\mathrm{P}(W>Ka_n).\end{aligned}$$

It is similar to (2.4) that, for $\delta a_n>x_0$,

$$\begin{aligned}n\mathrm{E}\left(\frac{W}{a_n}\right)^{2\alpha}I(W<\delta a_n)&\leqslant\frac{2\alpha n}{a_n^{2\alpha}}\int_0^{\delta a_n}x^{2\alpha-1}\mathrm{P}(W>x)dx\\&\leqslant\frac{2\alpha n}{a_n^{2\alpha}}\left(\frac{x_0^{2\alpha}}{(2\alpha)}+(c+\varepsilon)\int_{x_0}^{\delta a_n}x^{\alpha-1}dx\right)\\&\leqslant\frac{2n}{a_n^{2\alpha}}\left(\frac{x_0^{2\alpha}}{(2\alpha)}+\frac{c+\varepsilon}{\alpha}\delta^\alpha a_n^\alpha\right),\end{aligned}$$

which implies $\limsup_{\delta\to 0}\limsup_{n\to\infty}n\mathrm{E}(W/a_n)^2I(W<\delta a_n)=0$. By noting that

$$\limsup_{K\to\infty}\limsup_{n\to\infty}n\mathrm{P}(W>Ka_n)\leqslant\limsup_{K\to\infty}\limsup_{n\to\infty}n(c+\varepsilon)(Ka_n)^{-\alpha}=0,$$

we have

$$\limsup_{\delta\to 0,K\to\infty}\limsup_{n\to\infty}\mathrm{E}\left(\sum_{i=1}^n\left(h^c_{\delta,K}\left(\frac{W_i}{a_n}\right)-\mathrm{E}h^c_{\delta,K}\left(\frac{W_i}{a_n}\right)\right)\right)^2=0,$$

and then, for any $\varepsilon > 0$,

$$\limsup_{\delta\to 0, K\to\infty} \limsup_{n\to\infty} \mathrm{P}\left(\left|\sum_{i=1}^{n}\left(h_{\delta,K}^{c}\left(\frac{W_i}{a_n}\right) - \mathrm{E}h_{\delta,K}^{c}\left(\frac{W_i}{a_n}\right)\right)\right| \geqslant \varepsilon\right) = 0. \tag{2.15}$$

Now (2.11) follows from Lemma 2.3 and (2.13)–(2.15).

We next prove (2.12). Notice that if $\{i_1, j_1\} \cap \{i_2, j_2\} = \emptyset$, then $h_n(W_{i_1}, W_{j_1})$ and $h_n(W_{i_2}, W_{j_2})$ are independent random variables and $\mathrm{E}h_n(W_{i_1}, W_{j_1})h_n(W_{i_2}, W_{j_2}) = 0$. If $i_1 = i_2$ and $j_1 \neq j_2$, then W_{j_1} and W_{j_2} are conditionally independent given W_{i_1}, and

$$\mathrm{E}h_n(W_{i_1}, W_{j_1})h_n(W_{i_2}, W_{j_2}) = \mathrm{E}(\mathrm{E}(h_n(W_{i_1}, W_{j_1}) \,|\, W_{i_1})\mathrm{E}(h_n(W_{i_1}, W_{j_2}) \,|\, W_{i_1})) = 0.$$

Thus, we have

$$\frac{1}{n^2}\mathrm{E}\left(\sum_{1\leqslant i<j\leqslant n} h_n(W_i, W_j)\right)^2 = \frac{n-1}{2n}\mathrm{E}h_n^2(W_1, W_2). \tag{2.16}$$

Since $0 \leqslant W_1W_2/(a_n + W_1W_2) \leqslant 1$, by the Lebesgue dominated convergence theorem, we obtain that $\mathrm{E}(W_1W_2/(a_n + W_1W_2))^2 \to 0$ as $n \to \infty$. By conditional Jensen's inequality, $\mathrm{E}g_n^2(W_1) \leqslant \mathrm{E}(W_1W_2/(a_n + W_1W_2))^2 \to 0$. Hence,

$$\mathrm{E}h_n^2(W_1, W_2) \leqslant 4\left(\mathrm{E}\left(\frac{W_1W_2}{a_n + W_1W_2}\right)^2 + \mathrm{E}g_n^2(W_1) + \mathrm{E}g_n^2(W_2) + \theta_n^2\right) \to 0. \tag{2.17}$$

Then by using the Chebyshev inequality, (2.12) follows from (2.16) and (2.17).

The proof of Lemma 2.4 is completed. □

Now we turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. By the Chebyshev inequality and Lemmas 2.1 and 2.2, we obtain (1.3).

In the following, we will prove (1.4). Let I_{ij} denote the indicator when the edge ij is occupied. Then we have $e(G_n) = \sum_{1\leqslant i<j\leqslant n} I_{ij}$ and $\mathrm{E}I_{ij} = \theta_n$ for all $1 \leqslant i < j \leqslant n$, where θ_n is defined in (2.2). Furthermore, for any $t \in \mathbb{R}$, we have

$$\begin{aligned}\mathrm{E}\exp\left\{\mathrm{i}t\frac{e(G_n) - \mathrm{E}e(G_n)}{n}\right\} &= \mathrm{E}\left(\mathrm{E}\left(\exp\left\{\frac{\mathrm{i}t}{n}\sum_{1\leqslant i<j\leqslant n}(I_{ij} - \theta_n)\right\}\,\middle|\, W_1, \ldots, W_n\right)\right)\\ &= \mathrm{E}\left(\prod_{1\leqslant i<j\leqslant n} \mathrm{e}^{-\mathrm{i}t\theta_n/n}\left(1 + (\mathrm{e}^{\mathrm{i}t/n} - 1)\frac{W_iW_j}{a_n + W_iW_j}\right)\right) =: \mathrm{E}\mathrm{e}^{Y_n},\end{aligned}$$

where

$$Y_n = \sum_{1\leqslant i<j\leqslant n}\left(\log\left(1 + (\mathrm{e}^{\mathrm{i}t/n} - 1)\frac{W_iW_j}{a_n + W_iW_j}\right) - \frac{\mathrm{i}t\theta_n}{n}\right)$$

and $\log(\cdot)$ is the principal value of the complex logarithm function.

By using the Maclaurin series expansion of $\log(1 + x)$ for complex x with $|x| < 1$, we have

$$\frac{|\log(1 + x) - x|}{|x|^2} \longrightarrow \frac{1}{2}, \quad |x| \to 0.$$

Hence there exists some constant $c_0 > 0$ such that $|\log(1 + x) - x| \leqslant |x|^2$ holds for any $|x| \leqslant c_0$.

Then, for sufficiently large n, we have

$$\left|Y_n - \frac{\mathrm{i}t}{n}\sum_{1\leqslant i<j\leqslant n}\left(\frac{W_iW_j}{a_n + W_iW_j} - \theta_n\right)\right|$$

$$
\begin{aligned}
&\leqslant |\mathrm{e}^{\mathrm{i}t/n}-1|^2\sum_{1\leqslant i<j\leqslant n}\left(\frac{W_iW_j}{a_n+W_iW_j}\right)^2+\left|\mathrm{e}^{\mathrm{i}t/n}-1-\frac{\mathrm{i}t}{n}\right|\sum_{1\leqslant i<j\leqslant n}\frac{W_iW_j}{a_n+W_iW_j}\\
&\leqslant \frac{t^2}{n^2}\sum_{1\leqslant i<j\leqslant n}\left(\frac{W_iW_j}{a_n+W_iW_j}\right)^2+\frac{t^2}{2n^2}\sum_{1\leqslant i<j\leqslant n}\frac{W_iW_j}{a_n+W_iW_j},
\end{aligned}
\tag{2.18}
$$

where we have used the inequalities $|\mathrm{e}^{\mathrm{i}x}-1|\leqslant|x|$ and $|\mathrm{e}^{\mathrm{i}x}-1-\mathrm{i}x|\leqslant x^2/2$ for any $x\in\mathbb{R}$.

By the Lebesgue dominated convergence theorem, we have

$$
\frac{t^2}{n^2}\mathrm{E}\sum_{1\leqslant i<j\leqslant n}\left(\frac{W_iW_j}{a_n+W_iW_j}\right)^2\leqslant\frac{t^2}{2}\mathrm{E}\left(\frac{W_1W_2}{a_n+W_1W_2}\right)^2\to 0.
$$

Thus, by Chebyshev's inequality, we have

$$
\frac{t^2}{n^2}\sum_{1\leqslant i<j\leqslant n}\left(\frac{W_iW_j}{a_n+W_iW_j}\right)^2\xrightarrow{p}0.
\tag{2.19}
$$

Similarly, we have

$$
\frac{t^2}{2n^2}\sum_{1\leqslant i<j\leqslant n}\frac{W_iW_j}{a_n+W_iW_j}\xrightarrow{p}0.
\tag{2.20}
$$

Now, it follows from (2.18)–(2.20) and Lemma 2.4 that

$$
Y_n\xrightarrow{d}\mathrm{i}t\int_0^\infty h(x)(\eta(dx)-\nu(dx)).
$$

Hence, by noting that $|\mathrm{e}^{Y_n}|\leqslant 1$ and applying the dominated Lebesgue convergence theorem, we get that, for any $t\in\mathbb{R}$,

$$
\mathrm{E}\exp\left\{\mathrm{i}t\frac{e(G_n)-\mathrm{E}e(G_n)}{n}\right\}=\mathrm{E}\mathrm{e}^{Y_n}\to\mathrm{E}\exp\left\{\mathrm{i}t\int_0^\infty h(x)(\eta(dx)-\nu(dx))\right\}.
$$

Then we obtain (1.4) and the proof of Theorem 1.1 is completed. □

3 Number of edges in the generalized random graph model

In Theorem 1.1, we assume that L_n in the edge occupation probability (1.2) is a fixed constant. But in most of the existing literature, L_n is assumed to be the total weight of all vertices, i.e., $L_n=\sum_{k=1}^n W_k$. Hence, in this section, we shall extend Theorem 1.1 to the case where L_n is a random variable. We still use G_n to denote the generalized random graph model with the edge probability (1.2), and $e(G_n)$ is the number of edges in G_n. Except that L_n is a random variable, other conditions are the same as those in Theorem 1.1.

Theorem 3.1. *Assume that $n^{-1/\alpha}L_n\xrightarrow{d}Z$ holds for some positive random variable Z. Then we have*

$$
\frac{e(G_n)}{n\log n}\xrightarrow{d}\frac{c^2\alpha\pi}{2\sin(\alpha\pi)}Z^{-\alpha}.
$$

Remark 3.2. If $L_n=\sum_{k=1}^n W_k$, then, by Theorem 3.1, we have

$$
\frac{e(G_n)}{n\log n}\xrightarrow{d}\frac{c^2\alpha\pi}{2\sin(\alpha\pi)}S^{-\alpha},
$$

where S is defined in (1.1). The limit random variable has, apart from a scale factor, a Mittag-Leffler distribution with parameter α (see [4, Subsection 8.0.5]). Another case of interest is that $L_n=Ln^{1/\alpha}$, where L is a positive random variable. In this case, we get that

$$
\frac{e(G_n)}{n\log n}\xrightarrow{d}\frac{c^2\alpha\pi}{2\sin(\alpha\pi)}L^{-\alpha}.
$$

It is interesting that the limit distributions are the same for $L_n=\sum_{k=1}^n W_k$ and $L_n=n^{1/\alpha}S$.

Proof of Theorem 3.1. In the proof, we denote by $G_n(b_n)$ the inhomogeneous random graph obtained with the edge occupation probability (conditionally on the weights $\{W_i, i=1,\dots,n\}$)

$$p_{ij}^{b_n} = \frac{W_iW_j}{b_n + W_iW_j},$$

where b_n is a constant or random variable. Then, by suitable coupling, we may assume that $G_n(b_n) \supset G_n(b_n')$ if $b_n < b_n'$.

Since $L_n/a_n \xrightarrow{d} Z$ and Z is a positive random variable, for any $\varepsilon > 0$, there exists some constant $A > 0$ such that $\mathrm{P}(1/A \leqslant L_n/a_n \leqslant A) \geqslant 1-\varepsilon$. We choose $\delta > 0$ small enough such that

$$(c_0+\varepsilon)\left(\frac{k_0}{k_0+1}\right)^{\alpha} > c_0 + \frac{\varepsilon}{2}$$

holds true, where

$$c_0 := \frac{c^2\alpha\pi}{2\sin(\alpha\pi)}, \quad k_0 := [(\delta A)^{-1}].$$

Hence, for any $\varepsilon > 0$,

$$\begin{aligned}
&\mathrm{P}\left(\frac{e(G_n)}{n^2L_n^{-\alpha}\log n} > c_0+\varepsilon\right)\\
&\quad\leqslant \mathrm{P}\left(\frac{e(G_n)}{n^2L_n^{-\alpha}\log n} > c_0+\varepsilon, \frac{1}{A}\leqslant\frac{L_n}{a_n}\leqslant A\right) + \mathrm{P}\left(\frac{L_n}{a_n}\notin\left[\frac{1}{A}, A\right]\right)\\
&\quad\leqslant \sum_{k=k_0}^{k_1}\mathrm{P}\left(\frac{e(G_n)}{n^2L_n^{-\alpha}\log n} > c_0+\varepsilon, k\delta\leqslant\frac{L_n}{a_n}\leqslant(k+1)\delta\right)+\varepsilon\\
&\quad\leqslant \sum_{k=k_0}^{k_1}\mathrm{P}\left(\frac{e(G_n(k\delta a_n))}{n^2((k+1)\delta a_n)^{-\alpha}\log n} > c_0+\varepsilon\right)+\varepsilon\\
&\quad\leqslant \sum_{k=k_0}^{k_1}\mathrm{P}\left(\frac{e(G_n(k\delta a_n))}{n^2(k\delta a_n)^{-\alpha}\log n} > c_0+\frac{\varepsilon}{2}\right)+\varepsilon,
\end{aligned}$$

where $k_1 = [A/\delta]$. From the proof of Theorem 1.1, we can get that, for any fixed $k > 0$,

$$\frac{e(G_n(k\delta a_n))}{n^2(k\delta a_n)^{-\alpha}\log n} \xrightarrow{p} c_0.$$

Thus, for any fixed $k > 0$,

$$\mathrm{P}\left(\frac{e(G_n(k\delta a_n))}{n^2(k\delta a_n)^{-\alpha}\log n} > c_0+\frac{\varepsilon}{2}\right)\to 0,$$

and then

$$\limsup_{n\to\infty}\mathrm{P}\left(\frac{e(G_n)}{n^2L_n^{-\alpha}\log n} > c_0+\varepsilon\right)\leqslant\varepsilon.$$

Similarly

$$\limsup_{n\to\infty}\mathrm{P}\left(\frac{e(G_n)}{n^2L_n^{-\alpha}\log n} < c_0-\varepsilon\right)\leqslant\varepsilon.$$

Hence,

$$\frac{e(G_n)}{n^2L_n^{-\alpha}\log n}\xrightarrow{p} c_0.$$

Then Theorem 3.1 follows by Slutsky's theorem and the fact that $L_n/a_n \xrightarrow{d} Z$. □

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