SCIENCE CHINA
Mathematics

• ARTICLES •

June 2019 Vol. 62 No. 6: 1029–1040
https://doi.org/10.1007/s11425-018-9498-8

L^1-Poincaré and Sobolev inequalities for differential forms in Euclidean spaces

Dedicated to Professor Jean-Yves Chemin on the Occasion of His 60th Birthday

Annalisa Baldi[1], Bruno Franchi[1,*] & Pierre Pansu[2]

[1]*Dipartimento di Matematica, Università di Bologna, Bologna 40126, Italy;*
[2]*Laboratoire de Mathématiques d'Orsay, Université Paris-Sud, CNRS, Université Paris-Saclay, Orsay 91405, France*

Email: annalisa.baldi2@unibo.it, bruno.franchi@unibo.it, pierre.pansu@math.u-psud.fr

Received December 2, 2018; accepted February 21, 2019; published online March 27, 2019

Abstract In this paper, we prove Poincaré and Sobolev inequalities for differential forms in $L^1(\mathbb{R}^n)$. The singular integral estimates that it is possible to use for L^p, $p > 1$, are replaced here with inequalities which go back to Bourgain and Brezis (2007).

Keywords differential forms, Sobolev-Poincaré inequalities, homotopy formula

MSC(2010) 58A10, 26D15, 46E35

Citation: Baldi A, Franchi B, Pansu P. L^1-Poincaré and Sobolev inequalities for differential forms in Euclidean spaces. Sci China Math, 2019, 62: 1029–1040, https://doi.org/10.1007/s11425-018-9498-8

1 Introduction

The simplest form of the Poincaré inequality in an open set $B \subset \mathbb{R}^n$ can be stated as follows: if $1 \leqslant p < n$ there exists $C(B,p) > 0$ such that for any (say) smooth function u on $\mathbb{R}^n$ there exists a constant c_u such that

$$\|u - c_u\|_{L^q(B)} \leqslant C(n,p)\,\|\nabla u\|_{L^p(B)}$$

provided $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$. The Sobolev inequality is very similar, but in that case we are dealing with compactly supported functions, so that the constant c_u can be dropped. It is well known (see the Federer-Fleming theorem [5]) that for $p = 1$ the Sobolev inequality is equivalent to the classical isoperimetric inequality (whereas the Poincaré inequality corresponds to the classical *relative* isoperimetric inequality).

Let us restrict ourselves for a while to the case $B = \mathbb{R}^n$, to investigate generalizations of these inequalities to differential forms. It is easy to see that Sobolev and Poincaré inequalities are equivalent to the following problem: we ask whether, given a closed differential 1-form ω in $L^p(\mathbb{R}^n)$, there exists a 0-form ϕ in $L^q(\mathbb{R}^n)$ with $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$ such that

$$d\phi = \omega \quad \text{and} \quad \|\phi\|_q \leqslant C(n,p,h)\,\|\omega\|_p. \tag{1.1}$$

* Corresponding author

Clearly, this problem can be formulated in general for h-forms ω in $L^p(\mathbb{R}^n)$ and we are led to look for $(h-1)$-forms ϕ in $L^q(\mathbb{R}^n)$ such that (1.1) holds. This is the problem we have in mind when we speak about the Poincaré inequality for differential forms. When we speak about the Sobolev inequality, we have in mind compactly supported differential forms.

The case $p > 1$ has been fully understood on bounded convex sets by Iwaniec and Lutoborski [10]. On the other hand, in the full space $\mathbb{R}^n$ an easy proof consists in putting $\phi = d^*\Delta^{-1}\omega$. Here, Δ^{-1} denotes the inverse of the Hodge Laplacian $\Delta = d^*d + dd^*$ and d^* is the formal L^2-adjoint of d. The operator $d^*\Delta^{-1}$ is given by convolution with a homogeneous kernel of type 1 in the terminology of [6,7], and hence it is bounded from L^p to L^q if $p > 1$. Unfortunately, this argument does not suffice for $p = 1$ since, by [7, Theorem 6.10], $d^*\Delta^{-1}$ maps L^1 only into the weak Marcinkiewicz space $L^{n/(n-1),\infty}$. Upgrading from $L^{n/(n-1),\infty}$ to $L^{n/(n-1)}$ is possible for functions (see [8, 9, 13]), but the trick does not seem to generalize to differential forms.

Since the case $p = 1$ is the most relevant from a geometric point of view, we focus on that case. First of all, we notice that the Poincaré inequality with $p = 1$ fails in top degree unless a global integral inequality is satisfied. Indeed for $h = n$ forms belonging to L^1 and with nonvanishing integrals cannot be differentials of $L^{n/(n-1)}$ forms[1)]. In arbitrary degree, a similar integral obstruction takes the form $\int \omega \wedge \beta = 0$ for every constant coefficient form β of the complementary degree. Therefore we introduce the subspace L^1_0 of L^1-differential forms satisfying these conditions. However, in degree n assuming that the integral constraint is satisfied does not suffice, as we shall see in Section 4. On the other hand, for example it follows from [4] that the Poincaré inequality holds in degree $n-1$. We refer the reader to [2] for a discussion, in particular in connection with Van Schaftingen's [15] and Lanzani and Stein's [12] results.

We can state our main results. We have the following theorem.

Theorem 1.1 (Global Poincaré and Sobolev inequalities)**.** *Let $h = 1, \dots, n-1$ and set $q = n/(n-1)$. For every closed h-form $\alpha \in L^1_0(\mathbb{R}^n)$, there exists an $(h-1)$-form $\phi \in L^q(\mathbb{R}^n)$, such that*

$$d\phi = \alpha \quad \text{and} \quad \|\phi\|_q \leqslant C\,\|\alpha\|_1.$$

Furthermore, if α is compactly supported, so is ϕ.

We also prove a local version of this inequality.

Corollary 1.2. *For $h = 1, \dots, n-1$, let $q = n/(n-1)$. Let $B \subset \mathbb{R}^n$ be a bounded open convex set, and let B' be an open set, $B \Subset B'$. Then there exists $C = C(n, B, B')$ with the following property:*

(1) *The interior Poincaré inequality. For every closed h-form α in $L^1(B')$, there exists an $(h-1)$-form $\phi \in L^q(B)$, such that*

$$d\phi = \alpha_{|B} \quad \text{and} \quad \|\phi\|_{L^q(B)} \leqslant C\,\|\alpha\|_{L^1(B')}.$$

(2) *The Sobolev inequality. For every closed h-form $\alpha \in L^1$ with support in B, there exists an $(h-1)$-form $\phi \in L^q$, with support in B', such that*

$$d\phi = \alpha \quad \text{and} \quad \|\phi\|_{L^q(B')} \leqslant C\|\alpha\|_{L^1(B)}.$$

We shall refer to the above inequality as the interior Poincaré and interior Sobolev inequalities, respectively. The word "interior" is meant to stress the loss of the domain from B' to B.

Remarkably, most of the techniques developed here can be adapted, in combination with other *ad hoc* arguments to deal with Poincaré and Sobolev inequalities in the Rumin complex of Heisenberg groups (see [2]).

2 Kernels

Throughout the present paper our setting will be the Euclidean space $\mathbb{R}^n$ with $n > 2$.

1) Pansu P, Tripaldi F. Averages and the $\ell^{q,1}$-cohomology of Heisenberg groups. In preparation

If f is a real function defined in $\mathbb{R}^n$, we denote by ${}^{\mathrm{v}}f$ the function defined by ${}^{\mathrm{v}}f(p) := f(-p)$, and, if $T \in \mathcal{D}'(\mathbb{R}^n)$, then ${}^{\mathrm{v}}T$ is the distribution defined by $\langle {}^{\mathrm{v}}T \,|\, \phi\rangle := \langle T \,|\, {}^{\mathrm{v}}\phi\rangle$ for any test function ϕ.

We remind also that the convolution $f * g$ is well-defined when $f, g \in \mathcal{D}'(\mathbb{R}^n)$, provided at least one of them has compact support. In this case the following identities hold:

$$\langle f * g \,|\, \phi\rangle = \langle g \,|\, {}^{\mathrm{v}}f * \phi\rangle \quad \text{and} \quad \langle f * g \,|\, \phi\rangle = \langle f \,|\, \phi * {}^{\mathrm{v}}g\rangle \tag{2.1}$$

for any test function ϕ.

Following [6, Definition 5.3], we recall now the notion of *kernel of type* μ and some properties stated below in Proposition 2.2.

Definition 2.1. A kernel of type μ is a distribution $K \in \mathcal{S}'(\mathbb{R}^n)$, homogeneous of degree $\mu - n$ that is smooth outside the origin.

The convolution operator with a kernel of type μ,

$$f \to f * K,$$

is still called an operator of type μ.

Proposition 2.2. *Let $K \in \mathcal{S}'(\mathbb{R}^r)$ be a kernel of type μ and let D_j denote the j-th partial derivative in $\mathbb{R}^n$.*

(i) *${}^{\mathrm{v}}K$ is again a kernel of type μ.*

(ii) *D_jK and KD_j are associated with kernels of type $\mu - 1$ for $j = 1, \ldots, n$.*

(iii) *If $\mu > 0$, then $K \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

Lemma 2.3. *Let g be a kernel of type $\mu > 0$, and let $\psi \in \mathcal{D}(\mathbb{R}^n)$ be a test function. Then $\psi * g$ is smooth on $\mathbb{R}^n$.*

If, in addition, $R = R(D)$ is a homogeneous polynomial of degree $\ell \geqslant 0$ in

$$D := (D_1, \ldots, D_n),$$

we have

$$R(\psi * g)(p) = O(|p|^{\mu-n-\ell}) \quad \textit{as} \quad p \to \infty.$$

In particular, if $\psi \in \mathcal{D}(\mathbb{R}^n)$, and K is a kernel of type $\mu < n$, then both $\psi * K$ and all its derivatives belong to $L^\infty(\mathbb{R}^n)$.

Corollary 2.4. *If K is a kernel of type $\mu \in (0, n)$, $u \in L^1(\mathbb{R}^n)$ and $\psi \in \mathcal{D}(\mathbb{R}^n)$, then*

$$\langle u * K \,|\, \psi\rangle = \langle u \,|\, \psi * {}^{\mathrm{v}}K\rangle. \tag{2.2}$$

In this equation, the left-hand side is the action of a distribution on a test function (see Formula (3.5)) and the right-hand side is the inner product of an L^1 vector-valued function with an L^∞ vector-valued function.

Remark 2.5. The conclusion of Corollary 2.4 still holds if we assume $K \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, provided $u \in L^1(\mathbb{R}^n)$ is compactly supported.

Lemma 2.6. *Let K be a kernel of type $\alpha \in (0, n)$. Then for any $f \in L^1(\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n} f(y)\, dy = 0,$$

we have

$$R^{-\alpha} \int_{B(0,2R)\setminus B(0,R)} |K * f| dx \to 0 \quad \textit{as} \quad R \to \infty.$$

Proof. If $R > 1$, then we have

$$\begin{aligned}R^{-\alpha}\int_{R<|x|<2R}|K*f|dx &= R^{-\alpha}\int_{R<|x|<2R}dx\left|\int K(x-y)f(y)dy\right|\\
&= R^{-\alpha}\int_{R<|x|<2R}dx\left|\int [K(x-y)-K(x)]f(y)dy\right|\\
&\leqslant R^{-\alpha}\int |f(y)|\left(\int_{R<|x|<2R}|K(x-y)-K(x)|dx\right)dy\\
&= R^{-\alpha}\int_{|y|<\frac{1}{2}R}|f(y)|(\cdots)dy + R^{-\alpha}\int_{4R>|y|>\frac{1}{2}R}|f(y)|(\cdots)dy\\
&\quad + R^{-\alpha}\int_{|y|>4R}|f(y)|(\cdots)dy\\
&=: R^{-\alpha}I_1(R)+R^{-\alpha}I_2(R)+R^{-\alpha}I_3(R).\end{aligned}$$

Consider first the third term above. By homogeneity we have

$$I_3(R) \leqslant C_K\int_{|y|>4R}|f(y)|\left(\int_{R<|x|<2R}(|x-y|^{-n+\alpha}+|x|^{-n+\alpha})dx\right)dy.$$

Notice now that, if $|y| > 4R$ and $R < |x| < 2R$, then

$$|x-y| \geqslant |y|-|x| \geqslant 4R-R \geqslant \frac{3}{2}|x|.$$

Therefore

$$|x-y|^{-n+\alpha}+|x|^{-n+\alpha} \leqslant \left\{\left(\frac{2}{3}\right)^{n-\alpha}+1\right\}|x|^{-n+\alpha},$$

and then

$$\int_{R<|x|<2R}(|x-y|^{-n+\alpha}+|x|^{-n+\alpha})\,dx \leqslant C_\alpha\,R^\alpha.$$

Thus

$$R^{-\alpha}I_3(R) \leqslant C_{K,\alpha}\int_{|y|>4R}|f(y)|\,dy \to 0$$

as $R\to\infty$.

Consider now the second term. Again we have

$$I_2(R) \leqslant C_K\int_{\frac{1}{2}R<|y|<4R}|f(y)|\left(\int_{R<|x|<2R}(|x-y|^{-n+\alpha}+|x|^{-n+\alpha})\,dx\right)dy.$$

Obviously, as above,

$$\int_{R<|x|<2R}|x|^{-n+\alpha}\,dx \leqslant CR^\alpha.$$

Notice now that, if

$$\frac{1}{2}R < |y| < 4R \quad \text{and} \quad R < |x| < 2R,$$

then

$$|x-y| \leqslant |x|+|y| < 6R.$$

Hence

$$\int_{\frac{1}{2}R<|y|<4R}|f(y)|\left(\int_{|x-y|<6R}|x-y|^{-n+\alpha}\,dx\right)dy \leqslant CR^\alpha.$$

Therefore

$$R^{-\alpha}I_2(R) \leqslant C_K\int_{\frac{1}{2}R<|y|<4R}|f(y)|\,dy \to 0$$

as $R \to \infty$. Finally, if $|y| < \frac{R}{2}$ and $R < |x| < 2R$ we have $|y| < \frac{1}{2}|x|$ so that, by [7, Proposition 1.7 and Corollary 1.16],

$$\begin{aligned}
R^{-\alpha} I_1(R) &\leqslant C_K \int_{|y|<\frac{1}{2}R} |f(y)| \left(\int_{R<|x|<2R} \frac{|y|}{|x|^{n-\alpha+1}}\, dx \right) dy \\
&= C_K \int_{\mathbb{R}^n} |f(y)||y|\chi_{[0,\frac{1}{2}R]}(|y|) \left(R^{-\alpha} \int_{R<|x|<2R} \frac{1}{|x|^{n-\alpha+1}}\, dx \right) dy \\
&\leqslant C_K \int_{\mathbb{R}^n} |f(y)||y|\chi_{[0,\frac{1}{2}R]}(|y|) R^{-1}\, dy =: C_K \int_{\mathbb{R}^n} |f(y)| H_R(|y|)\, dy.
\end{aligned}$$

Obviously, for any fixed $y \in \mathbb{H}^n$ we have $(|y|)H_R(|y|) \to 0$ as $R \to \infty$. On the other hand,

$$|f(y)| H_R(|y|) \leqslant \frac{1}{2} |f(y)|,$$

so that, by the dominated convergence theorem,

$$R^{-\alpha} I_1(R) \to 0$$

as $R \to \infty$.

This completes the proof of the lemma. □

Definition 2.7. Let f be a measurable function on $\mathbb{R}^n$. If $t > 0$ we set

$$\lambda_f(t) = |\{|f| > t\}|.$$

If $1 \leqslant p \leqslant \infty$ and

$$\sup_{t>0} \lambda_f^p(t) < \infty,$$

we say that $f \in L^{p,\infty}(\mathbb{R}^n)$.

Definition 2.8. Following [3, Definition A.1], if $1 < p < \infty$, we set

$$\|u\|_{M^p} := \inf \left\{ C \geqslant 0; \int_K |u|\, dx \leqslant C|K|^{1/p'} \text{ for all } L\text{-measurable sets } K \subset \mathbb{R}^n \right\}.$$

By [3, Lemma A.2], we obtain the following lemma.

Lemma 2.9. *If $1 < p < \infty$, then*

$$\frac{(p-1)^p}{p^{p+1}} \|u\|_{M^p}^p \leqslant \sup_{\lambda>0} \{\lambda^p |\{|u| > \lambda\}|\} \leqslant \|u\|_{M^p}^p.$$

In particular, if $1 < p < \infty$, then $M^p = L^{p,\infty}(\mathbb{R}^n)$.

Corollary 2.10. *If $1 \leqslant s < p$, then $M^p \subset L^s_{\mathrm{loc}}(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof. If $u \in M^p$ then $|u|^s \in M^{p/s}$, and we can conclude thanks to Definition 2.8. □

Lemma 2.11. *Let E be a kernel of type $\alpha \in (0, n)$. Then for all $f \in L^1(\mathbb{R}^n)$ we have $f * E \in M^{n/(n-\alpha)}$ and there exists $C > 0$ such that*

$$\|f * E\|_{M^{n/(n-\alpha)}} \leqslant C \|f\|_{L^1(\mathbb{R}^n)}$$

for all $f \in L^1(\mathbb{R}^n)$. In particular, by Corollary 2.10, $f * E \in L^1_{\mathrm{loc}}$.

As in [1, Lemma 4.4 and Remark 4.5], we have the following remark.

Remark 2.12. Suppose $0 < \alpha < n$. If K is a kernel of type α and $\psi \in \mathcal{D}(\mathbb{R}^n)$, $\psi \equiv 1$ in a neighborhood of the origin, then the statements of Lemma 2.11 still hold if we replace K by $(1 - \psi)K$ or by ψK.

3 Differential forms and currents

Let $(dx_1, \ldots, dx_n)$ be the canonical basis of $(\mathbb{R}^n)^*$ and indicate as $\langle \cdot, \cdot \rangle$ the inner product in $(\mathbb{R}^n)^*$ that makes $(dx_1, \ldots, dx_n)$ an orthonormal basis. We put $\bigwedge^0(\mathbb{R}^n) := \mathbb{R}$ and, for $1 \leqslant h \leqslant n$,

$$\bigwedge^h(\mathbb{R}^n) := \operatorname{span}\{dx_{i_1} \wedge \cdots \wedge dx_{i_h} : 1 \leqslant i_1 < \cdots < i_h \leqslant n\}$$

the linear space of the alternating h-forms on $\mathbb{R}^n$. If $I := (i_1, \ldots, i_h)$ with $1 \leqslant i_1 < \cdots < i_h \leqslant n$, we set $|I| := h$ and

$$dx^I := dx_{i_1} \wedge \cdots \wedge dx_{i_h}.$$

We indicate as $\langle \cdot, \cdot \rangle$ also the inner product in $\bigwedge^h(\mathbb{R}^n)$ that makes $(dx_1, \ldots, dx_n)$ an orthonormal basis.

By translation, $\bigwedge^h(\mathbb{R}^n)$ defines a fibre bundle over $\mathbb{R}^n$, still denoted by $\bigwedge^h(\mathbb{R}^n)$. A differential form on $\mathbb{R}^n$ is a section of this fibre bundle.

Through this paper, if $0 \leqslant h \leqslant n$ and $\mathcal{U} \subset \mathbb{R}^n$ is an open set, we denote by $\Omega^h(\mathcal{U})$ the space of differential h-forms on $\mathcal{U}$, and by $d : \Omega^h(\mathcal{U}) \to \Omega^{h+1}(\mathcal{U})$ the exterior differential. Thus $(\Omega^\bullet(\mathcal{U}), d)$ is the de Rham complex in $\mathcal{U}$ and any $u \in \Omega^h$ can be written as

$$u = \sum_{|I|=h} u_I dx^I.$$

Definition 3.1. If $\mathcal{U} \subset \mathbb{R}^n$ is an open set and $0 \leqslant h \leqslant n$, we say that T is an h-current on $\mathcal{U}$ if T is a continuous linear functional on $\mathcal{D}(\mathcal{U}, \bigwedge^h(\mathbb{R}^n))$ endowed with the usual topology. We write $T \in \mathcal{D}'(\mathcal{U}, \bigwedge^h(\mathbb{R}^n))$. If $u \in L^1_{\mathrm{loc}}(\mathcal{U}, \bigwedge^h(\mathbb{R}^n))$, then u can be identified canonically with an h-current T_u through the formula

$$\langle T_u \,|\, \varphi \rangle := \int_{\mathcal{U}} u \wedge *\varphi = \int_{\mathcal{U}} \langle u, \varphi \rangle \, dx$$

for any $\varphi \in \mathcal{D}(\mathcal{U}, \bigwedge^h(\mathbb{R}^n))$.

From now on, if there is no way of misunderstandings, and $u \in L^1_{\mathrm{loc}}(\mathcal{U}, \bigwedge^h(\mathbb{R}^n))$, we shall write u instead of T_u.

Suppose now u is sufficiently smooth (take for example $u \in C^\infty(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$. If $\phi \in \mathcal{D}(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$, then by the Green formula,

$$\int_{\mathbb{R}^n} \langle du, \phi \rangle \, dx = \int_{\mathbb{R}^n} \langle u, d^*\phi \rangle \, dx.$$

Thus, if $T \in \mathcal{D}'(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$, it is natural to set

$$\langle dT \,|\, \phi \rangle = \langle T \,|\, d^*\phi \rangle$$

for any $\phi \in \mathcal{D}(\mathbb{R}^n, \bigwedge^{h+1}(\mathbb{R}^n))$.

Analogously, if $T \in \mathcal{D}'(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$, we set

$$\langle d^*T \,|\, \phi \rangle = \langle T \,|\, d\phi \rangle$$

for any $\phi \in \mathcal{D}(\mathbb{R}^n, \bigwedge^{h-1}(\mathbb{R}^n))$.

Notice that, if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$,

$$\langle u \,|\, d^*\phi \rangle = \int_{\mathbb{R}^n} u \wedge *d^*\varphi = (-1)^{h+1} \int_{\mathbb{R}^n} u \wedge d^*(*\varphi).$$

A straightforward approximation argument yields the following identity.

Lemma 3.2. *Let $u \in L^1(\mathbb{R}^n, \bigwedge^{h+1}(\mathbb{R}^n))$ be a closed form, and let K be a kernel of type $\mu \in (0, n)$. If $\psi \in \mathcal{D}(\mathbb{R}^n, \Omega^h)$, then*

$$\int \langle u, d^*(\psi * K) \rangle \, dx = 0. \tag{3.1}$$

Definition 3.3. In $\mathbb{R}^n$, we define the Laplace-Beltrami operator Δ_h on Ω^h by

$$\Delta_h = dd^* + d^*d.$$

Notice that

$$-\Delta_0 = \sum_{j=1}^{2n} \partial_j^2$$

is the usual Laplacian of $\mathbb{R}^n$.

Proposition 3.4 (See [11, (2.1.28)]). *If $u = \sum_{|I|=h} u_I dx^I$, then*

$$\Delta u = -\sum_{|I|=h} (\Delta u_I) dx^I.$$

For the sake of simplicity, since a basis of $\bigwedge^h(\mathbb{R}^n)$ is fixed, the operator Δ_h can be identified with a diagonal matrix-valued map, still denoted by Δ_h,

$$\Delta_h = -(\delta_{ij}\Delta)_{i,j=1,\dots,\dim \bigwedge^h(\mathbb{R}^n)} : \mathcal{D}'\Big(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n)\Big) \to D'\Big(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n)\Big), \tag{3.2}$$

where $D'(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$ is the space of vector-valued distributions on $\mathbb{R}^n$.

If we denote by Δ^{-1} the matrix valued kernel

$$\Delta_h^{-1} = -(\delta_{ij}\Delta^{-1})_{i,j=1,\dots,\dim \bigwedge^h(\mathbb{R}^n)} : \mathcal{D}'\Big(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n)\Big) \to D'\Big(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n)\Big), \tag{3.3}$$

then Δ_h^{-1} is a matrix-valued kernel of type 2 and

$$\Delta_h^{-1}\Delta_h \alpha = \Delta_h \Delta_h^{-1}\alpha = \alpha \quad \text{for all} \quad \alpha \in \mathcal{D}\Big(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n)\Big).$$

We notice that, if $n > 1$, since Δ_h^{-1} is associated with a kernel of type 2, $\Delta_h^{-1} f$ is well-defined when $f \in L^1(\mathbb{H}^n, E_0^h)$. More precisely, by Lemma 2.11 we have the following lemma.

Lemma 3.5. *If $1 \leqslant h < n$, and $R = R(D)$ is a homogeneous polynomial of degree $\ell = 1$ in $D_1, \dots, D_n$, we have*

$$\|f * R(D)\Delta_h^{-1}\|_{M^{n/(n-1)}} \leqslant C\|f\|_{L^1(\mathbb{R}^n)}$$

for all $f \in L^1(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$.

By Corollary 2.10*, in both cases,*

$$f * R(D)\Delta_h^{-1} \in L^1_{\mathrm{loc}}\Big(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n)\Big).$$

In particular, the map

$$\Delta_h^{-1} : L^1\Big(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n)\Big) \to L^1_{\mathrm{loc}}\Big(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n)\Big) \tag{3.4}$$

is continuous.

Remark 3.6. By Corollary 2.4, if $u \in L^1(\mathbb{R}^n, \bigwedge^{h+1}(\mathbb{R}^n))$ and $\psi \in \mathcal{D}(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$, then

$$\langle \Delta_h^{-1} u \,|\, \psi \rangle = \langle u \,|\, \Delta_h^{-1}\psi \rangle. \tag{3.5}$$

In this equation, the left-hand side is the action of a matrix-valued distribution on a vector-valued test function (see (3.3)), whereas the right-hand side is the inner product of an L^1 vector-valued function with an L^∞ vector-valued function.

A standard argument yields the following identities.

Lemma 3.7 (See [1, Lemma 4.11]). *If $\alpha \in \mathcal{D}(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$, then*

(i) $d\Delta_h^{-1}\alpha = \Delta_{h+1}^{-1} d\alpha$, $h = 0, 1, \ldots, n-1$;

(ii) $d^*\Delta_{\mathbb{R},h}^{-1}\alpha = \Delta_{\mathbb{R},h-1}^{-1} d^*\alpha$, $h = 1, \ldots, n$.

Lemma 3.8. *If $\alpha \in L^1(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$, then $\Delta_h^{-1}\alpha$ is well-defined and belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$. If in addition $d\alpha = 0$ in the distributional sense, then the following result holds*:

$$d\Delta_h^{-1}\alpha = 0.$$

Proof. Let $\phi \in \mathcal{D}(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$ be arbitrarily given. By Lemma 3.7, $d\Delta_h^{-1}\phi = \Delta_h^{-1} d\phi$. Thus Remark 3.6 and Lemma 3.2 yield

$$\langle d\Delta_h^{-1}\alpha \,|\, \phi\rangle = \langle \Delta_h^{-1}\alpha \,|\, d\phi\rangle = \langle \alpha \,|\, \Delta_h^{-1} d\phi\rangle = \langle \alpha \,|\, d\Delta_h^{-1}\phi\rangle = 0.$$

This completes the proof. □

4 n-parabolicity

Recall that a noncompact Riemannian manifold M is *p-parabolic* if for every compact subset K and every $\epsilon > 0$, there exists a smooth compactly supported function χ on M such that $\chi \geqslant 1$ on K and

$$\int_M |d\chi|^p < \epsilon.$$

It is well known that the Euclidean n-space is n-parabolic (the relevant functions χ can be taken to be piecewise affine functions of $\log r$, where r is the distance to the origin). It follows that the Sobolev inequality in L^n cannot hold, and, as we saw in Section 1, that the Poincaré inequality on n-forms fails as well.

Here, we explain another consequence of n-parabolicity.

Proposition 4.1. *Let ω be a k-form in $L^1(\mathbb{R}^n)$. Assume that $\omega = d\phi$ where $\phi \in L^{n/(n-1)}(\mathbb{R}^n)$. Then, for every constant coefficient $(n-k)$-form β,*

$$\int_{\mathbb{R}^n} \omega \wedge \beta = 0.$$

Proof. Let χ_R be a smooth compactly supported function on $\mathbb{R}^n$ such that $\chi_R = 1$ on $B(R)$ and

$$\int |d\chi_R|^n \leqslant \frac{1}{R}.$$

Let $\omega_R = d(\chi_R\phi)$. Then, since $\chi_R\phi \wedge \beta$ is compactly supported,

$$\int_{\mathbb{R}^n} \omega_R \wedge \beta = \int_{\mathbb{R}^n} d(\chi_R\phi \wedge \beta) = 0.$$

Write

$$\omega_R = d\chi_R \wedge \phi + \chi_R\omega.$$

Since

$$\left| \int_{\mathbb{R}^n} d\chi_R \wedge \phi \wedge \beta \right| \leqslant \|d\chi_R\|_n \|\phi\|_{n/(n-1)} \|\beta\|_\infty \leqslant \frac{C}{R^{1/n}}$$

tends to 0,

$$\begin{aligned}\int_{\mathbb{R}^n} \omega \wedge \beta &= \lim_{R\to\infty} \int_{\mathbb{R}^n} \chi_R\omega \wedge \beta \\ &= -\lim_{R\to\infty} \int_{\mathbb{R}^n} \omega_R \wedge \beta = 0.\end{aligned}$$

This completes the proof. □

In other words, the vanishing of all integrals $\int \omega \wedge \beta$ is a necessary condition for an L^1 k-form to be the differential of an $L^{n/(n-1)}$ $(k-1)$-form.

5 Main results

The following estimate provides primitives for globally defined closed L^1-forms, and can be derived from the Lanzani-Stein inequality [12], approximating closed forms in $L^1_0((\mathbb{R}^n),\bigwedge^h(\mathbb{R}^n))$ by means of closed compactly supported smooth forms. The convergence of the approximation is guaranteed by Lemma 2.6.

Proposition 5.1. *Denote by* $L^1_0((\mathbb{R}^n),\bigwedge^h(\mathbb{R}^n))$ *the subspace of* $L^1(\mathbb{R}^n,\bigwedge^h(\mathbb{R}^n))$ *of forms with vanishing average, and by* $\mathcal{H}^1(\mathbb{R}^n)$ *the classic real Hardy space (see* [14, *Chapter* 3]*). We have*

(i) *if* $h<n$, *then*

$$\|d^*\Delta_h^{-1}u\|_{L^{n/(n-1)}(\mathbb{R}^n)}\leqslant C\|u\|_{L^1(\mathbb{R}^n)}\quad \textit{for all}\quad u\in L^1_0(\mathbb{R}^n,\textstyle\bigwedge^h(\mathbb{R}^n))\cap\ker d;$$

(ii) *if* $h=n$, *then*

$$\|d^*\Delta_n^{-1}u\|_{L^{n/(n-1)}(\mathbb{R}^n)}\leqslant C\|u\|_{\mathcal{H}^1(\mathbb{R}^n)}\quad \textit{for all}\quad u\in \mathcal{H}^1(\mathbb{R}^n)\cap\ker d.$$

We stress that the vanishing average assumption is necessary (see Proposition 4.1*).*

A standard approximation argument (akin to that of the classical Meyers-Serrin theorem) yields the following density result.

Lemma 5.2. *Let* $B\subset\mathbb{R}^n$ *be an open set. If* $0\leqslant h\leqslant n$, *we set*

$$(L^1\cap d^{-1}L^1)\left(B,\bigwedge^h(\mathbb{R}^n)\right):=\left\{\alpha\in L^1\left(B,\bigwedge^h(\mathbb{R}^n)\right);\,d\alpha\in L^1\left(B,\bigwedge^{h+1}(\mathbb{R}^n)\right)\right\},$$

endowed with the graph norm. Then $C^\infty(B,\bigwedge^h(\mathbb{R}^n))$ *is dense in* $(L^1\cap d^{-1}L^1)(B,\bigwedge^h(\mathbb{R}^n))$.

Again through an approximation argument we can prove the following two lemmas.

Lemma 5.3. *If* $K=d^*\Delta_c^{-1}$, *then*

- K *is a kernel of type* 1;
- *if* χ *is a smooth function with compact support in* B, *then the identity*

$$\chi=dK\chi+Kd\chi$$

holds on the space $(L^1\cap d^{-1}L^1)(B,\bigwedge^\bullet(\mathbb{R}^n))$.

Lemma 5.4. *If* $1\leqslant h<n$, *let* $\psi\in L^1(\mathbb{R}^n,\bigwedge^h(\mathbb{R}^n))$ *be a compactly supported form with* $d\psi\in L^1(\mathbb{R}^n,\bigwedge^{h+1}(\mathbb{R}^n))$, *and let* $\xi\in\bigwedge^{2n-h}$ *be a constant coefficient form. Then*

$$\int_{\mathbb{R}^n}d\psi\wedge\xi=0.$$

We are now able to prove the following (approximate) homotopy formula for closed forms.

Proposition 5.5. *Let* $B\Subset B'$ *be open sets in* $\mathbb{R}^n$. *For* $h=1,\dots,n-1$, *take* $q=n/(n-1)$. *Then there exists a smoothing operator*

$$S:L^1\left(B',\bigwedge^h(\mathbb{R}^n)\right)\to W^{s,q}\left(B,\bigwedge^h(\mathbb{R}^n)\right)$$

for every $s\in\mathbb{N}$, *and a bounded operator*

$$T:L^1\left(B',\bigwedge^h(\mathbb{R}^n)\right)\to L^q\left(B,\bigwedge^{h-1}(\mathbb{R}^n)\right)$$

such that, for closed L^1*-forms* α *on* B',

$$\alpha=dT\alpha+S\alpha\quad \textit{on}\quad B.\tag{5.1}$$

In particular, $S\alpha$ *is closed.*

Furthermore, T *and* S *merely enlarge by a small amount the support of compactly supported differential forms.*

Proof. Le us fix two open sets B_0 and B_1 with

$$B \Subset B_0 \Subset B_1 \Subset B',$$

and a cut-off function $\chi \in \mathcal{D}(B_1)$, $\chi \equiv 1$ on B_0. If $\alpha \in (L^1 \cap d^{-1})(B', \bigwedge^\bullet(\mathbb{R}^n))$, we set $\alpha_0 = \chi\alpha$, continued by zero outside B_1. Denote by k the kernel associated with K in Lemma 5.3. We consider a cut-off function ψ_R supported in an R-neighborhood of the origin, such that $\psi_R \equiv 1$ near the origin. Then we can write $k = k\psi_R + (1-\psi_R)k$. Thus, let us denote by K_R the convolution operator associated with $\psi_R k$. By Lemma 5.3,

$$\alpha_0 = dK\alpha_0 + K_d\alpha_0 = dK_R\alpha_0 + K_Rd\alpha_0 + S\alpha_0, \tag{5.2}$$

where S_0 is defined by

$$S\alpha_0 := d((1-\psi_R)k * \alpha_0) + (1-\psi_R)k * d\alpha_0.$$

We set

$$T_1\alpha := K_R\alpha_0, \quad S_1\alpha := S\alpha_0.$$

If $\beta \in L^1(B_1, \bigwedge^h(\mathbb{R}^n))$, we set

$$T_1\beta := K_R(\chi\beta)_{\,|B}, \quad S_1\alpha := S\alpha_{0\,|B}.$$

We notice that, provided $R > 0$ is small enough, the values of $T_1\beta$ do not depend on the continuation of β outside B_1. Moreover,

$$K_Rd\alpha_{0\,|B} = K_Rd(\chi\alpha)_{\,|B} = K_R(\chi d\alpha)_{\,|B} = T_1(d\alpha),$$

since $d(\chi\alpha) \equiv \chi d\alpha$ on B_0. Thus, by (5.2),

$$\alpha = dT_1\alpha + T_1d\alpha + S_1\alpha \quad \text{in} \quad B.$$

Assume now that $d\alpha = 0$. Then

$$\alpha = dT_1\alpha + S_1\alpha \quad \text{in} \quad B.$$

Write $\phi = T_1\alpha \in L^1(B_0, \bigwedge^{h-1}(\mathbb{R}^n))$. By difference,

$$d\phi = \alpha - S_1\alpha \in L^1\left(B_0, \bigwedge^{h-1}(\mathbb{R}^n)\right).$$

The next step will consist of proving that $\phi \in L^q((B_0), \bigwedge^{h-1}(\mathbb{R}^n))$, "iterating" the previous argument. Let us sketch how this iteration will work: let ζ be a cut-off function supported in B_0, identically equal to 1 in a neighborhood $\mathcal{U}$ of B, and set $\omega = d(\zeta\phi)$. Obviously, the form $\zeta\phi$ (and therefore also ω) is defined on all $\mathbb{R}^n$ and is compactly supported in B_0. In addition, ω is closed. Suppose for a while we are able to prove that

(a) $\omega \in L^1(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$;

(b) $\|K_0\omega\|_{L^q(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))} \leqslant C\|\alpha\|_{L^1(B', \bigwedge^h(\mathbb{R}^n))}$, and let us show how the argument can be carried out.

First we stress that, if R is small enough, then when $x \in B$, $K_R\omega(x)$ depends only on the restriction of $d\phi$ to $\mathcal{U}$, so that the map

$$\alpha \to K_R\omega\,|_B$$

is linear.

In addition, notice that $\omega = \chi\omega$, so that, by (5.2),

$$d(\zeta\phi) = \omega = dK_R\omega + S\omega.$$

Therefore in B,

$$\alpha - S_1\alpha = d\phi = d(\zeta\phi) = dK_R\omega + S_0\omega,$$

and then in B,

$$\begin{aligned}\alpha &= d(K_R\omega\,|_B) + S_1\alpha\,|_B + S\omega\,|_B \\ &=: d(K_R(\chi\omega)\,|_B) + S\alpha = dT\alpha + S\alpha.\end{aligned}$$

First notice that the map $\alpha \to \omega = \omega(\alpha)$ is linear, and hence T and S are linear maps. In addition, by (b),

$$\|T\alpha\|_{L^q(B,\bigwedge^{h-1}(\mathbb{R}^n))} \leqslant \|K_R(\chi\omega)\|_{L^q(\mathbb{R}^n,\bigwedge^h(\mathbb{R}^n))} = \|K_R(\omega)\|_{L^q(\mathbb{R}^n,\bigwedge^h(\mathbb{R}^n))} \leqslant C\,\|\alpha\|_{L^1(B',\bigwedge^h(\mathbb{R}^n))}.$$

As for the map $\alpha \to S\alpha$ we just point out that, when $x \in B$, $S\alpha(x)$ can be written as the convolution of α_0 with a smooth kernel with bounded derivatives of any order, the proof is completed. □

Interior Poincaré and Sobolev inequalities follow now from the approximate homotopy formula for the closed forms (5.1).

Corollary 5.6 (Interior Poincaré and Sobolev inequalities). *Let $B \Subset B'$ be open sets in $\mathbb{R}^n$, and assume B is convex. For $h = 1, \dots, n-1$, let $q = n/(n-1)$. Then for every closed form $\alpha \in L^1(B', \bigwedge^h(\mathbb{R}^n))$, there exists an $(h-1)$-form $\phi \in L^q(B, \bigwedge^{h-1}(\mathbb{R}^n))$, such that*

$$d\phi = \alpha_{|B} \quad \text{and} \quad \|\phi\|_{L^q(B,\bigwedge^{h-1}(\mathbb{R}^n))} \leqslant C\,\|\alpha\|_{L^1(B',\bigwedge^h(\mathbb{R}^n))}.$$

Furthermore, if α is compactly supported, so is ϕ.

Proof. By Proposition 5.5, the h-form $S\alpha$ defined in (5.1) is closed and belongs to $L^q(B, \bigwedge^h(\mathbb{R}^n))$, with the norm controlled by the L^1-norm of α. Thus we can apply Iwaniec-Lutoborski's homotopy [10, Proposition 4.1] to obtain a differential $(h-1)$-form γ on B with the norm in $W^{1,q}(B, \bigwedge^{h-1}(\mathbb{R}^n))$ controlled by the L^q-norm of $S\alpha$ and therefore from the L^1-norm of α. Set $\phi := T\alpha + \gamma$. Clearly,

$$d\phi = dT\alpha + d\gamma = dT\alpha + S\alpha = \alpha.$$

Then, by Proposition 5.5,

$$\|\phi\|_{L^q(B,\bigwedge^{h-1}(\mathbb{R}^n))} \leqslant C(\|\alpha\|_{L^1(B,\bigwedge^h(\mathbb{R}^n))} + \|S\alpha\|_{L^q(B,\bigwedge^h(\mathbb{R}^n))}) \leqslant C\|\alpha\|_{L^1(B,\bigwedge^h(\mathbb{R}^n))}.$$

This completes the proof. □

Acknowledgements The first author and the second author were supported by Funds for Selected Research Topics from the University of Bologna, MAnET Marie Curie Initial Training Network, GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica "F. Severi"), Italy, and PRIN (Progetti di Ricerca di Rilevante Interesse Nazionale) of the MIUR (Ministero dell'Istruzione dell'Università e della Ricerca), Italy. The third author was supported by MAnET Marie Curie Initial Training Network, Agence Nationale de la Recherche (Grant Nos. ANR-10-BLAN 116-01 GGAA and ANR-15-CE40-0018 SRGI), and thanks the hospitality of Isaac Newton Institute, of EPSRC (Engineering and Physical Sciences Research Council) (Grant No. EP/K032208/1) and Simons Foundation.

References

1 Baldi A, Franchi B, Pansu P. Poincaré and Sobolev inequalities for differential forms in Heisenberg groups. ArXiv:1711.09786, 2017

2 Baldi A, Franchi B, Pansu P. L^1-Poincaré inequalities for differential forms on Euclidean spaces and Heisenberg groups. ArXiv:1902.04819, 2019

3 Benilan P, Brezis H, Crandall M G. A semilinear equation in $L^1(\mathbb{R}^N)$. Ann Sc Norm Super Pisa Cl Sci (5), 1975, 2: 523–555

4 Bourgain J, Brezis H. New estimates for elliptic equations and Hodge type systems. J Eur Math Soc (JEMS), 2007, 9: 277–315

5 Federer H, Fleming W H. Normal and integral currents. Ann of Math (2), 1960, 72: 458–520

6 Folland G B. Lectures on Partial Differential Equations. Tata Institute of Fundamental Research. Lectures on Mathematics and Physics, vol. 70. Berlin: Springer-Verlag, 1983

7 Folland G B, Stein E M. Hardy Spaces on Homogeneous Groups. Mathematical Notes, vol. 28. Princeton: Princeton University Press, 1982

8 Franchi B, Gallot S, Wheeden R L. Sobolev and isoperimetric inequalities for degenerate metrics. Math Ann, 1994, 300: 557–571

9 Franchi B, Lu G, Wheeden R L. Representation formulas and weighted Poincaré inequalities for Hörmander vector fields. Ann Inst Fourier (Grenoble), 1995, 45: 577–604

10 Iwaniec T, Lutoborski A. Integral estimates for null Lagrangians. Arch Ration Mech Anal, 1993, 145: 25–79

11 Jost J. Riemannian Geometry and Geometric Analysis, 5th ed. Berlin: Springer-Verlag, 2008

12 Lanzani L, Stein E M. A note on div curl inequalities. Math Res Lett, 2005, 12: 57–61

13 Long R L, Nie F S. Weighted Sobolev inequality and eigenvalue estimates of Schrödinger operators. In: Lecture Notes in Mathematics, vol. 1494. Berlin: Springer, 1991, 131–141

14 Stein E M. Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton Mathematical Series, vol. 43. Princeton: Princeton University Press, 1993

15 Van Schaftingen J. Limiting Bourgain-Brezis estimates for systems of linear differential equations: Theme and variations. J Fixed Point Theory Appl, 2014, 15: 273–297