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Mehler's formula and functional calculus

Dedicated to Professor Jean-Yves Chemin on the Occasion of His 60th Birthday

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Abstract We show that Mehler's formula can be used to handle several formulas involving the quantization of singular Hamiltonians. In particular, we diagonalize in the Hermite basis the Weyl quantization of the characteristic function of several domains of the phase space.

Keywords Mehler's formula, quantization, rough Hamiltonians

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1 Quantization of radial functions

1.1 Basic formulas

In this section, we work in one dimension and consider a function F in the Schwartz class of $\mathbb{R}$. We want to calculate somewhat explicitly the Weyl quantization of $F(x^2+\xi^2)$ (see our Appendix A), denoted by

$$(F(x^2+\xi^2))^w,$$

and also extend that computation to the case where F is merely $L^\infty(\mathbb{R})$. We have, say for F in the Wiener algebra $\mathcal{W}(\mathbb{R}) = \text{Fourier}(L^1(\mathbb{R}))$,

$$(F(x^2+\xi^2))^w = \int_{\mathbb{R}} \hat{F}(\tau)(\mathrm{e}^{2\mathrm{i}\pi\tau(x^2+\xi^2)})^w d\tau,$$

as an absolutely converging integral of a function defined on $\mathbb{R}$ (equipped with the Lebesgue measure) valued in $\mathcal{B}(L^2(\mathbb{R}))$ (bounded endomorphisms of $L^2(\mathbb{R})$). In fact applying Mehler's formula (B.2), we find

$$\underbrace{(\mathrm{e}^{2\mathrm{i}\pi\tau(x^2+\xi^2)})^w}_{\substack{\text{operator with Weyl symbol}\\ \mathrm{e}^{2\mathrm{i}\pi\tau(x^2+\xi^2)}}} = \cos(\arctan\tau)\ \underbrace{\mathrm{e}^{2\mathrm{i}\pi(\arctan\tau)(x^2+\xi^2)^w}}_{\substack{\text{exponential } \mathrm{e}^{\mathrm{i}M},\\ \text{with } M \text{ selfadjoint operator}\\ =2\pi(\arctan\tau)(x^2+\xi^2)^w}},$$

so that, using the spectral decomposition (A.22) of the harmonic oscillator

$$\pi(x^2+\xi^2)^w,$$

we get

$$\begin{aligned}(F(x^2+\xi^2))^w &= \int_{\mathbb{R}} \hat{F}(\tau) \sum_{k\geqslant 0} \mathrm{e}^{2\mathrm{i}(\arctan\tau)(k+\frac{1}{2})}\mathbb{P}_k \frac{d\tau}{\sqrt{1+\tau^2}} \\ &= \sum_{k\geqslant 0}\int_{\mathbb{R}} \hat{F}(\tau)\mathrm{e}^{2\mathrm{i}(k+\frac{1}{2})\arctan\tau}\frac{d\tau}{\sqrt{1+\tau^2}}\mathbb{P}_k,\end{aligned}$$

where the use of Fubini's theorem is justified by

$$\int_{\mathbb{R}} |\hat{F}(\tau)|\frac{d\tau}{\sqrt{1+\tau^2}} < +\infty, \quad \mathbb{P}_k \geqslant 0, \quad \sum_k \mathbb{P}_k = \mathrm{Id}\,.$$

We have

$$\int_{\mathbb{R}} \hat{F}(\tau)\mathrm{e}^{2\mathrm{i}(k+\frac{1}{2})\arctan\tau}\frac{d\tau}{\sqrt{1+\tau^2}} = \int_{\mathbb{R}} \hat{F}(\tau)(\cos(\arctan\tau)+\mathrm{i}\sin(\arctan\tau))^{2k+1}\frac{d\tau}{\sqrt{1+\tau^2}},$$

and, using Appendix A, we get

$$\int_{\mathbb{R}} \hat{F}(\tau)\mathrm{e}^{2\mathrm{i}(k+\frac{1}{2})\arctan\tau}\frac{d\tau}{\sqrt{1+\tau^2}} = \int_{\mathbb{R}} \hat{F}(\tau)(1+\mathrm{i}\tau)^{2k+1}\frac{d\tau}{(1+\tau^2)^{k+1}}.$$

We have proved the following lemma.

Lemma 1.1. *Let F be a tempered distribution on $\mathbb{R}$ such that $\hat{F}$ is locally integrable and such that*

$$\int_{\mathbb{R}} |\hat{F}(\tau)|\frac{d\tau}{\sqrt{1+\tau^2}} < +\infty. \tag{1.1}$$

Then the operator $(F(x^2+\xi^2))^w$ has the spectral decomposition

$$(F(x^2+\xi^2))^w = \sum_{k\geqslant 0}\int_{\mathbb{R}} \frac{\hat{F}(\tau)(1+\mathrm{i}\tau)^{2k+1}}{(1+\tau^2)^{k+1}}d\tau\mathbb{P}_k. \tag{1.2}$$

We notice that the regularity requirement (1.1) is quite mild and is satisfied in particular when $F = \mathbf{1}_{[a,b]}$ with a, b being real numbers. Our first example of rough Hamiltonian is $\mathbf{1}_{[a,b]}(x^2+\xi^2)$, which we would like to quantize (by the Weyl formula). We know from the above lemma that the Weyl quantization of that Hamiltonian is diagonal in the Hermite basis and we shall need only to calculate the integrals occurring on the right-hand side of (1.2).

1.2 Indicatrix of a disc

Let us assume now that, with some $a \geqslant 0$,

$$F = \mathbf{1}_{[-\frac{a}{2\pi},\frac{a}{2\pi}]}, \quad \text{so that} \quad F(x^2+\xi^2) = \mathbf{1}\{2\pi(x^2+\xi^2) \leqslant a\}.$$

According to Appendix A, we have

$$\hat{F}(\tau) = \frac{\sin a\tau}{\pi\tau},$$

so that (1.1) holds true. We find in this case,

$$(F(x^2+\xi^2))^w = \sum_{k\geqslant 0} F_k(a)\mathbb{P}_k, \quad F_k(a) = \int_{\mathbb{R}} \frac{\sin a\tau}{\pi\tau}\frac{(1+\mathrm{i}\tau)^k}{(1-\mathrm{i}\tau)^{k+1}}d\tau, \tag{1.3}$$

so that (note that $F_k(a)$ is real-valued since F is real-valued and thus the operator $(F(x^2+\xi^2))^w$ is selfadjoint) for $a > 0$, using the result (A.29), we obtain

$$F_k'(a) = \frac{1}{\pi}\int_{\mathbb{R}} \cos a\tau \frac{(1+\mathrm{i}\tau)^k}{(1-\mathrm{i}\tau)^{k+1}}d\tau$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}a\tau}\left\{\frac{(1+\mathrm{i}\tau)^k}{(1-\mathrm{i}\tau)^{k+1}} + \frac{(1-\mathrm{i}\tau)^k}{(1+\mathrm{i}\tau)^{k+1}}\right\}d\tau$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}a\tau}\left\{\frac{\mathrm{i}^k(\tau-\mathrm{i})^k}{(-\mathrm{i})^{k+1}(\tau+\mathrm{i})^{k+1}} + \frac{(-\mathrm{i})^k(\tau+\mathrm{i})^k}{\mathrm{i}^{k+1}(\tau-\mathrm{i})^{k+1}}\right\}d\tau$$
$$= \frac{(-1)^k}{2\mathrm{i}\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}a\tau}\left\{-\frac{(\tau-\mathrm{i})^k}{(\tau+\mathrm{i})^{k+1}} + \frac{(\tau+\mathrm{i})^k}{(\tau-\mathrm{i})^{k+1}}\right\}d\tau.$$

We shall now calculate explicitly both integrals above: let $1 < R$ be given and let us consider the closed path (see Figure 1)

$$\gamma_R = [-R,R] \cup \underbrace{\{R\mathrm{e}^{\mathrm{i}\theta}\}_{0\leqslant\theta\leqslant\pi}}_{\gamma_{2;R}}. \tag{1.4}$$

We have

$$\frac{1}{2\mathrm{i}\pi}\int_{\gamma_R} \mathrm{e}^{\mathrm{i}a\tau}\left\{-\frac{(\tau-\mathrm{i})^k}{(\tau+\mathrm{i})^{k+1}} + \frac{(\tau+\mathrm{i})^k}{(\tau-\mathrm{i})^{k+1}}\right\}d\tau = \mathrm{Res}\left(\mathrm{e}^{\mathrm{i}a\tau}\frac{(\tau+\mathrm{i})^k}{(\tau-\mathrm{i})^{k+1}};\mathrm{i}\right)$$
$$= \frac{1}{k!}\left(\frac{d}{d\tau}\right)^k\{\mathrm{e}^{\mathrm{i}a\tau}(\tau+\mathrm{i})^k\}_{|\tau=\mathrm{i}},$$

and we note that, for $a > 0$,

$$\lim_{R\to+\infty}\int_{\gamma_{2;R}} \mathrm{e}^{\mathrm{i}a\tau}\left\{-\frac{(\tau-\mathrm{i})^k}{(\tau+\mathrm{i})^{k+1}} + \frac{(\tau+\mathrm{i})^k}{(\tau-\mathrm{i})^{k+1}}\right\}d\tau = 0,$$

since for $R \geqslant 2$,

$$\int_0^\pi |\mathrm{e}^{\mathrm{i}aR\mathrm{e}^{\mathrm{i}\theta}}|\left|-\frac{(R\mathrm{e}^{\mathrm{i}\theta}-\mathrm{i})^k}{(R\mathrm{e}^{\mathrm{i}\theta}+\mathrm{i})^{k+1}} + \frac{(R\mathrm{e}^{\mathrm{i}\theta}+\mathrm{i})^k}{(R\mathrm{e}^{\mathrm{i}\theta}-\mathrm{i})^{k+1}}\right||\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}|d\theta$$
$$\leqslant \int_0^\pi \mathrm{e}^{-aR\sin\theta}\left|-\frac{(\mathrm{e}^{\mathrm{i}\theta}-\mathrm{i}R^{-1})^k}{(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{i}R^{-1})^{k+1}} + \frac{(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{i}R^{-1})^k}{(\mathrm{e}^{\mathrm{i}\theta}-\mathrm{i}R^{-1})^{k+1}}\right|d\theta$$
$$\leqslant \int_0^\pi \mathrm{e}^{-aR\sin\theta}d\theta \sup_{0\leqslant\rho\leqslant1/2}\left\{\frac{(1+\rho)^k}{(1-\rho)^{k+1}} + \frac{(1+\rho)^k}{(1-\rho)^{k+1}}\right\}.$$

For $a > 0$, we obtain

$$\lim_{R\to+\infty}\int_0^\pi \mathrm{e}^{-aR\sin\theta}d\theta = 0,$$

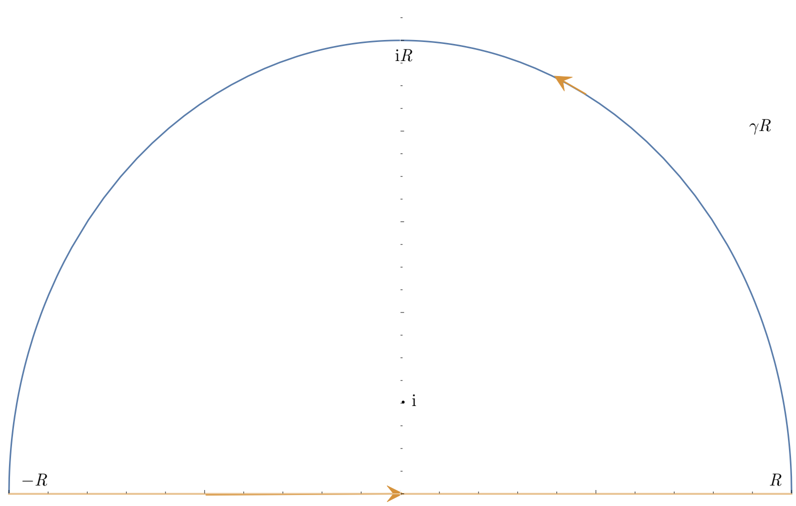

Figure 1 (Color online) $\gamma_R = [-R,R] \cup \{R\mathrm{e}^{\mathrm{i}\theta}\}_{0\leqslant\theta\leqslant\pi}$

by dominated convergence. As a result, we get

$$F_k'(a)=(-1)^k\frac{1}{k!}\left(\frac{d}{d\tau}\right)^k\{\mathrm{e}^{\mathrm{i}a\tau}(\tau+\mathrm{i})^k\}_{|\tau=\mathrm{i}}=(-1)^k\frac{1}{k!}\left(\frac{d}{\frac{\mathrm{i}}{a}d\epsilon}\right)^k\left\{\mathrm{e}^{-a-\epsilon}\left(\mathrm{i}+\mathrm{i}\frac{\epsilon}{a}+\mathrm{i}\right)^k\right\}_{|\epsilon=0},$$

i.e.,

$$F_k'(a)=\frac{(-1)^k}{k!}\mathrm{e}^{-a}\left(\frac{d}{d\epsilon}\right)^k\{\mathrm{e}^{-\epsilon}(2a+\epsilon)^k\}_{|\epsilon=0}.$$

We note that F_k' belongs to $L^1(\mathbb{R}_+)$ as the product of e^{-a} by a polynomial. We have also that

$$\lim_{a\to+\infty}F_k(a)=1\quad\text{(see Appendix on page 1162)},\tag{1.5}$$

and this yields

$$F_k(a)=1+\int_{+\infty}^a F_k'(b)db=1-\int_a^{+\infty}\frac{(-1)^k}{k!}\mathrm{e}^{-b}\left(\frac{d}{d\epsilon}\right)^k\{\mathrm{e}^{-\epsilon}(2b+\epsilon)^k\}_{|\epsilon=0}db,$$

so that

$$F_k(a)=1-\mathrm{e}^{-a}P_k(a)\tag{1.6}$$

with

$$\begin{aligned}P_k(a)&=\frac{(-1)^k}{k!}\int_0^{+\infty}\mathrm{e}^{-t}\left(\frac{d}{d\epsilon}\right)^k\{\mathrm{e}^{-2\epsilon}(a+t+\epsilon)^k\}_{|\epsilon=0}dt\\&=\frac{(-1)^k}{k!}\int_0^{+\infty}\mathrm{e}^{t}\left(\frac{d}{d\epsilon}\right)^k\{\mathrm{e}^{-2\epsilon-2t}(a+t+\epsilon)^k\}_{|\epsilon=0}dt\\&=\frac{(-1)^k}{k!}\int_0^{+\infty}\mathrm{e}^{t}\left(\frac{d}{dt}\right)^k\{\mathrm{e}^{-2t}(a+t)^k\}dt.\end{aligned}\tag{1.7}$$

We see that P_k is a polynomial with the leading monomial $\frac{2^ka^k}{k!}$ (by a direct computation) and $P_k(0)=1$ (since $0=F_k(0)=1-P_k(0)$) and moreover, using Laguerre polynomials (see, e.g., (C.1) in our Appendix C), we obtain

$$P_k(a)=\frac{(-1)^k}{k!}\int_0^{+\infty}\mathrm{e}^{-t}\mathrm{e}^{2t+2a}\left(\frac{d}{2dt}\right)^k\{\mathrm{e}^{-2t-2a}(2a+2t)^k\}dt\tag{1.8a}$$

$$=(-1)^k\int_0^{+\infty}\mathrm{e}^{-t}L_k(2t+2a)dt,\tag{1.8b}$$

and this gives in particular

$$\begin{aligned}P_k'(a)&=(-1)^k\int_0^{+\infty}\mathrm{e}^{-t}2L_k'(2t+2a)dt\\&=(-1)^k\left\{[\mathrm{e}^{-t}L_k(2t+2a)]_{t=0}^{t=+\infty}+\int_0^{+\infty}\mathrm{e}^{-t}L_k(2t+2a)dt\right\}\\&=(-1)^{k+1}L_k(2a)+P_k(a).\end{aligned}\tag{1.9}$$

Moreover we have from (1.7), for $k\geqslant 1$,

$$\begin{aligned}P_k'(a)&=\frac{(-1)^k}{k!}\int_0^{+\infty}\mathrm{e}^{t}\left(\frac{d}{dt}\right)^k\{\mathrm{e}^{-2t}k(a+t)^{k-1}\}dt\\&=\frac{(-1)^k}{k!}\int_0^{+\infty}\mathrm{e}^{t}\frac{d}{dt}\left(\frac{d}{dt}\right)^{k-1}\{\mathrm{e}^{-2t}k(a+t)^{k-1}\}dt\\&=\frac{(-1)^k}{k!}\left\{\left[\mathrm{e}^{t}\left(\frac{d}{dt}\right)^{k-1}\{\mathrm{e}^{-2t}k(a+t)^{k-1}\}\right]_{t=0}^{t=+\infty}-\int_0^{+\infty}\mathrm{e}^{t}\left(\frac{d}{dt}\right)^{k-1}\{\mathrm{e}^{-2t}k(a+t)^{k-1}\}dt\right\}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{d}{dt}\right)^{k-1}\{e^{-2t}(a+t)^{k-1}\}_{|t=0} + \frac{(-1)^{k-1}}{(k-1)!}\int_0^{+\infty} e^{t}\left(\frac{d}{dt}\right)^{k-1}\{e^{-2t}(a+t)^{k-1}\}dt \\
&= \frac{(-1)^{k-1}}{(k-1)!} e^{2t+2a}\left(\frac{d}{2dt}\right)^{k-1}\{e^{-2t-2a}(2a+2t)^{k-1}\}_{|t=0} \\
&\quad + \frac{(-1)^{k-1}}{(k-1)!}\int_0^{+\infty} e^{t}\left(\frac{d}{dt}\right)^{k-1}\{e^{-2t}(a+t)^{k-1}\}dt \\
&= (-1)^{k-1}L_{k-1}(2a) + P_{k-1}(a),
\end{aligned}
$$

so that

$$
\forall\, k \geqslant 1, \quad P_k'(a) = (-1)^{k-1}L_{k-1}(2a) + P_{k-1}(a) = (-1)^{k+1}L_k(2a) + P_k(a). \tag{1.10}
$$

This implies for $N \geqslant 1$,

$$
\sum_{1\leqslant k\leqslant N} P_k(a) - \sum_{1\leqslant k\leqslant N}(-1)^k L_k(2a) = \sum_{0\leqslant k\leqslant N-1} P_k(a) + \sum_{0\leqslant k\leqslant N-1}(-1)^k L_k(2a),
$$

yielding

$$
P_N(a) - \underbrace{P_0(a)}_{=1=L_0(a)} = \sum_{1\leqslant k\leqslant N}(-1)^k L_k(2a) + \sum_{0\leqslant k\leqslant N-1}(-1)^k L_k(2a),
$$

and

$$
P_N(a) = \sum_{0\leqslant k\leqslant N}(-1)^k L_k(2a) + \sum_{0\leqslant k\leqslant N-1}(-1)^k L_k(2a). \tag{1.11}
$$

Note that the previous formula holds as well for $N=0$, since $P_0 = 1 = L_0$.

Although the function $\mathbb{R}_+ \ni a \mapsto F_k(a)$ has no monotonicity properties, we prove below that $\mathbb{R}_+ \ni a \mapsto P_k(a)$ is indeed increasing. For that purpose, let us use (1.10), which implies

$$
\begin{aligned}
&P_k'(a) = (-1)^{k-1}L_{k-1}(2a) + P_{k-1}(a), \quad k \geqslant 1, \\
&P_{k-1}(a) = P_{k-2}(a) + (-1)^{k-2}L_{k-2}(2a) + (-1)^{k-1}L_{k-1}(2a), \quad k \geqslant 2, \\
&P_k'(a) = 2(-1)^{k-1}L_{k-1}(2a) + (-1)^{k-2}L_{k-2}(2a) + P_{k-2}(a), \quad k \geqslant 2.
\end{aligned}
$$

We claim that for $k \geqslant 1$,

$$
P_k'(a) = 2\sum_{0\leqslant l\leqslant k-1}(-1)^l L_l(2a). \tag{1.12}
$$

That property holds for $k=1$ since $P_1(a) = 1+2a$: we check $P_1'(a) = 2$. Moreover we have

$$
\begin{aligned}
P_{k+1}'(a) &= (-1)^k L_k(2a) + P_k(a) \quad \text{(from the first equation in (1.10))} \\
&\overset{\text{(using (1.11))}}{=} (-1)^k L_k(2a) + \sum_{0\leqslant l\leqslant k}(-1)^l L_l(2a) + \sum_{0\leqslant l\leqslant k-1}(-1)^l L_l(2a) \\
&= 2\sum_{0\leqslant l\leqslant k}(-1)^l L_l(2a).
\end{aligned}
$$

As a byproduct we find from (C.3),

$$
\forall\, a \geqslant 0, \quad P_k'(a) \geqslant 0, \tag{1.13}
$$

which implies that for $a \geqslant 0$, $P_k(a) \geqslant P_k(0) = 1$. We have proven the following lemma.

Lemma 1.2. *The polynomial* $P_k(a) = e^a(1 - F_k(a))$ *is increasing on* $\mathbb{R}_+$ *and* $P_k(0) = 1$.

Let us take a look at the first P_k: we have

$$
\begin{aligned}
&P_0(a) = 1, \\
&P_1(a) = 1 + 2a, \\
&P_2(a) = 1 + 2a^2,
\end{aligned}
$$

$$
\begin{aligned}
P_3(a) &= 1 + 2a - 2a^2 + \frac{4a^3}{3},\\
P_4(a) &= 1 + 4a^2 - \frac{8a^3}{3} + \frac{2a^4}{3},\\
P_5(a) &= 1 + 2a - 4a^2 + \frac{16a^3}{3} - 2a^4 + \frac{4a^5}{15},\\
P_6(a) &= 1 + 6a^2 - 8a^3 + \frac{14a^4}{3} - \frac{16a^5}{15} + \frac{4a^6}{45},\\
P_7(a) &= 1 + 2a - 6a^2 + 12a^3 - \frac{26a^4}{3} + \frac{44a^5}{15} - \frac{4a^6}{9} + \frac{8a^7}{315},\\
P_8(a) &= 1 + 8a^2 - 16a^3 + \frac{44a^4}{3} - \frac{32a^5}{5} + \frac{64a^6}{45} - \frac{16a^7}{105} + \frac{2a^8}{315},\\
P_9(a) &= 1 + 2a - 8a^2 + \frac{64a^3}{3} - \frac{68a^4}{3} + \frac{184a^5}{15} - \frac{32a^6}{9} + \frac{176a^7}{315} - \frac{2a^8}{45} + \frac{4a^9}{2835},\\
P_{10}(a) &= 1 + 10a^2 - \frac{80a^3}{3} + \frac{100a^4}{3} - \frac{64a^5}{3} + \frac{344a^6}{45} - \frac{496a^7}{315} + \frac{58a^8}{315} - \frac{32a^9}{2835} + \frac{4a^{10}}{14175}.
\end{aligned}
$$

We note as well that

$$
P_k(x) = \sum_{0\leqslant m\leqslant k} \frac{x^m}{m!} \sum_{m\leqslant l\leqslant k} 2^l(-1)^{k-l}\binom{k}{l}, \tag{1.14}
$$

since from (1.7),

$$
\begin{aligned}
P_k(a) &= \frac{(-1)^k}{k!}\int_0^{+\infty} \mathrm{e}^t\left(\frac{d}{dt}\right)^k\{\mathrm{e}^{-2t}(a+t)^k\}dt\\
&= (-1)^k \sum_{0\leqslant m\leqslant k}\int_0^{+\infty} \mathrm{e}^{-t}\frac{(-2)^{k-m}}{(k-m)!}\frac{k!}{(k-m)!m!}(a+t)^{k-m}dt\\
&= (-1)^k \sum_{0\leqslant m\leqslant k}\int_0^{+\infty} \mathrm{e}^{-t}\frac{(-2)^{k-m}}{(k-m)!}\frac{k!}{(k-m)!m!}\sum_{0\leqslant l\leqslant k-m} a^l t^{k-l-m}\binom{k-m}{l}dt\\
&= (-1)^k \sum_{\substack{0\leqslant m\leqslant k\\ 0\leqslant l\leqslant k-m}} \frac{(-2)^{k-m}}{(k-m)!}\frac{k!}{(k-m)!m!}a^l(k-l-m)!\binom{k-m}{l}\\
&= \sum_{0\leqslant l+m\leqslant k} \frac{(-1)^m 2^{k-m}}{(k-m)!}\frac{k!}{m!}a^l\frac{1}{l!}\\
&= \sum_{0\leqslant l\leqslant k}\frac{a^l}{l!}\sum_{l\leqslant m'\leqslant k}(-1)^{k-m'}2^{m'}\binom{k}{m'}.
\end{aligned}
$$

Lemma 1.3. *With the polynomial P_k defined by* (1.8b), *we have*

$$
\begin{cases}
P_k(a) = 2\displaystyle\sum_{0\leqslant l\leqslant k-1}(-1)^l L_l(2a) + (-1)^k L_k(2a),\\
P_k'(a) = 2\displaystyle\sum_{0\leqslant l\leqslant k-1}(-1)^l L_l(2a).
\end{cases} \tag{1.15}
$$

Proof. We may use the already proved (1.11) and (1.12), but we may also prove this directly by induction on k. □

Proposition 1.4. *Let F_k be given by* (1.6) *with P_k defined by* (1.7). *We have*

$$
F_k(a) = 1 - \mathrm{e}^{-a}P_k(a) \leqslant 1 - \mathrm{e}^{-a} = F_0(a) \quad \textit{for} \quad a \geqslant 0, \tag{1.16}
$$

$$
F_k'(a) = \mathrm{e}^{-a}(P_k(a) - P_k'(a)) = \mathrm{e}^{-a}(-1)^k L_k(2a), \tag{1.17}
$$

$$
F_k'(0) = (-1)^k, \quad \lim_{a\to+\infty} F_k'(a) = 0_+, \quad F_k(0) = 0, \quad \lim_{a\to+\infty} F_k(a) = 1_-. \tag{1.18}
$$

Proof. We use (1.6), (1.12) and (1.11) for the three first equalities, and Lemma 1.2 for the first inequality. The fourth equality follows from $L_k(0) = 1$, while the fifth is due to the fact that the leading monomial of $(-1)^k L_k(2a)$ is $2^k a^k/k!$. The last two equalities are a consequence of the first line. □

Remark 1.5. The zeros of F_k' on the positive half-line are the positive zeros of the Laguerre polynomial L_k divided by 2. When k is even (resp. odd) the function F_k is positive increasing (resp. negative decreasing) near 0, and then oscillates with changes of monotonicity at each a such that $L_k(2a) = 0$ and when $2a$ is larger than the largest zero of L_k, the function F_k is increasing, smaller than 1, with limit 1 at infinity.

Typically we have $F_{2l}(0) = 0, F_{2l}'(0) = +1$,

$$0 < a_{1,2l} < a_{2,2l} < \cdots < a_{2l-1,2l} < a_{2l,2l} \quad \text{the zeros of } L_{2l}(2a), \tag{1.19}$$

F_{2l} vanishes simply at $b_0 = 0$ and at $b_j \in (a_j, a_{j+1})$ for $1 \leqslant j \leqslant 2l-1$, also at $b_{2l} > a_{2l}$: $2l+1$ zeros with a positive (resp. negative) derivative at $b_0, b_2, \dots, b_{2l}$ (resp. at $b_1, b_3, \dots, b_{2l-1}$).

Moreover, we have $F_{2l+1}(0) = 0, F_{2l+1}'(0) = -1$,

$$0 < a_{1,2l+1} < a_{2,2l+1} < \cdots < a_{2l,2l+1} < a_{2l+1,2l+1} \quad \text{the zeros of } L_{2l+1}(2a), \tag{1.20}$$

F_{2l+1} vanishes simply at $b_0 = 0$ and at $b_j \in (a_j, a_{j+1})$ for $1 \leqslant j \leqslant 2l$, also at $b_{2l+1} > a_{2l+1}$: $2l+2$ zeros with a positive (resp. negative) derivative at $b_1, b_3, \dots, b_{2l+1}$ (resp. at $b_0, b_2, \dots, b_{2l}$).

1.3 Curves

Let us display some curves of $\mathbb{R}_+ \ni a \mapsto F_k(a) = 1 - \mathrm{e}^{-a} P_k(a)$ (see Figures 2 and 3).

We note as well that a consequence of the previous remark is that

$$\min_{a \geqslant 0} F_{2l}(a) = \min_{1 \leqslant j \leqslant l} \{F_{2l}(a_{2j,2l})\}, \tag{1.21}$$

$$\min_{a \geqslant 0} F_{2l+1}(a) = \min_{0 \leqslant j \leqslant l} \{F_{2l+1}(a_{2j+1,2l+1})\}, \tag{1.22}$$

where $(a_{p,k})_{1 \leqslant p \leqslant k}$ are defined in (1.19) and (1.20).

Theorem 1.6. *Let $a \geqslant 0$ be given and let $D_a = \{(x, \xi) \in \mathbb{R}^2, x^2 + \xi^2 \leqslant \frac{a}{2\pi}\}$. Then we have*

$$\mathbf{1}_{D_a}^w = \sum_{k \geqslant 0} F_k(a) \mathbb{P}_k \leqslant 1 - \mathrm{e}^{-a}. \tag{1.23}$$

Proof. It is an immediate consequence of (1.3) and (1.16). Note that the inequality in the above theorem is due to Flandrin [4] (see also [5,6]). □

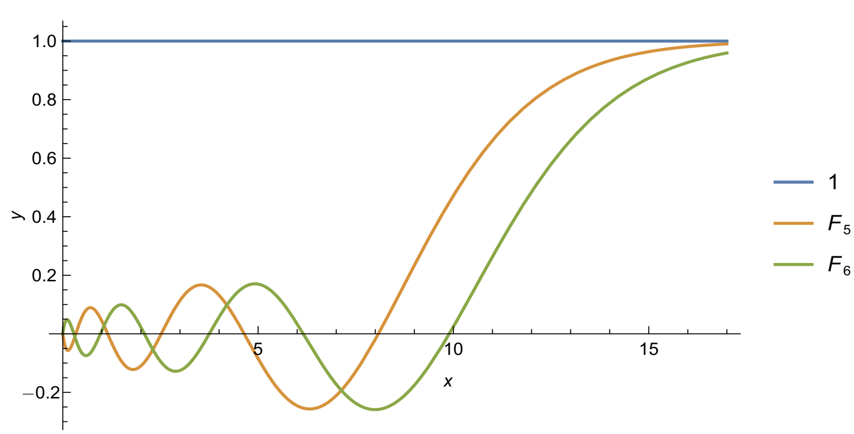

Figure 2 (Color online) Functions F_5 and F_6

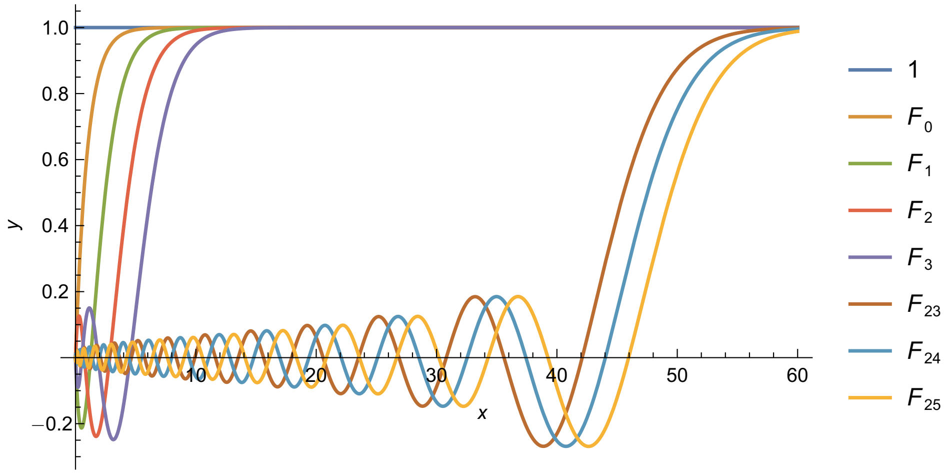

Figure 3 (Color online) Functions F_k

2 The n-dimensional case

2.1 Basics

Lemma 2.1. *Let F be a tempered distribution on $\mathbb{R}$ such that $\hat{F}$ is locally integrable and such that*

$$\int_{\mathbb{R}} |\hat{F}(\tau)| \frac{d\tau}{(1+\tau^2)^{\frac{n}{2}}} < +\infty. \tag{2.1}$$

With $|x|^2 + |\xi|^2$ standing for the Euclidean norm on $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$, the operator

$$(F(|x|^2 + |\xi|^2))^w \quad \textit{is bounded on} \ L^2(\mathbb{R}^n)$$

and has the spectral decomposition

$$(F(|x|^2 + |\xi|^2))^w = \sum_{k \geqslant 0} \int_{\mathbb{R}} \frac{\hat{F}(\tau)(1+\mathrm{i}\tau)^{2k+n}}{(1+\tau^2)^{k+n}} d\tau \mathbb{P}_k, \tag{2.2}$$

where $\mathbb{P}_k$ is the orthogonal projection onto $\mathcal{E}_{k,n}$ as defined by (A.23).

Proof. Using the n-dimensional Mehler's formula (B.2), we find

$$\begin{aligned}(F(|x|^2 + |\xi|^2))^w &= \int_{\mathbb{R}} \hat{F}(\tau) \sum_{k \geqslant 0} \mathrm{e}^{2\mathrm{i}(\arctan \tau)(k+\frac{n}{2})} \mathbb{P}_{k,n} \frac{d\tau}{(\sqrt{1+\tau^2})^n} \\ &= \sum_{k \geqslant 0} \int_{\mathbb{R}} \hat{F}(\tau)(1+\mathrm{i}\tau)^{2k+n} \frac{d\tau}{(1+\tau^2)^{k+n}},\end{aligned}$$

where the use of Fubini's theorem is justified by

$$\int_{\mathbb{R}} |\hat{F}(\tau)| \frac{d\tau}{(1+\tau^2)^{n/2}} < +\infty, \quad \mathbb{P}_{k,n} \geqslant 0, \quad \sum_k \mathbb{P}_{k,n} = \mathrm{Id}\,.$$

This completes the proof. □

2.2 Indicatrix of a Euclidean ball

The following result displays an explicit spectral decomposition on the Hermite basis for the Weyl quantization of the characteristic function of Euclidean balls.

Theorem 2.2. *Let $a \geqslant 0$ be given and let $\mathcal{Q}_{a,n} = (\mathbf{1}\{2\pi(|x|^2 + |\xi|^2) \leqslant a\})^w$ be the Weyl quantization of the characteristic function of the Euclidean ball of $\mathbb{R}^{2n}$ with center 0 and radius $\sqrt{a/(2\pi)}$. Then we have*

$$\mathcal{Q}_{a,n} = \sum_{k\geqslant 0} F_{k,n}(a)\mathbb{P}_{k,n}, \tag{2.3}$$

with

$$\mathbb{P}_{k,n} = \sum_{\alpha\in\mathbb{N}^n, |\alpha|=k} \mathbb{P}_\alpha,$$

where $\mathbb{P}_\alpha$ is the orthogonal projection onto Ψ_α (defined in (A.23)), with

$$|\alpha| = \sum_{1\leqslant j\leqslant n} \alpha_j = k$$

and

$$F_{k,n}(a) = \int_{\mathbb{R}} \frac{\sin a\tau}{\pi\tau} \frac{(1+\mathrm{i}\tau)^k}{(1-\mathrm{i}\tau)^{k+n}} d\tau. \tag{2.4}$$

The spectral decomposition of the previous theorem allows a simple recovery of the result of the article [10] by Lieb and Ostrover.

Theorem 2.3. *Let $a \geqslant 0, \mathcal{Q}_a, F_{k,n}$ be defined above. Then we have*

$$F_{k,n}(a) \leqslant 1 - \frac{1}{\Gamma(n)} \int_a^{+\infty} \mathrm{e}^{-t} t^{n-1} dt = 1 - \frac{\Gamma(n,a)}{\Gamma(n)}, \tag{2.5}$$

and thus we have

$$\mathcal{Q}_a \leqslant 1 - \frac{\Gamma(n,a)}{\Gamma(n)}, \tag{2.6}$$

where the incomplete Gamma function $\Gamma(\cdot,\cdot)$ is defined in (A.33).

Proofs of Theorems 2.2 *and* 2.3. We use the results of (the previous) Subsection 2.1: let us assume now that, with some $a \geqslant 0$,

$$F = \mathbf{1}_{[-\frac{a}{2\pi}, \frac{a}{2\pi}]}, \quad \text{so that} \quad F(|x|^2 + |\xi|^2) = \mathbf{1}\{2\pi(x^2+\xi^2) \leqslant a\}.$$

According to Appendix A, we have

$$\hat{F}(\tau) = \frac{\sin a\tau}{\pi\tau},$$

so that (1.1) holds true. We find in this case, following our calculations in Subsection 1.1,

$$(F(|x|^2 + |\xi|^2))^w = \sum_{k\geqslant 0} F_{k,n}(a)\mathbb{P}_{k,n}, \quad \mathbb{P}_{k,n} = \sum_{\alpha\in\mathbb{N}^n, |\alpha|=k} \mathbb{P}_\alpha, \tag{2.7}$$

$$F_{k,n}(a) = \int_{\mathbb{R}} \frac{\sin a\tau}{\pi\tau} \frac{(1+\mathrm{i}\tau)^k}{(1-\mathrm{i}\tau)^{k+n}} d\tau, \tag{2.8}$$

where $\mathbb{P}_\alpha$ is the orthogonal projection onto Ψ_α (defined in (A.23)), with

$$|\alpha| = \sum_{1\leqslant j\leqslant n} \alpha_j = k.$$

This completes the proof of Theorem 2.2. We have also the following result.

Lemma 2.4. *Let $(k,n) \in \mathbb{N} \times \mathbb{N}^*$. With $F_{k,n}(a)$ given by (2.8), we have*

$$F_{k,n}(a) = 1 - \mathrm{e}^{-a} P_{k,n}(a), \quad \textit{where } P_{k,n} \textit{ is the polynomial,} \tag{2.9}$$

$$P_{k,n}(a) = \frac{(-1)^{k+n-1}}{(k+n-1)!} \int_0^{+\infty} \mathrm{e}^{-t}(t+a)^{n-1} \left\{\mathrm{e}^s \left(\frac{d}{ds}\right)^{n+k-1} [s^k \mathrm{e}^{-s}]\right\}_{|s=2t+2a} dt, \tag{2.10}$$

$$P_{k,n}(a) = \frac{(-1)^{k+n-1}}{(k+n-1)!2^{n-1}} \int_0^{+\infty} (t+a)^{n-1} \mathrm{e}^t \left(\frac{d}{dt}\right)^{n+k-1} \{(t+a)^k \mathrm{e}^{-2t}\} dt. \tag{2.11}$$

Proof. The lemma holds true for $n=1$ from Proposition 1.4. We have for $a>0$, $n\geqslant 2$,

$$\begin{aligned}
F'_{k,n}(a) &= \frac{1}{\pi}\int_{\mathbb{R}} \cos a\tau \frac{(1+\mathrm{i}\tau)^k}{(1-\mathrm{i}\tau)^{k+n}}d\tau \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}a\tau} \frac{(1+\mathrm{i}\tau)^k}{(1-\mathrm{i}\tau)^{k+n}}d\tau + \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}a\tau} \frac{(1-\mathrm{i}\tau)^k}{(1+\mathrm{i}\tau)^{k+n}}d\tau \\
&= \frac{\mathrm{i}}{2\mathrm{i}\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}a\tau} \frac{\mathrm{i}^k(\tau-\mathrm{i})^k}{(-\mathrm{i})^{k+n}(\tau+\mathrm{i})^{k+n}}d\tau + \frac{\mathrm{i}}{2\mathrm{i}\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}a\tau} \frac{(-\mathrm{i})^k(\tau+\mathrm{i})^k}{\mathrm{i}^{k+n}(\tau-\mathrm{i})^{k+n}}d\tau,
\end{aligned}$$

so that

$$\begin{aligned}
F'_{k,n}(a) &= \mathrm{i}^{1-n}(-1)^k \mathrm{Res}\bigg(\mathrm{e}^{\mathrm{i}a\tau}\frac{(\tau+\mathrm{i})^k}{(\tau-\mathrm{i})^{k+n}};\mathrm{i}\bigg) \\
&= \frac{\mathrm{i}^{1-n}(-1)^k}{(k+n-1)!}\bigg(\frac{d}{d\tau}\bigg)^{k+n-1}\{\mathrm{e}^{\mathrm{i}a\tau}(\tau+\mathrm{i})^k\}_{|\tau=\mathrm{i}}
\end{aligned}$$

and thus

$$\begin{aligned}
F'_{k,n}(a) &= \frac{\mathrm{i}^{1-n}(-1)^k}{(k+n-1)!}\bigg(\frac{d}{\frac{\mathrm{i}}{a}d\epsilon}\bigg)^{k+n-1}\bigg\{\mathrm{e}^{-a-\epsilon}\bigg(\mathrm{i}+\mathrm{i}\frac{\epsilon}{a}+\mathrm{i}\bigg)^k\bigg\}_{|\epsilon=0} \\
&= \frac{\mathrm{i}^{1-n}(-1)^k a^{n-1}}{\mathrm{i}^{n-1}(k+n-1)!}\bigg(\frac{d}{d\epsilon}\bigg)^{k+n-1}\{\mathrm{e}^{-a-\epsilon}(2a+\epsilon)^k\}_{|\epsilon=0} \\
&= \mathrm{e}^a\frac{(-1)^{k+n-1}a^{n-1}}{(k+n-1)!}\bigg(\frac{d}{2d\epsilon}\bigg)^{k+n-1}\{\mathrm{e}^{-2a-2\epsilon}(2a+2\epsilon)^k\}_{|\epsilon=0},
\end{aligned}$$

i.e.,

$$\begin{aligned}
F'_{k,n}(t) &= \frac{(-1)^{k+n-1}}{(k+n-1)!}\mathrm{e}^t t^{n-1}\bigg(\frac{d}{ds}\bigg)^{k+n-1}\{\mathrm{e}^{-s}s^k\}_{|s=2t} \\
&= \frac{(-1)^{k+n-1}}{(k+n-1)!2^{n-1}}\mathrm{e}^t t^{n-1}\bigg(\frac{d}{dt}\bigg)^{k+n-1}\{\mathrm{e}^{-2t}t^k\}.
\end{aligned}$$

We have also that $\lim_{a\to+\infty}F_{k,n}(a)=1$ (following the arguments of Subsection 1.2) and this yields

$$\begin{aligned}
F_{k,n}(a) &= 1-\frac{(-1)^{k+n-1}}{(k+n-1)!2^{n-1}}\int_a^{+\infty}\mathrm{e}^t t^{n-1}\bigg(\frac{d}{dt}\bigg)^{k+n-1}\{\mathrm{e}^{-2t}t^k\}dt \\
&= 1-\mathrm{e}^{-a}\frac{(-1)^{k+n-1}}{(k+n-1)!2^{n-1}}\int_0^{+\infty}(t+a)^{n-1}\mathrm{e}^t\bigg(\frac{d}{dt}\bigg)^{k+n-1}\{\mathrm{e}^{-2t}(t+a)^k\}dt,
\end{aligned}$$

concluding the proof of the lemma. □

Formulas (2.7)–(2.8) imply that for any $\alpha\in\mathbb{N}^n, |\alpha|=k$, we have

$$F_{k,n}(a)=\langle (F(|x|^2+|\xi|^2))^w\Psi_\alpha,\Psi_\alpha\rangle.$$

We may then choose

$$\alpha=(k,\underbrace{0,\ldots,0}_{n-1}),$$

and obtain

$$F_{k,n}(a)=\iint_{\{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n,2\pi(|x|^2+|\xi|^2)\leqslant a\}}\mathcal{H}(\Psi_\alpha,\Psi_\alpha)(x,\xi)dxd\xi.$$

Since the Wigner functions respect the tensor product structure, we find that, with

$$(x,\xi)=(x_1,x',\xi_1,\xi')\in\mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R}^{n-1},$$

$$\mathcal{H}_n(\Psi_\alpha,\Psi_\alpha)(x,\xi)=\mathcal{H}_1(\psi_k,\psi_k)(x_1,\xi_1)\mathcal{H}_{n-1}(\Psi_{(0)},\Psi_{(0)})(x',\xi'),\quad (0)=\underbrace{(0,\dots,0)}_{n-1}.$$

Using (A.17), we have

$$\mathcal{H}_{n-1}(\Psi_{(0)},\Psi_{(0)})(x',\xi')=2^{n-1}\mathrm{e}^{-2\pi(|x'|^2+|\xi'|^2)},$$

and consequently

$$F_{k,n}(a)=\iint_{2\pi(|x'|^2+|\xi'|^2)\leqslant a}2^{n-1}\mathrm{e}^{-2\pi(|x'|^2+|\xi'|^2)}\iint_{\substack{2\pi(x_1^2+\xi_1^2)\\ \leqslant a-2\pi(|x'|^2+|\xi'|^2)}}\mathcal{H}_1(\psi_k,\psi_k)(x_1,\xi_1)dx_1d\xi_1dx'd\xi'.$$

This entails

$$\begin{aligned}F_{k,n}(a)=&\iint_{2\pi(|x'|^2+|\xi'|^2)\leqslant a}2^{n-1}\mathrm{e}^{-2\pi(|x'|^2+|\xi'|^2)}\\&\times\{1-\mathrm{e}^{-(a-2\pi(|x'|^2+|\xi'|^2))}P_{k,1}(a-2\pi(|x'|^2+|\xi'|^2))\}dx'd\xi',\end{aligned}$$

and thus

$$\begin{aligned}F_{k,n}(a)=&\,2^{n-1}|\mathbb{S}^{2n-3}|\int_0^{(a/2\pi)^{1/2}}\mathrm{e}^{-2\pi r^2}r^{2n-3}dr\\&-2^{n-1}|\mathbb{S}^{2n-3}|\mathrm{e}^{-a}\int_0^{(a/2\pi)^{1/2}}r^{2n-3}P_{k,1}(a-2\pi r^2)dr.\end{aligned}$$

We obtain that

$$\begin{aligned}&2^{n-1}|\mathbb{S}^{2n-3}|\int_0^{(a/2\pi)^{1/2}}\mathrm{e}^{-2\pi r^2}r^{2n-3}dr\\&=2^{n-1}|\mathbb{S}^{2n-3}|\int_0^{+\infty}\mathrm{e}^{-2\pi r^2}r^{2n-3}dr-2^{n-1}|\mathbb{S}^{2n-3}|\int_{(a/2\pi)^{1/2}}^{+\infty}\mathrm{e}^{-2\pi r^2}r^{2n-3}dr\\&=2^{n-1}\frac{2\pi^{n-1}}{\Gamma(n-1)}(2^{-n}\pi^{-n+1}\Gamma(n-1))-2^n\frac{\pi^{n-1}}{\Gamma(n-1)}\int_{(a/2\pi)^{1/2}}^{+\infty}\mathrm{e}^{-2\pi r^2}r^{2n-3}dr\end{aligned}$$

and thus

$$\begin{aligned}F_{k,n}(a)=1-2^n\frac{\pi^{n-1}}{\Gamma(n-1)}\bigg\{&\int_{(a/2\pi)^{1/2}}^{+\infty}\mathrm{e}^{-2\pi r^2}r^{2n-3}dr\\&+\mathrm{e}^{-a}\int_0^{(a/2\pi)^{1/2}}r^{2n-3}P_{k,1}(a-2\pi r^2)dr\bigg\}.\end{aligned}\tag{2.12}$$

Since $P_{k,1}(b)\geqslant P_{k,1}(0)=1$ for $b\geqslant 0$ from Lemma 1.2, we find that

$$\begin{aligned}F_{k,n}(a)&\leqslant 1-2^n\frac{\pi^{n-1}}{\Gamma(n-1)}\bigg\{\int_{(a/2\pi)^{1/2}}^{+\infty}\mathrm{e}^{-2\pi r^2}r^{2n-3}dr+\mathrm{e}^{-a}\int_0^{(a/2\pi)^{1/2}}r^{2n-3}dr\bigg\}\\&=1-2^n\frac{\pi^{n-1}}{\Gamma(n-1)}\bigg\{\int_{(a/2\pi)^{1/2}}^{+\infty}\mathrm{e}^{-2\pi r^2}r^{2n-3}dr+\mathrm{e}^{-a}\int_0^{(a/2\pi)^{1/2}}r^{2n-3}dr\bigg\}\\&\overset{(r=\sqrt{t/(2\pi)})}{=}1-2^n\frac{\pi^{n-1}}{\Gamma(n-1)}\bigg\{\int_a^{+\infty}\mathrm{e}^{-t}t^{n-3/2}\frac{1}{2}t^{-1/2}dt(2\pi)^{-n+3/2-1/2}\\&\quad+\mathrm{e}^{-a}\frac{a^{n-1}(2\pi)^{-n+1}}{2n-2}\bigg\}\\&=1-\frac{1}{\Gamma(n-1)}\bigg(\int_a^{+\infty}\mathrm{e}^{-t}t^{n-2}dt+\frac{\mathrm{e}^{-a}a^{n-1}}{n-1}\bigg)\\&=1-\frac{1}{\Gamma(n-1)}\bigg(\int_a^{+\infty}\mathrm{e}^{-t}\frac{t^{n-1}}{n-1}dt+\bigg[\mathrm{e}^{-t}\frac{t^{n-1}}{n-1}\bigg]_{t=a}^{t=+\infty}+\frac{\mathrm{e}^{-a}a^{n-1}}{n-1}\bigg)\end{aligned}$$

$$= 1 - \frac{1}{\Gamma(n)} \int_a^{+\infty} \mathrm{e}^{-t} t^{n-1} dt$$
$$= 1 - \frac{\Gamma(n,a)}{\Gamma(n)}.$$

This completes the proof of Theorem 2.3. □

Lemma 2.5. *With $P_{k,n}$ defined in Lemma 2.4, we have*

$$\forall\, a \geqslant 0, \quad P'_{k,n}(a) \geqslant \begin{cases} \dfrac{\mathrm{e}^{-a}\Gamma(n-2,a)}{\Gamma(n-2)} + \dfrac{a^{n-2}}{\Gamma(n-1)} & \text{for } n \geqslant 3, \\ 1 & \text{for } n = 2. \end{cases}$$

Proof. We have from (2.12),

$$P_{k,n}(a) = 2^n \frac{\pi^{n-1}}{\Gamma(n-1)} \left\{ \mathrm{e}^a \int_{(a/2\pi)^{1/2}}^{+\infty} \mathrm{e}^{-2\pi r^2} r^{2n-3} dr + \int_0^{(a/2\pi)^{1/2}} r^{2n-3} P_{k,1}(a - 2\pi r^2) dr \right\},$$

i.e.,

$$P_{k,n}(a) = \frac{1}{\Gamma(n-1)} \mathrm{e}^a \int_a^{+\infty} \mathrm{e}^{-t} t^{n-3/2-1/2} dt + \frac{1}{\Gamma(n-1)} \int_0^a P_{k,1}(a-t) t^{n-2} dt,$$

i.e.,

$$P_{k,n}(a) = \frac{1}{\Gamma(n-1)} \left(\int_0^{+\infty} \mathrm{e}^{-t} (t+a)^{n-2} dt + \int_0^a P_{k,1}(a-t) t^{n-2} dt \right), \tag{2.13}$$

which implies for $n \geqslant 2$,

$$\Gamma(n-1) P'_{k,n}(a) = \int_0^{+\infty} (n-2) \mathrm{e}^{-t} (t+a)^{n-3} dt + \int_0^a P'_{k,1}(a-t) t^{n-2} dt + P_{k,1}(0) a^{n-2}$$

and since $P'_{k,1}(b) \geqslant 0$ for $b \geqslant 0$, we obtain

$$\Gamma(n-1) P'_{k,n}(a) \geqslant (n-2) \mathrm{e}^{-a} \int_a^{+\infty} \mathrm{e}^{-t} t^{n-3} dt + a^{n-2},$$

i.e., for $n \geqslant 3, a \geqslant 0$,

$$P'_{k,n}(a) \geqslant \frac{\mathrm{e}^{-a}\Gamma(n-2,a)}{\Gamma(n-2)} + \frac{a^{n-2}}{\Gamma(n-1)} \geqslant 0, \tag{2.14}$$

where the incomplete Gamma function $\Gamma(n,x)$ is given by (A.33). □

2.3 A monotonicity result

Theorem 2.6. *Let F be as in Lemma 1.1. We have then for $k \in \mathbb{N}$,*

$$\int_{\mathbb{R}} \frac{\hat{F}(\tau)(1+\mathrm{i}\tau)^{2k+1}}{(1+\tau^2)^{k+1}} d\tau = \int_0^{+\infty} F\left(\frac{t}{2\pi}\right) \mathrm{e}^{-t} dt + \int_0^{+\infty} F'\left(\frac{t}{2\pi}\right) \frac{1}{2\pi} \mathrm{e}^{-t} (P_k(t) - 1) dt, \tag{2.15}$$

where P_k is the polynomial defined by (1.8b).

Proof. Using Plancherel's formula and (C.4)–(C.5), we have

$$\begin{aligned} \int_{\mathbb{R}} \frac{\hat{F}(\tau)(1+\mathrm{i}\tau)^{2k+1}}{(1+\tau^2)^{k+1}} d\tau &= \int_{\mathbb{R}} \hat{F}(\tau) \widehat{G_k}(-\tau/(2\pi)) d\tau = \int_{\mathbb{R}} F(t) G_k(2\pi t) dt 2\pi \\ &= 2\pi \int_0^{+\infty} F(t) \mathrm{e}^{-2\pi t} ((-1)^k L_k(4\pi t) - 1) dt + \int_0^{+\infty} F\left(\frac{t}{2\pi}\right) \mathrm{e}^{-t} dt \\ &= \int_0^{+\infty} F\left(\frac{t}{2\pi}\right) \mathrm{e}^{-t} dt + \int_0^{+\infty} \underbrace{F\left(\frac{t}{2\pi}\right)}_{u(t)} \underbrace{\mathrm{e}^{-t}((-1)^k L_k(2t) - 1)}_{v'_k(t)} dt, \end{aligned}$$

with

$$v_k(t) = \int_0^t \mathrm{e}^{-s}((-1)^k L_k(2s) - 1)ds.$$

We note that $v_k(0) = 0$ and according to (1.8a), (1.8b) and Lemma 1.2,

$$\begin{aligned}
v_k(+\infty) &= \int_0^{+\infty} \mathrm{e}^{-t}(-1)^k L_k(2t)dt - \int_0^{+\infty} \mathrm{e}^{-s}ds = P_k(0) - 1 = 0,\\
-v_k(t) &= \int_t^0 \mathrm{e}^{-s}(-1)^k L_k(2s)ds - \mathrm{e}^{-t} + 1\\
&= \int_t^{+\infty} \mathrm{e}^{-s}(-1)^k L_k(2s)ds + \int_{+\infty}^0 \mathrm{e}^{-s}(-1)^k L_k(2s)ds - \mathrm{e}^{-t} + 1\\
&= \mathrm{e}^{-t}\int_0^{+\infty} \mathrm{e}^{-s}(-1)^k L_k(2s+2t)ds - P_k(0) - \mathrm{e}^{-t} + 1\\
&= \mathrm{e}^{-t}(P_k(t) - 1).
\end{aligned}$$

This implies that

$$\int_{\mathbb{R}} \frac{\hat{F}(\tau)(1+\mathrm{i}\tau)^{2k+1}}{(1+\tau^2)^{k+1}}d\tau = \int_0^{+\infty} F\left(\frac{t}{2\pi}\right)\mathrm{e}^{-t}dt - \int_0^{+\infty} F'\left(\frac{t}{2\pi}\right)\frac{1}{2\pi}v_k(t)dt,$$

and thus

$$\int_{\mathbb{R}} \frac{\hat{F}(\tau)(1+\mathrm{i}\tau)^{2k+1}}{(1+\tau^2)^{k+1}}d\tau = \int_0^{+\infty} F\left(\frac{t}{2\pi}\right)\mathrm{e}^{-t}dt + \int_0^{+\infty} F'\left(\frac{t}{2\pi}\right)\frac{1}{2\pi}\mathrm{e}^{-t}(P_k(t) - 1)dt.$$

This completes the proof. □

Corollary 2.7 (See [1]). *Let F as in Lemma* 1.1 *be real-valued and non-decreasing. Then the operator with Weyl symbol $F(x^2+\xi^2)$ is selfadjoint and such that*

$$(F(x^2+\xi^2))^w \geqslant \int_0^{+\infty} F\left(\frac{t}{2\pi}\right)\mathrm{e}^{-t}dt.$$

Proof. From Lemma 1.2, we find that for $t \geqslant 0$, $P_k(t) - 1 \geqslant 0$, thus (2.15) and (1.2) imply the result. □

Remark 2.8. The normalization in the article [1] is not the same as ours. Take a symbol a and their Weyl quantization is defined as $a^{\tilde{w}}$ with

$$\begin{aligned}
a^{\tilde{w}}u(x) &= \iint \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right)u(y)dyd\xi(2\pi)^{-n}\\
&= \iint \mathrm{e}^{\mathrm{i}2\pi(x-y)\cdot\xi} a\left(\frac{x+y}{2}, 2\pi\xi\right)u(y)dyd\xi = (a(x, 2\pi\xi))^w u,
\end{aligned}$$

an operator which is unitarily equivalent to $(a(x\sqrt{2\pi}, \xi\sqrt{2\pi}))^w$. As a result, we have

$$(\Phi(x^2+\xi^2))^{\tilde{w}} \equiv (\Phi(2\pi(x^2+\xi^2)))^w = (F(x^2+\xi^2))^w,$$

with $F(s) = \Phi(2\pi s)$. The lower bound obtained in Corollary 2.7 is thus

$$\int_0^{+\infty} \mathrm{e}^{-t}\Phi(t)dt,$$

the same as in [1].

Remark 2.9. In higher dimensions it is possible to use Mehler's formula to tackle for example,

$$\{(x,\xi) \in \mathbb{R}^4, x_1^2 + \xi_1^2 + (x_2^2 + \xi_2^2)^2 \leqslant a\}.$$

In fact we have for a function F defined on $\mathbb{R}^n$,

$$F(x_1^2+\xi_1^2,\dots,x_n^2+\xi_n^2)=\int_{\mathbb{R}^n} \mathrm{e}^{2\mathrm{i}\pi\sum_{1\leqslant j\leqslant n}(x_j^2+\xi_j^2)\tau_j}\hat{F}(\tau)d\tau,$$

and Weyl-quantifying that identity, we find an expression of

$$(F(x_1^2+\xi_1^2,\dots,x_n^2+\xi_n^2))^{\mathrm{Weyl}}=\int_{\mathbb{R}^n}\hat{F}(\tau)\prod_{1\leqslant j\leqslant n}\frac{(1+\mathrm{i}\tau_j)^{2k_j+1}}{(1+\tau_j^2)^{k_j+1}}\mathbb{P}_{k_j,1;j}d\tau,$$

and thus an explicit spectral decomposition for the operator under scope.

Some remarks on ellipsoids. We provide below a couple of remarks on ellipsoids in higher dimensions. Let us first recall a particular case of [8, Theorem 21.5.3].

Theorem 2.10 (Symplectic reduction of quadratic forms)**.** *Let q be a positive definite quadratic form on $\mathbb{R}^n\times\mathbb{R}^n$ equipped with the canonical symplectic form. Then there exist S in the symplectic group of $\mathbb{R}^{2n}$ and $\mu_1,\dots,\mu_n$ positive such that for all $X=(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$,*

$$q(SX)=\sum_{1\leqslant j\leqslant n}\mu_j(x_j^2+\xi_j^2). \tag{2.16}$$

Note that an interesting consequence of this theorem is that, considering a general ellipsoid in $\mathbb{R}^{2n}$,

$$\mathbb{E}=\{X\in\mathbb{R}^{2n},q(X)\leqslant 1\},$$

where q is a positive definite quadratic form, we are able to find symplectic coordinates such that q is given by (2.16). Note however that no further simplification is possible and that the μ_j are symplectic invariants of $\mathbb{E}$. In particular the volume of $\mathbb{E}$ is given by

$$|\mathbb{E}|_{2n}=\frac{\pi^n}{n!\mu_1\cdots\mu_n}.$$

Spectral decomposition for the quantization of the characteristic function of the ellipsoid. Let $a_1,\dots,a_n$ be positive numbers. We consider the ellipsoid $E(a_1,\dots,a_n)$ given by

$$E(a)=E(a_1,\dots,a_n)=\left\{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n,2\pi\sum_{1\leqslant j\leqslant n}\frac{x_j^2+\xi_j^2}{a_j}\leqslant 1\right\}.$$

We define the function

$$F(X_1,\dots,X_n)=\mathbf{1}_{[-1,1]}\left(\frac{2\pi}{a_1}X_1+\cdots+\frac{2\pi}{a_n}X_n\right)$$

and we have

$$\begin{aligned}(\mathbf{1}_{E(a)})^w=(F(x_1^2+\xi_1^2,\dots,x_n^2+\xi_n^2))^w&=\int_{\mathbb{R}^n}\hat{F}(\tau)(\mathrm{e}^{2\mathrm{i}\pi\sum_j\tau_j(x_j^2+\xi_j^2)})^w d\tau\\ &=\sum_{\alpha\in\mathbb{N}^n}\int_{\mathbb{R}^n}\hat{F}(\tau)\prod_{1\leqslant j\leqslant n}\frac{(1+\mathrm{i}\tau_j)^{2\alpha_j+1}}{(1+\tau_j^2)^{\alpha_j+1}}d\tau\mathbb{P}_{\alpha,n}\\ &=\sum_{\alpha\in\mathbb{N}^n}\int_{\mathbb{R}^n}\hat{F}(\tau)\prod_{1\leqslant j\leqslant n}\frac{(1+\mathrm{i}\tau_j)^{\alpha_j}}{(1-\mathrm{i}\tau_j)^{\alpha_j+1}}d\tau\mathbb{P}_{\alpha,n},\end{aligned}$$

where $\mathbb{P}_{\alpha,n}$ is defined in (A.26). On the other hand we have

$$\begin{aligned}\hat{F}(\tau)&=\int\mathrm{e}^{-2\mathrm{i}\pi\tau\cdot x}\mathbf{1}_{[-1,1]}\left(\frac{2\pi}{a_1}x_1+\cdots+\frac{2\pi}{a_n}x_n\right)dx_1\cdots dx_n\\ &=a_1\cdots a_n(2\pi)^{-n}\int\mathrm{e}^{-\mathrm{i}\sum_j\tau_ja_jy_j}\mathbf{1}_{[-1,1]}\left(\sum y_j\right)dy,\end{aligned}$$

so that, with $\mathbb{P}_\alpha = \mathbb{P}_{\alpha,n}$,

$$\begin{aligned}(\mathbf{1}_{E(a)})^w &= a_1\cdots a_n \sum_{\alpha\in\mathbb{N}^n} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}2\pi\sum_j \tau_j a_j y_j} \mathbf{1}_{[-1,1]}\Big(\sum y_j\Big) dy \prod_{1\leqslant j\leqslant n} \frac{(1+\mathrm{i}2\pi\tau_j)^{\alpha_j}}{(1-\mathrm{i}2\pi\tau_j)^{\alpha_j+1}} d\tau \mathbb{P}_\alpha \\ &= a_1\cdots a_n \sum_{\alpha\in\mathbb{N}^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}2\pi\sum_j \tau_j a_j y_j} \mathbf{1}_{[-1,1]}\Big(\sum y_j\Big) dy \prod_{1\leqslant j\leqslant n} \overline{\hat{G}_{\alpha_j}(\tau_j)} d\tau \mathbb{P}_\alpha \\ &= a_1\cdots a_n \int_{\mathbb{R}^n} \mathbf{1}_{[-1,1]}\Big(\sum y_j\Big) \prod_{1\leqslant j\leqslant n} G_{\alpha_j}(a_j y_j) dy \mathbb{P}_\alpha \\ &= \int_{\mathbb{R}^n} \mathbf{1}_{[-1,1]}\Big(\sum t_j/a_j\Big) \prod_{1\leqslant j\leqslant n} (-1)^{\alpha_j} H(t_j)\mathrm{e}^{-t_j} L_{\alpha_j}(2t_j) dt \mathbb{P}_\alpha,\end{aligned}$$

with

$$\begin{aligned}F_\alpha(a) &= \int_{\mathbb{R}^n} \Big(1-\mathbf{1}_{[1,+\infty]}\Big(\sum t_j/a_j\Big)\Big) \prod_{1\leqslant j\leqslant n} (-1)^{\alpha_j} H(t_j)\mathrm{e}^{-t_j} L_{\alpha_j}(2t_j) dt \\ &= 1 - \int_{\mathbb{R}^n} \mathbf{1}_{[1,+\infty]}\Big(\sum t_j/a_j\Big) \prod_{1\leqslant j\leqslant n} (-1)^{\alpha_j} H(t_j)\mathrm{e}^{-t_j} L_{\alpha_j}(2t_j) dt,\end{aligned}$$

so that setting

$$K_\alpha(a) = \int_{\substack{\sum t_j/a_j\geqslant 1\\ t_j\geqslant 0}} \mathrm{e}^{-(t_1+\cdots+t_n)} \prod_{1\leqslant j\leqslant n} (-1)^{\alpha_j} L_{\alpha_j}(2t_j) dt,$$

we have $F_\alpha(a) = 1 - K_\alpha(a)$. The domain of integration is

$$\Big\{\frac{t_1}{a_1}+\cdots+\frac{t_{n-1}}{a_{n-1}} \geqslant 1 - \frac{t_n}{a_n}, t_j\geqslant 0, 0\leqslant \frac{t_n}{a_n}\leqslant 1\Big\} \cup \Big\{\frac{t_n}{a_n}\geqslant 1, t_j\geqslant 0, 1\leqslant j\leqslant n-1\Big\},$$

so that

$$\begin{aligned}K_{\alpha_1,\ldots,\alpha_n}(a_1,\ldots,a_n) &= \mathrm{e}^{-a_n} P_{\alpha_n}(a_n) \\ &\quad + \int_0^{a_n} (-1)^{\alpha_n} L_{\alpha_n}(2t_n)\mathrm{e}^{-t_n} K_{\alpha_1,\ldots,\alpha_{n-1}}(a_1(1-t_n/a_n),\ldots,a_{n-1}(1-t_n/a_n)) dt_n \\ &= \mathrm{e}^{-a_n} P_{\alpha_n}(a_n) \\ &\quad + \int_0^1 (-1)^{\alpha_n} L_{\alpha_n}(2a_n\theta)\mathrm{e}^{-\theta a_n} K_{\alpha_1,\ldots,\alpha_{n-1}}(a_1(1-\theta),\ldots,a_{n-1}(1-\theta)) d\theta a_n.\end{aligned}$$

We have $K_{\alpha_1}(a_1) = \mathrm{e}^{-a_1} P_{\alpha_1}(a_1)$ and thus if $n=2$, we get using Lemma 1.3,

$$\begin{aligned}K_{\alpha_1,\alpha_2}(a_1,a_2) &= \mathrm{e}^{-a_2} P_{\alpha_2}(a_2) + \int_0^1 (-1)^{\alpha_2} L_{\alpha_2}(2a_2\theta)\mathrm{e}^{-\theta a_2} P_{\alpha_1}(a_1(1-\theta))\mathrm{e}^{-a_1(1-\theta)} d\theta a_2 \\ &= \mathrm{e}^{-a_2} P_{\alpha_2}(a_2) + a_2\mathrm{e}^{-a_1} \int_0^1 (-1)^{\alpha_2} L_{\alpha_2}(2a_2\theta)\mathrm{e}^{-\theta(a_2-a_1)} P_{\alpha_1}(a_1(1-\theta)) d\theta.\end{aligned}$$

Question. *It is not difficult to calculate explicitly* K_{α_1,α_2} *when* $\min(\alpha_1,\alpha_2)\leqslant 1$*, but a general "explicit" formula for* $K_{\alpha_1,\ldots,\alpha_n}(a_1,\ldots,a_n)$ *would be interesting as well as the proof of*

$$K_{\alpha_1,\ldots,\alpha_n}(a_1,\ldots,a_n) \geqslant \mathrm{e}^{-\min_{1\leqslant j\leqslant n} a_j}. \tag{2.17}$$

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Appendix A The Fourier transform, Weyl quantization, harmonic oscillator

The Fourier transform. We use in this paper the following normalization for the Fourier transform and inversion formula: for $u \in \mathscr{S}(\mathbb{R}^n)$,

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\mathrm{i}\pi x\cdot\xi} u(x)dx, \quad u(x) = \int_{\mathbb{R}^n} \mathrm{e}^{2\mathrm{i}\pi x\cdot\xi}\hat{u}(\xi)d\xi, \tag{A.1}$$

a formula that can be extended to $u \in \mathscr{S}'(\mathbb{R}^n)$, with defining the distribution $\hat{u}$ by the duality bracket

$$\langle \hat{u}, \phi\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle u, \hat{\phi}\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)}. \tag{A.2}$$

Checking (A.1) for $u \in \mathscr{S}'(\mathbb{R}^n)$ is then easy, i.e.,

$$\check{\hat{\hat{u}}} = u, \tag{A.3}$$

where the distribution $\check{u}$ is defined by

$$\langle \check{u}, \phi\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle u, \check{\phi}\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)}, \quad \text{with} \quad \check{\phi}(x) = \phi(-x). \tag{A.4}$$

It is useful to notice that for $u \in \mathscr{S}'(\mathbb{R}^n)$,

$$\check{\hat{u}} = \hat{\check{u}}. \tag{A.5}$$

This normalization yields simple formulas for the Fourier transform of Gaussian functions: for A a real-valued symmetric positive definite $n \times n$ matrix, we define the function v_A in the Schwartz space by

$$v_A(x) = \mathrm{e}^{-\pi\langle Ax,x\rangle}, \quad \text{and we have} \quad \widehat{v_A}(\xi) = (\det A)^{-1/2}\mathrm{e}^{-\pi\langle A^{-1}\xi,\xi\rangle}. \tag{A.6}$$

Similarly when B is a real-valued symmetric non-singular $n \times n$ matrix, the function w_B defined by

$$w_B(x) = \mathrm{e}^{\mathrm{i}\pi\langle Bx,x\rangle}$$

is in $L^\infty(\mathbb{R}^n)$ and thus a tempered distribution and we have

$$\widehat{w_B}(\xi) = |\det B|^{-1/2}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\operatorname{sign} B}\mathrm{e}^{-\mathrm{i}\pi\langle B^{-1}\xi,\xi\rangle}, \tag{A.7}$$

where $\operatorname{sign} B$ stands for the *signature of* B, i.e., with E the set of eigenvalues of B (which are real and non-zero),

$$\operatorname{sign} B = \operatorname{Card}(E \cap \mathbb{R}_+) - \operatorname{Card}(E \cap \mathbb{R}_-). \tag{A.8}$$

With H standing for the characteristic function of $\mathbb{R}_+$, we have

$$1 = H + \check{H}, \quad \delta_0 = \hat{H} + \hat{\check{H}},$$

$$D\,\mathrm{sign} = \frac{\delta_0}{\mathrm{i}\pi}, \quad \widehat{D\,\mathrm{sign}} = \frac{1}{\mathrm{i}\pi}, \quad \xi\widehat{\mathrm{sign}} = \frac{1}{\mathrm{i}\pi}, \quad \widehat{\mathrm{sign}} = \frac{1}{\mathrm{i}\pi}\mathrm{pv}\frac{1}{\xi}, \quad \text{(principal value)}$$

the latter formula following from the fact that

$$\xi\left(\widehat{\mathrm{sign}} - \mathrm{pv}\frac{1}{\mathrm{i}\pi\xi}\right) = 0, \quad \text{which implies} \quad \widehat{\mathrm{sign}} - \mathrm{pv}\frac{1}{\mathrm{i}\pi\xi} = c\delta_0 = 0,$$

since $\widehat{\mathrm{sign}} - \frac{1}{\mathrm{i}\pi\xi}$ is odd. We infer from that

$$\hat{H} - \hat{\check{H}} = \widehat{\mathrm{sign}} = \mathrm{pv}\frac{1}{\mathrm{i}\pi\xi},$$

and

$$\hat{H} = \frac{\delta_0}{2} + \mathrm{pv}\frac{1}{2\mathrm{i}\pi\xi}. \tag{A.9}$$

The Weyl quantization. Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$. We define the operator a^w, continuous from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$, given by the formula

$$(a^w u)(x) = \iint \mathrm{e}^{2\mathrm{i}\pi(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi, \tag{A.10}$$

to be understood weakly as

$$\langle a^w u, \bar{v}\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle a, \mathcal{H}(u,v)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}, \tag{A.11}$$

where the so-called Wigner function $\mathcal{H}(u,v)$ is defined for $u, v \in \mathscr{S}(\mathbb{R}^n)$ by

$$\mathcal{H}(u,v)(x,\xi) = \int \mathrm{e}^{-2\mathrm{i}\pi z\cdot\xi} u\left(x + \frac{z}{2}\right)\bar{v}\left(x - \frac{z}{2}\right) dz. \tag{A.12}$$

We note that the sesquilinear mapping

$$\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \ni (u,v) \mapsto \mathcal{H}(u,v) \in \mathscr{S}(\mathbb{R}^{2n})$$

is continuous so that the above bracket of the duality

$$\langle a, \mathcal{H}(u,v)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})},$$

makes sense. We note as well that a tempered distribution $a \in \mathscr{S}'(\mathbb{R}^{2n})$ gets quantized by a continuous operator a^w from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$. Moreover, for $a \in \mathscr{S}'(\mathbb{R}^{2n})$ and b a polynomial in $\mathbb{C}[x,\xi]$, we have the composition formula,

$$a^w b^w = (a\sharp b)^w, \tag{A.13}$$

$$(a\sharp b)(x,\xi) = \sum_{k\geqslant 0} \frac{1}{(4\mathrm{i}\pi)^k} \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{|\beta|}}{\alpha!\beta!} (\partial_\xi^\alpha \partial_x^\beta a)(x,\xi)(\partial_x^\alpha \partial_\xi^\beta b)(x,\xi), \tag{A.14}$$

which involves here a finite sum. This follows from [9, (2.1.26)] where several generalizations can be found.

Also, we find that $\mathcal{H}(u,u)$ is real-valued since

$$\overline{\mathcal{H}(u,u)(x,\xi)} = \int \mathrm{e}^{2\mathrm{i}\pi z\cdot\xi}\bar{u}\left(x + \frac{z}{2}\right)u\left(x - \frac{z}{2}\right)dz = \int \mathrm{e}^{-2\mathrm{i}\pi z\cdot\xi}\bar{u}\left(x - \frac{z}{2}\right)u\left(x + \frac{z}{2}\right)dz = \mathcal{H}(u,u)(x,\xi).$$

A particular case of Segal's formula (see, e.g., [9, Theorem 2.1.2]) is with F standing for the Fourier transformation,

$$F^* a^w F = a(\xi, -x)^w. \tag{A.15}$$

Some explicit computations. We may also calculate with

$$
\begin{aligned}
&u_a(x) = (2a)^{1/4}\mathrm{e}^{-\pi a x^2}, \quad a>0, && \text{(A.16)}\\
&\mathcal{H}(u_a,u_a)(x,\xi) = (2a)^{1/2}\int \mathrm{e}^{-2\mathrm{i}\pi z\cdot\xi}\mathrm{e}^{-\pi a|x-\frac{z}{2}|^2}\mathrm{e}^{-\pi a|x+\frac{z}{2}|^2}dz\\
&\qquad = (2a)^{1/2}\int \mathrm{e}^{-2\mathrm{i}\pi z\cdot\xi}\mathrm{e}^{-2\pi a x^2}\mathrm{e}^{-\pi a z^2/2}dz\\
&\qquad = (2a)^{1/2}\mathrm{e}^{-2\pi a x^2}2^{1/2}a^{-1/2}\mathrm{e}^{-\pi\frac{2}{a}\xi^2}\\
&\qquad = 2\mathrm{e}^{-2\pi(ax^2+a^{-1}\xi^2)}, && \text{(A.17)}
\end{aligned}
$$

which is also a Gaussian function on the phase space (and the positive function). The calculation of $\mathcal{H}(u_a',u_a')(x,\xi)$ is interesting since we have

$$4\pi^2\langle D_x b^w D_x u_a, \bar{u}_a\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle b^w u_a', \bar{u}_a'\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle b, \mathcal{H}(u_a',u_a')\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})},$$

and for $b(x,\xi)$ real-valued we have

$$\xi\sharp b\sharp\xi = \left(\xi b + \frac{b_x'}{4\mathrm{i}\pi}\right)\sharp\xi = \xi^2 b + \frac{b_x'\xi}{4\mathrm{i}\pi} - \frac{\partial_x}{4\mathrm{i}\pi}\left(\xi b + \frac{b_x'}{4\mathrm{i}\pi}\right) = \xi^2 b + \frac{b_{xx}''}{16\pi^2},$$

so that

$$4\pi^2\iint 2\mathrm{e}^{-2\pi(ax^2+a^{-1}\xi^2)}\left(\xi^2 b + \frac{b_{xx}''}{16\pi^2}\right)dxd\xi = \langle b, \mathcal{H}(u_a',u_a')\rangle,$$

proving that

$$
\begin{aligned}
\mathcal{H}(u_a',u_a')(x,\xi) &= 2\mathrm{e}^{-2\pi(ax^2+a^{-1}\xi^2)}4\pi^2\xi^2 + \frac{1}{4}2\partial_x^2(\mathrm{e}^{-2\pi(ax^2+a^{-1}\xi^2)})\\
&= 2\mathrm{e}^{-2\pi(ax^2+a^{-1}\xi^2)}\left(4\pi^2\xi^2 + \frac{1}{4}((-4\pi a x)^2 - 4\pi a)\right)\\
&= 8\pi^2\mathrm{e}^{-2\pi(ax^2+a^{-1}\xi^2)}a\left(a^{-1}\xi^2 + ax^2 - \frac{1}{4\pi}\right).
\end{aligned}
$$

We obtain that the function $\mathcal{H}(u_a',u_a')$ is negative on

$$a^{-1}\xi^2 + ax^2 < \frac{1}{4\pi},$$

which has area 1/4. We may note as well for consistency that for u_a given by (A.16), we have

$$u_a' = (2a)^{1/4}(-2\pi a x)\mathrm{e}^{-\pi a x^2}, \quad \|u_a'\|_{L^2}^2 = \pi a,$$

and

$$\iint \mathcal{H}(u_a',u_a')(x,\xi)dxd\xi = 8\pi^2 a\iint \mathrm{e}^{-2\pi(y^2+\eta^2)}\left(y^2+\eta^2-\frac{1}{4\pi}\right)dyd\eta = \frac{8\pi^2 a}{8\pi} = \pi a = \|u_a'\|_{L^2}^2.$$

For $\lambda>0$ and $a\in\mathscr{S}'(\mathbb{R}^{2n})$, we define

$$a_\lambda(x,\xi) = a(\lambda^{-1}x,\lambda\xi), \tag{A.18}$$

and we find that

$$
\begin{aligned}
&(a_\lambda)^w = U_\lambda^* a^w U_\lambda, && \text{(A.19)}\\
&\text{for } f\in\mathscr{S}(\mathbb{R}^n),\quad (U_\lambda f)(x) = f(\lambda x)\lambda^{n/2},\quad U_\lambda^* = U_{\lambda^{-1}} = (U_\lambda)^{-1}. && \text{(A.20)}
\end{aligned}
$$

We note that the above formula is a particular case of *Segal's formula* (see, e.g., [9, Theorem 2.1.2]).

The harmonic oscillator. The harmonic oscillator $\mathcal{H}_n$ in n dimensions is defined as the operator with Weyl symbol $\pi(|x|^2+|\xi|^2)$ and thus from (A.19), we find that

$$\mathcal{H}=U_{\sqrt{2\pi}}\,\frac{1}{2}(|x|^2+4\pi^2|\xi|^2)^w U^*_{\sqrt{2\pi}}=U_{\sqrt{2\pi}}\,\frac{1}{2}(-\Delta+|x|^2)U^*_{\sqrt{2\pi}}.$$

We shall define in one dimension the Hermite function of level $k\in\mathbb{N}$, by

$$\psi_k(x)=\frac{(-1)^k}{2^k\sqrt{k!}}2^{1/4}\mathrm{e}^{\pi x^2}\left(\frac{d}{\sqrt{\pi}dx}\right)^k(\mathrm{e}^{-2\pi x^2}), \tag{A.21}$$

and we find that $(\psi_k)_{k\in\mathbb{N}}$ is a Hilbertian orthonormal basis on $L^2(\mathbb{R})$. The one-dimensional harmonic oscillator can be written as

$$\mathcal{H}=\sum_{k\geqslant 0}\left(\frac{1}{2}+k\right)\mathbb{P}_k, \tag{A.22}$$

where $\mathbb{P}_k$ is the orthogonal projection onto ψ_k.

In n dimensions, we consider a multi-index $(\alpha_1,\dots,\alpha_n)=\alpha\in\mathbb{N}^n$ and we define on $\mathbb{R}^n$, using the one-dimensional (A.21),

$$\Psi_\alpha(x)=\prod_{1\leqslant j\leqslant n}\psi_{\alpha_j}(x_j),\quad \mathcal{E}_k=\mathrm{Vect}\{\Psi_\alpha\}_{\alpha\in\mathbb{N}^n,|\alpha|=k},\quad |\alpha|=\sum_{1\leqslant j\leqslant n}\alpha_j. \tag{A.23}$$

We note that

$$\text{the dimension of } \mathcal{E}_{k,n} \text{ is } \quad \binom{k+n-1}{n-1} \tag{A.24}$$

and that (A.22) holds with $\mathbb{P}_{k,n}$ standing for the orthogonal projection onto $\mathcal{E}_{k,n}$; the lowest eigenvalue of $\mathcal{H}$ is $n/2$ and the corresponding eigenspace is one-dimensional in all dimensions, although in two and more dimensions, the eigenspaces corresponding to the eigenvalue $\frac{n}{2}+k, k\geqslant 1$ are multi-dimensional with dimension

$$\binom{k+n-1}{n-1}.$$

The n-dimensional harmonic oscillator can be written as

$$\mathcal{H}_n=\sum_{k\geqslant 0}\left(\frac{n}{2}+k\right)\mathbb{P}_{k,n}, \tag{A.25}$$

where $\mathbb{P}_{k,n}$ stands for the orthogonal projection onto $\mathcal{E}_{k,n}$ defined above. We have in particular

$$\mathbb{P}_{k,n}=\sum_{\alpha\in\mathbb{N}^n,|\alpha|=k}\mathbb{P}_{\alpha,n},\quad \text{where } \mathbb{P}_{\alpha,n} \text{ is the orthogonal projection onto } \Psi_\alpha. \tag{A.26}$$

Some elementary formulas. We define for $\tau\in\mathbb{R}$,

$$\arctan\tau=\int_0^\tau\frac{dt}{1+t^2}, \tag{A.27}$$

and we note that $\arctan\tau\in(-\pi/2,\pi/2)$,

$$\forall\,\tau\in\mathbb{R},\quad \tan(\arctan\tau)=\tau,\quad \forall\,\theta\in(-\pi/2,\pi/2),\quad \arctan(\tan\theta)=\theta.$$

Moreover we have for $\tau\in\mathbb{R}$,

$$\mathrm{e}^{\mathrm{i}\arctan\tau}=\frac{1}{\sqrt{1+\tau^2}}(1+\mathrm{i}\tau), \tag{A.28}$$

since for $\theta\in(-\pi/2,\pi/2)$, $\tau=\tan\theta$, we have $1+\tau^2=\frac{1}{\cos^2\theta}$ and thus

$$\cos\theta>0\Rightarrow\cos\theta=\frac{1}{\sqrt{1+\tau^2}}\Rightarrow-\sin\theta=-\frac{1}{2}(1+\tau^2)^{-3/2}2\tau(1+\tau^2),$$

so that

$$e^{i\theta} = \frac{1}{\sqrt{1+\tau^2}}(1+i\tau).$$

Some Fourier transform. Let $a \in \mathbb{R}_+$ be given. The Fourier transform of $\mathbf{1}_{[-a,a]}$ is

$$\int_{-a}^{a} e^{-2i\pi x\xi} dx = 2\int_0^a \cos(2\pi x\xi)dx = \frac{2}{2\pi\xi}[\sin(2\pi x\xi)]_{x=0}^{x=a} = \frac{\sin(2\pi a\xi)}{\pi\xi}.$$

Taking the derivative of F_k on $\mathbb{R}_+$. We have, using a parity argument,

$$F_k(a) = \int_{\mathbb{R}} \frac{\sin a\tau}{\pi\tau}\frac{(1+i\tau)^{2k+1}}{(1+\tau^2)^{k+1}}d\tau = \sum_{0\leqslant 2l\leqslant 2k}\int_{\mathbb{R}} \frac{\sin a\tau}{\pi\tau}\frac{\binom{2k+1}{2l}(-1)^l\tau^{2l}}{(1+\tau^2)^{k+1}}d\tau.$$

We see also that

$$1+2k+2-2l = 2k+3-2l \geqslant 3$$

so that we can take the derivative of F_k and get

$$F_k'(a) = \sum_{0\leqslant 2l\leqslant 2k}\int_{\mathbb{R}} \frac{\cos a\tau}{\pi}\frac{\binom{2k+1}{2l}(-1)^l\tau^{2l}}{(1+\tau^2)^{k+1}}d\tau = \frac{1}{\pi}\int_{\mathbb{R}}(\cos a\tau)\,\mathrm{Re}\left(\frac{(1+i\tau)^k}{(1-i\tau)^{k+1}}\right)d\tau,$$

with absolutely converging integrals. For $a > 0$, we have

$$F_k'(a) = \frac{1}{\pi}\int_{\mathbb{R}}(\cos a\tau)\frac{(1+i\tau)^k}{(1-i\tau)^{k+1}}d\tau, \tag{A.29}$$

since

$$\lim_{\lambda\to+\infty}\int_{-\lambda}^{\lambda}\frac{\tau^j\cos(a\tau)}{(1+\tau^2)^{k+1}}d\tau \quad \text{makes sense for} \quad j\leqslant 2k+1 \quad \text{(and vanishes for } j \text{ odd)}. \tag{A.30}$$

Proof of the weak limit (1.5). We have for $u \in \mathscr{S}(\mathbb{R}^n)$, according to (A.11),

$$\langle(\mathbf{1}\{2\pi(x^2+\xi^2)\leqslant a\})^w u, u\rangle = \iint_{2\pi(x^2+\xi^2)\leqslant a}\mathcal{H}(u,u)(x,\xi)dxd\xi,$$

so that (1.3) implies

$$\sum_{k\geqslant 0}F_k(a)\langle\mathbb{P}_k u, u\rangle_{L^2(\mathbb{R}^n)} = \iint_{2\pi(x^2+\xi^2)\leqslant a}\mathcal{H}(u,u)(x,\xi)dxd\xi.$$

Choosing now $u = u_k$ as a normalized eigenfunction of the harmonic oscillator with eigenvalue $k+1/2$, we obtain

$$F_k(a) = \iint_{2\pi(x^2+\xi^2)\leqslant a}\mathcal{H}(u_k,u_k)(x,\xi)dxd\xi.$$

Since the function $(x,\xi)\mapsto\mathcal{H}(u_k,u_k)(x,\xi)$ belongs to the Schwartz class of $\mathbb{R}^{2n}$, we find that

$$\lim_{a\to+\infty}F_k(a) = \iint_{\mathbb{R}^{2n}}\mathcal{H}(u_k,u_k)(x,\xi)dxd\xi = \|u_k\|^2_{L^2(\mathbb{R}^n)} = 1. \qquad \square$$

A different normalization for the Wigner function. The paper [10] is using a different normalization for the Wigner distribution in n dimensions with

$$\mathcal{W}(u,v)(x,\xi) = (2\pi)^{-n}\int_{\mathbb{R}^n}u\left(x+\frac{z}{2}\right)\bar{v}\left(x-\frac{z}{2}\right)e^{-iz\cdot\xi}dz. \tag{A.31}$$

The relationship with our definition (A.12) is

$$\mathcal{W}(u,v)(x,\xi) = \mathcal{H}(u,v)\left(x,\frac{\xi}{2\pi}\right)(2\pi)^{-n}. \tag{A.32}$$

As a result, we find that

$$\mathcal{E}_{lo}(\mathbb{B}^{2n}(R)) = \sup_{\|u\|_{L^2(\mathbb{R}^n)}=1} \iint_{|x|^2+|\xi|^2\leqslant R^2} \mathcal{W}(u,u)(x,\xi)dxd\xi$$

is equal to

$$\sup_{\|u\|_{L^2(\mathbb{R}^n)}=1} \iint_{|x|^2+4\pi^2|\xi|^2\leqslant R^2} \mathcal{H}(u,u)(x,\xi)dxd\xi = \sup_{\|u\|_{L^2(\mathbb{R}^n)}=1} \iint_{2\pi(|x|^2+|\xi|^2)\leqslant R^2} \mathcal{H}(u,u)(x,\xi)dxd\xi$$

and we have proven here that for $u \in L^2(\mathbb{R}^n)$ with norm 1,

$$\iint_{|x|^2+|\xi|^2\leqslant \frac{a}{2\pi}=\frac{R^2}{2\pi}} \mathcal{H}(u,u)(x,\xi)dxd\xi \leqslant 1 - \frac{1}{(n-1)!}\int_a^{+\infty} \mathrm{e}^{-t}t^{n-1}dt = 1 - \frac{\Gamma(n,R^2)}{\Gamma(n)},$$

where the upper incomplete Gamma function $\Gamma(z,x)$ is given by

$$\Gamma(z,x) = \int_x^{+\infty} t^{z-1}\mathrm{e}^{-t}dt. \tag{A.33}$$

This is indeed the result of [10, Theorem 1].

Appendix B Mehler's formula

We provide first a proof of a particular case of the results of [7].

Lemma B.1. *For* $\mathrm{Re}\, t \geqslant 0$, $t \notin \mathrm{i}\pi(2\mathbb{Z}+1)$, *we have in* n *dimensions,*

$$(\cosh(t/2))^n \exp -t\pi(|x|^2+|\xi|^2)^w = (\mathrm{e}^{-2\tanh(\frac{t}{2})\pi(x^2+\xi^2)})^w. \tag{B.1}$$

Proof. By tensorisation, it is enough to prove that formula for $n=1$, which we assume from now on. To prove that formula, we need only to consider the one-dimensional case. We define

$$L = \xi + \mathrm{i}x, \quad \bar{L} = \xi - \mathrm{i}x, \quad M(t) = \beta(t)(\mathrm{e}^{-\alpha(t)\pi L\bar{L}})^w,$$

where α and β are smooth functions of t to be chosen below. Assuming $\beta(0)=1, \alpha(0)=0$, we find that $M(0) = \mathrm{Id}$ and

$$\dot{M} + \pi(|L|^2)^w M = (\dot{\beta}\mathrm{e}^{-\alpha\pi|L|^2} - \beta\dot{\alpha}\pi|L|^2\mathrm{e}^{-\alpha\pi|L|^2} + \pi(|L|^2)\sharp\beta\mathrm{e}^{-\alpha\pi|L|^2})^w.$$

We have from (A.14), since $\partial_x\partial_\xi|L|^2 = 0$,

$$\begin{aligned}
|L|^2\sharp\mathrm{e}^{-\alpha\pi|L|^2} &= |L|^2\mathrm{e}^{-\alpha\pi|L|^2} + \frac{1}{4\mathrm{i}\pi}\overbrace{\{|L|^2, \mathrm{e}^{-\alpha\pi|L|^2}\}}^{=0} \\
&\quad + \frac{1}{(4\mathrm{i}\pi)^2}\frac{1}{2}(\partial_\xi^2(|L|^2)\partial_x^2\mathrm{e}^{-\alpha\pi|L|^2} + \partial_x^2(|L|^2)\partial_\xi^2\mathrm{e}^{-\alpha\pi|L|^2}) \\
&= |L|^2\mathrm{e}^{-\alpha\pi|L|^2} + \frac{1}{(4\mathrm{i}\pi)^2}\frac{1}{2}\mathrm{e}^{-\alpha\pi|L|^2}(2((-2\alpha\pi x)^2 - 2\alpha\pi) + 2((-2\alpha\pi\xi)^2 - 2\alpha\pi)) \\
&= |L|^2\mathrm{e}^{-\alpha\pi|L|^2}\left(1 - \frac{4\alpha^2\pi^2}{16\pi^2}\right) + \frac{\alpha\pi}{4\pi^2}\mathrm{e}^{-\alpha\pi|L|^2},
\end{aligned}$$

so that

$$\begin{aligned}
&\dot{M} + \pi(|L|^2)^w M \\
&= \left(\dot{\beta}\mathrm{e}^{-\alpha\pi|L|^2} - \beta\dot{\alpha}\pi|L|^2\mathrm{e}^{-\alpha\pi|L|^2} + \pi\beta|L|^2\mathrm{e}^{-\alpha\pi|L|^2}\left(1 - \frac{4\alpha^2\pi^2}{16\pi^2}\right) + \frac{\alpha\pi\beta}{4\pi}\mathrm{e}^{-\alpha\pi|L|^2}\right)^w
\end{aligned}$$

$$= \left(\mathrm{e}^{-\alpha\pi|L|^2}\left\{|L|^2\left(-\pi\dot{\alpha}\beta+\pi\beta\left(1-\frac{\alpha^2}{4}\right)\right)+\dot{\beta}+\frac{\alpha\beta}{4}\right\}\right)^w.$$

We solve now

$$\dot{\alpha}=1-\frac{\alpha^2}{4},\quad \alpha(0)=0 \Leftrightarrow \alpha(t)=2\tanh(t/2),$$

and

$$4\dot{\beta}+\alpha\beta=0,\quad \beta(0)=1 \Leftrightarrow \beta(t)=\frac{1}{\cosh(t/2)}.$$

We obtain that

$$\dot{M}+\pi(|L|^2)^w M=0,\quad M(0)=\mathrm{Id},$$

and this implies

$$\beta(t)(\mathrm{e}^{-\alpha(t)\pi L\bar{L}})^w=M(t)=\exp -t\pi(|L|^2)^w,$$

which proves (B.1). □

In particular, for $t=-2\mathrm{i}s$, $s\in\mathbb{R}$, $s\notin\frac{\pi}{2}(1+2\mathbb{Z})$, we have in n dimensions,

$$(\cos s)^n\exp(2\mathrm{i}\pi s(|x|^2+|\xi|^2)^w)=(\mathrm{e}^{2\mathrm{i}\pi\tan s(|x|^2+|\xi|^2)})^w. \tag{B.2}$$

Lemma B.2. *For any $z\in\mathbb{C}, \operatorname{Re} z\geqslant 0$, we have in n dimensions,*

$$[\exp -(2z\pi(|\xi|^2+|x|^2))]^w=\frac{1}{(1+z)^n}\sum_{k\geqslant 0}\left(\frac{1-z}{1+z}\right)^k\mathbb{P}_{k,n}, \tag{B.3}$$

where $\mathbb{P}_{k,n}$ is defined in Appendix A and the equality holds between $L^2(\mathbb{R}^n)$-bounded operators.

Proof. Starting from (B.2), we get for $\tau\in\mathbb{R}$, in n dimensions,

$$(\cos(\arctan\tau))^n\exp(2\mathrm{i}\pi\arctan\tau(|x|^2+|\xi|^2)^w)=(\mathrm{e}^{2\mathrm{i}\pi\tau(|x|^2+|\xi|^2)})^w,$$

so that using the spectral decomposition of the (n-dimensional) harmonic oscillator and (A.28), we get

$$(1+\tau^2)^{-n/2}\sum_{k\geqslant 0}\mathrm{e}^{2\mathrm{i}(\arctan\tau)(k+\frac{n}{2})}\mathbb{P}_{k,n}=(\mathrm{e}^{2\mathrm{i}\pi\tau(|x|^2+|\xi|^2)})^w,$$

which implies

$$(1+\tau^2)^{-n/2}\sum_{k\geqslant 0}\frac{(1+\mathrm{i}\tau)^{2k+n}}{(1+\tau^2)^{k+\frac{n}{2}}}\mathbb{P}_{k,n}=(\mathrm{e}^{2\mathrm{i}\pi\tau(|x|^2+|\xi|^2)})^w,$$

entailing

$$\sum_{k\geqslant 0}\frac{(1+\mathrm{i}\tau)^k}{(1-\mathrm{i}\tau)^{k+n}}\mathbb{P}_{k,n}=(\mathrm{e}^{2\mathrm{i}\pi\tau(|x|^2+|\xi|^2)})^w,$$

proving the lemma by analytic continuation (we may refer the reader as well to [11, pp. 204–205] and note that for any $z\in\mathbb{C}$, $\operatorname{Re} z\geqslant 0$, we have $|\frac{1-z}{1+z}|\leqslant 1$). □

Appendix C Laguerre polynomials

The Laguerre polynomials $\{L_k\}_{k\in\mathbb{N}}$ are defined by

$$L_k(x)=\sum_{0\leqslant l\leqslant k}\frac{(-1)^l}{l!}\binom{k}{l}x^l=\mathrm{e}^x\frac{1}{k!}\left(\frac{d}{dx}\right)^k\{x^k\mathrm{e}^{-x}\}=\left(\frac{d}{dx}-1\right)^k\left\{\frac{x^k}{k!}\right\}, \tag{C.1}$$

and we have

$$L_0=1,$$

$$
\begin{aligned}
L_1 &= -X+1,\\
L_2 &= \frac{1}{2}(X^2-4X+2),\\
L_3 &= \frac{1}{6}(-X^3+9X^2-18X+6),\\
L_4 &= \frac{1}{24}(X^4-16X^3+72X^2-96X+24),\\
L_5 &= \frac{1}{120}(-X^5+25X^4-200X^3+600X^2-600X+120),\\
L_6 &= \frac{1}{720}(X^6-36X^5+450X^4-2400X^3+5400X^2-4320X+720),\\
L_7 &= \frac{-X^7+49X^6-882X^5+7350X^4-29400X^3+52920X^2-35280X+5040}{5040}.
\end{aligned}
$$

We get also easily from the above definition that

$$L'_{k+1} = L'_k - L_k, \tag{C.2}$$

since with $T = d/dX - 1$,

$$L'_k - L_k = TL_k = T^{k+1}\left(\frac{X^k}{k!}\right) = T^{k+1}\left(\frac{d}{dX}\frac{X^{k+1}}{(k+1)!}\right) = \frac{d}{dX}L_{k+1}.$$

Formula (6.8) and Theorem 12 in [2] provided the inequalities

$$\forall\, k\in\mathbb{N}, \quad \forall\, x\geqslant 0, \quad \sum_{0\leqslant l\leqslant k}(-1)^l L_l(x)\geqslant 0. \tag{C.3}$$

This result follows as well from [3, (73)] in 1940 by Feldheim.

Let us calculate the Fourier transform of the Laguerre polynomials: we have

$$L_k(x) = \left(\frac{d}{dx}-1\right)^k\left\{\frac{x^k}{k!}\right\},$$

so that

$$\widehat{L_k}(\xi) = (2\mathrm{i}\pi\xi-1)^k\left(\frac{-1}{2\mathrm{i}\pi}\right)^k\frac{\delta_0^{(k)}}{k!} = \frac{(-1)^k}{k!}\left(\xi-\frac{1}{2\mathrm{i}\pi}\right)^k\delta_0^{(k)}(\xi).$$

As a result, defining for $k\in\mathbb{N}, t\in\mathbb{R}$,

$$G_k(t) = (-1)^k H(t)\mathrm{e}^{-t}L_k(2t), \quad H = \mathbf{1}_{\mathbb{R}_+}, \tag{C.4}$$

we find, using the homogeneity of degree $-k-1$ of $\delta_0^{(k)}$,

$$
\begin{aligned}
\widehat{G_k}(\tau) &= \frac{1}{2}\frac{(-1)^k}{k!}\left(\frac{\tau}{2}-\frac{1}{2\mathrm{i}\pi}\right)^k\delta_0^{(k)}\left(\frac{\tau}{2}\right) * \frac{(-1)^k}{1+2\mathrm{i}\pi\tau}\\
&= (-1)^k\left(\frac{d}{d\sigma}\right)^k\left\{\frac{(\sigma-\frac{1}{\mathrm{i}\pi})^k/k!}{1+2\mathrm{i}\pi(\tau-\sigma)}\right\}_{|\sigma=0},\\
\widehat{G_k}(\tau) &= \sum_l(-1)^k\binom{k}{l}\frac{(\sigma-\frac{1}{\mathrm{i}\pi})^{k-l}}{(k-l)!}\frac{(k-l)!(2\mathrm{i}\pi)^{k-l}}{(1+2\mathrm{i}\pi(\tau-\sigma))^{1+k-l}}_{|\sigma=0}\\
&= \sum_l(-1)^k\binom{k}{l}\frac{(-2)^{k-l}}{(1+2\mathrm{i}\pi\tau)^{1+k-l}}\\
&= \frac{(-1)^k}{(1+2\mathrm{i}\pi\tau)}\sum_l\binom{k}{l}\frac{(-2)^{k-l}}{(1+2\mathrm{i}\pi\tau)^{k-l}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(-1)^k}{(1+2\mathrm{i}\pi\tau)}\left(1-\frac{2}{(1+2\mathrm{i}\pi\tau)}\right)^k\\
&= \frac{(-1)^k}{(1+2\mathrm{i}\pi\tau)}\left(\frac{-1+2\mathrm{i}\pi\tau}{1+2\mathrm{i}\pi\tau}\right)^k\\
&= \frac{1}{(1+2\mathrm{i}\pi\tau)}\left(\frac{1-2\mathrm{i}\pi\tau}{1+2\mathrm{i}\pi\tau}\right)^k
\end{aligned}
$$

so that

$$
\widehat{G_k}(\tau) = \frac{(1-2\mathrm{i}\pi\tau)^k}{(1+2\mathrm{i}\pi\tau)^{k+1}} = \frac{(1-2\mathrm{i}\pi\tau)^{2k+1}}{(1+4\pi^2\tau^2)^{k+1}}. \tag{C.5}
$$