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Sharp heat kernel estimates for spectral fractional Laplacian perturbed by gradients

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Abstract Using Duhamel's formula, we prove sharp two-sided estimates for the spectral fractional Laplacian's heat kernel with time-dependent gradient perturbation in bounded $C^{1,1}$ domains. In addition, we obtain a gradient estimate as well as the Hölder continuity of the heat kernel's gradient.

Keywords spectral fractional Laplacian, Dirichlet heat kernel, Kato class, gradient estimate

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1 Introduction

Let W_t be a Brownian motion in $\mathbb{R}^d$ ($d \geqslant 1$) with the generator Δ, and let T_t be an independent $\alpha/2$-stable subordinator where $\alpha \in (0,2)$. Then, the subordinate process $X_t := W_{T_t}$ is an isotropic α-stable process, and its infinitesimal generator is the fractional Laplacian operator $-(-\Delta^{\alpha/2})$, given by

$$-(-\Delta^{\alpha/2})f(x) := \int_{\mathbb{R}^d} [f(x+z) - f(x) - 1_{|z|\leqslant 1} z \cdot \nabla f(x)] \frac{c_{d,\alpha}}{|z|^{d+\alpha}} dz$$

for $f \in C_c^2(\mathbb{R}^d)$, where $c_{d,\alpha}$ is a positive constant. It is well known that the heat kernel $p(t,x,y)$ of $-(-\Delta^{\alpha/2})$, which is also the transition density of $X := (X_t)_{t\geqslant 0}$, has the following estimate: for every $t > 0$ and $x, y \in \mathbb{R}^d$,

$$p(t,x,y) \asymp \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right). \tag{1.1}$$

Here and in the following sections, for two non-negative functions, f and g, the notation $f \asymp g$ expresses the presence of positive constants, c_1 and c_2, such that $c_1 g(x) \leqslant f(x) \leqslant c_2 g(x)$ in the common domain of the definitions of f and g.

In [2], Bogdan and Jakubowski used Duhamel's formula to study the following gradient perturbation of $-(-\Delta^{\alpha/2})$:

$$\mathscr{L}^b := -(-\Delta^{\alpha/2}) + b(x) \cdot \nabla, \quad \alpha \in (1,2),$$

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where $b = (b^1, \dots, b^d) : \mathbb{R}^d \to \mathbb{R}^d$ with b^j, $j = 1, \dots, d$, belonging to the Kato class $\mathbf{K}_d^{\alpha-1}$, which is defined as follows: for $\gamma > 0$,

$$\mathbf{K}_d^{\gamma} := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^d) : \lim_{r\downarrow 0} \sup_{x\in\mathbb{R}^d} \int_{B(x,r)} \frac{|f(y)|}{|x-y|^{d-\gamma}} dy = 0 \right\}. \tag{1.2}$$

Here and below, $B(x,r)$ denotes the open ball centered at $x \in \mathbb{R}^d$ with the radius denoted as r. Let $p^b(t,x,y)$ be the heat kernel of $\mathscr{L}^b$. Small-time sharp two-sided estimates for $p^b(t,x,y)$ of the form (1.1) have been established in [2, Theorems 1 and 2]. In [2], the authors' perturbation method includes two key components. First, an accurate estimate for $\nabla_x p(t,x,y)$ is known, and second, the following 3-P inequality concerning $p(t,x,y)$ holds: there exists $C_0 > 0$ such that for any $0 < s < t$ and $x, y, z \in \mathbb{R}^d$,

$$\frac{p(t-s,x,z)p(s,z,y)}{p(t,x,y)} \leqslant C_0(p(t-s,x,z) + p(s,z,y)). \tag{1.3}$$

See also [5, 9, 13, 14, 22, 23] and the references therein for two-sided heat kernel estimates of more general non-local operators in the whole space $\mathbb{R}^d$.

Let D be an open subset of $\mathbb{R}^d$; hence, the process X can be killed upon exiting D and a subprocess, X^D, known as the killed isotropic α-stable process may be obtained. The infinitesimal generator of X^D is the Dirichlet fractional Laplacian, $-(-\Delta)^{\alpha/2}|_D$, i.e., the fractional Laplacian with zero exterior conditions. By owing to complications near the boundary, two-sided estimates for the Dirichlet heat kernel of $-(-\Delta)^{\alpha/2}|_D$ (i.e., the transition density of X^D) are extremely difficult to obtain. To state related results, we first recall that an open set D in $\mathbb{R}^d$ is said to be $C^{1,1}$ if there exist $r_0 > 0$ and $\Lambda > 0$ such that for every $Q \in \partial D$, there exist a $C^{1,1}$-function $\phi = \phi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$, which satisfies $\phi(0) = \nabla\phi(0) = 0$, $\|\nabla\phi\|_\infty \leqslant \Lambda$, $|\nabla\phi(x) - \nabla\phi(z)| \leqslant \Lambda|x-z|$ and an orthonormal coordinate system $y = (y_1, \dots, y_{d-1}, y_d) =: (\tilde{y}, y_d)$ such that

$$B(Q, r_0) \cap D = B(Q, r_0) \cap \{y : y_d > \phi(\tilde{y})\}.$$

The pair (r_0, Λ) represents characteristics of the $C^{1,1}$ open set D. In [6], Chen et al. proved that when D is a $C^{1,1}$ open set in $\mathbb{R}^d$, the heat kernel $p^D(t,x,y)$ of $-(-\Delta)^{\alpha/2}|_D$ has the following two-sided estimates: for every $T > 0$ and $(t,x,y) \in (0,T] \times D \times D$,

$$p^D(t,x,y) \asymp \left(1 \wedge \frac{\rho(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\rho(y)^{\alpha/2}}{\sqrt{t}}\right) p(t,x,y), \tag{1.4}$$

where $\rho(x)$ denotes the distance between x and D^c.

By reversing the order of subordination and killing, one can obtain a process Y^D, which is different from X^D. More precisely, in this study, we first kill the Brownian motion W at τ_D (i.e., the first exit time of W from D) and then subordinate the killed Brownian motion W^D using the independent $\alpha/2$-stable subordinator T_t. Thus, $Y^D := (W^D)_{T_t}$ is defined as

$$Y_t^D := \begin{cases} W_{T_t}, & T_t < \tau_D, \\ \partial, & T_t \geqslant \tau_D \end{cases} = \begin{cases} W_{T_t}, & t < A_{\tau_D}, \\ \partial, & t \geqslant A_{\tau_D}, \end{cases}$$

where ∂ is a cemetery state, $A_t := \inf\{s > 0 : T_s \geqslant t\}$ is the inverse of T, and the last equality follows from the fact that $\{T_t < \tau_D\} = \{t < A_{\tau_D}\}$. The process Y^D is called a subordinate killed Brownian motion. To understand the relations between the processes X^D and Y^D, see [20]. The infinitesimal generator of Y^D is the spectral fractional Laplacian $-(-\Delta|_D)^{\alpha/2}$, which is defined as a fractional power of the negative Dirichlet Laplacian. This operator is a very useful object in analysis and partial differential equations (see [3, 17, 21]) and has been intensively studied (see [1, 10, 12, 19] and the references therein). When D is a bounded $C^{1,1}$ domain, the following sharp estimates for the heat kernel $r^D(t,x,y)$ of $-(-\Delta|_D)^{\alpha/2}$ (which is also the transition density of Y^D) were obtained in [18, Theorem 4.7]: for every $T > 0$ and $(t,x,y) \in (0,T] \times D \times D$,

$$r^D(t,x,y) \asymp \left(1 \wedge \frac{\rho(x)\rho(y)}{(|x-y| + t^{1/\alpha})^2}\right) p(t,x,y).$$

In Lemma 2.1 below, we will provide the following alternative form of the above-mentioned estimates: for $(t,x,y)\in(0,T]\times D\times D$,

$$r^D(t,x,y)\asymp\left(1\wedge\frac{\rho(x)}{|x-y|+t^{1/\alpha}}\right)\left(1\wedge\frac{\rho(y)}{|x-y|+t^{1/\alpha}}\right)p(t,x,y),$$

which is more convenient to use.

Gradient perturbations of Dirichlet operators have also been widely studied in recent years. In [7], Chen et al. studied the following perturbation of the Dirichlet fractional Laplacian by a gradient operator:

$$\mathscr{L}^{b,D}:=(-(-\Delta)^{\alpha/2}+b(x)\cdot\nabla)\,|_D,\quad \alpha\in(1,2).$$

Under the condition that $b\in\mathbf{K}_d^{\alpha-1}$ (see (1.2)) and D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with $d\geqslant 2$, Chen et al. [7, Theorem 1.3] demonstrated that the heat kernel $p^{b,D}(t,x,y)$ of $\mathscr{L}^{b,D}$ has the same estimate as that in (1.4). This result was generalized to unbounded $C^{1,1}$ open sets by [15]. Unlike the whole space case, there was no good estimate on $\nabla_x p^D(t,x,y)$; thus, Chen et al. [7] and Kim and Song [15] used Duhamel's formula for the Green function and the probabilistic road-map designed in [6] for establishing the estimates (1.4).

In a recent study [16], Kulczycki and Ryznar proved the following gradient estimate for $p^D(t,x,y)$: for any $T>0$, there exists a constant $C=C(d,T)>0$ such that for any $(t,x,y)\in(0,T]\times D\times D$,

$$|\nabla_x p^D(t,x,y)|\leqslant\frac{C}{\rho(x)\wedge t^{1/\alpha}}p^D(t,x,y).$$

Following this, in a recent study [4], we complete a direct proof of the main results in [7, 15] using Duhamel's formula, with drift $b=(b^1,\dots,b^d):D\to\mathbb{R}^d$, where each b^j, $j=1,\dots,d$, belongs to the following Kato class:

$$\mathbf{K}_D^{\alpha-1}:=\left\{f\in L^1_{\rm loc}(D):\lim_{r\downarrow 0}\sup_{x\in D}\int_{D\cap B(x,r)}\frac{|f(y)|}{|x-y|^{d+1-\alpha}}dy=0\right\}.$$

Moreover, we also obtain a gradient estimate for $p^{b,D}(t,x,y)$. Notice that by using Hölder's inequality, $L^p(D)\subseteq\mathbf{K}_D^{\alpha-1}$ provided $d/(\alpha-1)<p\leqslant\infty$.

The aim of this article is to study the following spectral fractional Laplacian perturbed by a time-dependent gradient operator:

$$\mathscr{L}^{D,b}:=-(-\Delta\,|_D)^{\frac{\alpha}{2}}+b(t,x)\cdot\nabla,\quad \alpha\in(1,2)$$

with $b(t,x)=(b^1(t,x),\dots,b^d(t,x)):(0,\infty)\times D\to\mathbb{R}^d$ satisfying certain conditions which will be specified below. Herein, we derive sharp two-sided estimates for the heat kernel $r^{D,b}(s,x;t,y)$ of $\mathscr{L}^{D,b}$ in bounded $C^{1,1}$ domains. Moreover, we also obtain a gradient estimate and the Hölder continuity of the gradient of $r^{D,b}(s,x;t,y)$, which are of independent interest.

To state our main result, let us first introduce our local Kato class of space-time functions used herein.

Definition 1.1. Let D be a domain in $\mathbb{R}^d$ and $\gamma\geqslant 0$. For a real-valued function f on $(0,\infty)\times D$ and every $\delta>0$, we define

$$\begin{aligned}K_f^\gamma(\delta):=\sup_{t>0,x\in D}\delta^{\gamma/\alpha}\int_0^\delta\!\!\int_D[s^{-\gamma/\alpha}+(\delta-s)^{-\gamma/\alpha}]\left(1\wedge\frac{\rho(y)}{|x-y|+s^{1/\alpha}}\right)\\ \times\frac{s}{(|x-y|+s^{1/\alpha})^{d+\alpha+1}}\cdot|f(t\pm s,y)|dyds.\end{aligned}$$

We declare that the function f belongs to the Kato class $\mathbb{K}_D^\gamma$ if $\lim_{\delta\downarrow 0}K_f^\gamma(\delta)=0$.

Remark 1.2. We note that our Kato class is time-dependent, which is crucial when considering parabolic problems [13, 24]. Moreover, the boundary behavior of the heat kernel is involved in the definition. One can easily check that if $0\leqslant\gamma_1<\gamma_2$, then $\mathbb{K}_D^{\gamma_2}\subseteq\mathbb{K}_D^{\gamma_1}$. Based on Lemma 3.1, we know that $\mathbf{K}_D^{\alpha-1}\subset\mathbb{K}_D^0$ and that, for $1<p,q\leqslant\infty$, $L^q(\mathbb{R};L^p(D))\subseteq\mathbb{K}_D^\gamma$ provided $\frac{d}{\alpha p}+\frac{1}{q}<1-\frac{1+\gamma}{\alpha}$.

In the remainder of this paper, we consistently assume that $b = (b^1, \ldots, b^d) : (0,\infty) \times D \to \mathbb{R}^d$ and each b^j, $j = 1, \ldots, d$, belongs to $\mathbb{K}_D^0$.

According to Duhamel's formula, the heat kernel $r^{D,b}(s,x;t,y)$ of $\mathscr{L}^{D,b}$ should satisfy the following integral equation: for $0 \leqslant s < t$ and $x, y \in D$,

$$r^{D,b}(s,x;t,y) = r^D(t-s,x,y) + \int_s^t\!\!\int_D r^{D,b}(s,x;r,z)b(r,z) \cdot \nabla_z r^D(t-r,z,y)dzdr, \tag{1.5}$$

or

$$r^{D,b}(s,x;t,y) = r^D(t-s,x,y) + \int_s^t\!\!\int_D r^D(r-s,x,z)b(r,z) \cdot \nabla_z r^{D,b}(r,z;t,y)dzdr. \tag{1.6}$$

Notice that in (1.5) the derivative of the unknown heat kernel is not involved, and hence, it is easier to solve, while, (1.6) is directly connected to the mild solutions of the corresponding parabolic equations, from which one can easily derive the Hölder continuity of the gradient of the unknown heat kernel. For convenience, for $t > 0$ and $x, y \in D$, we define

$$q^D(t,x,y) := \left(1 \wedge \frac{\rho(x)}{|x-y| + t^{1/\alpha}}\right)\left(1 \wedge \frac{\rho(y)}{|x-y| + t^{1/\alpha}}\right)p(t,x,y). \tag{1.7}$$

The following is the main result of this study.

Theorem 1.3. *Let D be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and $b \in \mathbb{K}_D^0$. Then, there exists a unique function $r^{D,b}(s,x;t,y)$ on $(0,\infty) \times D \times D$ satisfying* (1.5) *such that*:

(i) (*two-sided estimates*) *for any $\delta > 0$, there exists a constant $C_1 > 1$ such that for all $0 \leqslant s < t \leqslant s+\delta$ and $x, y \in D$, we have*

$$C_1^{-1} q^D(t-s,x,y) \leqslant r^{D,b}(s,x;t,y) \leqslant C_1 q^D(t-s,x,y); \tag{1.8}$$

(ii) (*gradient estimate*) *for any $\delta > 0$, there exists a constant $C_2 > 0$ such that for all $0 \leqslant s < t \leqslant s+\delta$ and $x, y \in D$,*

$$|\nabla_x r^{D,b}(s,x;t,y)| \leqslant C_2 \frac{1}{\rho(x) \wedge (|x-y| + (t-s)^{1/\alpha})} q^D(t-s,x,y), \tag{1.9}$$

and $r^{D,b}(s,x;t,y)$ also satisfies (1.6);

(iii) (*C-K equation*) *for all $0 \leqslant s < r < t$ and $x, y \in D$, the following Chapman-Kolmogorov* (*C-K*) *equation holds*:

$$\int_D r^{D,b}(s,x;r,z) r^{D,b}(r,z;t,y)dz = r^{D,b}(s,x;t,y); \tag{1.10}$$

(iv) (*generator*) *for any $f \in C_c^2(D)$, we have*

$$R_{s,t}^{D,b} f(x) = f(x) + \int_s^t R_{s,r}^{D,b} \mathcal{L}^{D,b} f(x)dr, \tag{1.11}$$

where $R_{s,t}^{D,b} f(x) := \int_D r^{D,b}(s,x;t,y)f(y)dy$;

(v) (*continuity*) *for any uniformly continuous function $f(x)$ with compact supports, we have*

$$\lim_{t \downarrow s} \|R_{s,t}^{D,b} f - f\|_\infty = 0; \tag{1.12}$$

(vi) (*Hölder continuity*) *if we further assume that $b \in \mathbb{K}_D^\gamma$ for some $\gamma \in (0, \alpha-1)$, then for any $\delta > 0$, there exists a constant $C_3 > 0$ such that for any $0 \leqslant s < t \leqslant s+\delta$ and $x, x', y \in D$, we have*

$$|\nabla_x r^{D,b}(s,x;t,y) - \nabla_x r^{D,b}(s,x';t,y)| \leqslant C_3 |x-x'|^\gamma (t-s)^{-\gamma/\alpha}$$

$$\times \frac{1}{\rho(\widetilde{x}) \wedge (|\widetilde{x}-y|+(t-s)^{1/\alpha})} q^D(t-s,\widetilde{x},y), \tag{1.13}$$

where $\widetilde{x}$ denotes the point between x and x' which is closer to y.

Notably, the gradient estimates (1.9) and (1.13) are new even in the case $b \equiv 0$. We now briefly describe the main idea of our argument. By owing to differences between the processes Y^D and X^D, the method used in [7, 15] does not work for $\mathscr{L}^{D,b}$. Instead, we use Duhamel's formula (1.5) to obtain the sharp two-sided estimates of the heat kernel. As mentioned before, the following two key components are necessary: the gradient estimate for $r^D(t,x,y)$ and the corresponding 3-P inequality, both of which are currently unknown. In fact, by Remark 2.2, we shall see that the 3-P inequality of the form (1.3) does not hold for the heat kernel $r^D(t,x,y)$. Hence, we will first derive an estimate on $\nabla_x r^D(t,x,y)$, and then establish a generalized 3-P type inequality for $r^D(t,x,y)$. It turns out that, in the process of deriving an estimate on $\nabla_x r^D(t,x,y)$, we also slightly improve the estimates concerning the heat kernel of the Dirichlet Laplacian operator $\Delta|_D$, which has already been intensively studied (see Lemmas 2.6 and 2.7). The gradient and the Hölder estimates for $r^{D,b}(s,x;t,y)$ follow as easy by-products of our perturbation argument.

The rest of this paper is organized as follows. In Section 2, we prepare some important inequalities for $r^D(t,x,y)$ and derive its first and second order gradient estimates. The proof of the main result, Theorem 1.3, is presented in Section 3.

We conclude this introduction by defining some conventions that will be used throughout this paper. The letter C with or without subscripts will denote an unimportant constant, and $f \preceq g$ represents $f \leqslant Cg$ for some $C \geqslant 1$. The letter $\mathbb{N}$ will denote the collection of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We will use the symbol := to denote a definition, and we assume that all the functions considered in this paper are Borel measurable.

2 Estimates for $r^D(t,x,y)$

In the rest of this paper, D denotes a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. For simplicity, we first introduce some functions for later use. Given $d \geqslant 1$, $\vartheta \in \mathbb{R}$, $\alpha \in (0,2]$, $t > 0$, and $x, y \in D$, we define

$$\varrho_d^{\vartheta}(t,x) := \frac{t^{\vartheta}}{(|x|+t^{1/\alpha})^{d+\alpha}}$$

and

$$\hat{q}_\alpha(t,x,y) := 1 \wedge \frac{\rho(x)}{|x-y|+t^{1/\alpha}}, \quad q_\alpha(t,x,y) := \hat{q}_\alpha(t,x,y)\hat{q}_\alpha(t,y,x). \tag{2.1}$$

Then, we have $p(t,x,y) \asymp \varrho_d^1(t,x-y)$ and $q^D(t,x,y) = q_\alpha(t,x,y)p(t,x,y)$.

We will first establish a generalized 3-P type inequality for $r^D(t,x,y)$, and then derive its first and second order gradient estimates, which will be essential in constructing the solution to the integral equation (1.5).

2.1 Generalized 3-P inequality

Let $T > 0$ be fixed. Recall that $r^D(t,x,y)$ is the heat kernel of $-(-\Delta|_D)^{\frac{\alpha}{2}}$, and for any $t \in (0,T]$ and $x, y \in D$, we have

$$r^D(t,x,y) \asymp \left(1 \wedge \frac{\rho(x)\rho(y)}{(|x-y|+t^{1/\alpha})^2}\right)\varrho_d^1(t,x-y).$$

The estimates provided above are not particularly convenient for our application as $\rho(x)$ and $\rho(y)$ are intertwined together. Hence, we prove the following result.

Lemma 2.1. *For any $t \in (0,T]$ and $x,y \in D$, we have*

$$r^D(t,x,y) \asymp q_\alpha(t,x,y)\varrho_d^1(t,x-y) \asymp q^D(t,x,y). \tag{2.2}$$

Proof. The second comparison follows from the fact that $p(t,x,y) \asymp \varrho_d^1(t,x-y)$ and $q^D(t,x,y) = q_\alpha(t,x,y)p(t,x,y)$. Hence, we will only prove the first comparison. It is clear that

$$q_\alpha(t,x,y) \leqslant 1 \wedge \frac{\rho(x)\rho(y)}{(|x-y|+t^{1/\alpha})^2}.$$

Thus, we only show that

$$q_\alpha(t,x,y) \succeq 1 \wedge \frac{\rho(x)\rho(y)}{(|x-y|+t^{1/\alpha})^2}. \tag{2.3}$$

One can see that the above inequality holds when

$$\rho(x) \vee \rho(y) \leqslant |x-y|+t^{1/\alpha} \quad \text{or} \quad \rho(x) \wedge \rho(y) \geqslant |x-y|+t^{1/\alpha}.$$

By symmetry, it suffices to prove (2.3) in the case when

$$\rho(x) \leqslant |x-y|+t^{1/\alpha} \leqslant \rho(y).$$

Based on the fact that $\rho(y) \leqslant \rho(x)+|x-y|$, we can deduce

$$\begin{aligned}
1 \wedge \frac{\rho(x)\rho(y)}{(|x-y|+t^{1/\alpha})^2} &\leqslant 1 \wedge \frac{\rho(x)(\rho(x)+|x-y|)}{(|x-y|+t^{1/\alpha})^2} \\
&\leqslant 1 \wedge \frac{\rho(x)^2}{(|x-y|+t^{1/\alpha})^2} + 1 \wedge \frac{\rho(x)\cdot|x-y|}{(|x-y|+t^{1/\alpha})^2} \\
&\preceq 1 \wedge \frac{\rho(x)}{|x-y|+t^{1/\alpha}},
\end{aligned}$$

which implies the desired result. □

Remark 2.2. Using (2.2) and the same argument in [8, Remark 2.3], one can understand that for all $t/4 < s < 3t/4$ and $x,y,z \in D$ with $2|x-y| \geqslant |x-z|+|z-y|$, the following expression holds:

$$\begin{aligned}
&\frac{r^D(t+s,x,y)[r^D(t,x,z)+r^D(s,z,y)]}{r^D(t,x,z)r^D(s,z,y)} \\
&\quad \preceq \left(\frac{\rho(x)[\rho(z)+|x-y|+(t+s)^{1/\alpha}]}{\rho(z)[\rho(x)+|x-y|+(t+s)^{1/\alpha}]}\right) \\
&\qquad + \left(\frac{\rho(y)[\rho(z)+|x-y|+(t+s)^{1/\alpha}]}{\rho(z)[\rho(y)+|x-y|+(t+s)^{1/\alpha}]}\right),
\end{aligned}$$

which goes to zero as $\rho(x) = \rho(y) \to 0$. This means that for fixed z and t, unlike (1.3), the inequality

$$\frac{r^D(t,x,z)r^D(s,z,y)}{r^D(t+s,x,y)} \preceq r^D(t,x,z)+r^D(s,z,y)$$

cannot be true for all $t,s > 0$ and $x,y,z \in D$.

We now proceed to prove a generalized 3-P type inequality for $r^D(t,x,y)$. Let us begin with the following result.

Lemma 2.3. *For any $t,s \geqslant 0$ and $x,y,z \in D$, we have*

$$\frac{q_\alpha(t,x,z)q_\alpha(s,z,y)}{q_\alpha(t+s,x,y)} \preceq [\hat{q}_\alpha(t,z,x)]^2 + [\hat{q}_\alpha(s,z,y)]^2. \tag{2.4}$$

Proof. Note that, for any $a, b > 0$, it holds that

$$1 \wedge \frac{a}{b} \asymp \frac{a}{a+b}. \tag{2.5}$$

Thus,

$$\begin{aligned}
&\frac{\hat{q}_\alpha(t,x,z)\hat{q}_\alpha(s,y,z)}{q_\alpha(t+s,x,y)} \\
&\quad\asymp \frac{((t+s)^{1/\alpha}+|x-y|+\rho(x))((t+s)^{1/\alpha}+|x-y|+\rho(y))}{(t^{1/\alpha}+|x-z|+\rho(x))(s^{1/\alpha}+|z-y|+\rho(y))} \\
&\quad\preceq 1 + \frac{t^{1/\alpha}+|x-z|}{s^{1/\alpha}+|z-y|+\rho(y)} + \frac{s^{1/\alpha}+|z-y|}{t^{1/\alpha}+|x-z|+\rho(x)}.
\end{aligned}$$

Based on (2.1), we have

$$\begin{aligned}
\mathcal{I} &:= \frac{q_\alpha(t,x,z)q_\alpha(s,z,y)}{q_\alpha(t+s,x,y)} \\
&= \frac{\hat{q}_\alpha(t,x,z)\hat{q}_\alpha(s,y,z)}{q_\alpha(t+s,x,y)}\hat{q}_\alpha(t,z,x)\hat{q}_\alpha(s,z,y) \\
&\preceq \hat{q}_\alpha(t,z,x)\hat{q}_\alpha(s,z,y) + \frac{\rho(z)}{s^{1/\alpha}+|z-y|+\rho(y)}\hat{q}_\alpha(s,z,y) \\
&\quad + \frac{\rho(z)}{t^{1/\alpha}+|x-z|+\rho(x)}\hat{q}_\alpha(t,z,x).
\end{aligned}$$

Using the following expression $\rho(x)+|x-z| \asymp \rho(z)+|x-z|$, we further calculate that

$$\frac{\rho(z)}{t^{1/\alpha}+|x-z|+\rho(x)} \asymp \frac{\rho(z)}{t^{1/\alpha}+|x-z|+\rho(z)} \asymp \hat{q}_\alpha(t,z,x),$$

and similarly,

$$\frac{\rho(z)}{s^{1/\alpha}+|z-y|+\rho(y)} \asymp \frac{\rho(z)}{s^{1/\alpha}+|z-y|+\rho(z)} \asymp \hat{q}_\alpha(s,z,y).$$

Thus, we have

$$\begin{aligned}
\mathcal{I} &\preceq \hat{q}_\alpha(t,z,x)\hat{q}_\alpha(s,z,y) + [\hat{q}_\alpha(t,z,x)]^2 + [\hat{q}_\alpha(s,z,y)]^2 \\
&\preceq [\hat{q}_\alpha(t,z,x)]^2 + [\hat{q}_\alpha(s,z,y)]^2.
\end{aligned}$$

The proof is finished. □

As a direct consequence, we can obtain the following generalized 3-P type inequality for $r^D(t,x,y)$.

Lemma 2.4. *Let $T > 0$. For any $0 \leqslant s, t \leqslant T$ and $x, y, z \in D$, it holds that*

$$\begin{aligned}
\frac{r^D(t,x,z)r^D(s,z,y)}{r^D(t+s,x,y)} &\preceq (t \wedge s)([\hat{q}_\alpha(t,z,x)]^2\varrho_d^0(t,x-z) \\
&\quad + [\hat{q}_\alpha(s,z,y)]^2\varrho_d^0(s,z-y)).
\end{aligned} \tag{2.6}$$

Proof. Combining (2.2) and (2.4), we obtain

$$\begin{aligned}
\mathcal{J} &:= \frac{r^D(t,x,z)r^D(s,z,y)}{r^D(t+s,x,y)} \\
&\preceq ([\hat{q}_\alpha(t,z,x)]^2 + [\hat{q}_\alpha(s,z,y)]^2)\frac{\varrho_d^1(t,x-z)\varrho_d^1(s,z-y)}{\varrho_d^1(t+s,x-y)}.
\end{aligned}$$

Note that

$$(|x-y|+(t+s)^{1/\alpha})^{d+\alpha} \preceq (|x-z|+t^{1/\alpha})^{d+\alpha} + (|z-y|+s^{1/\alpha})^{d+\alpha}.$$

Thus,

$$\frac{\varrho_d^1(t,x-z)\varrho_d^1(s,z-y)}{\varrho_d^1(t+s,x-y)} = \frac{t\cdot s}{t+s}\cdot\frac{\varrho_d^0(t,x-z)\varrho_d^0(s,z-y)}{\varrho_d^0(t+s,x-y)} \preceq (t\wedge s)(\varrho_d^0(t,x-z)+\varrho_d^0(s,z-y)). \tag{2.7}$$

Hence,

$$\begin{aligned}\mathcal{J} &\preceq (t\wedge s)([\hat{q}_\alpha(t,z,x)]^2+[\hat{q}_\alpha(s,z,y)]^2)(\varrho_d^0(t,x-z)+\varrho_d^0(s,z-y))\\ &\preceq (t\wedge s)([\hat{q}_\alpha(t,z,x)]^2\varrho_d^0(t,x-z)+[\hat{q}_\alpha(s,z,y)]^2\varrho_d^0(s,z-y)),\end{aligned}$$

where, in the last inequality, we have used the following expression:

$$\hat{q}_\alpha(t,z,x)\preceq\hat{q}_\alpha(s,z,y)\Leftrightarrow\varrho_d^0(t,x-z)\preceq\varrho_d^0(s,z-y) \tag{2.8}$$

and the symmetry in x and y. The proof is finished. □

2.2 Gradient estimates

In this subsection, we derive gradient estimates for $r^D(t,x,y)$. Recall that $r^D(t,x,y)$ is the transition density of Y^D. Based on the construction of Y^D, it holds (see [18, (2.2)]) that

$$r^D(t,x,y)=\int_0^\infty p_2^D(s,x,y)\mu(t,s)ds, \tag{2.9}$$

where $p_2^D(t,x,y)$ is the Dirichlet heat kernel of $\Delta|_D$, and $\mu(t,s)$ is the density of the subordinator T_t. To derive gradient estimates for $r^D(t,x,y)$, we must recall some estimates for $p_2^D(t,x,y)$.

For any $\gamma,\lambda\in\mathbb{R}$ and $(t,x)\in(0,\infty)\times\mathbb{R}^d$, we define

$$\xi_\lambda^\gamma(t,x):=t^{-(d+\gamma)/2}\mathrm{e}^{-\lambda|x|^2/t}.$$

It is known (see, for example, [25] or [18, Theorems 3.1 and 3.2]) that there exist constants $\lambda_1,\lambda_2>0$, $C_1>1$, and $C_2<1$ such that for all $(t,x,y)\in(0,\infty)\times D\times D$,

$$p_2^D(t,x,y)\leqslant C_1\left(1\wedge\frac{\rho(x)\rho(y)}{t}\right)\xi_{\lambda_1}^0(t,x-y) \tag{2.10}$$

and for all $(t,x,y)\in(0,1]\times D\times D$,

$$p_2^D(t,x,y)\geqslant C_2\left(1\wedge\frac{\rho(x)\rho(y)}{t}\right)\xi_{\lambda_2}^0(t,x-y). \tag{2.11}$$

Moreover, it follows from [26, Theorem 2.1] that, for any $T>0$, there exists a constant, $C_T>0$, such that for all $t\in(0,T]$ and $x,y\in D$,

$$|\nabla_x p_2^D(t,x,y)|\leqslant\begin{cases}\dfrac{C_T}{\rho(x)}p_2^D(t,x,y), & \text{if } \rho(x)\leqslant\sqrt{t},\\ \dfrac{C_T}{\sqrt{t}}\left(1+\dfrac{|x-y|}{\sqrt{t}}\right)p_2^D(t,x,y), & \text{if } \rho(x)>\sqrt{t}.\end{cases} \tag{2.12}$$

It turns out that (2.10)–(2.12) are not very convenient to use since the expressions roll $\rho(x)$ and $\rho(y)$ together. To generate more practical forms of the above-mentioned estimates, we first conduct some manipulations on $p_2^D(t,x,y)$. We want to separate the terms $\rho(x)$ and $\rho(y)$. The following elementary observation will be important.

Lemma 2.5. *For any $\lambda_2>\lambda_1>0$ and $\gamma\in\mathbb{R}$, the following expression holds for all $t>0$ and $x,y\in D$:*

$$\left(1\wedge\frac{\rho(x)\rho(y)}{t}\right)\xi_{\lambda_2}^\gamma(t,x-y)\preceq\left(1\wedge\frac{\rho(x)}{\sqrt{t}}\right)\left(1\wedge\frac{\rho(y)}{\sqrt{t}}\right)\xi_{\lambda_1}^\gamma(t,x-y). \tag{2.13}$$

Proof. In light of (2.5), it suffices to show that for any $\lambda_0 > 0$,

$$(\rho(x) + \sqrt{t})(\rho(y) + \sqrt{t}) \preceq (\rho(x)\rho(y) + t)\mathrm{e}^{\lambda_0 \frac{|x-y|^2}{t}}.$$

In fact, by using symmetry and the elementary inequality

$$\rho(x) \leqslant \rho(y) + |x - y|,$$

we have

$$\rho(x)^2 + \rho(y)^2 \preceq \rho(x)\rho(y) + |x - y|^2.$$

Thus, we can deduce that

$$\begin{aligned}(\rho(x) + \sqrt{t})(\rho(y) + \sqrt{t}) &\preceq \rho(x)\rho(y) + t + \rho(x)^2 + \rho(y)^2 \\ &\preceq \rho(x)\rho(y) + t + |x - y|^2.\end{aligned}$$

Note that for any $\lambda_0 > 0$, we have

$$|x - y|^2 \preceq t \cdot \mathrm{e}^{\lambda_0 \frac{|x-y|^2}{t}}.$$

The desired result follows immediately. □

Recall the definition of $q_\alpha(t, x, y)$ in (2.1). We provide a more appropriate form of (2.10) and (2.11) as follows.

Lemma 2.6. *There exist constants $\lambda_1, \lambda_2 > 0$, $C_1 > 1$, and $C_2 < 1$ such that*

$$p_2^D(t, x, y) \leqslant C_1 q_2(t, x, y)\xi^0_{\lambda_1}(t, x - y), \quad (t, x, y) \in (0, \infty) \times D \times D, \tag{2.14}$$

$$p_2^D(t, x, y) \geqslant C_2 q_2(t, x, y)\xi^0_{\lambda_2}(t, x - y), \quad (t, x, y) \in (0, 1] \times D \times D. \tag{2.15}$$

Proof. The lower bound (2.15) is implied; we need only prove the upper bound (2.14). Combining (2.10) and (2.11) with (2.13), we understand that for $\lambda_0 > 0$,

$$p_2^D(t, x, y) \preceq \left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right)\xi^0_{\lambda_0}(t, x - y).$$

Thus, (2.14) is true when $|x - y| \leqslant \sqrt{t}$. On the other hand, notably for $0 < \tilde{\lambda}_0 < \lambda_0$, we have

$$\frac{\rho(x)}{\sqrt{t}}\mathrm{e}^{-\lambda_0 \frac{|x-y|^2}{t}} = \frac{\rho(x)}{|x - y|} \cdot \frac{|x - y|}{\sqrt{t}}\mathrm{e}^{-\lambda_0 \frac{|x-y|^2}{t}} \preceq \frac{\rho(x)}{|x - y|}\mathrm{e}^{-\tilde{\lambda}_0 \frac{|x-y|^2}{t}}. \tag{2.16}$$

Combining (2.16) with (2.10) gives the desired result for $|x - y| > \sqrt{t}$. □

Now we prove the first- and second-order gradient estimates for $p_2^D(t, x, y)$.

Lemma 2.7. *Let $T > 0$. There exist constants $C_T, \lambda_3 > 0$ such that for $j = 1, 2$,*

(i) *for all $t \in (0, T]$ and $x, y \in D$,*

$$|\nabla_x^j p_2^D(t, x, y)| \leqslant C_T \hat{q}_2(t, y, x)\xi^j_{\lambda_3}(t, x - y); \tag{2.17}$$

(ii) *for all $t \in (T, \infty)$ and $x, y \in D$,*

$$|\nabla_x^j p_2^D(t, x, y)| \leqslant \frac{C_T}{T^{j/2}}\hat{q}_2(t, y, x)\xi^0_{\lambda_3}(t, x - y), \tag{2.18}$$

where ∇_x^j denotes the j-order derivative with respect to the x variable.

Proof. For (2.17), we need only show that there exist $\lambda_3 > 0$ and $C_T > 0$ such that for every $t \in (0, T]$ and $x, y \in D$,

$$|\nabla_x^j p_2^D(t, x, y)| \leqslant C_T\left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right)\xi^j_{\lambda_3}(t, x - y).$$

Then, applying (2.16), we can obtain (2.17). Based on [11, Chapter VI, Section 2, Theorem 2.1], we know that for every $t \in (0, T]$ and $x, y \in D$,

$$|\nabla_x^j p_2^D(t, x, y)| \preceq \xi_{\lambda_3}^j(t, x - y).$$

Using the Chapman-Kolmogorov equation, we obtain

$$\begin{aligned}|\nabla_x^j p_2^D(t, x, y)| &\leqslant \int_D \left|\nabla_x^j p_2^D\left(\frac{t}{2}, x, z\right)\right| \cdot p_2^D\left(\frac{t}{2}, z, y\right) dz \\ &\preceq \left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right) \int_D \xi_{\lambda_3}^j\left(\frac{t}{2}, x - z\right) \xi_{\lambda_1}^0\left(\frac{t}{2}, z - y\right) dz \\ &\preceq \left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right) \xi_{\lambda_3}^j(t, x - y).\end{aligned}$$

Thus, (2.17) is valid. We now prove (2.18). Similarly, it suffices to show that for every $t > T$ and $x, y \in D$,

$$|\nabla_x^j p_2^D(t, x, y)| \leqslant \frac{C_T}{T^{j/2}} \left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right) \xi_{\lambda_3}^0(t, x - y).$$

According to (2.14), (2.17), and the Chapman-Kolmogorov equation, for $t > T$, we have

$$\begin{aligned}|\nabla_x^j p_2^D(t, x, y)| &\leqslant \int_D \left|\nabla_x^j p_2^D\left(\frac{T}{2}, x, z\right)\right| \cdot p_2^D\left(t - \frac{T}{2}, z, y\right) dz \\ &\preceq \int_{\mathbb{R}^d} \xi_{\lambda_3}^j\left(\frac{T}{2}, x - z\right)\left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right) \xi_{\lambda_0}^0\left(t - \frac{T}{2}, z - y\right) dz \\ &\leqslant \frac{C_T}{T^{j/2}}\left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right) \xi_{\lambda_3}^0(t, x - y).\end{aligned}$$

The proof is finished. □

Remark 2.8. In fact, in the form of (2.12), our result means that for every $t \in (0, T]$,

$$|\nabla_x^j p_2^D(t, x, y)| \leqslant C_T \frac{(|x - y| + \sqrt{t})^{1-j}}{\rho(x) \wedge (|x - y| + \sqrt{t})} q_2(t, x, y) \xi_{\lambda_3}^0(t, x - y).$$

Compared with (2.10)–(2.12), the additional term $|x - y|$ in (2.14)–(2.15) and (2.17)–(2.18) is of critical importance in our derivation of the gradient estimates of $r^D(t, x, y)$.

Recall the definition of $q^D(t, x, y)$ in (1.7). We are now ready to derive the following gradient estimates for the Dirichlet heat kernel, $r^D(t, x, y)$.

Lemma 2.9. *Let $T > 0$. There exists a constant $C_T > 0$ such that for $j = 1, 2$, all $t \in (0, T]$ and $x, y \in D$,*

$$|\nabla_x^j r^D(t, x, y)| \leqslant C_T \frac{(|x - y| + t^{1/\alpha})^{1-j}}{\rho(x) \wedge (|x - y| + t^{1/\alpha})} q^D(t, x, y). \tag{2.19}$$

Moreover, for any $\vartheta \in (0, 1)$ and $t \in (0, T]$, $x, x', y \in D$, we have

$$|\nabla_x r^D(t, x, y) - \nabla_x r^D(t, x', y)| \leqslant C_T |x - x'|^\vartheta \hat{q}_\alpha(t, y, \widetilde{x}) \varrho_{d+1+\vartheta}^1(t, \widetilde{x} - y), \tag{2.20}$$

where $\widetilde{x}$ is the point between x and x' which is closer to y.

Proof. We claim that for $j = 1, 2$,

$$|\nabla_x^j r^D(t, x, y)| \preceq \hat{q}_\alpha(t, y, x) \varrho_{d+j}^1(t, x - y). \tag{2.21}$$

As a consequence of this claim, we obtain

$$|\nabla_x^j r^D(t, x, y)| \preceq \frac{1}{(|x - y| + t^{1/\alpha})^j \hat{q}_\alpha(t, x, y)} \hat{q}_\alpha(t, x, y) \hat{q}_\alpha(t, y, x) \varrho_d^1(t, x - y)$$

$$\asymp \frac{(|x-y|+t^{1/\alpha})^{1-j}}{\rho(x)\wedge(|x-y|+t^{1/\alpha})}q^D(t,x,y).$$

Next, we prove the claim (2.21). Based on [18, (4.1)], we know that for all $\xi\in\mathbb{R}^d$,

$$\int_0^\infty s^{-d/2}\mathrm{e}^{-\frac{|\xi|^2}{s}}\mu(t,s)ds\asymp\varrho_d^1(t,\xi).$$

Combining this formula (applied to d and $d+j$) with (2.9), (2.17), and (2.18), we can obtain

$$\begin{aligned}|\nabla_x^j r^D(t,x,y)| &\leqslant \int_0^1 |\nabla_x^j p_2^D(s,x,y)|\mu(t,s)ds + \int_1^\infty |\nabla_x^j p_2^D(s,x,y)|\mu(t,s)ds\\ &\preceq \left(1\wedge\frac{\rho(y)}{|x-y|}\right)\left[\int_0^\infty \xi_{\lambda_3}^j(s,x-y)\mu(t,s)ds\right.\\ &\quad\left.+\int_0^\infty \xi_{\lambda_3}^0(s,x-y)\mu(t,s)ds\right]\\ &\asymp \left(1\wedge\frac{\rho(y)}{|x-y|}\right)[\varrho_{d+j}^1(t,x-y)+\varrho_d^1(t,x-y)]\\ &\preceq \left(1\wedge\frac{\rho(y)}{|x-y|}\right)\varrho_{d+j}^1(t,x-y),\end{aligned}\tag{2.22}$$

where, in the last inequality, we have used the fact that D is bounded and $t\in(0,T]$. Thus, (2.21) is true when $|x-y|\geqslant t^{1/\alpha}$. For the case when $|x-y|<t^{1/\alpha}$, we may argue similarly to obtain

$$\begin{aligned}|\nabla_x^j r^D(t,x,y)| &\preceq \rho(y)\left[\int_0^\infty \xi_{\lambda_3}^{j+1}(s,x-y)\mu(t,s)ds\right.\\ &\quad\left.+\int_0^\infty \xi_{\lambda_3}^1(s,x-y)\mu(t,s)ds\right]\\ &\asymp \rho(y)[\varrho_{d+j+1}^1(t,x-y)+\varrho_{d+1}^1(t,x-y)]\\ &\preceq \frac{\rho(y)}{t^{1/\alpha}}\varrho_{d+j}^1(t,x-y).\end{aligned}$$

This, together with the estimate (2.22), implies (2.21).

For (2.20), without loss of generality, we may assume that $|x-y|\leqslant|x'-y|$. Using (2.21) with $j=1$, we find that when $|x-x'|\geqslant(|x-y|+t^{1/\alpha})/2$,

$$\begin{aligned}\mathscr{Q} &:= |\nabla_x r^D(t,x,y)-\nabla_x r^D(t,x',y)|\\ &\leqslant C_T|x-x'|^\vartheta(|x-y|+t^{1/\alpha})^{-\vartheta}(\hat{q}_\alpha(t,y,x)\varrho_{d+1}^1(t,x-y)\\ &\quad+\hat{q}_\alpha(t,y,x')\varrho_{d+1}^1(t,x'-y))\\ &\leqslant C_T|x-x'|^\vartheta\hat{q}_\alpha(t,y,x)\varrho_{d+1+\vartheta}^1(t,x-y).\end{aligned}$$

When $|x-x'|<(|x-y|+t^{1/\alpha})/2$, we have, according to the mean value theorem and (2.21) with $j=2$, for some $\varepsilon\in[0,1]$,

$$\begin{aligned}\mathscr{Q} &\leqslant C_T|x-x'|\hat{q}_\alpha(t,y,x+\varepsilon(x'-x))\varrho_{d+2}^1(t,x+\varepsilon(x'-x)-y)\\ &\leqslant C_T|x-x'|\hat{q}_\alpha(t,y,x)\varrho_{d+2}^1(t,x-y)\\ &\leqslant C_T|x-x'|^\vartheta\hat{q}_\alpha(t,y,x)\varrho_{d+1+\vartheta}^1(t,x-y).\end{aligned}$$

The proof is finished. □

3 Proof of Theorem 1.3

Let

$$\widehat{\mathbf{K}}_D^{\alpha-1} := \left\{f\in L^1_{\mathrm{loc}}(D):\lim_{t\downarrow0}\sup_{x\in D}\int_D\left(1\wedge\frac{\rho(y)}{|x-y|}\right)\right.$$

$$\times\left(\frac{1}{|x-y|^{d+1-\alpha}}\wedge\frac{t^2}{|x-y|^{d+\alpha+1}}\right)|f(y)|dy=0\Bigg\}.$$

We first provide the following result regarding our Kato class.

Lemma 3.1. *We have* $\mathbf{K}_D^{\alpha-1}\subset\widehat{\mathbf{K}}_D^{\alpha-1}\subset\mathbb{K}_D^0$. *Moreover, for any* $\gamma\geqslant 0$, *if* $1<p,q\leqslant\infty$ *satisfies*

$$\frac{d}{\alpha p}+\frac{1}{q}<1-\frac{1+\gamma}{\alpha},\tag{3.1}$$

then, $L^q(\mathbb{R};L^p(D))\subseteq\mathbb{K}_D^\gamma$.

Proof. It follows from [4, Lemma 2.1], which follows from [2, Corollary 12], that a real-valued function, f, belongs to $\mathbf{K}_D^{\alpha-1}$ if and only if

$$\lim_{t\to0}\sup_{x\in D}\int_D\left(\frac{1}{|x-y|^{d+1-\alpha}}\wedge\frac{t^2}{|x-y|^{d+\alpha+1}}\right)|f(y)|dy=0.$$

Thus, the first inclusion is obvious. To show that a real-valued, time-independent function f on D belongs to $\mathbb{K}_D^0$, it suffices to show that

$$\int_0^t\varrho_{d+1}^1(s,x-y)ds\preceq\frac{1}{|x-y|^{d+1-\alpha}}\wedge\frac{t^2}{|x-y|^{d+\alpha+1}}.$$

This directly follows from [4, Lemma 2.3] with $\gamma=1$. Thus, the second inclusion is valid. Now we prove the third inclusion. According to Hölder's inequality, we obtain

$$K_f^\gamma(\delta)\leqslant\left(\int_{\mathbb{R}}\left(\int_D|f(s,y)|^pdy\right)^{\frac{q}{p}}ds\right)^{\frac{1}{q}}I_{\alpha,\gamma}(\delta),$$

where

$$I_{\alpha,\gamma}(\delta):=\delta^{\frac{\gamma}{\alpha}}\left(\int_0^\delta[s^{-\gamma/\alpha}+(\delta-s)^{-\gamma/\alpha}]^{q^*}\left(\int_{\mathbb{R}^d}\frac{s^{p^*}}{(|y|+s^{1/\alpha})^{(d+\alpha+1)p^*}}dy\right)^{\frac{q^*}{p^*}}ds\right)^{\frac{1}{q^*}}$$

with $q^*:=\frac{q}{q-1}$ and $p^*:=\frac{p}{p-1}$. Noticing that

$$\int_{\mathbb{R}^d}\frac{s^{p^*}}{(|y|+s^{1/\alpha})^{(d+\alpha+1)p^*}}dy\leqslant s^{p^*}\left(\int_{|y|\leqslant s^{1/\alpha}}s^{-\frac{(d+\alpha+1)p^*}{\alpha}}dy+\int_{|y|>s^{1/\alpha}}\frac{dy}{|y|^{(d+\alpha+1)p^*}}\right)\preceq s^{\frac{d-(d+1)p^*}{\alpha}},$$

we have

$$I_{\alpha,\gamma}(\delta)\preceq\delta^{\frac{\gamma}{\alpha}}\left(\int_0^\delta[s^{-\gamma/\alpha}+(\delta-s)^{-\gamma/\alpha}]^{q^*}s^{\frac{dq^*}{\alpha p^*}-\frac{(d+1)q^*}{\alpha}}ds\right)^{\frac{1}{q^*}}.$$

Thus, $I_{\alpha,\gamma}(\delta)$ converges to zero as $\delta\to0$, provided that

$$-\frac{\gamma q^*}{\alpha}+\frac{dq^*}{\alpha p^*}-\frac{d+1}{\alpha}q^*+1>0\Leftrightarrow(3.1).$$

The desired result follows. □

The following lemma is related to the smallness of $b\cdot\nabla$ as a perturbation of $-(-\Delta|_D)^{\alpha/2}$, which plays an important role in proving our main result.

Lemma 3.2. *Let* $\delta>0$ *and* $b\in\mathbb{K}_D^0$. *Then, for all* $0\leqslant s<t\leqslant s+\delta$ *and* $x,y\in D$, *we have*

$$\int_s^t\int_D r^D(r-s,x,z)|b(r,z)|\cdot|\nabla_zr^D(t-r,z,y)|dzdr\leqslant C(\delta)r^D(t-s,x,y),$$

where $C(\delta)$ *is a positive constant with* $C(\delta)\to0$ *as* $\delta\downarrow0$.

Proof. In this proof, we consistently assume that $0 \leqslant s < t \leqslant s+\delta$ and $x, y \in D$. For brevity, we write

$$\mathcal{W} := \frac{r^D(r-s,x,z)|\nabla_z r^D(t-r,z,y)|}{r^D(t-s,x,y)}.$$

It follows from (2.19) that

$$\begin{aligned}\mathcal{W} &\preceq \frac{r^D(r-s,x,z)r^D(t-r,z,y)}{r^D(t-s,x,y)} \cdot \frac{1}{\rho(z) \wedge (|z-y| + (t-r)^{1/\alpha})} \\ &\preceq \frac{r^D(r-s,x,z)r^D(t-r,z,y)}{r^D(t-s,x,y)} \left(\frac{1}{\rho(z)} + \frac{1}{|z-y| + (t-r)^{1/\alpha}} \right) \\ &=: \mathcal{W}_1 + \mathcal{W}_2.\end{aligned}$$

Based on (2.6), we have

$$\begin{aligned}\mathcal{W}_1 &\preceq ((r-s) \wedge (t-r))([\hat{q}_\alpha(r-s,z,x)]^2 \varrho_d^0(r-s,x-z) \\ &\quad + [\hat{q}_\alpha(t-r,z,y)]^2 \varrho_d^0(t-r,z-y)) \frac{1}{\rho(z)} \\ &\preceq \hat{q}_\alpha(r-s,z,x)\varrho_{d+1}^1(r-s,x-z) + \hat{q}_\alpha(t-r,z,y)\varrho_{d+1}^1(t-r,z-y). \qquad (3.2)\end{aligned}$$

Again using (2.6), we have

$$\begin{aligned}\mathcal{W}_2 &\preceq ((r-s) \wedge (t-r))([\hat{q}_\alpha(r-s,z,x)]^2 \varrho_d^0(r-s,x-z) \\ &\quad + [\hat{q}_\alpha(t-r,z,y)]^2 \varrho_d^0(t-r,z-y)) \frac{1}{|z-y| + (t-r)^{1/\alpha}}.\end{aligned}$$

According to the same argument as that in (2.8), in the case

$$|x-z| + (r-s)^{1/\alpha} \leqslant |z-y| + (t-r)^{1/\alpha},$$

we have

$$\begin{aligned}\mathcal{W}_2 &\preceq [\hat{q}_\alpha(r-s,z,x)]^2 \varrho_d^1(r-s,x-z) \frac{1}{|x-z| + (r-s)^{1/\alpha}} \\ &\preceq \hat{q}_\alpha(r-s,z,x)\varrho_{d+1}^1(r-s,x-z).\end{aligned}$$

In the case $|x-z| + (r-s)^{1/\alpha} > |z-y| + (t-r)^{1/\alpha}$, we have

$$\begin{aligned}\mathcal{W}_2 &\preceq [\hat{q}_\alpha(t-r,z,y)]^2 \varrho_d^1(t-r,z-y) \frac{1}{|z-y| + (t-r)^{1/\alpha}} \\ &\preceq \hat{q}_\alpha(t-r,z,y)\varrho_{d+1}^1(t-r,z-y).\end{aligned}$$

Hence,

$$\mathcal{W}_2 \preceq \hat{q}_\alpha(r-s,z,x)\varrho_{d+1}^1(r-s,x-z) + \hat{q}_\alpha(t-r,z,y)\varrho_{d+1}^1(t-r,z-y),$$

which, together with (3.2), yields that

$$\mathcal{W} \preceq \hat{q}_\alpha(r-s,z,x)\varrho_{d+1}^1(r-s,x-z) + \hat{q}_\alpha(t-r,z,y)\varrho_{d+1}^1(t-r,z-y).$$

Consequently, by the definition of the Kato class $\mathbb{K}_D^0$, it holds that

$$\begin{aligned}&\int_s^t \int_D r^D(r-s,x,z)|b(r,z)| \cdot |\nabla_z r^D(t-r,z,y)| dz dr \\ &\preceq \int_s^t \int_D \hat{q}_\alpha(r-s,z,x)\varrho_{d+1}^1(r-s,x-z)|b(r,z)| dz dr \cdot r^D(t-s,x,y) \\ &\quad + \int_s^t \int_D \hat{q}_\alpha(t-r,z,y)\varrho_{d+1}^1(t-r,z-y)|b(r,z)| dz dr \cdot r^D(t-s,x,y) \\ &\leqslant 2K_b^0(\delta) r^D(t-s,x,y),\end{aligned}$$

where $K_b^0(\delta)$ is defined in Definition 1.1. The proof is thus finished. □

To derive the gradient estimate of the Dirichlet heat kernel, we also need the following result.

Lemma 3.3. *Let $\delta > 0$ and $b \in \mathbb{K}_D^0$. Then, for all $0 \leqslant s < t \leqslant s + \delta$ and $x, y \in D$, we have*

$$\int_s^t \int_D |\nabla_x r^D(r-s,x,z)||b(r,z)| \cdot |\nabla_z r^D(t-r,z,y)| dz dr \\ \leqslant \hat{C}(\delta) \frac{1}{\rho(x) \wedge (|x-y| + (t-s)^{1/\alpha})} r^D(t-s,x,y),$$

where $\hat{C}(\delta)$ is a positive constant with $\hat{C}(\delta) \to 0$ as $\delta \downarrow 0$.

Proof. In this proof, we consistently assume that $0 \leqslant s < t \leqslant s + \delta$ and $x, y \in D$. Define

$$\mathcal{V} := \frac{|\nabla_x r^D(r-s,x,z)| \cdot |\nabla_z r^D(t-r,z,y)|}{\hat{q}_\alpha(t-s,y,x) \varrho_{d+1}^1(t-s,x-y)}.$$

It follows from (2.21) that

$$\mathcal{V} \preceq \mathcal{Q} \cdot \frac{\varrho_{d+1}^1(r-s,x-z) \varrho_{d+1}^1(t-r,z-y)}{\varrho_{d+1}^1(t-s,x-y)},$$

where

$$\mathcal{Q} := \frac{\hat{q}_\alpha(r-s,z,x) \hat{q}_\alpha(t-r,y,z)}{\hat{q}_\alpha(t-s,y,x)}.$$

Using (2.16), we obtain

$$\begin{aligned} \mathcal{Q} &\asymp \rho(z) \cdot \frac{\rho(y) + |x-y| + (t-s)^{1/\alpha}}{(\rho(z) + |x-z| + (r-s)^{1/\alpha})(\rho(y) + |z-y| + (t-r)^{1/\alpha})} \\ &\preceq \rho(z) \cdot \frac{\rho(z) + |x-z| + (r-s)^{1/\alpha} + \rho(z) + |z-y| + (t-r)^{1/\alpha}}{(\rho(z) + |x-z| + (r-s)^{1/\alpha})(\rho(z) + |z-y| + (t-r)^{1/\alpha})} \\ &= \frac{\rho(z)}{\rho(z) + |x-z| + (r-s)^{1/\alpha}} + \frac{\rho(z)}{\rho(z) + |z-y| + (t-r)^{1/\alpha}} \\ &\asymp (\hat{q}_\alpha(r-s,z,x) + \hat{q}_\alpha(t-r,z,y)). \end{aligned}$$

Combining this with (2.7), and based on the same argument as in (2.8), we further obtain

$$\begin{aligned} \mathcal{V} &\preceq [(r-s) \wedge (t-r)](\hat{q}_\alpha(r-s,z,x) + \hat{q}_\alpha(t-r,z,y)) \\ &\quad \times (\varrho_{d+1}^0(r-s,x-z) + \varrho_{d+1}^0(t-r,z-y)) \\ &\preceq \hat{q}_\alpha(r-s,z,x) \varrho_{d+1}^1(r-s,x-z) + \hat{q}_\alpha(t-r,z,y) \varrho_{d+1}^1(t-r,z-y). \end{aligned} \tag{3.3}$$

Hence,

$$\begin{aligned} &\int_s^t \int_D |\nabla_x r^D(r-s,x,z)||b(r,z)| \cdot |\nabla_z r^D(t-r,z,y)| dz dr \\ &\quad \preceq K_b^0(\delta) \hat{q}_\alpha(t-s,y,x) \varrho_{d+1}^1(t-s,x-y) \\ &\quad \leqslant K_b^0(\delta) \frac{1}{\rho(x) \wedge (|x-y| + (t-s)^{1/\alpha})} r^D(t-s,x,y), \end{aligned}$$

which yields the desired result. The proof is finished. □

We now proceed to solve the integral equation (1.5). For all $0 \leqslant s < t$ and $x, y \in D$, set $r_0(s,x;t,y) := r^D(t-s,x,y)$, and define for $k \geqslant 1$ that

$$r_k(s,x;t,y) := \int_s^t \int_D r_{k-1}(s,x;r,z) b(r,z) \cdot \nabla_z r_0(r,z;t,y) dz dr. \tag{3.4}$$

The following result is an easy consequence of Lemmas 3.2 and 3.3.

Lemma 3.4. *Let $\delta > 0$ and $b \in \mathbb{K}_D^0$. Then, for all $k \geqslant 1$, $0 \leqslant s < t \leqslant s + \delta$ and $x, y \in D$, we have*

$$|r_k(s,x;t,y)| \leqslant [C(\delta)]^k r^D(t-s,x,y) \tag{3.5}$$

and

$$|\nabla_x r_k(s,x;t,y)| \leqslant [\hat{C}(\delta)]^k \frac{1}{\rho(x)\wedge(|x-y|+(t-s)^{1/\alpha})} r^D(t-s,x,y), \tag{3.6}$$

where $C(\delta)$ and $\hat{C}(\delta)$ are the constants in Lemmas 3.2 and 3.3, respectively. Moreover, it holds that

$$r_k(s,x;t,y) = \int_s^t \int_D r_0(s,x;r,z) b(r,z) \cdot \nabla_z r_{k-1}(r,z;t,y) dz dr. \tag{3.7}$$

Proof. We first prove (3.5) via induction. By Lemma 3.2 and (3.4), we know that (3.5) holds for $k = 1$. Now, suppose that (3.5) holds for $k > 1$. Then, by definition and using Lemmas 3.2 and 2.1, we obtain

$$\begin{aligned}
|r_{k+1}(s,x;t,y)| &\leqslant \int_s^t \int_D |r_k(s,x;r,z)| \cdot |b(r,z)| \cdot |\nabla_z r_0(r,z;t,y)| dz dr \\
&\leqslant [C(\delta)]^k \int_s^t \int_D r^D(s,x;r,z) |b(r,z)| \cdot |\nabla_z r_0(r,z;t,y)| dz dr \\
&\leqslant [C(\delta)]^{k+1} r^D(t-s,x,y).
\end{aligned}$$

To prove (3.6), let e_i be the i-th unit coordinate vector in $\mathbb{R}^d$. Then, for $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{\varepsilon} |r_1(s,x+\varepsilon e_i;t,y) - r_1(s,x;t,y)| \\
&\quad \leqslant \frac{1}{\varepsilon} \int_0^\varepsilon d\theta \bigg(\int_s^t \int_D |\nabla_x r^D(r-s,x+\theta\varepsilon e_i,z)| \cdot |b(r,z)| \cdot |\nabla_z r_0(r,z;t,y)| dz dr \bigg) \\
&\quad \leqslant \hat{C}(\delta) \frac{1}{\varepsilon} \int_0^\varepsilon \frac{1}{\rho(x+\theta\varepsilon e_i) \wedge (|x+\theta\varepsilon e_i - y| + (t-s)^{1/\alpha})} r^D(t-s,x+\theta\varepsilon e_i,y) d\theta,
\end{aligned}$$

where, in the last inequality, we used Lemma 3.3. Letting $\varepsilon \to 0$, we obtain that

$$|\partial_{x_i} r_1(s,x;t,y)| \leqslant \hat{C}(\delta) \frac{1}{\rho(x) \wedge (|x-y| + (t-s)^{1/\alpha})} r^D(t-s,x,y),$$

which in turn implies that (3.6) holds for $k = 1$. Following the same argument as before, we can show that (3.6) is true for every $k \geqslant 1$. We proceed to prove (3.7). It is obvious that (3.7) holds for $k = 1$. Suppose that (3.7) is true for $k > 1$. Then, according to (3.4) and Fubini's theorem, we have

$$\begin{aligned}
r_{k+1}(s,x;t,y) &= \int_s^t \int_D r_k(s,x;r,z) b(r,z) \cdot \nabla_z r_0(r,z;t,y) dz dr \\
&= \int_s^t \int_D \int_s^r \int_D r_0(s,x;r',z') b(r',z') \cdot \nabla_{z'} r_{k-1}(r',z';r,z) dz' dr' \\
&\quad \times b(r,z) \cdot \nabla_z r_0(r,z;t,y) dz dr \\
&= \int_s^t \int_D r_0(s,x;r',z') b(r',z') \cdot \int_{r'}^t \int_D \nabla_{z'} r_{k-1}(r',z';r,z) \\
&\quad \times b(r,z) \cdot \nabla_z r_0(r,z;t,y) dz dr dz' dr' \\
&= \int_s^t \int_D r_0(s,x;r',z') b(r',z') \cdot \nabla_{z'} r_k(r',z';t,y) dz' dr'.
\end{aligned}$$

The proof is completed. □

Thus, we are ready to provide the following proof.

Proof of Theorem 1.3. Let r_k be defined by (3.4). For $\delta > 0$, define

$$\mathbb{D}_\delta := \{(s,x;t,y) : x,y \in D \text{ and } 0 \leqslant s < t \leqslant s+\delta\}.$$

It follows from Lemma 3.2 that there exists a $\delta_0 \in (0,1]$ such that for all $0 \leqslant s < t \leqslant s+\delta_0$, we have $C(\delta_0) < 1/4$, where $C(\delta_0)$ is the constant in Lemma 3.4. Hence,

$$\sum_{k=0}^{\infty} |r_k(s,x;t,y)| \leqslant \frac{4}{3} r^D(t-s,x,y) \quad \text{on } \mathbb{D}_{\delta_0}, \tag{3.8}$$

which means that the series $\sum_{k=0}^{\infty} r_k(s,x;t,y)$ converges on $\mathbb{D}_{\delta_0}$. Define

$$r^{D,b}(s,x;t,y) := \sum_{k=0}^{\infty} r_k(s,x;t,y) \quad \text{on } \mathbb{D}_{\delta_0}.$$

Based on (3.4), we have

$$\sum_{k=0}^{n+1} r_k(s,x;t,y) = r_0(s,x;t,y) + \int_s^t\!\!\int_D \sum_{k=0}^{n} r_k(s,x;r,z) b(r,z) \cdot \nabla_z r_0(r,z;t,y) dz dr.$$

Letting $n \to \infty$ on both sides, we obtain (1.5).

(i) The upper bound on $\mathbb{D}_{\delta_0}$ follows from (3.8). As for the lower bound on $\mathbb{D}_{\delta_0}$, we have

$$r^{D,b}(s,x;t,y) \geqslant r^D(t-s,x,y) - \sum_{k=1}^{\infty} |r_k(s,x;t,y)| \geqslant \frac{2}{3} r^D(t-s,x,y).$$

Thus, (1.8) is valid on $\mathbb{D}_{\delta_0}$.

Now let $\tilde{r}^{D,b}(s,x;t,y)$ be another solution to the integral equation (1.5) satisfying (1.8) on $\mathbb{D}_{\delta_0}$. We claim that for every $k \in \mathbb{N}$, there exists a constant C_0 such that on $\mathbb{D}_{\delta_0}$,

$$|r^{D,b}(s,x;t,y) - \tilde{r}^{D,b}(s,x;t,y)| \leqslant C_0 [C(\delta_0)]^k r^D(t-s,x,y). \tag{3.9}$$

Indeed, for $k=1$, using (1.5), (1.8) and Lemma 3.2, we have

$$\begin{aligned}
&|r^{D,b}(s,x;t,y) - \tilde{r}^{D,b}(s,x;t,y)| \\
&\quad \leqslant \int_s^t\!\!\int_D (|r^{D,b}(s,x;r,z)| + |\tilde{r}^{D,b}(s,x;r,z)|) \cdot |b(r,z)| \cdot |\nabla_z r^D(t-r,z,y)| dz dr \\
&\quad \leqslant C_0 \int_s^t\!\!\int_D r^D(r-s,x,z) \cdot |b(r,z)| \cdot |\nabla_z r^D(t-r,z,y)| dz dr \\
&\quad \leqslant C_0 C(\delta_0) r^D(t-s,x,y).
\end{aligned}$$

Suppose that (3.9) holds for some $k \in \mathbb{N}$. Based on (1.5), Lemma 3.2, and the induction hypothesis, we have

$$\begin{aligned}
&|r^{D,b}(s,x;t,y) - \tilde{r}^{D,b}(s,x;t,y)| \\
&\quad \leqslant \int_s^t\!\!\int_D |r^{D,b}(s,x;r,z) - \tilde{r}^{D,b}(s,x;r,z)| \cdot |b(r,z)| \cdot |\nabla_z r^D(r,z;t,y)| dz dr \\
&\quad \leqslant C_0 [C(\delta_0)]^k \int_s^t\!\!\int_D r^D(r-s,x,z) \cdot |b(r,z)| \cdot |\nabla_z r^D(t-r,z,y)| dz dr \\
&\quad \leqslant C_0 [C(\delta_0)]^{k+1} r^D(t-s,x,y).
\end{aligned}$$

As $C(\delta_0) < 1$, letting $k \to \infty$, we obtain the uniqueness.

(ii) By choosing δ_0, smaller if necessary, we can assume that $\hat{C}(\delta_0) < 1$ for $0 \leqslant s < t \leqslant s + \delta_0$, where $\widehat{C}(\delta_0)$ is the constant from Lemma 3.4. It then follows from (3.6) that on $\mathbb{D}_{\delta_0}$,

$$\left|\sum_{k=0}^{\infty} \nabla_x r_k(s,x;t,y)\right| \preceq \frac{1}{\rho(x) \wedge (|x-y| + (t-s)^{1/\alpha})} r^D(t-s,x,y),$$

which means that (1.9) is true. Moreover, according to (3.7) and Fubini's theorem, we have

$$\begin{aligned}
r^{D,b}(s,x;t,y) &= \sum_{k=0}^{\infty} r_k(s,x;t,y) \\
&= r^D(s,x;t,y) + \sum_{k=0}^{\infty} \int_s^t \int_D r_0(s,x;r,z) b(r,z) \cdot \nabla_z r_k(r,z;t,y) dz dr \\
&= r^D(s,x;t,y) + \int_s^t \int_D r_0(s,x;r,z) b(r,z) \cdot \nabla_z r^{D,b}(r,z;t,y) dz dr.
\end{aligned}$$

This yields (1.6).

(iii) According to Fubini's theorem, we have

$$\int_D r^{D,b}(s,x;r,z) r^{D,b}(r,z;t,y) dz = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \int_D r_m(s,x;r,z) r_{n-m}(r,z;t,y) dz.$$

Thus, for proving (1.10), it suffices to show that for each $n \in \mathbb{N}_0$,

$$\sum_{m=0}^{n} \int_D r_m(s,x;r,z) r_{n-m}(r,z;t,y) dz = r_n(s,x;t,y). \tag{3.10}$$

It is clear that the above equality holds for $n = 0$. Suppose that it holds for some $n \in \mathbb{N}$. Hence, we have

$$\sum_{m=0}^{n+1} \int_D r_m(s,x;r,z) r_{n+1-m}(r,z;t,y) dz =: \mathcal{J}_1 + \mathcal{J}_2,$$

where

$$\mathcal{J}_1 := \int_D r_{n+1}(s,x;r,z) p_0(r,z;t,y) dz$$

and

$$\mathcal{J}_2 := \sum_{m=0}^{n} \int_D r_m(s,x;r,z) p_{n+1-m}(r,z;t,y) dz.$$

According to (3.4) and Fubini's theorem, we have

$$\begin{aligned}
\mathcal{J}_1 &= \int_D \left(\int_s^r \int_D r_n(s,x;r',z') b(r',z') \cdot \nabla_{z'} r_0(r',z';r,z) dz' dr' \right) r_0(r,z;t,y) dz \\
&= \int_s^r \int_D r_n(s,x;r',z') b(r',z') \cdot \left(\int_D \nabla_{z'} r_0(r',z';r,z) r_0(r,z;t,y) dz \right) dz' dr' \\
&= \int_s^r \int_D r_n(s,x;r',z') b(r',z') \cdot \nabla_{z'} r_0(r',z';t,y) dz' dr'.
\end{aligned}$$

Similarly, based on (3.4) and the induction hypothesis, we have

$$\mathcal{J}_2 = \int_r^t \int_D r_n(s,x;r',z') b(r',z') \cdot \nabla_{z'} r_0(r',z';t,y) dz' dr'.$$

Hence,

$$\mathcal{J}_1 + \mathcal{J}_2 = \int_s^t \int_D r_n(s,x;r',z') b(r',z') \cdot \nabla_{z'} r_0(r',z';t,y) dz' dr' = r_{n+1}(s,x;t,y),$$

which provides (3.10).

Now, we can extend $r^{D,b}(s,x;t,y)$ from $\mathbb{D}_{\delta_0}$ to the set $\{(s,x;t,y): x,y \in D \text{ and } 0 \leqslant s < t < \infty\}$. Then, it is routine to extend the above assertions on $\mathbb{D}_{\delta_0}$ to $\mathbb{D}_\delta$ for any $\delta > 0$. Moreover, for $s+\delta_0 < t \leqslant s+2\delta_0$, based on (1.10), we have

$$\begin{aligned} r^{D,b}(s,x;t,y) &= \int_D r^{D,b}(s,x;s+\delta_0,u) r^{D,b}(s+\delta_0,u;t,y)du \\ &= \int_D r^{D,b}(s,x;s+\delta_0,u)\bigg[r^D(t-s-\delta_0,u;y) + \int_{s+\delta_0}^t \int_D r^{D,b}(s+\delta_0,u;r,z) \\ &\quad \times b(r,z)\cdot\nabla_z r^D(t-r,z,y)dzdr\bigg]du \\ &= \int_D r^{D,b}(s,x;s+\delta_0,u) r^D(t-s-\delta_0,u;y)du \\ &\quad + \int_{s+\delta_0}^t \int_D r^{D,b}(s,x;r,z) b(r,z)\cdot\nabla_z r^D(t-r,z,y)dzdr, \end{aligned} \tag{3.11}$$

where, in the second equality, we have used the fact that $r^{D,b}(s+\delta_0,u;t,y)$ satisfies (1.5). Similarly, we have

$$\begin{aligned} &\int_D r^{D,b}(s,x;s+\delta_0,u) r^D(t-s-\delta_0,u;y)du \\ &\quad = r^D(t-s,x,y) + \int_s^{s+\delta_0}\int_D r^{D,b}(s,x;r,z)b(r,z)\cdot\nabla_z r^D(t-r,z,y)dzdr, \end{aligned}$$

which in conjunction with (3.11) yields that $r^{D,b}(s,x;t,y)$ satisfies (1.5) with $s+\delta_0 < t \leqslant s+2\delta_0$. Using induction, we can show that $r^{D,b}(s,x;t,y)$ satisfies (1.5) on $\mathbb{D}_\delta$ for any $\delta > 0$.

(iv) Let $R_{s,t}f(x) := \int_D r^D(t-s,x,y)f(y)dy$. According to (1.5), for any $f \in C_b(D)$, we have

$$R^{D,b}_{s,t}f(x) = R_{s,t}f(x) + \int_s^t R^{D,b}_{s,r}(b\cdot\nabla R_{r,t}f)(x)dr. \tag{3.12}$$

It then follows that for all $f \in C_c^2(D)$,

$$\begin{aligned} R^{D,b}_{s,t}f(x) - f(x) &= R_{s,t}f(x) - f(x) + \int_s^t R^{D,b}_{s,r}(b\cdot\nabla R_{r,t}f)(x)dr \\ &= \int_s^t R_{s,r}(-(-\Delta|_D)^{\alpha/2})f(x)dr + \int_s^t R^{D,b}_{s,r}(b\cdot\nabla R_{r,t}f)(x)dr, \end{aligned} \tag{3.13}$$

and, based on (3.12) and Fubini's theorem,

$$\begin{aligned} &\int_s^t R^{D,b}_{s,r}(-(-\Delta|_D)^{\alpha/2})f(x)ds - \int_s^t R_{s,r}(-(-\Delta|_D)^{\alpha/2})f(x)dr \\ &\quad = \int_s^t\int_s^r R^{D,b}_{s,u}(b\cdot\nabla R_{u,r}(-(-\Delta|_D)^{\alpha/2})f)(x)dudr \\ &\quad = \int_s^t R^{D,b}_{s,u} b\cdot\nabla\bigg(\int_u^t R_{u,r}(-(-\Delta|_D)^{\alpha/2})f(x)dr\bigg)du \\ &\quad = \int_s^t R^{D,b}_{s,u} b\cdot\nabla(R_{u,t}f(x) - f(x))du. \end{aligned}$$

Combining this with (3.13), we obtain

$$R^{D,b}_{s,t}f(x) - f(x) = \int_s^t R^{D,b}_{s,r}\mathscr{L}^{D,b}f(x)dr,$$

which provides (1.11).

(v) Because $r^D(t,x,y)$ is the transition density of the process Y^D, for any uniformly continuous function $f(x)$ with compact supports, we have

$$\lim_{t\downarrow s}\|R_{s,t}f-f\|_\infty=0.$$

Furthermore, according to (1.8) and Lemma 3.2, we have

$$\begin{aligned}
&\left|\int_D\left(\int_s^t\!\!\int_D r_\alpha^{D,b}(s,x;r,z)b(r,z)\cdot\nabla_z r^D(t-r,z,y)dzdr\right)f(y)dy\right|\\
&\quad\preceq\|f\|_\infty\int_D\left(\int_s^t\!\!\int_D r^D(r-s,x,z)|b(r,z)|\cdot|\nabla_z r^D(t-r,z,y)|dzdr\right)dy\\
&\quad\leqslant C(\delta)\|f\|_\infty\int_D r^D(t-s,x,y)dy\leqslant C(\delta)\|f\|_\infty,
\end{aligned}$$

which yields (1.12) using (1.5).

(vi) Set

$$\Phi(s,x;t,y):=\int_s^t\!\!\int_D r^D(r-s,x,z)b(r,z)\cdot\nabla_z r^{D,b}(r,z;t,y)dzdr.$$

If we further assume that for $\gamma\in(0,\alpha-1)$, $b\in\mathbb{K}_D^\gamma$, then using (2.20) for any $x,x',y\in D$, we have

$$\begin{aligned}
&|\nabla_x\Phi(s,x;t,y)-\nabla_x\Phi(s,x';t,y)|\\
&\quad\preceq|x-x'|^\gamma\int_s^t\!\!\int_D(\hat q_\alpha(r-s,z,x)\varrho^1_{d+1+\gamma}(r-s,x-z)\\
&\qquad+\hat q_\alpha(r-s,z,x')\varrho^1_{d+1+\gamma}(r-s,x'-z))\cdot|b(r,z)|\hat q_\alpha(t-r,y,z)\varrho^1_{d+1}(t-r,z-y)dzdr\\
&\quad\preceq|x-x'|^\gamma\int_s^t\!\!\int_D(r-s)^{-\gamma/\alpha}(\hat q_\alpha(r-s,z,x)\varrho^1_{d+1}(r-s,x-z)\\
&\qquad+\hat q_\alpha(r-s,z,x')\varrho^1_{d+1}(r-s,x'-z))\cdot|b(r,z)|\hat q_\alpha(t-r,y,z)\varrho^1_{d+1}(t-r,z-y)dzdr\\
&\quad\preceq|x-x'|^\gamma\hat q_\alpha(t-s,y,\tilde x)\varrho^1_{d+1}(t-s,\tilde x-y)\int_s^t\!\!\int_D(r-s)^{-\gamma/\alpha}|b(r,z)|\\
&\qquad\times(\hat q_\alpha(r-s,z,x)\varrho^1_{d+1}(r-s,x-z)+\hat q_\alpha(r-s,z,x')\varrho^1_{d+1}(r-s,x'-z)\\
&\qquad+\hat q_\alpha(t-r,z,y)\varrho^1_{d+1}(t-r,z-y))dzdr\\
&\quad\preceq|x-x'|^\gamma(t-s)^{-\gamma/\alpha}\hat q_\alpha(t-s,y,\tilde x)\varrho^1_{d+1}(t-s,\tilde x-y),
\end{aligned}$$

where the third inequality is due to (3.3), and $\widetilde{x}$ is the point between x and x' which is closer to y, and the last inequality follows from the definition of $\mathbb{K}_D^\gamma$. When this finding is combined with (1.6) and (2.20), we obtain the desired result. The proof is completed. □

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