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Multi-window dilation-and-modulation frames on the half real line

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Abstract Wavelet and Gabor systems are based on translation-and-dilation and translation-and-modulation operators, respectively, and have been studied extensively. However, dilation-and-modulation systems cannot be derived from wavelet or Gabor systems. This study aims to investigate a class of dilation-and-modulation systems in the causal signal space $L^2(\mathbb{R}_+)$. $L^2(\mathbb{R}_+)$ can be identified as a subspace of $L^2(\mathbb{R})$, which consists of all $L^2(\mathbb{R})$-functions supported on $\mathbb{R}_+$ but not closed under the Fourier transform. Therefore, the Fourier transform method does not work in $L^2(\mathbb{R}_+)$. Herein, we introduce the notion of Θ_a-transform in $L^2(\mathbb{R}_+)$ and characterize the dilation-and-modulation frames and dual frames in $L^2(\mathbb{R}_+)$ using the Θ_a-transform; and present an explicit expression of all duals with the same structure for a general dilation-and-modulation frame for $L^2(\mathbb{R}_+)$. Furthermore, it has been proven that an arbitrary frame of this form is always nonredundant whenever the number of the generators is 1 and is always redundant whenever the number is greater than 1. Finally, some examples are provided to illustrate the generality of our results.

Keywords frame, wavelet frame, Gabor frame, dilation-and-modulation frame, multi-window dilation-and-modulation frame

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1 Introduction

It is well known that translation, modulation and dilation are fundamental operations in wavelet analysis. The translation operator T_{x_0}, modulation operator M_{x_0} with $x_0 \in \mathbb{R}$, and dilation operator D_c with $0 < c \neq 1$ are defined by

$$T_{x_0} f(\cdot) = f(\cdot - x_0), \quad M_{x_0} f(\cdot) = \mathrm{e}^{2\pi \mathrm{i} x_0 \cdot} f(\cdot) \quad \text{and} \quad D_c f(\cdot) = \sqrt{c} f(c\cdot)$$

for $f \in L^2(\mathbb{R})$, respectively. Given a finite subset Ψ of $L^2(\mathbb{R})$, Gabor frames of the form

$$\{M_{mb} T_{na} \psi : m, n \in \mathbb{Z}, \psi \in \Psi\} \tag{1.1}$$

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and wavelet frames of the form

$$\{D_{a^j}T_{bk}\psi : j,\, k \in \mathbb{Z},\, \psi \in \Psi\} \tag{1.2}$$

with $a, b > 0$ have been extensively studied (see [4, 13, 14, 22, 26–28]). However, dilation-and-modulation frames of the form

$$\{M_{mb}D_{a^j}\psi : m,\, j \in \mathbb{Z},\, \psi \in \Psi\} \quad \text{with} \quad a, b > 0 \tag{1.3}$$

have not been studied sufficiently. It has been found that the Fourier transform of (1.3) which is

$$\{T_{mb}D_{a^j}\hat{\psi} : m,\, j \in \mathbb{Z},\, \psi \in \Psi\} \tag{1.4}$$

does not fall into the framework of the above wavelet and Gabor systems. Herein, our focus is on a class of dilation-and-modulation frames for $L^2(\mathbb{R}_+)$ with $\mathbb{R}_+ = (0, \infty)$. $L^2(\mathbb{R}_+)$ can be considered as a closed subspace of $L^2(\mathbb{R})$ comprising all functions in $L^2(\mathbb{R})$ that vanish outside $\mathbb{R}_+$ and can model a causal signal space.

For more details on subspace Gabor and wavelet frames of the forms (1.1) and (1.2), respectively, see, e.g., [2, 6–8, 15–18], [19, 23, 24, 31, 32, 36, 38, 39], [40, 44, 48, 50, 51] and the references therein. It is easy to check that there exists no nonzero function ψ such that

$$T_{nc}\psi(\cdot) = 0 \quad \text{on} \quad (-\infty, 0)$$

for some $c > 0$ and for all $n \in \mathbb{Z}$. This implies that $L^2(\mathbb{R}_+)$ admits no frame of the form (1.1), (1.2) or (1.4). Therefore, constructing frames for $L^2(\mathbb{R}_+)$ with good structures is important. Two methods are known for this purpose. The first is to construct frames for $L^2(\mathbb{R}_+)$ comprising a subsystem of (1.2) and some inhomogeneous refinable function-based "boundary wavelets". For more details, see, e.g., [3, 5, 29, 30, 35, 41, 46, 47] and the references therein. The other is to use the Cantor group operation and Walsh series theory to introduce the notion of (frame) multiresolution analysis in $L^2(\mathbb{R}_+)$, and then derive wavelet frames similar to the case of $L^2(\mathbb{R})$. For more details, see, e.g., [1, 10–12, 33, 34, 42, 43, 45] and the references therein. In [20], numerical experiments were presented to establish that the nonnegative integer shifts of the Gaussian function form a Riesz sequence in $L^2(\mathbb{R}_+)$, and in [21], a sufficient condition was obtained to determine whether or not the nonnegative translations of a given function form a Riesz sequence on $L^2(\mathbb{R}_+)$.

Given $a > 1$, a measurable function h defined on $\mathbb{R}_+$ is said to be *a-dilation periodic* if $h(a\cdot) = h(\cdot)$ a.e. on $\mathbb{R}_+$. Throughout this paper, we denote by $\{\Lambda_m\}_{m\in\mathbb{Z}}$ the sequence of a-dilation periodic functions defined by

$$\Lambda_m(\cdot) = \frac{1}{\sqrt{a-1}} \mathrm{e}^{\frac{2\pi \mathrm{i} m\cdot}{a-1}} \quad \text{on} \quad [1, a) \quad \text{for each} \quad m \in \mathbb{Z}. \tag{1.5}$$

Motivated by the above works, herein, we aim to investigate the dilation-and-modulation systems in $L^2(\mathbb{R}_+)$ of the form:

$$\mathcal{MD}(\Psi, a) = \{\Lambda_m D_{a^j}\psi_l : m,\, j \in \mathbb{Z},\, 1 \leqslant l \leqslant L\} \tag{1.6}$$

under the following general setup:

General setup. (i) a is a fixed positive number greater than 1.

(ii) $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\}$ is a finite subset of $L^2(\mathbb{R}_+)$ with cardinality L.

As in multi-window Gabor analysis, throughout this paper, we say a set $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\}$ is a finite subset of $L^2(\mathbb{R}_+)$ with cardinality L, which means that it is a finite sequence in $L^2(\mathbb{R}_+)$ with L terms, i.e., we do not require that $\psi_1, \psi_2, \ldots, \psi_L$ are pairwise different.

For $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_L\}$, we define $\mathcal{MD}(\Phi, a)$, as defined in (1.6). The system $\mathcal{MD}(\Psi, a)$ differs from (1.3). The modulation factor $\mathrm{e}^{2\pi \mathrm{i} mb\cdot}$ in (1.1) is $\frac{1}{b}\mathbb{Z}$-periodic under addition, while Λ_m in (1.6) is a-dilation periodic. The motivation of introducing $\mathcal{MD}(\Psi, a)$ in $L^2(\mathbb{R}_+)$ is from the group structure of $\mathbb{R}_+$. Since $\mathbb{R}$ is a group under addition, one chooses addition periodic prefactors M_{mb} to match shift-invariant systems, and obtain Gabor systems of the form (1.1). However, $\mathbb{R}_+$ is a group under multiplication instead of addition. Therefore, we choose dilation periodic prefactors to match dilation-invariant systems and therefore, we study $\mathcal{MD}(\Psi, a)$ in $L^2(\mathbb{R}_+)$ of the form (1.6).

Let a be a fixed positive number greater than 1, and $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\}$ be a finite subset of $L^2(\mathbb{R}_+)$ with cardinality L. The system $\mathcal{MD}(\Psi, a)$ is called a *frame* for $L^2(\mathbb{R}_+)$ if there exist $0 < C_1 \leqslant C_2 < \infty$ such that

$$C_1\|f\|^2_{L^2(\mathbb{R}_+)} \leqslant \sum_{l=1}^{L}\sum_{m,j\in\mathbb{Z}} |\langle f, \Lambda_m D_{a^j}\psi_l\rangle_{L^2(\mathbb{R}_+)}|^2 \leqslant C_2\|f\|^2_{L^2(\mathbb{R}_+)} \quad \text{for} \quad f \in L^2(\mathbb{R}_+), \tag{1.7}$$

where C_1 and C_2 are called *frame bounds*; it is called a *Bessel sequence* in $L^2(\mathbb{R}_+)$ if the right-hand side inequality in (1.7) holds, where C_2 is called a *Bessel bound*. Particularly, it is called a *Parseval frame* if in (1.7), $C_1 = C_2 = 1$. Given a frame $\mathcal{MD}(\Psi, a)$ for $L^2(\mathbb{R}_+)$, a sequence $\mathcal{MD}(\Phi, a)$ is called a *dual* (or an $\mathcal{MD}$*-dual*) of $\mathcal{MD}(\Psi, a)$ if it is a frame such that

$$f = \sum_{l=1}^{L}\sum_{m,j\in\mathbb{Z}} \langle f, \Lambda_m D_{a^j}\varphi_l\rangle_{L^2(\mathbb{R}_+)}\Lambda_m D_{a^j}\psi_l \quad \text{for} \quad f \in L^2(\mathbb{R}_+). \tag{1.8}$$

It is easy to check that $\mathcal{MD}(\Psi, a)$ is also a dual of $\mathcal{MD}(\Phi, a)$ if $\mathcal{MD}(\Phi, a)$ is a dual of $\mathcal{MD}(\Psi, a)$. Therefore, in this case, we say $\mathcal{MD}(\Psi, a)$ and $\mathcal{MD}(\Phi, a)$ form a pair of dual frames for $L^2(\mathbb{R}_+)$. By the knowledge of frame theory, $\mathcal{MD}(\Psi, a)$ and $\mathcal{MD}(\Phi, a)$ form a pair of dual frames for $L^2(\mathbb{R}_+)$ if they are Bessel sequences and satisfy (1.8). The fundamentals of frames can be found in [4, 9, 27, 49]. Observe that $L^2(\mathbb{R}_+)$ is the Fourier transform of the Hardy space $H^2(\mathbb{R})$ which is a reducing subspace of $L^2(\mathbb{R})$ defined by

$$H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \hat{f}(\cdot) = 0 \text{ a.e. on } (-\infty, 0)\}.$$

Wavelet frames in $H^2(\mathbb{R})$ of the form (1.2) were studied in [28, 44, 48]. By the Plancherel theorem, an $H^2(\mathbb{R})$-frame, which is given by $\{D_{a^j}T_{bk}\psi : j, k \in \mathbb{Z}, \psi \in \Psi\}$, leads to an $L^2(\mathbb{R}_+)$-frame given by

$$\{\mathrm{e}^{-2\pi\mathrm{i}a^jk\cdot}\hat{\psi}(a^j\cdot) : j, k \in \mathbb{Z}, \psi \in \Psi\}. \tag{1.9}$$

In (1.9), $\mathrm{e}^{-2\pi\mathrm{i}a^jk\cdot}$ is $a^{-j}\mathbb{Z}$-periodic with respect to addition, and the period varies with j. However, Λ_m in (1.6) is a-dilation periodic, and is unrelated to j. Therefore, the system (1.6) differs from (1.9), and it is of independent interest.

This study focuses on the theory of $L^2(\mathbb{R}_+)$-frames of the form (1.6). It cannot be derived from the well-known wavelet and Gabor systems, and its operation is more intuitive when compared with the Cantor group and Walsh series-based systems in [1, 10–12, 33, 34, 42, 43, 45]. Also $L^2(\mathbb{R}_+)$ is not closed under the Fourier transform. In particular, the Fourier transform of a compactly supported nonzero function in $L^2(\mathbb{R}_+)$ lies outside this space. Therefore, the Fourier transform cannot be used in our setting and thus, it is desirable to find a new method.

The system (1.6) is related to a kind of function-valued frames in [25] by Hasankhani and Dehghan. They introduced the notion of function-valued frame as follows. Given $a > 1$ and $f, g \in L^2(\mathbb{R}_+)$, define the function-valued inner product $\langle f, g\rangle_a$ of f and g by

$$\langle f, g\rangle_a(\cdot) = \sum_{j\in\mathbb{Z}} a^j f(a^j\cdot)\overline{g(a^j\cdot)}, \quad \text{and} \quad \|f\|_a(\cdot) = \sqrt{\langle f, f\rangle_a(\cdot)}.$$

A sequence $\{f_j\}_{j\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$ is called a function-valued frame for $L^2(\mathbb{R}_+)$ with respect to a, if there exist positive constants A and B, such that

$$A\|f\|^2_a(\cdot) \leqslant \sum_{j\in\mathbb{Z}} |\langle f, f_j\rangle_a(\cdot)|^2 \leqslant B\|f\|^2_a(\cdot) \quad \text{a.e. on} \ \ [1, a]$$

for $f \in L^2(\mathbb{R}_+)$. Take $f_j = D_{a^j}\psi$ for some $\psi \in L^2(\mathbb{R}_+)$. Then, applying [25, Theorem 4.8], we have that $\{D_{a^j}\psi\}_{j\in\mathbb{Z}}$ is a function-valued frame for $L^2(\mathbb{R}_+)$ with respect to a if and only if $\mathcal{MD}(\psi, a)$ (i.e., $L = 1$ in (1.6)) is a frame for $L^2(\mathbb{R}_+)$. We characterized frames of the form $\mathcal{MD}(\psi, a)$ in [37] in terms of the bi-infinite matrix-valued function $\mathcal{G}(\psi, \cdot) = (\overline{D_{a^{j+l}}\psi(\cdot)})_{j,l\in\mathbb{Z}}$, where the notion of Θ_a-transform was not formally formulated. In this paper, we will introduce the Θ_a-transform, and use the Θ_a-transform method to study multi-window frames $\mathcal{MD}(\Psi, a)$ of the form (1.6) and their duals. By Theorem 3.9 below, frames in [37] are all Riesz bases but frames in this paper are redundant ones if $L > 1$.

The rest of this paper is organized as follows. In Section 2, we introduce the notion of Θ_a-transform, and give a Θ_a-transform domain characterization for a dilation-and-modulation system $\mathcal{MD}(\Psi, a)$ to be complete, a Bessel sequence, and a frame in $L^2(\mathbb{R}_+)$, accordingly. In Section 3, using the Θ_a-transform we characterize dual frame pairs of the form $(\mathcal{MD}(\Psi, a),\, \mathcal{MD}(\Phi, a))$ and obtain an explicit expression of all $\mathcal{MD}$-duals of a general frame $\mathcal{MD}(\Psi, a)$ for $L^2(\mathbb{R}_+)$. We also prove that an arbitrary frame $\mathcal{MD}(\Psi, a)$ is a Riesz basis if and only if $L = 1$. In Section 4, we give some examples of $\mathcal{MD}$-dual frame pairs for $L^2(\mathbb{R}_+)$ to illustrate the generality of our results.

2 Θ_a-transform domain frame characterization

Let a be a fixed positive number greater than 1, and $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\}$ be a finite subset of $L^2(\mathbb{R}_+)$ with cardinality L. In this section, by introducing the Θ_a-transform we give the conditions of completeness, Bessel sequence and frame of $\mathcal{MD}(\Psi, a)$ in $L^2(\mathbb{R}_+)$, accordingly.

Definition 2.1. Let a be a fixed positive number greater than 1. For $f \in L^2(\mathbb{R}_+)$, we define

$$\Theta_a f(x, \xi) = \sum_{l\in\mathbb{Z}} a^{\frac{l}{2}} f(a^l x) \mathrm{e}^{-2\pi \mathrm{i} l\xi} \tag{2.1}$$

for a.e. $(x, \xi) \in \mathbb{R}_+ \times \mathbb{R}$.

Remark 2.2. Observe that, given $f \in L^2(\mathbb{R}_+)$,

$$\int_{a^j}^{a^{j+1}} \sum_{l\in\mathbb{Z}} a^l |f(a^l x)|^2 dx = \|f\|^2_{L^2(\mathbb{R}_+)} < \infty \quad \text{for} \quad j \in \mathbb{Z}.$$

This implies that $\sum_{l\in\mathbb{Z}} a^l |f(a^l \cdot)|^2 < \infty$ a.e. on $\mathbb{R}_+$ by the arbitrariness of j. Therefore, (2.1) is well defined.

Lemma 2.3. *Let a be a fixed positive number greater than 1. For $m,\, j \in \mathbb{Z}$, define Λ_m as in (1.5), and $e_{m,j}$ by*

$$e_{m,j}(x, \xi) = \Lambda_m(x) \mathrm{e}^{2\pi \mathrm{i} j\xi} \quad \textit{for} \quad (x, \xi) \in \mathbb{R}_+ \times \mathbb{R}.$$

Then

(i) *$\{\Lambda_m : m \in \mathbb{Z}\}$ and $\{e_{m,j} : m,\, j \in \mathbb{Z}\}$ are orthonormal bases for $L^2([1, a))$ and $L^2([1, a) \times [0, 1))$, respectively;*

(ii) *the Θ_a-transform has the following quasi-periodicity: given $f \in L^2(\mathbb{R}_+)$,*

$$\Theta_a f(a^j x, \xi + m) = \mathrm{e}^{2\pi \mathrm{i} j\xi} a^{-\frac{j}{2}} \Theta_a f(x, \xi)$$

for $j,\, m \in \mathbb{Z}$ and a.e. $(x, \xi) \in \mathbb{R}_+ \times \mathbb{R}$;

(iii) *for $j, m \in \mathbb{Z}$, $f \in L^2(\mathbb{R}_+)$,*

$$\Theta_a(\Lambda_m D_{a^j} f)(x, \xi) = e_{m,j}(x, \xi) \Theta_a f(x, \xi) \quad \textit{for a.e.} \quad (x, \xi) \in \mathbb{R}_+ \times \mathbb{R};$$

(iv) *the Θ_a-transform is a unitary operator from $L^2(\mathbb{R}_+)$ onto $L^2([1, a) \times [0, 1))$;*

(v)

$$\int_{[1,a)\times[0,1)} |f(x, \xi)|^2 dx d\xi = \sum_{m,j\in\mathbb{Z}} \left| \int_{[1,a)\times[0,1)} f(x, \xi) \overline{e_{m,j}(x, \xi)} dx d\xi \right|^2 \tag{2.2}$$

for $f \in L^1([1, a) \times [0, 1))$.

Proof. By a standard argument, we have (i)–(iii). Next, we prove (iv) and (v).

(iv) It is obvious that the Θ_a-transform is linear. We only need to prove that it is norm-preserving and onto. For $f \in L^2(\mathbb{R}_+)$, we have

$$\begin{aligned}\|\Theta_a f\|^2_{L^2([1,a)\times[0,1))} &= \int_1^a dx \int_0^1 \Big|\sum_{l\in\mathbb{Z}} a^{\frac{l}{2}} f(a^l x)\mathrm{e}^{-2\pi \mathrm{i} l\xi}\Big|^2 d\xi \\ &= \int_1^a \sum_{l\in\mathbb{Z}} a^l |f(a^l x)|^2 dx \\ &= \|f\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}$$

This implies that the Θ_a-transform is norm-preserving. Next, we prove that it is onto. Let $F \in L^2([1,a) \times [0,1))$. Then there exists a unique $\{c_{m,j}\}_{m,j\in\mathbb{Z}} \in l^2(\mathbb{Z}^2)$ such that

$$F(x,\xi) = \sum_{m,j\in\mathbb{Z}} c_{m,j} e_{m,j}(x,\xi) = \sum_{j\in\mathbb{Z}} \Big(\sum_{m\in\mathbb{Z}} c_{m,\,j}\Lambda_m(x)\Big)\mathrm{e}^{2\pi \mathrm{i} j\xi}$$

for a.e. $(x,\xi) \in [1,a) \times [0,1)$ by (i). Define f on $\mathbb{R}_+$ by

$$f(a^j x) = a^{-\frac{j}{2}} \sum_{m\in\mathbb{Z}} c_{m,\,-j}\Lambda_m(x) \quad \text{for} \quad j \in \mathbb{Z} \quad \text{and a.e.} \quad x \in [1,a).$$

Then

$$\|f\|^2_{L^2(\mathbb{R}_+)} = \sum_{j\in\mathbb{Z}} \int_1^a a^j |f(a^j x)|^2 dx = \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |c_{m,\,-j}|^2 = \sum_{m,j\in\mathbb{Z}} |c_{m,j}|^2 < \infty$$

by (i), and we have

$$\Theta_a f(x,\xi) = F(x,\xi) \quad \text{for a.e.} \quad (x,\xi) \in [1,a) \times [0,1).$$

Hence, the Θ_a-transform is onto.

(v) By (i), (2.2) holds if $f \in L^2([1,a) \times [0,1))$. When $f \in L^1([1,a) \times [0,1)) \setminus L^2([1,a) \times [0,1))$, the left-hand side of (2.2) is infinite. Now we prove by contradiction that the right-hand side of (2.2) is also infinite. Suppose it is finite. Then the function

$$g = \sum_{m,j\in\mathbb{Z}} \Big(\int_{[1,a)\times[0,1)} f(x,\xi)\overline{e_{m,j}(x,\xi)}dxd\xi\Big)e_{m,j}$$

belongs to $L^2([1,a) \times [0,1))$ by (i), and thus it belongs to $L^1([1,a) \times [0,1))$. Since it has the same Fourier coefficients as f, by the uniqueness of Fourier coefficients, $f = g$. Thus $f \in L^2([1,a) \times [0,1))$, which leads to a contradiction. The proof is completed. □

Lemma 2.4. *Let a be a fixed positive number greater than 1, and $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\}$ be a finite subset of $L^2(\mathbb{R}_+)$ with cardinality L. Then*

$$\sum_{l=1}^L \sum_{m,j\in\mathbb{Z}} |\langle f,\, \Lambda_m D_{a^j}\psi_l\rangle_{L^2(\mathbb{R}_+)}|^2 = \int_{[1,a)\times[0,1)} \Big(\sum_{l=1}^L |\Theta_a\psi_l(x,\xi)|^2\Big)|\Theta_a f(x,\xi)|^2 dxd\xi \quad \textit{for} \quad f \in L^2(\mathbb{R}_+).$$

Proof. Fix $f \in L^2(\mathbb{R}_+)$. By Lemmas 2.3(iii) and 2.3(iv), we have

$$\begin{aligned}\sum_{l=1}^L \sum_{m,j\in\mathbb{Z}} |\langle f,\, \Lambda_m D_{a^j}\psi_l\rangle_{L^2(\mathbb{R}_+)}|^2 &= \sum_{l=1}^L \sum_{m,j\in\mathbb{Z}} |\langle \Theta_a f,\, \Theta_a\Lambda_m D_{a^j}\psi_l\rangle_{L^2([1,a)\times[0,1))}|^2 \\ &= \sum_{l=1}^L \sum_{m,j\in\mathbb{Z}} |\langle \Theta_a f,\, e_{m,j}\Theta_a\psi_l\rangle_{L^2([1,a)\times[0,1))}|^2 \\ &= \sum_{l=1}^L \sum_{m,j\in\mathbb{Z}} \Big|\int_{[1,a)\times[0,1)} \overline{\Theta_a\psi_l(x,\xi)}\Theta_a f(x,\xi)\overline{e_{m,j}(x,\xi)}dxd\xi\Big|^2.\end{aligned}$$

Again applying Lemma 2.3(v) to $\overline{\Theta_a\psi_l(x,\xi)}\Theta_a f(x,\xi)$ leads to

$$\sum_{l=1}^{L}\sum_{m,j\in\mathbb{Z}}|\langle f,\,\Lambda_m D_{a^j}\psi_l\rangle_{L^2(\mathbb{R}_+)}|^2=\sum_{l=1}^{L}\int_{[1,a)\times[0,1)}|\overline{\Theta_a\psi_l(x,\xi)}\Theta_a f(x,\xi)|^2dxd\xi$$
$$=\int_{[1,a)\times[0,1)}\left(\sum_{l=1}^{L}|\Theta_a\psi_l(x,\xi)|^2\right)|\Theta_a f(x,\xi)|^2dxd\xi.$$

This completes the proof. □

Theorem 2.5. *Let a be a fixed positive number greater than 1, and $\Psi=\{\psi_1,\psi_2,\dots,\psi_L\}$ be a finite subset of $L^2(\mathbb{R}_+)$ with cardinality L. Then $\mathcal{MD}(\Psi,a)$ is complete in $L^2(\mathbb{R}_+)$ if and only if*

$$\sum_{l=1}^{L}|\Theta_a\psi_l(x,\xi)|^2\neq 0\quad \textit{for a.e.}\quad (x,\xi)\in[1,a)\times[0,1).\tag{2.3}$$

Proof. By Lemma 2.4, for $f\in L^2(\mathbb{R}_+)$,

$$\sum_{l=1}^{L}\sum_{m,j\in\mathbb{Z}}|\langle f,\,\Lambda_m D_{a^j}\psi_l\rangle_{L^2(\mathbb{R}_+)}|^2=0\quad\text{a.e. on}\quad\mathbb{R}_+\tag{2.4}$$

if and only if

$$\left(\sum_{l=1}^{L}|\Theta_a\psi_l(x,\xi)|^2\right)|\Theta_a f(x,\xi)|^2=0\quad\text{for a.e.}\quad(x,\xi)\in[1,a)\times[0,1).\tag{2.5}$$

Observe that $\mathcal{MD}(\Psi,a)$ is complete in $L^2(\mathbb{R}_+)$ if and only if $f=0$ is the unique solution to (2.4) in $L^2(\mathbb{R}_+)$. It follows that the completeness of $\mathcal{MD}(\Psi,a)$ in $L^2(\mathbb{R}_+)$ is equivalent to $f=0$ being the unique solution to (2.5) in $L^2(\mathbb{R}_+)$. This is in turn equivalent to the fact that $\Theta_a f=0$ is the unique solution to (2.5) in $L^2([1,a)\times[0,1))$ by Lemma 2.3(iv), which is as well equivalent to (2.3). This completes the proof. □

Theorem 2.6. *Let a be a fixed positive number greater than 1, and $\Psi=\{\psi_1,\psi_2,\dots,\psi_L\}$ be a finite subset of $L^2(\mathbb{R}_+)$ with cardinality L. Then $\mathcal{MD}(\Psi,a)$ is a Bessel sequence in $L^2(\mathbb{R}_+)$ with the Bessel bound B if and only if*

$$\sum_{l=1}^{L}|\Theta_a\psi_l(x,\xi)|^2\leqslant B\quad \textit{for a.e.}\quad (x,\xi)\in[1,a)\times[0,1).\tag{2.6}$$

Proof. By Lemmas 2.4 and 2.3(v), we have

$$\sum_{l=1}^{L}\sum_{m,j\in\mathbb{Z}}|\langle f,\,\Lambda_m D_{a^j}\psi_l\rangle_{L^2(\mathbb{R}_+)}|^2=\int_{[1,a)\times[0,1)}\left(\sum_{l=1}^{L}|\Theta_a\psi_l(x,\xi)|^2\right)|\Theta_a f(x,\xi)|^2dxd\xi,\tag{2.7}$$

and

$$\int_{[1,a)\times[0,1)}|\Theta_a f(x,\xi)|^2dxd\xi=\|f\|^2_{L^2(\mathbb{R}_+,\,\mathbb{C}^L)}\tag{2.8}$$

for $f\in L^2(\mathbb{R}_+,\,\mathbb{C}^L)$. So by Lemma 2.3(iv), (2.6) implies that

$$\sum_{l=1}^{L}\sum_{m,j\in\mathbb{Z}}|\langle f,\,\Lambda_m D_{a^j}\psi_l\rangle_{L^2(\mathbb{R}_+)}|^2\leqslant B\|f\|^2_{L^2(\mathbb{R}_+,\,\mathbb{C}^L)}\tag{2.9}$$

for $f\in L^2(\mathbb{R}_+)$. Thus, $\mathcal{MD}(\Psi,a)$ is a Bessel sequence in $L^2(\mathbb{R}_+)$ with the Bessel bound B.

Now we prove the converse implication by contradiction. Suppose $\mathcal{MD}(\Psi,a)$ is a Bessel sequence in $L^2(\mathbb{R}_+)$ with the Bessel bound B, and $\sum_{l=1}^{L}|\Theta_a\psi(\cdot,\cdot)|^2>B$ on some $E\subset[1,a)\times[0,1)$ with $|E|>0$. Define f by

$$\Theta_a f(\cdot,\cdot)=\chi_E(\cdot,\cdot)\quad\text{on}\quad[1,a)\times[0,1)$$

in (2.7), where χ_E denotes the characteristic function of E. Then f is well defined,

$$\|f\|^2_{L^2(\mathbb{R}_+)} = \int_{[1,a)\times[0,1)} |\Theta_a f(x,\xi)|^2 dx d\xi = |E|$$

by Lemma 2.3(iv), and

$$\sum_{l=1}^{L} \sum_{m,j\in\mathbb{Z}} |\langle f, \Lambda_m D_{a^j}\psi_l\rangle_{L^2(\mathbb{R}_+)}|^2 > B|E| = B\|f\|_{L^2_{(\mathbb{R}_+)}}.$$

This contradicts the fact that $\mathcal{MD}(\Psi,a)$ is a Bessel sequence in $L^2(\mathbb{R}_+)$ with the Bessel bound B. This completes the proof. □

By a similar argument to that in Theorem 2.6, we obtain the following theorem.

Theorem 2.7. *Let a be a fixed positive number greater than 1, and $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\}$ be a finite subset of $L^2(\mathbb{R}_+)$ with cardinality L. Then $\mathcal{MD}(\Psi,a)$ is a frame in $L^2(\mathbb{R}_+)$ with frame bounds A and B if and only if*

$$A \leqslant \sum_{l=1}^{L} |\Theta_a\psi_l(x,\xi)|^2 \leqslant B$$

for a.e. $(x,\xi) \in [1,a)\times[0,1)$.

3 Θ_a-transform domain expression of duals

In this section, we characterize and express $\mathcal{MD}$-duals of a general frame $\mathcal{MD}(\Psi,a)$ for $L^2(\mathbb{R}_+)$ and also, we study the redundancy of a general frame $\mathcal{MD}(\Psi,a)$ for $L^2(\mathbb{R}_+)$. Interestingly, we prove that an arbitrary frame $\mathcal{MD}(\Psi,a)$ for $L^2(\mathbb{R}_+)$ is always nonredundant if $L=1$, and is always redundant if $L>1$ (see Theorem 3.9 below).

For the ease and convenience, we write

$$\mathfrak{D} = \{f \in L^2(\mathbb{R}_+) : \Theta_a f \in L^\infty([1,a)\times[0,1))\}. \tag{3.1}$$

Then using Lemma 2.3(iv) and the fact that $L^\infty([1,a)\times[0,1))$ is dense in $L^2([1,a)\times[0,1))$, we have that $\mathfrak{D}$ is dense in $L^2(\mathbb{R}_+)$. This fact will be frequently used in what follows.

Let a be a fixed positive number greater than 1, $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\}$ be a finite subset of $L^2(\mathbb{R}_+)$ with cardinality L, and $\mathcal{MD}(\Psi,a)$ be a Bessel sequence in $L^2(\mathbb{R}_+)$. We denote by S its frame operator, i.e.,

$$Sf = \sum_{l=1}^{L} \sum_{m,j\in\mathbb{Z}} \langle f, \Lambda_m D_{a^j}\psi_l\rangle_{L^2(\mathbb{R}_+)} \Lambda_m D_{a^j}\psi_l \quad \text{for} \quad f \in L^2(\mathbb{R}_+).$$

By a standard argument, we have the following lemma that shows that S commutes with the modulation and dilation operators.

Lemma 3.1. *Let a be a fixed positive number greater than 1, and $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\}$ be a finite subset of $L^2(\mathbb{R}_+)$ with cardinality L. Assume that $\mathcal{MD}(\Psi,a)$ is a Bessel sequence in $L^2(\mathbb{R}_+)$, and that S is its frame operator. Then*

$$S\Lambda_m f = \Lambda_m Sf, \quad SD_{a^j}f = D_{a^j}Sf,$$

and thus $S\Lambda_m D_{a^j} f = \Lambda_m D_{a^j} Sf$ for $f \in L^2(\mathbb{R}_+)$ and $m, j \in \mathbb{Z}$.

Lemma 3.2. *Let a be a fixed positive number greater than 1, $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\}$ be a finite subset of $L^2(\mathbb{R}_+)$ with cardinality L, and $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_L\} \subset L^2(\mathbb{R}_+)$. Then*

$$\sum_{l=1}^{L} \sum_{m,j\in\mathbb{Z}} \langle f, \Lambda_m D_{a^j}\psi_l\rangle_{L^2(\mathbb{R}_+)} \langle \Lambda_m D_{a^j}\varphi_l, g\rangle_{L^2(\mathbb{R}_+)} = \int_{[1,a)\times[0,1)} \Omega(x,\xi)\Theta_a f(x,\xi)\overline{\Theta_a g(x,\xi)} dx d\xi \tag{3.2}$$

for $f,\, g \in \mathfrak{D}$*, where*

$$\Omega(x,\xi) = \sum_{l=1}^{L} \Theta_a \varphi_l(x,\xi)\overline{\Theta_a \psi_l(x,\xi)}.$$

Proof. Let $f,\, g \in \mathfrak{D}$ be fixed. Then by Lemma 2.4, we have

$$\sum_{l=1}^{L} \sum_{m,j\in\mathbb{Z}} |\langle f,\, \Lambda_m D_{a^j}\psi_l\rangle|^2 < \infty, \quad \text{and} \quad \sum_{l=1}^{L} \sum_{m,j\in\mathbb{Z}} |\langle g,\, \Lambda_m D_{a^j}\varphi_l\rangle|^2 < \infty.$$

Thus, the series

$$\sum_{l=1}^{L} \sum_{m,j\in\mathbb{Z}} \langle f,\, \Lambda_m D_{a^j}\psi_l\rangle_{L^2(\mathbb{R}_+)} \langle \Lambda_m D_{a^j}\varphi_l,\, g\rangle_{L^2(\mathbb{R}_+)}$$

is well defined and converges absolutely. By Lemmas 2.3(i), 2.3(iii) and 2.3(iv), we see that

$$\begin{aligned}
&\sum_{l=1}^{L} \sum_{m,j\in\mathbb{Z}} \langle f,\, \Lambda_m D_{a^j}\psi_l\rangle_{L^2(\mathbb{R}_+)} \langle \Lambda_m D_{a^j}\varphi_l,\, g\rangle_{L^2(\mathbb{R}_+)} \\
&\quad= \sum_{l=1}^{L} \sum_{m,j\in\mathbb{Z}} \langle \Theta_a f,\, \Theta_a\Lambda_m D_{a^j}\psi_l\rangle_{L^2([1,a)\times[0,1))} \langle \Theta_a\Lambda_m D_{a^j}\varphi_l,\, \Theta_a g\rangle_{L^2([1,a)\times[0,1))} \\
&\quad= \sum_{l=1}^{L} \sum_{m,j\in\mathbb{Z}} \langle \overline{\Theta_a\psi_l}\Theta_a f,\, e_{m,j}\rangle_{L^2([1,a)\times[0,1))} \langle e_{m,j},\, \overline{\Theta_a\varphi_l}\Theta_a g\rangle_{L^2([1,a)\times[0,1))} \\
&\quad= \sum_{l=1}^{L} \langle \Theta_a f\overline{\Theta_a\psi_l},\, \Theta_a g\overline{\Theta_a\varphi_l}\rangle_{L^2([1,a)\times[0,1))} \\
&\quad= \int_{[1,a)\times[0,1)} \Omega(x,\xi)\Theta_a f(x,\xi)\overline{\Theta_a g(x,\xi)}dxd\xi.
\end{aligned}$$

This completes the proof. □

Lemma 3.3. *Let* a *be a fixed positive number greater than* 1*, and* $\Psi = \{\psi_1,\, \psi_2, \ldots, \psi_L\}$ *be a finite subset of* $L^2(\mathbb{R}_+)$ *with cardinality* L*. Assume that* $\mathcal{MD}(\Psi,a)$ *is a Bessel sequence in* $L^2(\mathbb{R}_+)$*, and that* S *is its frame operator. Then, for* $f \in L^2(\mathbb{R}_+)$*,*

$$\Theta_a Sf(\cdot,\cdot) = \left(\sum_{l=1}^{L} |\Theta_a\psi_l(\cdot,\cdot)|^2\right)\Theta_a f(\cdot,\cdot) \tag{3.3}$$

a.e. on $[1,a)\times[0,1)$*.*

Proof. By Lemma 3.2, we have

$$\langle Sf,\, g\rangle_{L^2(\mathbb{R}_+)} = \int_{[1,a)\times[0,1)} \left(\sum_{l=1}^{L} |\Theta_a\psi_l(x,\xi)|^2\right)\Theta_a f(x,\xi)\overline{\Theta_a g(x,\xi)}dxd\xi$$

for $f,\, g \in \mathfrak{D}$. Since $\mathfrak{D}$ is dense in $L^2(\mathbb{R}_+)$ and $\mathcal{MD}(\Psi,a)$ is a Bessel sequence, by Theorem 2.6 and a standard argument, it follows that

$$\langle Sf,\, g\rangle_{L^2(\mathbb{R}_+)} = \left\langle \left(\sum_{l=1}^{L} |\Theta_a\psi_l(x,\xi)|^2\right)\Theta_a f,\, \Theta_a g\right\rangle_{L^2([1,a)\times[0,1))}$$

for $f,\, g \in L^2(\mathbb{R}_+)$. Also observing that

$$\langle Sf,\, g\rangle_{L^2(\mathbb{R}_+)} = \langle \Theta_a Sf,\, \Theta_a g\rangle_{L^2([1,a)\times[0,1))}$$

by Lemma 2.3(iv), we have that

$$\langle \Theta_a Sf, \Theta_a g\rangle_{L^2([1,a)\times[0,1))} = \left\langle \left(\sum_{l=1}^{L} |\Theta_a \psi_l|^2\right)\Theta_a f, \Theta_a g\right\rangle_{L^2([1,a)\times[0,1))}$$

for f, $g \in L^2(\mathbb{R}_+)$. This implies (3.3) by Lemma 2.3(iv). This completes the proof. □

Lemma 3.4. *Let a be a fixed positive number greater than 1, and $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\}$ be a finite subset of $L^2(\mathbb{R}_+)$ with cardinality L. Then $\mathcal{MD}(\Psi, a)$ cannot be a Riesz sequence in $L^2(\mathbb{R}_+)$ whenever $L > 1$.*

Proof. We proceed by contradiction. Suppose $L > 1$ and $\mathcal{MD}(\Psi, a)$ is a Riesz sequence in $L^2(\mathbb{R}_+)$. Let S be its frame operator. Then by Lemma 3.1, it commutes with $\Lambda_m D_{a^j}$ for all m, $j \in \mathbb{Z}$. Since S is self-adjoint, invertible and bounded, it follows that

$$S^{-\frac{1}{2}}\Lambda_m D_{a^j}\psi_l = \Lambda_m D_{a^j} S^{-\frac{1}{2}}\psi_l \quad \text{for} \quad m, j \in \mathbb{Z} \quad \text{and} \quad 1 \leqslant l \leqslant L.$$

Hence, $\mathcal{MD}(S^{-\frac{1}{2}}(\Psi), a)$ is an orthonormal system in $L^2(\mathbb{R}_+)$. Write $S^{-\frac{1}{2}}\psi_l = \varphi_l$ for $1 \leqslant l \leqslant L$. Then for m_1, m_2, j_1, $j_2 \in \mathbb{Z}$ and $1 \leqslant l_1, l_2 \leqslant L$, we have

$$\langle \Lambda_{m_1} D_{a^{j_1}}\varphi_{l_1}, \Lambda_{m_2} D_{a^{j_2}}\varphi_{l_2}\rangle_{L^2(\mathbb{R}_+)} = \delta_{m_1,m_2}\delta_{j_1,j_2}\delta_{l_1,l_2},$$

where the Kronecker delta is defined by

$$\delta_{n,m} = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}$$

By Lemmas 2.3(iii) and 2.3(iv), it is equivalent to

$$\langle e_{m_1,j_1}\Theta_a\varphi_{l_1}, e_{m_2,j_2}\Theta_a\varphi_{l_2}\rangle_{L^2([1,a)\times[0,1))} = \delta_{m_1,m_2}\delta_{j_1,j_2}\delta_{l_1,l_2}$$

for m_1, m_2, j_1, $j_2 \in \mathbb{Z}$ and $1 \leqslant l_1, l_2 \leqslant L$, equivalently,

$$\frac{1}{\sqrt{a-1}}\int_{[1,a)\times[0,1)} \Theta_a\varphi_{l_1}(x,\xi)\overline{\Theta_a\varphi_{l_2}(x,\xi)e_{m,j}(x,\xi)}dxd\xi = \delta_{m,0}\delta_{j,0}\delta_{l_1,l_2}$$

for m, $j \in \mathbb{Z}$ and $1 \leqslant l_1, l_2 \leqslant L$. This in turn is equivalent to

$$\Theta_a\varphi_{l_1}(\cdot,\cdot)\overline{\Theta_a\varphi_{l_2}(\cdot,\cdot)} = \delta_{l_1,l_2} \quad \text{a.e. on} \quad [1,a)\times[0,1)$$

for $1 \leqslant l_1, l_2 \leqslant L$ by the uniqueness of Fourier coefficients. In particular, it implies that

$$|\Theta_a\varphi_1(\cdot,\cdot)| = |\Theta_a\varphi_2(\cdot,\cdot)| = 1$$

and

$$\Theta_a\varphi_1(\cdot,\cdot)\overline{\Theta_a\varphi_2(\cdot,\cdot)} = 0$$

a.e. on $[1,a)\times[0,1)$. This leads to a contradiction. Hence, this completes the proof. □

The following lemma is extracted from [37, Corollary 3.1].

Lemma 3.5. *Let a be a fixed positive number greater than 1, and Ψ be a singleton in $L^2(\mathbb{R}_+)$. Then $\mathcal{MD}(\Psi, a)$ is a Parseval frame for $L^2(\mathbb{R}_+)$ if and only if it is an orthonormal basis for $L^2(\mathbb{R}_+)$.*

Theorem 3.6. *Let a be a fixed positive number greater than 1, $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\}$ be a finite subset of $L^2(\mathbb{R}_+)$ with cardinality L, and $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_L\} \subset L^2(\mathbb{R}_+)$. Assume that $\mathcal{MD}(\Psi, a)$ and $\mathcal{MD}(\Phi, a)$ are Bessel sequences in $L^2(\mathbb{R}_+)$. Then $\mathcal{MD}(\Psi, a)$ and $\mathcal{MD}(\Phi, a)$ form a pair of dual frames for $L^2(\mathbb{R}_+)$ if and only if*

$$\sum_{l=1}^{L}\Theta_a\varphi_l(x,\xi)\overline{\Theta_a\psi_l(x,\xi)} = 1 \quad \textit{for a.e.} \quad (x,\xi) \in [1,a)\times[0,1). \tag{3.4}$$

Proof. Since $\mathcal{MD}(\Psi,a)$ and $\mathcal{MD}(\Phi,a)$ are Bessel sequences in $L^2(\mathbb{R}_+)$, and $\mathfrak{D}$ is dense in $L^2(\mathbb{R}_+)$, we have that $\mathcal{MD}(\Psi,a)$ and $\mathcal{MD}(\Phi,a)$ form a pair of dual frames for $L^2(\mathbb{R}_+)$ if and only if

$$\sum_{l=1}^{L}\sum_{m,j\in\mathbb{Z}}\langle f,\,\Lambda_m D_{a^j}\psi_l\rangle_{L^2(\mathbb{R}_+)}\langle\Lambda_m D_{a^j}\varphi_l,\,g\rangle_{L^2(\mathbb{R}_+)}=\langle f,\,g\rangle_{L^2(\mathbb{R}_+)} \tag{3.5}$$

for $f,\,g\in\mathfrak{D}$. By Lemmas 3.2 and 2.3(iv), (3.5) is equivalent to

$$\begin{aligned}&\int_{[1,a)\times[0,1)}\left(\sum_{l=1}^{L}\Theta_a\varphi_l(x,\xi)\overline{\Theta_a\psi_l(x,\xi)}\right)\Theta_a f(x,\xi)\overline{\Theta_a g(x,\xi)}dxd\xi\\&\quad=\int_{[1,a)\times[0,1)}\Theta_a f(x,\xi)\overline{\Theta_a g(x,\xi)}dxd\xi\end{aligned} \tag{3.6}$$

for $f,\,g\in\mathfrak{D}$. Obviously, (3.4) implies (3.6). Now we prove the converse implication to finish the proof. Suppose (3.6) holds. By Theorem 2.6 and the Cauchy-Schwarz inequality, we have

$$\sum_{l=1}^{L}\Theta_a\varphi_l\overline{\Theta_a\psi_l}\in L^\infty([1,a)\times(0,1)).$$

This implies that almost every point in $(1,a)\times(0,1)$ is a Lebesgue point of $\sum_{l=1}^{L}\Theta_a\varphi_l\overline{\Theta_a\psi_l}$. Arbitrarily fix such a point $(x_0,\xi_0)\in(1,a)\times(0,1)$, and take $f,\,g\in\mathfrak{D}$ in (3.6) such that

$$\Theta_a f=\Theta_a g=\frac{1}{\sqrt{|B((x_0,\xi_0),\varepsilon)|}}\chi_{B((x_0,\xi_0),\varepsilon)}$$

on $[1,a)\times[0,1)$ with $B((x_0,\xi_0),\varepsilon)\subset(1,a)\times(0,1)$ and $\varepsilon>0$, where $B((x_0,\xi_0),\varepsilon)$ denotes the ε-neighborhood of (x_0,ξ_0). Then by Lemma 2.3(iv), f and g are well defined and thus, we obtain that

$$\frac{1}{|B((x_0,\xi_0),\varepsilon)|}\int_{B((x_0,\xi_0),\varepsilon)}\sum_{l=1}^{L}\Theta_a\varphi_l(x,\xi)\overline{\Theta_a\psi_l(x,\xi)}dxd\xi=1. \tag{3.7}$$

Letting $\varepsilon\to0$ in (3.7) leads to

$$\sum_{l=1}^{L}\Theta_a\varphi_l(x_0,\xi_0)\overline{\Theta_a\psi_l(x_0,\xi_0)}=1.$$

This implies (3.4) by the arbitrariness of (x_0,ξ_0). This completes the proof. □

Now, we turn to the expression of $\mathcal{MD}$-duals. Let a be a fixed positive number greater than 1, $\Psi=\{\psi_1,\,\psi_2,\dots,\psi_L\}$ be a finite subset of $L^2(\mathbb{R}_+)$ with cardinality L, $\mathcal{MD}(\Psi,a)$ be a frame for $L^2(\mathbb{R}_+)$, and S be its frame operator. By Lemma 3.1, $S\Lambda_m D_{a^j}=\Lambda_m D_{a^j}S$, and thus $S^{-1}\Lambda_m D_{a^j}=\Lambda_m D_{a^j}S^{-1}$ for $m,\,j\in\mathbb{Z}$. So $\mathcal{MD}(\Psi,a)$ and its canonical dual $S^{-1}(\mathcal{MD}(\Psi,a))$ share the same dilation-and-modulation structure, i.e.,

$$S^{-1}(\mathcal{MD}(\Psi,a))=\mathcal{MD}(S^{-1}(\Psi),\,a).$$

The following theorem gives its canonical dual window and all $\mathcal{MD}$-dual windows in the Θ_a transform domain.

Theorem 3.7. *Let a be a fixed positive number greater than 1, $\Psi=\{\psi_1,\,\psi_2,\dots,\psi_L\}$ be a finite subset of $L^2(\mathbb{R}_+)$ with cardinality L, and $\mathcal{MD}(\Psi,a)$ be a frame for $L^2(\mathbb{R}_+)$. Then*

(i) *its canonical dual $\mathcal{MD}(S^{-1}(\Psi),\,a)$ is given by*

$$\Theta_a S^{-1}\psi_l(\cdot,\cdot)=\frac{\Theta_a\psi_l(\cdot,\cdot)}{\sum_{l=1}^{L}|\Theta_a\psi_l(\cdot,\cdot)|^2}\quad a.e.\ on\ \ [1,a)\times[0,1)\quad for\quad 1\leqslant l\leqslant L;$$

(ii) *a dilation-and-modulation system* $\mathcal{MD}(\Phi, a)$ *with* $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_L\}$ *is a dual frame of* $\mathcal{MD}(\Psi, a)$ *if and only if* Φ *is defined by*

$$\Theta_a \varphi_l(\cdot,\cdot) = \frac{\Theta_a \psi_l(\cdot,\cdot)(1 - \sum_{l=1}^L \overline{\Theta_a \psi_l(\cdot,\cdot)} X_l(\cdot,\cdot))}{\sum_{l=1}^L |\Theta_a \psi_l(\cdot,\cdot)|^2} + X_l(\cdot,\cdot) \quad \text{a.e. on } [1,a)\times[0,1), \tag{3.8}$$

where $X_l \in L^\infty([1,a)\times[0,1))$ *with* $1 \leqslant l \leqslant L$.

Proof. (i) Since S is an invertible and bounded operator on $L^2(\mathbb{R}_+)$, by Lemma 3.3, we have

$$\Theta_a f(\cdot,\cdot) = \left(\sum_{l=1}^L |\Theta_a \psi_l(\cdot,\cdot)|^2\right) \Theta_a S^{-1} f(\cdot,\cdot) \quad \text{for } f \in L^2(\mathbb{R}_+). \tag{3.9}$$

Replacing f by ψ_l in (3.9) with $1 \leqslant l \leqslant L$, we have (i).

(ii) For sufficiency, suppose Φ is given by (3.8). Then by Theorem 2.6, $\mathcal{MD}(\Phi, a)$ is a Bessel sequence in $L^2(\mathbb{R}_+)$. By a simple computation, we obtain

$$\sum_{l=1}^L \Theta_a \varphi_l(\cdot,\cdot)\overline{\Theta_a \psi_l(\cdot,\cdot)} = 1 \quad \text{a.e. on } [1,a)\times[0,1).$$

It follows by Theorem 3.6 that $\mathcal{MD}(\Phi, a)$ is a dual frame of $\mathcal{MD}(\Psi, a)$.

For necessity, suppose $\mathcal{MD}(\Phi, a)$ is a dual frame of $\mathcal{MD}(\Psi, a)$. Then by Theorem 3.6, we have

$$\sum_{l=1}^L \overline{\Theta_a \psi_l(\cdot,\cdot)}\Theta_a \varphi_l(\cdot,\cdot) = 1 \quad \text{a.e. on } [1,a)\times[0,1),$$

and $\Theta_a \varphi_l \in L^\infty([1,a)\times[0,1))$. So we have (3.8) with $X_l = \Theta_a \varphi_l, 1 \leqslant i \leqslant L$. This completes the proof. □

Remark 3.8. Theorem 3.7 gives us much flexibility in constructing $\mathcal{MD}$-duals. For example, suppose $\mathcal{MD}(\Psi, a)$ is a Parseval frame for $L^2(\mathbb{R}_+)$. Take X_l as a fixed function X in $L^\infty([1,a)\times[0,1))$ for all $1 \leqslant l \leqslant L$. Then, by Theorems 2.7 and 3.7(ii), we obtain a dual frame $\mathcal{MD}(\Phi, a)$ with $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_L\}$,

$$\Theta_a \varphi_l(\cdot,\cdot) = \Theta_a \psi_l(\cdot,\cdot)\left(1 - X(\cdot,\cdot)\sum_{l=1}^L \overline{\Theta_a \psi_l(\cdot,\cdot)}\right) + X(\cdot,\cdot) \quad \text{a.e. on } [1,a)\times[0,1). \tag{3.10}$$

We can obtain Φ with properties similar to Ψ by choosing good X. Section 4 will focus on some other examples.

The following theorem shows that the cardinality L of Ψ determines whether or not a frame $\mathcal{MD}(\Psi, a)$ is redundant. If $L = 1$, there exists no redundant frame $\mathcal{MD}(\Psi, a)$ for $L^2(\mathbb{R}_+)$. If $L > 1$, there exists no nonredundant frame $\mathcal{MD}(\Psi, a)$ for $L^2(\mathbb{R}_+)$.

Theorem 3.9. *Let* a *be a fixed positive number greater than* 1, $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\}$ *be a finite subset of* $L^2(\mathbb{R}_+)$ *with cardinality* L, *and* $\mathcal{MD}(\Psi, a)$ *be a frame for* $L^2(\mathbb{R}_+)$. *Then* $\mathcal{MD}(\Psi, a)$ *is a Riesz basis for* $L^2(\mathbb{R}_+)$ *if and only if* $L = 1$.

Proof. The necessity is an immediate consequence of Lemma 3.4. Now we show the sufficiency. Suppose $L = 1$. From the proof of Lemma 3.4, we have

$$S^{-\frac{1}{2}}(\mathcal{MD}(\Psi, a)) = \mathcal{MD}(S^{-\frac{1}{2}}(\Psi), a).$$

So $\mathcal{MD}(S^{-\frac{1}{2}}(\Psi), a)$ is a Parseval frame for $L^2(\mathbb{R}_+)$ since $\mathcal{MD}(\Psi, a)$ is a frame for $L^2(\mathbb{R}_+)$. This implies by Lemma 3.5 that $\mathcal{MD}(S^{-\frac{1}{2}}(\Psi), a)$ is an orthonormal basis. This is equivalent to the fact that $\mathcal{MD}(\Psi, a)$ is a Riesz basis for $L^2(\mathbb{R}_+)$. Hence, this completes the proof. □

Now we conclude this section with the following remark on fast-converging series expansion associated with $\mathcal{MD}$-frames.

Remark 3.10. Let $\mathcal{MD}(\Psi, a)$ and $\mathcal{MD}(\Phi, a)$ form a pair of dual frames for $L^2(\mathbb{R}_+)$. Then

$$f = \sum_{l=1}^{L} \sum_{m,j\in\mathbb{Z}} \langle f,\, \Lambda_m D_{a^j} \varphi_l \rangle_{L^2(\mathbb{R}_+)} \Lambda_m D_{a^j} \psi_l, \tag{3.11}$$

and

$$\left\| f - \sum_{l=1}^{L} \sum_{|m|,|j|\leqslant N} \langle f,\, \Lambda_m D_{a^j} \varphi_l \rangle_{L^2(\mathbb{R}_+)} \Lambda_m D_{a^j} \psi_l \right\|^2 \leqslant B \sum_{l=1}^{L} \sum_{|m| \text{ or } |j|>N} |\langle f,\, \Lambda_m D_{a^j} \varphi_l \rangle_{L^2(\mathbb{R}_+)}|^2 \tag{3.12}$$

for $f \in L^2(\mathbb{R}_+)$, where B is the Bessel bound of $\mathcal{MD}(\Psi, a)$. So the fast-converging series expansion reduces to a fast decay of $\{\langle f, \Lambda_m D_{a^j} \varphi_l \rangle_{L^2(\mathbb{R}_+)}\}_{m,j\in\mathbb{Z}}$ with $1 \leqslant l \leqslant L$. By Lemmas 2.3(i) and 2.3(iv), we can understand the functions in $L^2(\mathbb{R}_+)$ in the Θ_a-transform domain, and every $f \in L^2(\mathbb{R}_+)$ corresponds to the unique representation

$$\Theta_a f(x,\xi) = \sum_{m,\, j\in\mathbb{Z}} c_{m,j} e_{m,j}(x,\xi) \quad \text{for a.e.} \quad (x,\xi) \in [1,a) \times [0,1) \tag{3.13}$$

with $c \in l^2(\mathbb{Z}^2)$. Suppose

$$\Theta_a \varphi_l(x,\xi) = \sum_{m,\, j\in\mathbb{Z}} d_{l,m,j} e_{m,j}(x,\xi) \quad \text{for a.e.} \quad (x,\xi) \in [1,a) \times [0,1) \tag{3.14}$$

with $d_l \in l^2(\mathbb{Z}^2)$. Again using Lemma 2.3(iii), we obtain that

$$\langle f,\, \Lambda_m D_{a^j} \varphi_l \rangle_{L^2(\mathbb{R}_+)} = \langle \Theta_a f,\, e_{m,j} \Theta_a \varphi_l \rangle_{L^2([1,a)\times[0,1))} = \sum_{n,\, k\in\mathbb{Z}} c_{n,k} \overline{d_{l,n-m,k-j}}.$$

It is the convolution of c and $\{\overline{d_{l,-m,-j}}\}_{m,j\in\mathbb{Z}}$. This implies that $\{\langle f, \Lambda_m D_{a^j} \varphi_l \rangle_{L^2(\mathbb{R}_+)}\}_{m,j\in\mathbb{Z}}$ with $1 \leqslant l \leqslant L$ have rapid decay if c and d_l with $1 \leqslant l \leqslant L$ have rapid decay. Therefore, we conclude that, if d_l with $1 \leqslant l \leqslant L$ in (3.14) have rapid decay, then the series expansion (3.11) has fast convergence for $f \in L^2(\mathbb{R}_+)$ satisfying (3.13) with c having rapid decay.

4 Some examples

Theorems 2.6, 2.7 and 3.7 provide us with an easy method to construct $\mathcal{MD}$-dual frame pairs for $L^2(\mathbb{R}_+)$. They show that we can construct $\mathcal{MD}$-dual frame pairs for $L^2(\mathbb{R}_+)$ with good properties such as dual windows having bounded supports and certain smoothness. This section focuses on presenting some examples.

Example 4.1. Let c be a finitely supported sequence defined on $\mathbb{Z}$ such that its Fourier transform

$$\hat{c}(\xi) = \sum_{l\in\mathbb{Z}} c_l \mathrm{e}^{-2\pi \mathrm{i} l\xi}$$

has no zero on $[0,1)$. Define $\psi \in L^2(\mathbb{R}_+)$ by

$$\Theta_a \psi(x,\xi) = \hat{c}(\xi) \quad \text{for} \quad (x,\xi) \in [1,a) \times [0,1).$$

Then by the definition of Θ_a, we have that ψ is a step function and of bounded support and by Theorems 2.7 and 3.9, we also have that $\mathcal{MD}(\psi, a)$ is a Riesz basis for $L^2(\mathbb{R}_+)$. It follows by Theorems 3.7 and 3.9 that $\mathcal{MD}(\psi, a)$ has the unique $\mathcal{MD}$-dual window $S^{-1}\psi$ defined by

$$\Theta_a S^{-1}\psi(x,\xi) = \frac{1}{\sum_{l\in\mathbb{Z}} \overline{c_l} \mathrm{e}^{2\pi \mathrm{i} l\xi}} \quad \text{for} \quad (x,\xi) \in [1,a) \times [0,1).$$

Observe that, if at least two c_l's are nonzero, we have

$$\frac{1}{\sum_{l\in\mathbb{Z}} \overline{c_l}\mathrm{e}^{2\pi\mathrm{i}l\xi}} = \sum_{l\in\mathbb{Z}} d_l \mathrm{e}^{-2\pi\mathrm{i}l\xi}$$

with d being infinitely supported. Although ψ is of bounded support, it follows by the definition of Θ_a that the dual window $S^{-1}\psi$ is of unbounded support.

The following example shows that it is possible for us to obtain multi-window $\mathcal{MD}$-dual frame pairs for $L^2(\mathbb{R}_+)$ with each window of bounded support.

Example 4.2. For $L > 1$, let $m_1, m_2, \dots, m_L$ be trigonometric polynomials satisfying

$$|m_1(\xi)|^2 + |m_2(\xi)|^2 + \cdots + |m_L(\xi)|^2 = 1 \quad \text{for} \;\; \xi \in [0,1).$$

Define $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\}$ by

$$\Theta_a\psi_l(x,\xi) = m_l(\xi) \quad \text{for} \;\; (x,\xi) \in [1,a) \times [0,1).$$

Then $\mathcal{MD}(\Psi, a)$ is a frame for $L^2(\mathbb{R}_+)$, and every ψ_l is of bounded support by an argument similar to that of Example 4.1. Define $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_L\}$ by

$$\Theta_a\phi_l(x,\xi) = m_l(\xi)\left(1 - \sum_{l=1}^{L} \overline{m_l(\xi)} X_l(x,\xi)\right) + X_l(x,\xi) \quad \text{for a.e.} \;\; (x,\xi) \in [1,a) \times [0,1) \tag{4.1}$$

with $X_l \in L^\infty([1,a) \times [0,1))$. Then by Theorem 3.7, $\mathcal{MD}(\Psi, a)$ and $\mathcal{MD}(\Phi, a)$ form a pair of dual frames for $L^2(\mathbb{R}_+)$. Let

$$X_l(x,\xi) = \sum_{j\in\mathbb{Z}} d_{l,j}(x)\mathrm{e}^{-2\pi\mathrm{i}j\xi} \quad \text{for a.e.} \;\; (x,\xi) \in [1,a) \times [0,1). \tag{4.2}$$

If, in addition, we require that every $\{d_{l,j}(\cdot)\}_{j\in\mathbb{Z}}$ with $1 \leqslant l \leqslant L$ is a finitely supported sequence of functions on $[1,a)$, then each φ_l with $1 \leqslant l \leqslant L$ is of bounded support by (4.1) and the definition of Θ_a.

Example 4.3. For $L \geqslant 1$, let $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\}$ be a finite subset of $L^2(\mathbb{R}_+)$, and $\operatorname{supp}(\psi_l) \subset [1,a)$. Assume that

$$\sum_{l=1}^{L} |\psi_l(x)|^2 = 1 \quad \text{for a.e.} \;\; x \in [1,a).$$

Define $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_L\}$ by

$$\Theta_a\varphi_l(x,\xi) = \psi_l(x)\left(1 - \sum_{l=1}^{L} \overline{\psi_l(x)} X_l(x,\xi)\right) + X_l(x,\xi) \quad \text{for a.e.} \;\; (x,\xi) \in [1,a) \times [0,1) \tag{4.3}$$

with $X_l \in L^\infty([1,a) \times [0,1))$. Then by Theorem 3.7, $\mathcal{MD}(\Psi, a)$ and $\mathcal{MD}(\Phi, a)$ form a pair of dual frames for $L^2(\mathbb{R}_+)$. In particular, if every X_l with $1 \leqslant l \leqslant L$ is as in (4.2) with $\{d_{l,j}(\cdot)\}_{j\in\mathbb{Z}}$ being a finitely supported sequence of functions on $[1,a)$, then each φ_l with $1 \leqslant l \leqslant L$ is of bounded support.

In Examples 4.2 and 4.3, $\Theta_a\psi_l, 1 \leqslant l \leqslant L$, are defined by univariate functions. Next, we give a more general example.

Example 4.4. Assume that $c_0(x)$ and $c_1(x)$ are two real-valued measurable functions defined on $[1,a]$, and that there exist two positive constants A and B such that

$$A \leqslant |c_0(x)| + |c_1(x)| \leqslant B \quad \text{for} \;\; x \in [1,a].$$

Define $\Psi = \{\psi_1, \psi_2\} \subset L^2(\mathbb{R}_+)$ by

$$\Theta_a\psi_1(x,\xi) = c_0(x) + c_1(x)\mathrm{e}^{-4\pi\mathrm{i}\xi},$$

$$\Theta_a\psi_2(x,\xi)=\begin{cases}2\mathrm{i}\sqrt{c_0(x)c_1(x)}\sin 2\pi\xi, & \text{if } c_0(x)c_1(x)\geqslant 0,\\ 2\sqrt{-c_0(x)c_1(x)}\cos 2\pi\xi, & \text{if } c_0(x)c_1(x)<0\end{cases}$$

for $(x,\xi)\in[1,\,a]\times[0,1)$. Then by simple computation and the definition of Θ_a, we have

$$\psi_1(x)=\begin{cases}c_0(x), & \text{if } 1\leqslant x\leqslant a,\\ a^{-1}c_1(a^{-2}x), & \text{if } a^2\leqslant x\leqslant a^3,\\ 0, & \text{otherwise},\end{cases}$$

$$\psi_2(x)=\begin{cases}a^{\frac12}\sqrt{c_0(ax)c_1(ax)}, & \text{if } \ a^{-1}\leqslant x\leqslant 1,\\ -a^{-\frac12}\sqrt{c_0(a^{-1}x)c_1(a^{-1}x)}, & \text{if } \ a\leqslant x\leqslant a^2 \text{ and } c_0(a^{-1}x)c_1(a^{-1}x)\geqslant 0,\\ a^{-\frac12}\sqrt{-c_0(a^{-1}x)c_1(a^{-1}x)}, & \text{if } \ a\leqslant x\leqslant a^2 \text{ and } c_0(a^{-1}x)c_1(a^{-1}x)<0,\\ 0, & \text{otherwise},\end{cases}$$

and

$$|\Theta_a\psi_1(x,\xi)|^2+|\Theta_a\psi_2(x,\xi)|^2=(|c_0(x)|+|c_1(x)|)^2.$$

It follows that

$$A^2\leqslant|\Theta_a\psi_1(x,\xi)|^2+|\Theta_a\psi_2(x,\xi)|^2\leqslant B^2 \tag{4.4}$$

for a.e. $(x,\xi)\in[1,\,a]\times[0,1)$ and thus by Theorem 2.7, we have that $\mathcal{MD}(\Psi,a)$ is a frame for $L^2(\mathbb{R}_+)$. Obviously, ψ_1 and ψ_2 are real-valued and of bounded support.

Now we check the $\mathcal{MD}$-duals of $\mathcal{MD}(\Psi,a)$. Define $\Phi=\{\varphi_1,\varphi_2\}$ by

$$\Theta_a\varphi_l(x,\xi)=\frac{\Theta_a\psi_l(x,\xi)(1-\overline{\Theta_a\psi_1(x,\xi)}X_1(x,\xi)-\overline{\Theta_a\psi_2(x,\xi)}X_2(x,\xi))}{(|c_0(x)|+|c_1(x)|)^2}+X_l(x,\xi)$$

for $1\leqslant l\leqslant 2$ and a.e. $(x,\xi)\in[1,a]\times[0,1)$ with $X_1,X_2\in L^\infty([1,a]\times[0,1))$. Then by Theorem 3.7, we have that $\mathcal{MD}(\Psi,a)$ and $\mathcal{MD}(\Phi,a)$ form a pair of dual frames for $L^2(\mathbb{R}_+)$. If every X_l with $1\leqslant l\leqslant 2$ is as in (4.2) with $\{d_{l,j}(\cdot)\}_{j\in\mathbb{Z}}$ being a finitely supported sequence of real-valued functions on $[1,a]$, then φ_1 and φ_2 are also real-valued and of bounded support. Also we can obtain Φ with good smoothness by choosing good X_1 and X_2. For example, let us make further assumption that $c_0(x)$, $c_1(x)$ and $\sqrt{|c_0(x)c_1(x)|}$ are k-th continuously differentiable on $(1,a)$, and that $c_0(1)c_1(1)=c_0(a)c_1(a)=0$, $c_0(x)c_1(x)>0$ for $x\in(1,a)$. Then ψ_1 and ψ_2 are continuous on $\mathbb{R}_+$ and k-th continuously differentiable on $(1,a)\cup(a^2,\,a^3)$ and $(a^{-1},\,1)\cup(a,\,a^2)$, respectively. In this case, if we further require that $|c_0(x)|+|c_1(x)|$ is a constant on $[1,\,a]$, and $X_1(x,\xi)$ and $X_2(x,\xi)$ satisfy

$$X_1(x,\xi)=\sum_{j\in\mathbb{Z}}d_{1,j}\mathrm{e}^{-2\pi\mathrm{i}j\xi}$$

and

$$X_2(x,\xi)=\sum_{j\in\mathbb{Z}}d_{2,j}\mathrm{e}^{-2\pi\mathrm{i}j\xi},$$

respectively, for $\xi\in[0,1)$ with $\{d_{1,j}\}_{j\in\mathbb{Z}}$ and $\{d_{2,j}\}_{j\in\mathbb{Z}}$ being two finitely supported real number sequences. Then φ_1 and φ_2 are real-valued and of bounded support, and have the same continuity and differentiability as ψ_1 and ψ_2.

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