SCIENCE CHINA
Mathematics

• ARTICLES •

March 2021 Vol. 64 No. 3: 507–518
https://doi.org/10.1007/s11425-018-9461-8

Diffusion and mixing in fluid flow via the resolvent estimate

Dongyi Wei[1,2]

[1]*School of Mathematical Sciences, Peking University, Beijing* 100871*, China;*
[2]*Beijing International Center for Mathematical Research, Peking University, Beijing* 100871*, China*

Email: jnwdyi@163.com

Received October 25, 2018; accepted January 7, 2019; published online August 28, 2019

Abstract In this paper, we first present a Gearhart-Prüss type theorem with a sharp bound for m-accretive operators. Then we give two applications: (1) we give a simple proof of the result proved by Constantin et al. on relaxation enhancement induced by incompressible flows; (2) we show that shear flows with a class of Weierstrass functions obey logarithmically fast dissipation time-scales.

Keywords mixing, resolvent estimate, shear flows, Weierstrass functions

MSC(2010) 35Q35

Citation: Wei D Y. Diffusion and mixing in fluid flow via the resolvent estimate. Sci China Math, 2021, 64: 507–518, https://doi.org/10.1007/s11425-018-9461-8

1 Introduction

Let X be a complex Hilbert space. We denote by $\|\cdot\|$ the norm and by $\langle\cdot,\cdot\rangle$ the inner product. Let H be a linear operator in X with the domain $D(H)$. The Hille-Yosida theorem gives a necessary and sufficient condition so that H generates a strongly continuous semigroup $S(t)=\mathrm{e}^{tH}$. We say that $S(t)$ satisfies $P(M,\omega)$ if

$$\|S(t)\|\leqslant M\mathrm{e}^{\omega t},\quad t\geqslant 0.$$

Theorem 1.1 (Hille-Yosida theorem)**.** *Let H be a linear operator in X with the domain $D(H)$. Let $\omega\in\mathbb{R}, M>0$. Then H generates a strongly continuous semigroup $S(t)=\mathrm{e}^{tH}$ satisfying $P(M,\omega)$ if and only if*

1. *H is closed and $D(H)$ is dense in X;*
2. *for all $\lambda>\omega$, λ belongs to the resolvent set $\rho(H)$ of H, and for all positive integers n,*

$$\|(\lambda-H)^{-n}\|\leqslant\frac{M}{(\lambda-\omega)^n}.$$

However, it is not easy to check the second property for all powers of the resolvent. In the special case of $M=1,\omega=0$ (i.e., $S(t)$ is a contraction semigroup), it is enough to check that for any $\lambda>0$,

$$\|(\lambda-H)^{-1}\|\leqslant\frac{1}{\lambda}.$$

Gearhart-Prüss theorem gives the semigroup bound via the resolvent estimate.

Theorem 1.2 (Gearhart-Prüss theorem). *Let H be a closed operator with a dense domain $D(H)$ generating a strongly continuous semigroup e^{tH}. Assume that $\|(z-H)^{-1}\|$ is uniformly bounded for $\mathrm{Re}\, z \geqslant \omega$. Then there exists $M > 0$ so that e^{tH} satisfies $P(M,\omega)$.*

Let us refer to [7] for more introductions.

Recently, Helffer and Sjöstrand [10] presented a quantitative version of Gearhart-Prüss theorem and gave some interesting applications to the complex Airy operator, complex harmonic oscillator and Fokker-Planck operator. Motivated by their work, we first present a Gearhart-Prüss type theorem with a sharp bound for m-accretive operators. A closed operator H in a Hilbert space X is called m-accretive if the left open half-plane is contained in the resolvent set $\rho(H)$ with

$$(H+\lambda)^{-1} \in \mathcal{B}(X), \quad \|(H+\lambda)^{-1}\| \leqslant (\mathrm{Re}\lambda)^{-1} \quad \text{for } \mathrm{Re}\lambda > 0.$$

Here $\mathcal{B}(X)$ is the set of bounded linear operators on X. An m-accretive operator H is accretive and densely defined (see [12, Section V-3.10]), i.e., $D(H)$ is dense in X and $\mathrm{Re}\langle Hf, f\rangle \geqslant 0$ for $f \in D(H)$, and $-H$ is a generator of a semigroup e^{-tH}. We denote

$$\Psi(H) = \inf\{\|(H - \mathrm{i}\lambda)f\| : f \in D(H),\ \lambda \in \mathbb{R},\ \|f\| = 1\}.$$

Let us state the following Gearhart-Prüss type theorem for accretive operators.

Theorem 1.3. *Let H be an m-accretive operator in a Hilbert space X. Then we have $\|\mathrm{e}^{-tH}\| \leqslant \mathrm{e}^{-t\Psi(H)+\pi/2}$ for all $t \geqslant 0$.*

We will give two applications of Theorem 1.3.

The first application of Theorem 1.3 is to give a simple proof of the result in [5] on relaxation enhancement induced by incompressible flows. More precisely, we consider the passive scalar equation

$$\phi_t^A(x,t) + Au \cdot \nabla \phi^A(x,t) - \Delta \phi^A(x,t) = 0, \quad \phi(x,0) = \phi_0(x), \tag{1.1}$$

in a smooth compact d-dimensional Riemannian manifold M. Here Δ is the Laplace-Beltrami operator on M, and u is a divergence free vector field. Roughly speaking, a velocity field u is relaxation-enhancing if by the diffusive time-scale $O(1)$ arbitrarily much energy is already dissipated for A large enough. The main result of [5] characterizes relaxation-enhancing flows in terms of the spectral properties of the operator $u \cdot \nabla$. Precisely, u is relaxation-enhancing if and only if the operator $u \cdot \nabla$ has no nontrivial eigenfunctions in $\dot{H}^1(M)$. The proof of this result is based on the so-called RAGE theorem. Thanks to Theorem 1.3, the proof of relaxation-enhancing can be reduced to the resolvent estimate of the operator $Au \cdot \nabla - \Delta$, avoiding the use of the RAGE theorem. See Section 3 for more details.

For $A > 0$, by time rescaling $\tau = At$, (1.1) becomes

$$\phi_\tau + u \cdot \nabla \phi - \nu \Delta \phi = 0, \quad \phi(x,0) = \phi_0(x), \tag{1.2}$$

with $\nu = 1/A$. Our second application focuses on the case when $M = \mathbb{T}^2$ and u is a shear flow. More precisely, we study the decay estimates in time of the linear evolution semigroup $S_\nu(t) : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$, with $\nu > 0$ a positive parameter, generated by the drift-diffusion scalar equation

$$\partial_t f - \nu \Delta f + u(y)\partial_x f = 0, \quad (x,y) \in \mathbb{T}^2, \quad t > 0, \tag{1.3}$$

and of its hypoelliptic counterpart $R_\nu(t) : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$, generated by

$$\partial_t f - \nu \partial_y^2 f + u(y)\partial_x f = 0, \quad (x,y) \in \mathbb{T}^2, \quad t > 0. \tag{1.4}$$

The case of general shear flows $(u(y), 0)$ with a finite number of critical points was treated in [2], where the enhanced dissipation time-scale was proved to be $O(\nu^{-p})$, for $p = p(n_0) = \frac{n_0+1}{n_0+3} \geqslant \frac{1}{3}$, where n_0 denotes the maximal order of vanishing of u' at the critical points. The proof there used the hypocoercivity method in [15]. The reader may also see [4] for an interesting application of the enhanced dissipation and related work [13].

Our proof is based on Theorem 1.3. To this end, we study the resolvent estimate of $H = -\partial_y^2 + \mathrm{i}u(y)$ with $X = L^2(\mathbb{T})$ and $D(H) = H^2(\mathbb{T})$ in Section 4. Set $\Psi_1(u) = \Psi(H)$ and

$$\Psi_0(u) = \inf\{\|Hf\|_{L^2} : f \in D(H),\ \|f\|_{L^2} = 1\}.$$

Then $\Psi_1(u) = \inf_{\lambda\in\mathbb{R}} \Psi_0(u-\lambda)$. We will give a lower bound of $\Psi_0(u)$ and $\Psi_1(u)$ in terms of the following quantities:

$$\omega_0(\delta, u) = \inf_{x,c\in\mathbb{R}} \int_{x-\delta}^{x+\delta} |\psi(y) - c|^2 dy,$$
$$\omega_1(\delta, u) = \inf_{c\in\mathbb{R}} \omega_0(\delta, u-c) = \inf_{x,c_1,c_2\in\mathbb{R}} \int_{x-\delta}^{x+\delta} |\psi(y) - c_1 - c_2 y|^2 dy.$$

Here $u(y) = \psi'(y)$ and we identify a function on $\mathbb{T}$ with a 2π-periodic function on $\mathbb{R}$. Notice that $\psi(y)$ may not be a periodic function. The quantities ω_0 and ω_1 are well defined since their values do not change if ψ is added by a constant. We will give the dependence of p on u via the quantity $\omega_1(\delta, u)$. More precisely, if $\omega_1(\delta, u) \geqslant C_1\delta^{2\alpha+3}$ for $\delta \in (0, 1)$ and some constants $\alpha > 0$ and $C_1 > 0$, then the enhanced dissipation time-scale is $O(\nu^{-\frac{\alpha}{\alpha+2}})$.

To show the effectiveness of this criterion, we will discuss the case when $u(y) = \sum_{n=1}^{\infty} a_n \sin(3^n y)$ is a Weierstrass function. Our result is stated as follows.

- If $a_n \in \mathbb{R}$, $3^{-n\alpha} \leqslant |a_n| \leqslant C3^{-n\alpha}$ for some $\alpha \in (0, 1)$, and $1 \leqslant |a_n|/|a_{n+1}| \leqslant 3$, then the enhanced dissipation time-scale is $O(\nu^{-\frac{\alpha}{\alpha+2}})$.
- If $a_n \in \mathbb{R}$, $n^{-\alpha} \leqslant |a_n| \leqslant Cn^{-\alpha}$ for some $\alpha \in (1, 2)$, and $1 \leqslant |a_n|/|a_{n+1}| \leqslant 3$, then the enhanced dissipation time-scale is $O(|\ln\nu|^{\alpha})$.

In fact the quantity $\omega_1(\delta, u)$ describes quantitatively the degeneracy of the critical point, and it is also related with the regularity of the shear flow. If the shear flow is smooth, the best possible value should be $\alpha = 1$, and the rate can become worse depending on the order of the degeneracy of the critical point: $\alpha = n_0 + 1$. The example of the Weierstrass function shows the stronger dissipation can be brought by nonsmooth shear flows. A key point is that we can give the lower bound of $\omega_1(\delta, u)$ in terms of a_n, and then the stronger dissipation follows from slower decay of a_n. See Section 5 for more details.

Recently, Coti Zelati et al. [6] proved the enhanced dissipation time-scale $O(|\ln\nu|^2)$ in (1.2) for all contact Anosov flows on a smooth $2d + 1$ dimensional connected compact Riemannian manifold. Their proof is based on the knowledge of mixing decay rates. Our result shows that a similar phenomenon also happens for a class of shear flows.

Let us also mention some important progress on the enhanced dissipation of the linearized Navier-Stokes equations around shear flows such as Couette flow and Kolmogorov flow [1, 3, 9, 11, 14, 16, 17].

Throughout this paper, we denote by C a constant independent of A, ν, t, which may be different from line to line.

2 Proof of Gearhart-Prüss type theorem

In this section, we prove Theorem 1.3. The proof is partially motivated by [10].

Proof of Theorem 1.3. Let $\Psi = \Psi(H)$. Since $D(H)$ is dense in X, we only need to prove that

$$\|\mathrm{e}^{-tH} f\| \leqslant \mathrm{e}^{-t\Psi+\pi/2}\|f\|, \quad \forall f \in D(H),\ t \geqslant 0. \tag{2.1}$$

For $f \in D(H)$, $t \geqslant 0$, let $g(t) = \|\mathrm{e}^{-tH} f\|^2$. Since H is accretive, $g(t)$ is decreasing for $t \geqslant 0$, and we only need to prove (2.1) for $t\Psi > \pi/2$. In this case, $\Psi > 0$. We denote

$$t_1 = \frac{\pi}{4\Psi}, \quad t_2 = t - \frac{\pi}{4\Psi}, \quad t_3 = t + \frac{\pi}{4\Psi}, \quad l = t + \frac{\pi}{2\Psi}.$$

For $\chi \in C^1[0, l]$, $\chi(0) = \chi(l) = 0$, set $f_1(s) = \chi(s)\mathrm{e}^{-sH} f$ and $f_2(s) = \chi'(s)\mathrm{e}^{-sH} f$. Then $\partial_t f_1 + Hf_1 = f_2$ in $[0, l]$. Take Fourier transform in t: $\widehat{f_j}(\lambda) = \int_0^l f_j(s)\mathrm{e}^{-\mathrm{i}\lambda s} ds$, for $j = 1, 2$, $\lambda \in \mathbb{R}$. Then $\widehat{f_2}(\lambda) =$

$(\mathrm{i}\lambda + H)\widehat{f_1}(\lambda)$. By the definition of Ψ, we have $\|\widehat{f_2}(\lambda)\| \geqslant \Psi\|\widehat{f_1}(\lambda)\|$. We use Plancherel's theorem to conclude

$$\|f_2\|_{L^2([0,l],X)} = (2\pi)^{-\frac{1}{2}}\|\widehat{f_2}\|_{L^2(\mathbb{R},X)} \geqslant (2\pi)^{-\frac{1}{2}}\Psi\|\widehat{f_1}\|_{L^2(\mathbb{R},X)} = \Psi\|f_1\|_{L^2([0,l],X)}.$$

By the definitions of f_1, f_2 and g, the above inequality becomes

$$\int_0^l \chi'(s)^2 g(s)ds \geqslant \Psi^2 \int_0^l \chi(s)^2 g(s)ds.$$

Now we choose χ as follows,

$$\chi(s) = \begin{cases} \sin \Psi s, & 0 \leqslant s \leqslant t_1, \\ \mathrm{e}^{\Psi s - \pi/4}/\sqrt{2}, & t_1 \leqslant s \leqslant t_2, \\ \mathrm{e}^{\Psi l - \pi} \sin(\Psi(l-s)), & t_2 \leqslant s \leqslant l. \end{cases}$$

Set $h(s) = \chi'(s)^2 - \Psi^2\chi(s)^2$. Then $\int_0^l h(s)g(s)ds \geqslant 0$, and

$$h(s) = \begin{cases} \Psi^2 \cos(2\Psi s), & 0 \leqslant s \leqslant t_1, \\ 0, & t_1 \leqslant s \leqslant t_2, \\ \Psi^2 \mathrm{e}^{2\Psi l - 2\pi} \cos(2\Psi(l-s)), & t_2 \leqslant s \leqslant l. \end{cases}$$

Therefore, $h(s) \geqslant 0$ for $0 \leqslant s \leqslant t_1$ or $t_3 \leqslant s \leqslant l$, $h(s) \leqslant 0$ for $t_2 \leqslant s \leqslant t_3$. Since g is decreasing, we have $h(s)g(s) \leqslant h(s)g(0)$ for $0 \leqslant s \leqslant t_1$, $h(s)g(s) \leqslant h(s)g(t)$ for $t_2 \leqslant s \leqslant t$, $h(s)g(s) \leqslant h(s)g(t_3)$ for $t \leqslant s \leqslant l$, and

$$\begin{aligned} 0 &\leqslant \int_0^l h(s)g(s)ds = \int_0^{t_1} h(s)g(s)ds + \int_{t_2}^l h(s)g(s)ds \\ &\leqslant \int_0^{t_1} h(s)g(0)ds + \int_{t_2}^t h(s)g(t)ds + \int_t^l h(s)g(t_3)ds \\ &= \frac{\Psi}{2}g(0) - \frac{\Psi}{2}\mathrm{e}^{2\Psi l - 2\pi}g(t) + 0. \end{aligned}$$

Therefore, $g(t) \leqslant \mathrm{e}^{-2\Psi l + 2\pi}g(0)$, which implies that

$$\|\mathrm{e}^{-tH}f\| \leqslant \mathrm{e}^{-\Psi l + \pi}\|f\| = \mathrm{e}^{-\Psi t + \pi/2}\|f\|.$$

This completes the proof. □

3 Diffusion and mixing in fluid flows

Let us recall the following definition from [5].

Definition 3.1. Let M be a smooth compact Riemannian manifold. The incompressible flow u on M is called relaxation-enhancing if for every $\tau > 0$ and $\delta > 0$, there exists $A(\tau, \delta)$ such that for any $A > A(\tau, \delta)$ and any $\phi_0 \in L^2(M)$, $\|\phi_0\|_{L^2(M)} = 1$,

$$\|\phi^A(\cdot, \tau) - \overline{\phi}\|_{L^2(M)} < \delta,$$

where $\phi^A(x,t)$ is the solution of (1.1) and $\overline{\phi}$ the average of ϕ_0.

We take $X = \{f \in L^2(M) : \int_M f = 0\}$ to be the subspace of mean zero functions, and $H = H_A = -\Delta + Au \cdot \nabla$ with $D(H) = H^2(M) \cap X$. Set $\Psi_2(A) = \Psi(H_A)$. Our result is as follows.

Theorem 3.2. *Let M be a smooth compact Riemannian manifold. A continuous incompressible flow u is relaxation-enhancing if and only if the operator $u \cdot \nabla$ has no eigenfunctions in $H^1(M)$, other than the constant function.*

Proof. The proof of the first part is the same as that in [5]. First of all, we have

$$\partial_t\|\phi^A(t)\|_{L^2}^2 = \langle \phi_t^A, \phi^A\rangle + \langle \phi^A, \phi_t^A\rangle = -2\|\nabla\phi^A(t)\|_{L^2}^2.$$

If $u\cdot\nabla$ has nonconstant eigenfunctions in $H^1(M)$, then $u\cdot\nabla$ has nonconstant eigenfunctions in $H^1(M)\cap X$. Assume that the initial datum $\phi_0 \in H^1(M)\cap X$ for (1.1) is an eigenvector of $u\cdot\nabla$ corresponding to an eigenvalue $\mathrm{i}\lambda$, normalized so that $\|\phi_0\|_{L^2} = 1$, and then $\overline{\phi} = 0$, $\lambda\in\mathbb{R}$. Taking the inner product of (1.1) with ϕ_0, we arrive at

$$\partial_t\langle\phi^A(t), \phi_0\rangle = -\mathrm{i}A\lambda\langle\phi^A(t), \phi_0\rangle + \langle\Delta\phi^A(t), \phi_0\rangle.$$

This along with the assumption $\phi_0\in H^1(M)$ leads to

$$|\partial_t(\mathrm{e}^{\mathrm{i}A\lambda t}\langle\phi^A(t), \phi_0\rangle)| = |\langle\nabla\phi^A(t), \nabla\phi_0\rangle| \leqslant \frac{1}{2}(\|\nabla\phi^A(t)\|_{L^2}^2 + \|\nabla\phi_0\|_{L^2}^2).$$

Note that $\int_0^\tau \|\nabla\phi^A(t)\|_{L^2}^2 dt = (\|\phi_0\|_{L^2}^2 - \|\phi^A(t)\|_{L^2}^2)/2 \leqslant 1/2$. Then for $0 < t \leqslant \tau = (2\|\nabla\phi_0\|_{L^2}^2)^{-1}$, we have $|\langle\phi^A(t), \phi_0\rangle| \geqslant 1/2$. Thus, $\|\phi^A(\tau)\|_{L^2} \geqslant 1/2$ uniformly in A, and u is not relaxation-enhancing.

Now we prove the converse, we first claim that $\lim_{A\to+\infty}\Psi_2(A) = +\infty$ implies relaxation-enhancing. In fact, since $\phi^A(\cdot,\tau) - \overline{\phi}\in X$ and $\phi^A(\cdot,\tau) - \overline{\phi} = \mathrm{e}^{-\tau H_A}(\phi_0 - \overline{\phi})$, by Theorem 1.3, we have

$$\|\phi^A(\cdot,\tau) - \overline{\phi}\|_{L^2(M)} \leqslant \mathrm{e}^{-\tau\Psi_2(A)+\pi/2}\|\phi_0 - \overline{\phi}\|_{L^2(M)} \leqslant \mathrm{e}^{-\tau\Psi_2(A)+\pi/2}.$$

If $\lim_{A\to+\infty}\Psi_2(A) = +\infty$, then we can find $A(\tau,\delta)$ such that for any $A > A(\tau,\delta)$, we have $\Psi_2(A) > (\pi/2 - \ln\delta)/\tau$, and thus $\|\phi^A(\cdot,\tau) - \overline{\phi}\|_{L^2(M)} < \delta$.

Next, we claim that $\liminf_{A\to+\infty}\Psi_2(A) < +\infty$ implies that $u\cdot\nabla$ has a nonzero eigenfunction in $H^1(M)\cap X$. In fact, in this case, there exist $A_n\to+\infty$ and $C_0\in\mathbb{R}$ such that $\Psi_2(A_n) < C_0$ and there exist $\lambda_n\in\mathbb{R}$ and $f_n\in X$ such that $\|f_n\|_{L^2(M)} = 1$ and $\|(H_{A_n} - \mathrm{i}\lambda_n)f_n\|_{L^2(M)} < C_0$. Then

$$\|\nabla f_n\|_{L^2(M)}^2 = \mathrm{Re}\langle f_n, g_n\rangle \leqslant \|f_n\|_{L^2(M)}\|g_n\|_{L^2(M)} < C_0,$$

where $g_n = (H_{A_n} - \mathrm{i}\lambda_n)f_n$. Thus, the sequence $\{f_n\}$ is bounded in $H^1(M)$ and there exist a subsequence of $\{f_n\}$ (still denoted by $\{f_n\}$) and $f_0\in H^1(M)$, such that $f_n\to f_0$ strongly in $L^2(M)$. Therefore, $\|f_0\|_{L^2(M)} = 1$, $f_0\in X$. For $f\in H^1(M)$, we have

$$\langle g_n, f\rangle = \langle\nabla f_n, \nabla f\rangle + A_n\langle u\cdot\nabla f_n, f\rangle - \mathrm{i}\lambda_n\langle f_n, f\rangle,$$

and

$$\langle u\cdot\nabla f_n, f\rangle - \mathrm{i}\frac{\lambda_n}{A_n}\langle f_n, f\rangle = \frac{\langle g_n, f\rangle - \langle\nabla f_n, \nabla f\rangle}{A_n} \to 0,$$

as $n\to+\infty$, where we used $A_n\to+\infty$ and

$$\begin{aligned}|\langle g_n, f\rangle - \langle\nabla f_n, \nabla f\rangle| &\leqslant \|g_n\|_{L^2(M)}\|f\|_{L^2(M)} + \|\nabla f_n\|_{L^2(M)}\|\nabla f\|_{L^2(M)}\\ &\leqslant C_0\|f\|_{L^2(M)} + C_0^{\frac{1}{2}}\|\nabla f\|_{L^2(M)}.\end{aligned}$$

Moreover,

$$\langle u\cdot\nabla f_n, f\rangle = -\langle f_n, u\cdot\nabla f\rangle \to -\langle f_0, u\cdot\nabla f\rangle = \langle u\cdot\nabla f_0, f\rangle,$$

and $\langle f_n, f\rangle\to\langle f_0, f\rangle$ as $n\to+\infty$. Therefore,

$$\lim_{n\to+\infty}\mathrm{i}\frac{\lambda_n}{A_n}\langle f_n, f\rangle = \langle u\cdot\nabla f_0, f\rangle.$$

If we take $f = f_0$ then we have $\langle f_n, f\rangle\to\langle f_0, f\rangle = \langle f_0, f_0\rangle = 1\neq 0$ and $\mathrm{i}\frac{\lambda_n}{A_n}\to\langle u\cdot\nabla f_0, f_0\rangle \stackrel{\text{def}}{=} \mathrm{i}\lambda$, as $n\to+\infty$. Therefore, for every $f\in H^1(M)$, we have

$$\mathrm{i}\lambda\langle f_0, f\rangle = \lim_{n\to+\infty}\mathrm{i}\frac{\lambda_n}{A_n}\langle f_n, f\rangle = \langle u\cdot\nabla f_0, f\rangle.$$

Since $H^1(M)$ is dense in $L^2(M)$, we have $\mathrm{i}\lambda f_0 = u\cdot\nabla f_0$, $f_0\neq 0$. This proves our claim.

With the above two claims, we prove the converse part. □

Compared with [5], we do not need to assume u to be Lipschitz continuous and our proof is easier. As in [5], with a slight modification, we can prove a more general result. Let Γ be a self-adjoint, positive, unbounded operator with a discrete spectrum on a separable Hilbert space H: Let $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ be the eigenvalues of Γ, and e_j the corresponding orthonormal eigenvectors forming a basis in H. The (homogeneous) Sobolev space $H^m(\Gamma)$ associated with Γ is formed by all vectors $\psi = \sum_j c_j e_j$ such that $\|\psi\|_m^2 = \sum_j \lambda_j^m |c_j|^2 < +\infty$. Note that $H^2(\Gamma) = D(\Gamma)$. Let L be a symmetric operator such that $H^1(\Gamma) \subseteq D(L)$ and $\|L\psi\|_0 \leqslant C\|\psi\|_1$ for $\psi \in H^1(\Gamma)$. Consider a solution $\phi^A(t)$ of the Bochner differential equation

$$\partial_t \phi^A(t) = \mathrm{i}AL\phi^A(t) - \Gamma\phi^A(t), \quad \phi^A(0) = \phi_0. \tag{3.1}$$

Theorem 3.3. *For Γ and L satisfying the above conditions, the following two statements are equivalent*:

- *For every $\tau > 0$ and $\delta > 0$, there exists $A(\tau, \delta)$ such that for any $A > A(\tau, \delta)$ and any $\phi_0 \in H$, $\|\phi_0\|_0 = 1$, the solution $\phi^A(t)$ of the equation* (3.1) *satisfies $\|\phi^A(\tau)\|_0 < \delta$.*
- *The operator L has no eigenvectors lying in $H^1(\Gamma)$.*

Here we do not assume that $\|\mathrm{e}^{\mathrm{i}Lt}\psi\|_1 \leqslant B(t)\|\psi\|_1$. Therefore, our result is applicable to the example given in [5]: $H = L^2(0,1)$, $\Gamma f(x) = \sum_n \mathrm{e}^{n^2} \widehat{f}(n)\mathrm{e}^{2\pi \mathrm{i}nx}$, and $Lf(x) = xf(x)$.

4 The resolvent estimate for the shear flows

In this section, we give the resolvent estimate of the operator $H = H_{(u)} = -\partial_y^2 + \mathrm{i}u(y)$. We start with a few basic observations concerning the operator $H_{(u)}$. As is well known, the operator $H_{(0)} = -\partial_y^2$ is self-adjoint in $L^2(\mathbb{T})$ with a compact resolvent, and its spectrum is a sequence of eigenvalues $\{\lambda_n^0\}_{n\in\mathbb{N}}$, where $\lambda_0^0 = 0$, $\lambda_{2n}^0 = \lambda_{2n-1}^0 = n^2$. By the classical perturbation theory [12], it follows that $H_{(u)}$ has a compact resolvent for any $u \in C(\mathbb{T}, \mathbb{R})$, and that its spectrum is again a sequence of eigenvalues $\{\lambda_n^{(u)}\}_{n\in\mathbb{N}}$, with $\mathrm{Re}(\lambda_n^{(u)}) \to +\infty$ as $n \to +\infty$.

Since $\mathrm{Re}\langle Hf, f\rangle_{L^2} = \|\partial_y f\|_{L^2}^2 \geqslant 0$ for $f \in D(H) = H^2(\mathbb{T})$, H is accretive, $\mathrm{Re}(\lambda_n^{(u)}) \geqslant 0$, and $\|(H+\lambda)u\|_{L^2}\|u\|_{L^2} \geqslant \mathrm{Re}\langle (H+\lambda)u, u\rangle_{L^2} \geqslant \mathrm{Re}\langle \lambda u, u\rangle_{L^2} = (\mathrm{Re}\lambda)\|u\|_{L^2}^2$, $\|(H+\lambda)u\|_{L^2} \geqslant (\mathrm{Re}\lambda)\|u\|_{L^2}$ for $\mathrm{Re}\lambda > 0$, which implies that H is m-accretive.

For $\lambda \in \mathbb{R}$, $H - \mathrm{i}\lambda$ is invertible if and only if

$$\inf\{\|(H - \mathrm{i}\lambda)f\|_{L^2} : f \in D(H),\ \|f\|_{L^2} = 1\} > 0.$$

If $\Psi(H) > 0$, then $H - \mathrm{i}\lambda$ is invertible for all $\lambda \in \mathbb{R}$, and

$$\Psi(H) = \Big(\sup_{\lambda\in\mathbb{R}} \|(H - \mathrm{i}\lambda)^{-1}\|\Big)^{-1} = \inf\{\|(H - \mathrm{i}\lambda)f\|_{L^2} : f \in D(H),\ \lambda \in \mathbb{R},\ \|f\|_{L^2} = 1\}.$$

Thus, our definition of $\Psi(H)$ is the same as that in [8]. We first give a lower bound of $\Psi_0(u)$ in terms of $\omega_0(\delta, u)$. Then the lower bound of $\Psi_1(u)$ follows by minimizing λ. Recall that

$$\Psi_0(u) = \inf\{\|Hf\|_{L^2} : f \in D(H),\ \|f\|_{L^2} = 1\}.$$

The following lemma shows the existence of the minimizer.

Lemma 4.1. *If $\mu = \Psi_0(u)$, then there exists $0 \neq f \in D(H)$ so that $Hf = \mu\overline{f}$.*

Proof. By the definition of $\mu = \Psi_0(u)$, we have $\|Hg\|_{L^2} \geqslant \mu\|g\|_{L^2}$ for all $g \in D(H)$, and we can take $f_n \in D(H)$ and $\|f_n\|_{L^2} = 1$ such that $\|Hf_n\|_{L^2} \to \mu$ as $n \to \infty$. Then the sequence $\{f_n\}$ is bounded in $H^2(\mathbb{T})$, and there exist a subsequence of $\{f_n\}$ (still denoted by $\{f_n\}$) and $f_0 \in D(H)$, such that $f_n \to f_0$ strongly in $L^2(\mathbb{T})$ and $Hf_n \rightharpoonup Hf_0$ weakly in $L^2(\mathbb{T})$ as $n \to \infty$. Therefore, $\|f_0\|_{L^2} = 1$, $\|Hf_0\|_{L^2} \leqslant \mu$. If $\mu = 0$, then we can take $f = f_0$. If $\mu > 0$, we have for all $g \in D(H)$, $t \in \mathbb{R}$, $\|Hf_0 + tHg\|_{L^2} \geqslant \mu\|f_0 + tg\|_{L^2}$, and the equality holds at $t = 0$, therefore,

$$0 = \frac{d}{dt}\bigg|_{t=0} (\|Hf_0 + tHg\|_{L^2}^2 - \mu^2\|f_0 + tg\|_{L^2}^2) = 2\mathrm{Re}\langle Hf_0, Hg\rangle - 2\mu^2\mathrm{Re}\langle f_0, g\rangle,$$

and we also have $0 = 2\mathrm{Re}\langle Hf_0, \mathrm{i}Hg\rangle - 2\mu^2\mathrm{Re}\langle f_0, \mathrm{i}g\rangle$. Thus, $2\langle Hf_0, Hg\rangle = 2\mu^2\langle f_0, g\rangle$. Set $Hf_0 = \mu g_0$, and then $\langle g_0, Hg\rangle = \mu\langle f_0, g\rangle$ for all $g \in D(H)$. This implies that $g_0 \in D(H^*)$, and $H^*g_0 = \mu f_0$. Here $H^* = -\partial_y^2 - \mathrm{i}u(y)$ and $D(H^*) = H^2(\mathbb{T})$. Therefore, $\overline{g_0} \in D(H)$ and $H\overline{g_0} = \mu\overline{f_0}$. Since $f_0 + \overline{g_0} \neq 0$ or $f_0 - \overline{g_0} \neq 0$, we can take $f = f_0 + \overline{g_0}$ or $\mathrm{i}(f_0 - \overline{g_0})$. □

Now we need to study the equation $Hf = \mu\overline{f}$. Set $u(y) = \psi'(y)$ and $\psi(y) \in \mathbb{R}$ for $y \in \mathbb{R}$. Now we can define $\omega_0(\delta, u)$ and $\omega_1(\delta, u)$ as in Section 1. Recall that

$$\omega_0(\delta, u) = \inf_{x,c\in\mathbb{R}} \int_{x-\delta}^{x+\delta} |\psi(y) - c|^2 dy.$$

Lemma 4.2. *If* $0 \neq f \in D(H)$, $Hf = \mu\overline{f}$, $\mu \geqslant 0$, $\delta > 0$, *then* $\sqrt{\mu} \geqslant \frac{\pi}{2\delta}$ *or* $36\sqrt{\mu}\tan(\sqrt{\mu}\delta) \geqslant \omega_0(\delta, u)$.

Proof. If $\mu = 0$, then $0 = \mathrm{Re}\langle Hf, f\rangle_{L^2} = \|\partial_y f\|_{L^2}^2$, f is constant, $u \equiv 0$, ψ is a constant, $\omega_0(\delta, u) = 0$, and the result is true. Now we assume $\mu > 0$.

In this case, we have $f \in C^2(\mathbb{T})$. We can normalize $\|f\|_{L^\infty} = 1$, and assume $|f(y_0)| = 1$ for some $y_0 \in \mathbb{R}$. Set $a = \sup\{y : f(y) = 0, y < y_0\}$ and $b = \inf\{y : f(y) = 0, y > y_0\}$. Then $-\infty \leqslant a < y_0 < b \leqslant +\infty$ and $f \neq 0$ in (a, b), $f(a) = 0$ if $a > -\infty$, $f(b) = 0$ if $b < +\infty$. Now we can find $g \in C^2(a, b)$ such that $f = \mathrm{e}^g$ in (a, b), and then $Hf = (-g'' - g'^2 + \mathrm{i}u)f$. Set $g = \rho + \mathrm{i}\theta$, $\rho, \theta \in \mathbb{R}$. Then $\overline{f} = \mathrm{e}^{-2\mathrm{i}\theta}f$ and the equation $Hf = \mu\overline{f}$ in (a, b) can be written as $-g'' - g'^2 + \mathrm{i}u = \mu\mathrm{e}^{-2\mathrm{i}\theta}$ or

$$-\rho'' - \rho'^2 + \theta'^2 = \mu\cos 2\theta, \qquad -\theta'' - 2\rho'\theta' + u = -\mu\sin 2\theta.$$

As $\|f\|_{L^\infty} = 1$, $|f(y_0)| = 1$, we have $\rho \leqslant 0$ in (a, b), $\rho(y_0) = 0$, and $\rho'(y_0) = 0$.

We first give the lower bounds of $y_0 - a$ and $b - y_0$. By the standard theory of ODEs, if $a > -\infty$, then $\lim_{y\to a_+} \rho(y) = -\infty$, while if $b < +\infty$, then $\lim_{y\to b_-} \rho(y) = -\infty$.

Since $\rho'' + \rho'^2 + \mu = \theta'^2 + 2\mu(\sin\theta)^2 \geqslant 0$, setting $\rho_1 = \arctan\frac{\rho'}{\sqrt{\mu}}$, we have $\rho_1(y_0) = 0$, $\frac{\rho_1'}{\sqrt{\mu}} + 1 = \frac{\rho''}{\rho'^2+\mu} + 1 \geqslant 0$, and $\rho_1(z) \geqslant \rho_1(y) + (y - z)\sqrt{\mu}$ for $a < y < z < b$.

For $y \in (a, b)$, if $b < y + (\rho_1(y) + \pi/2)/\sqrt{\mu}$, then

$$\inf_{(y,b)} \rho_1 \geqslant \rho_1(y) + (y - b)\sqrt{\mu} > -\frac{\pi}{2}, \quad \inf_{(y,b)} \rho' > -\infty, \quad \inf_{(y,b)} \rho > -\infty,$$

a contradiction. Therefore $b \geqslant y + (\rho_1(y) + \pi/2)/\sqrt{\mu}$. Similarly, $a \leqslant y + (\rho_1(y) - \pi/2)/\sqrt{\mu}$. In particular,

$$a \leqslant y_0 - \frac{\pi}{2\sqrt{\mu}} < y_0 + \frac{\pi}{2\sqrt{\mu}} \leqslant b.$$

Now we estimate $|\rho'(y)|$. For $y \in (a, b)$, we have $y_1 = y + \rho_1(y)/\sqrt{\mu} \in (a, b)$,

$$\begin{aligned} \rho(y_1) - \rho(y) &= \int_y^{y_1} \rho'(z)dz = \int_y^{y_1} \sqrt{\mu}\tan\rho_1(z)dz \\ &\geqslant \int_y^{y_1} \sqrt{\mu}\tan(\rho_1(y) + (y - z)\sqrt{\mu})dz \\ &= \ln\frac{1}{\cos\rho_1(y)}. \end{aligned}$$

Since $\rho(y_1) \leqslant 0$, we have $\mathrm{e}^{\rho(y)} \leqslant \cos\rho_1(y)$. On the other hand, if $|y - y_0| < \frac{\pi}{2\sqrt{\mu}}$, then

$$\begin{aligned} \rho(y_0) - \rho(y) &= \int_y^{y_0} \rho'(z)dz = \int_y^{y_0} \sqrt{\mu}\tan\rho_1(z)dz \\ &\leqslant \int_y^{y_0} \sqrt{\mu}\tan(\rho_1(y_0) + (y_0 - z)\sqrt{\mu})dz \\ &= \ln\frac{1}{\cos((y - y_0)\sqrt{\mu})}. \end{aligned}$$

Here we used $\rho_1(y_0)=\rho(y_0)=0$. Therefore, $\cos((y-y_0)\sqrt{\mu})\leqslant \mathrm{e}^{\rho(y)}\leqslant \cos\rho_1(y)$. Since $(y-y_0)\sqrt{\mu},\rho_1(y)\in(-\pi/2,\pi/2)$, we have $|y-y_0|\sqrt{\mu}\geqslant|\rho_1(y)|$ and $|\rho'(y)|=\sqrt{\mu}\tan|\rho_1(y)|\leqslant\sqrt{\mu}\tan(|y-y_0|\sqrt{\mu})$. Now if $\sqrt{\mu}<\frac{\pi}{2\delta}$, then $\delta<\frac{\pi}{2\sqrt{\mu}}$ and

$$\begin{aligned}\int_{y_0-\delta}^{y_0+\delta}|\theta'(z)|^2dz&\leqslant\int_{y_0-\delta}^{y_0+\delta}(\rho''+\rho'^2+\mu)dz\\&\leqslant\rho'|_{y_0-\delta}^{y_0+\delta}+\int_{y_0-\delta}^{y_0+\delta}(\mu\tan^2(|z-y_0|\sqrt{\mu})+\mu)dz\\&\leqslant 2\sqrt{\mu}\tan(\delta\sqrt{\mu})+2\sqrt{\mu}\tan(\delta\sqrt{\mu})\\&=4\sqrt{\mu}\tan(\delta\sqrt{\mu}).\end{aligned}$$

Here we used $|\rho'(y)|\leqslant\sqrt{\mu}\tan(|y-y_0|\sqrt{\mu})=\sqrt{\mu}\tan(\delta\sqrt{\mu})$ for $y=y_0\pm\delta$.

Now we estimate $\omega_0(\delta,u)$. Since $u(y)=\psi'(y)$, we have

$$-\theta''-2\rho'\theta'+\psi'=-\mu\sin 2\theta\quad\text{and}\quad\psi(y)-\theta'(y)-c=\int_{y_0}^{y}(2\rho'\theta'-\mu\sin 2\theta)dz$$

for $c=\psi(y_0)-\theta'(y_0)$. If $y_0<y<y_0+\delta$, then

$$\begin{aligned}|\psi(y)-\theta'(y)-c|&\leqslant\int_{y_0}^{y}\frac{2\rho'^2+\theta'^2+2\mu}{\sqrt{2}}dz\leqslant\int_{y_0}^{y}\frac{2\rho'^2+(\rho''+\rho'^2+\mu)+2\mu}{\sqrt{2}}dz\\&=\frac{\rho'|_{y_0}^{y}}{\sqrt{2}}+\frac{3}{\sqrt{2}}\int_{y_0}^{y}(\rho'^2+\mu)dz\leqslant\frac{\rho'(y)}{\sqrt{2}}+\frac{3}{\sqrt{2}}\int_{y_0}^{y}(\mu\tan^2(|z-y_0|\sqrt{\mu})+\mu)dz\\&\leqslant\sqrt{\mu/2}\tan((y-y_0)\sqrt{\mu})+3\sqrt{\mu/2}\tan((y-y_0)\sqrt{\mu})\\&=2\sqrt{2\mu}\tan((y-y_0)\sqrt{\mu}).\end{aligned}$$

Similarly, if $y_0-\delta<y<y_0$, then $|\psi(y)-\theta'(y)-c|\leqslant 2\sqrt{2\mu}\tan(|y-y_0|\sqrt{\mu})$. Therefore,

$$\begin{aligned}\omega_0(\delta,u)&\leqslant\int_{y_0-\delta}^{y_0+\delta}|\psi(y)-c|^2dy\leqslant 3\int_{y_0-\delta}^{y_0+\delta}|\theta'(z)|^2dz+\frac{3}{2}\int_{y_0-\delta}^{y_0+\delta}|\psi(y)-\theta'(y)-c|^2dy\\&\leqslant 3\cdot 4\sqrt{\mu}\tan(\delta\sqrt{\mu})+\frac{3}{2}\int_{y_0-\delta}^{y_0+\delta}8\mu\tan^2(|y-y_0|\sqrt{\mu})dy\\&=12\sqrt{\mu}\tan(\delta\sqrt{\mu})+24(\sqrt{\mu}\tan(\delta\sqrt{\mu})-\mu\delta)\\&\leqslant 36\sqrt{\mu}\tan(\delta\sqrt{\mu}).\end{aligned}$$

This completes the proof. □

Set $\varphi:[0,\pi/2)\to[0,+\infty)$, $\varphi(x)=36x\tan x$. Then φ is a one-to-one increasing function and we denote $\varphi^{-1}:[0,+\infty)\to[0,\pi/2)$ to be the inverse function.

Lemma 4.3. *For $\delta>0$, we have*

$$\Psi_0(u)\geqslant(\varphi^{-1}(\delta\omega_0(\delta,u))/\delta)^2,\quad\Psi_1(u)\geqslant(\varphi^{-1}(\delta\omega_1(\delta,u))/\delta)^2.$$

Proof. Let $\mu=\Psi_0(u)$. By Lemma 4.1, there exists $0\neq f\in D(H)$ such that $Hf=\mu\overline{f}$. By Lemma 4.2, we have $\sqrt{\mu}\geqslant\frac{\pi}{2\delta}$ or $36\sqrt{\mu}\tan(\sqrt{\mu}\delta)\geqslant\omega_0(\delta,u)$. Therefore, $\sqrt{\mu}\delta\geqslant\frac{\pi}{2}$ or $\varphi(\sqrt{\mu}\delta)=36\sqrt{\mu}\delta\tan(\sqrt{\mu}\delta)\geqslant\delta\omega_0(\delta,u)$.

Since $\varphi(x)=36x\tan x$ is a one-to-one increasing function, we have $\sqrt{\mu}\delta\geqslant\frac{\pi}{2}$ or $\sqrt{\mu}\delta\geqslant\varphi^{-1}(\delta\omega_0(\delta,u))$. As $\varphi^{-1}(\delta\omega_0(\delta,u))<\pi/2$, $\sqrt{\mu}\delta\geqslant\varphi^{-1}(\delta\omega_0(\delta,u))$ is always true, and $\sqrt{\mu}\geqslant\varphi^{-1}(\delta\omega_0(\delta,u))/\delta$, $\Psi_0(u)=(\sqrt{\mu})^2\geqslant(\varphi^{-1}(\delta\omega_0(\delta,u))/\delta)^2$. Now we have

$$\begin{aligned}\Psi_1(u)&=\inf_{\lambda\in\mathbb{R}}\Psi_0(u-\lambda)\geqslant\inf_{\lambda\in\mathbb{R}}(\varphi^{-1}(\delta\omega_0(\delta,u-\lambda))/\delta)^2\\&=\Big(\varphi^{-1}\Big(\delta\inf_{\lambda\in\mathbb{R}}\omega_0(\delta,u-\lambda)\Big)/\delta\Big)^2=(\varphi^{-1}(\delta\omega_1(\delta,u))/\delta)^2.\end{aligned}$$

This completes the proof. □

5 Enhanced dissipation for shear flows

As in [2], let $L_{k,\nu} = \mathrm{i}ku - \nu(\partial_y^2 - |k|^2)$ and $R_{k,\nu} = \mathrm{i}ku - \nu\partial_y^2$ be the linear operators associated with the k-th Fourier projections of (1.3) and (1.4), associated with the linear semigroups

$$\mathrm{e}^{-tL_{k,\nu}} = S_\nu(t)P_k, \quad \mathrm{e}^{-tR_{k,\nu}} = R_\nu(t)P_k.$$

For fixed ν and k, $L_{k,\nu}$ and $R_{k,\nu}$ are m-accretive. Notice that $R_{k,\nu} = \nu H_{(ku/\nu)}$ and $L_{k,\nu} = R_{k,\nu} + \nu|k|^2$. By Theorem 1.3, we have

$$\begin{aligned}\|\mathrm{e}^{-tL_{k,\nu}}\|_{L^2\to L^2} &= \|\mathrm{e}^{-\nu|k|^2t}\mathrm{e}^{-tR_{k,\nu}}\|_{L^2\to L^2} \leqslant \|\mathrm{e}^{-tR_{k,\nu}}\|_{L^2\to L^2} = \|\mathrm{e}^{-t\nu H_{(ku/\nu)}}\|_{L^2\to L^2}\\ &\leqslant \mathrm{e}^{-t\nu\Psi(H_{(ku/\nu)})+\pi/2} = \mathrm{e}^{-t\nu\Psi_1(ku/\nu)+\pi/2}, \quad \forall\, t \geqslant 0.\end{aligned}$$

Let us first give the decay rate in terms of $\omega_1(\delta, u)$.

Theorem 5.1. *For $\alpha > 0$, $u \in C(\mathbb{T}, \mathbb{R})$, assume that $\omega_1(\delta, u) \geqslant C_1\delta^{2\alpha+3}$ for $\delta \in (0, 1)$. Here C_1 is a positive constant. Then there exist positive constants ε and C such that for every $\nu > 0$ and every integer $k \neq 0$ satisfying $\nu|k|^{-1} \leqslant 1/2$,*

$$\|S_\nu(t)P_k\|_{L^2\to L^2} \leqslant C\mathrm{e}^{-\varepsilon\widetilde{\lambda}_{\nu,k}t}, \quad \|R_\nu(t)P_k\|_{L^2\to L^2} \leqslant C\mathrm{e}^{-\varepsilon\widetilde{\lambda}_{\nu,k}t}, \quad \forall\, t \geqslant 0, \tag{5.1}$$

where P_k denotes the projection to the k-th Fourier mode in x and $\widetilde{\lambda}_{\nu,k} = \nu^{\frac{\alpha}{\alpha+2}}|k|^{\frac{2}{\alpha+2}}$ is the decay rate.

Proof. By the definition, we have $\omega_1(\delta, u) \geqslant 0$ is increasing with respect to δ and homogeneous of degree 2 with respect to u, i.e., $\omega_1(\delta, Au) = A^2\omega_1(\delta, u)$ for every constant $A \in \mathbb{R}$. Since $\widetilde{\lambda}_{\nu,k} = \nu^{\frac{\alpha}{\alpha+2}}|k|^{\frac{2}{\alpha+2}}$, we take $\delta = (\nu/\widetilde{\lambda}_{\nu,k})^{1/2} = (\nu/|k|)^{\frac{1}{\alpha+2}} \in (0, 1)$. Then

$$\delta\omega_1(\delta, ku/\nu) = \delta(|k|/\nu)^2\omega_1(\delta, u) = \delta\delta^{-2(\alpha+2)}\omega_1(\delta, u) = \omega_1(\delta, u)/\delta^{2\alpha+3} \geqslant C_1.$$

By Lemma 4.3, for $\delta = (\nu/\widetilde{\lambda}_{\nu,k})^{1/2} > 0$, we have $\Psi_1(ku/\nu) \geqslant (\varphi^{-1}(\delta\omega_1(\delta, ku/\nu))/\delta)^2$, and

$$\begin{aligned}\nu\Psi_1(ku/\nu) &\geqslant \nu(\varphi^{-1}(\delta\omega_1(\delta, ku/\nu))/\delta)^2 \geqslant \nu(\varphi^{-1}(C_1)/\delta)^2\\ &= \nu(\varphi^{-1}(C_1))^2/((\nu/\widetilde{\lambda}_{\nu,k})^{1/2})^2 = (\varphi^{-1}(C_1))^2\widetilde{\lambda}_{\nu,k}.\end{aligned}$$

Thus,

$$\begin{aligned}\|S_\nu(t)P_k\|_{L^2\to L^2} &= \|\mathrm{e}^{-tL_{k,\nu}}\|_{L^2\to L^2} \leqslant \|R_\nu(t)P_k\|_{L^2\to L^2} = \|\mathrm{e}^{-tR_{k,\nu}}\|_{L^2\to L^2}\\ &\leqslant \mathrm{e}^{-t\nu\Psi_1(ku/\nu)+\pi/2} \leqslant \mathrm{e}^{-\varepsilon\widetilde{\lambda}_{\nu,k}t+\pi/2}, \quad \forall\, t \geqslant 0,\end{aligned}$$

where $\varepsilon = (\varphi^{-1}(C_1))^2 > 0$ is a constant. □

The following lemma gives the lower bound of $\omega_1(\delta, u)$ when $u(y)$ is a Weierstrass function.

Lemma 5.2. *If $u(y) = \sum_{n=1}^{\infty} a_n\sin(3^ny)$ is a Weierstrass function, $a_n \in \mathbb{R}\backslash\{0\}$, and $1 \leqslant |a_n|/|a_{n+1}| \leqslant 3$, $m \in \mathbb{Z}$, $m > 0$, then $\omega_1(3^{-m}\pi, u) \geqslant C^{-1}3^{-3m}a_m^2$.*

Proof. We can take $\psi(y) = -\sum_{n=1}^{\infty}\frac{a_n}{3^n}\cos(3^ny)$ such that $u(y) = \psi'(y)$. We introduce the difference operator $\triangle_hf(y) = f(y) - f(y+h)$ satisfying $\triangle_h^3f(y) = f(y) - 3f(y+h) + 3f(y+2h) - f(y+3h)$. Noticing that $\triangle_h^3(\mathrm{e}^{\mathrm{i}ny}) = \mathrm{e}^{\mathrm{i}ny}(1-\mathrm{e}^{\mathrm{i}nh})^3 = \mathrm{i}\mathrm{e}^{\mathrm{i}n(y+\frac{3}{2}h)}(2\sin\frac{nh}{2})^3$, we have $\triangle_h^3\cos(ny) = -\sin(n(y+\frac{3}{2}h))(2\sin\frac{nh}{2})^3$, $\triangle_h^3\psi(y) = \sum_{n=1}^{\infty}\frac{a_n}{3^n}\sin(3^n(y+\frac{3}{2}h))(2\sin\frac{3^nh}{2})^3$. Note that for $x, c_1, c_2 \in \mathbb{R}$, $h > 0$, we have $\triangle_h^3(\psi(y) - c_1 - c_2y) = \triangle_h^3\psi(y)$,

$$\int_{x-3h}^{x}(\triangle_h^3\psi(y))^2dy = \int_{x-3h}^{x}(\triangle_h^3(\psi(y) - c_1 - c_2y))^2dy \leqslant C\int_{x-3h}^{x+3h}(\psi(y) - c_1 - c_2y)^2dy,$$

and thus, $\inf_{x\in\mathbb{R}}\int_{x-3h}^{x}(\triangle_h^3\psi(y))^2dy \leqslant C\omega_1(3h, u)$. Now we take $h = 3^{-m-1}\pi$. Then we can write $\triangle_h^3\psi(y) = f_1(y) + f_2(y)$ with $f_1(y) = \sum_{n=1}^{m-1}\frac{a_n}{3^n}\sin(3^n(y+\frac{3}{2}h))(2\sin\frac{3^nh}{2})^3$ and

$$f_2(y) = \sum_{n=m}^{+\infty}\frac{a_n}{3^n}\sin\left(3^n\left(y+\frac{3}{2}h\right)\right)\left(2\sin\frac{3^nh}{2}\right)^3$$

$$= \sum_{n=m}^{+\infty} \frac{a_n}{3^n} \sin\left(3^n y + \frac{3^{n-m}\pi}{2}\right)\left(2\sin\frac{3^{n-m-1}\pi}{2}\right)^3$$
$$= \frac{a_m}{3^m}\cos(3^m y) - \sum_{n=m+1}^{+\infty} \frac{8a_n}{3^n}\cos(3^n y).$$

Thanks to $1 \leqslant |a_n|/|a_{n+1}| \leqslant 3$, we have $|a_n| \leqslant |a_m|3^{m-n}$ for $1 \leqslant n < m$ and

$$|f_1(y)| \leqslant \sum_{n=1}^{m-1} \frac{|a_n|}{3^n}\left|2\sin\frac{3^n h}{2}\right|^3 \leqslant \sum_{n=1}^{m-1}\frac{|a_n|}{3^n}|3^n h|^3 \leqslant \sum_{n=1}^{m-1}\frac{|a_m|3^{m-n}}{3^n}|3^n h|^3$$
$$= |a_m|3^m \sum_{n=1}^{m-1} 3^n h^3 \leqslant |a_m|3^m \frac{3^m h^3}{2} = |a_m|3^{2m}\frac{(3^{-m-1}\pi)^3}{2} = \frac{|a_m|}{3^m}\frac{\pi^3}{54}.$$

Notice that the functions $\{\cos(3^n y)\}_{n\in\mathbb{Z},n\geqslant m}$ are orthogonal in $L^2(x-3h,x) = L^2(x-3^{-m}\pi,x)$ for every $x\in\mathbb{R}$, and $|a_n| \geqslant |a_m|3^{m-n}$ for $n \geqslant m$. Then we have

$$\|f_2\|^2_{L^2(x-3h,x)} = \frac{|a_m|^2}{3^{2m}}\|\cos(3^m y)\|^2_{L^2(x-3h,x)} + \sum_{n=m+1}^{+\infty}\frac{(8|a_n|)^2}{3^{2n}}\|\cos(3^n y)\|^2_{L^2(x-3h,x)}$$
$$= \frac{|a_m|^2}{3^{2m}}\frac{3h}{2} + \sum_{n=m+1}^{+\infty}\frac{(8|a_n|)^2}{3^{2n}}\frac{3h}{2} \geqslant \frac{|a_m|^2}{3^{2m}}\frac{3h}{2} + \sum_{n=m+1}^{+\infty}\frac{(8|a_m|3^{m-n})^2}{3^{2n}}\frac{3h}{2}$$
$$= \frac{|a_m|^2}{3^{2m}}\frac{3h}{2}\left(1 + \sum_{n=m+1}^{+\infty}\frac{8^2}{3^{4(n-m)}}\right) = \frac{|a_m|^2}{3^{2m}}\frac{3h}{2}\frac{9}{5}.$$

Therefore,

$$\|\triangle_h^3\psi\|_{L^2(x-3h,x)} \geqslant \|f_2\|_{L^2(x-3h,x)} - \|f_1\|_{L^2(x-3h,x)}$$
$$\geqslant \left(\frac{|a_m|^2}{3^{2m}}\frac{3h}{2}\frac{9}{5}\right)^{\frac{1}{2}} - \left\|\frac{|a_m|}{3^m}\frac{\pi^3}{54}\right\|_{L^2(x-3h,x)}$$
$$= \frac{|a_m|}{3^m}\left(\frac{3h}{2}\frac{9}{5}\right)^{\frac{1}{2}} - \frac{|a_m|}{3^m}\frac{\pi^3}{54}(3h)^{\frac{1}{2}}$$
$$= \frac{|a_m|}{3^m}\left(\left(\frac{9}{10}\right)^{\frac{1}{2}} - \frac{\pi^3}{54}\right)(3h)^{\frac{1}{2}}$$
$$\geqslant \frac{|a_m|}{3^m}\frac{3}{10}(3h)^{\frac{1}{2}}.$$

Here we used $(\frac{9}{10})^{\frac{1}{2}} - \frac{\pi^3}{54} \geqslant \frac{9}{10} - \frac{32}{54} \geqslant \frac{9}{10} - \frac{6}{10} = \frac{3}{10}$. Therefore,

$$\inf_{x\in\mathbb{R}}\|\triangle_h^3\psi\|^2_{L^2(x-3h,x)} \geqslant \left(\frac{|a_m|}{3^m}\frac{3}{10}\right)^2(3h) = \left(\frac{|a_m|}{3^m}\frac{3}{10}\right)^2 3^{-m}\pi \geqslant \frac{|a_m|^2}{C3^{3m}},$$

and

$$\omega_1(3^{-m}\pi,u) = \omega_1(3h,u) \geqslant C^{-1}\inf_{x\in\mathbb{R}}\|\triangle_h^3\psi\|^2_{L^2(x-3h,x)} \geqslant C^{-1}3^{-3m}|a_m|^2.$$

This completes the proof. □

Now we are in a position to give some examples of shear flows that induce an enhanced dissipation time-scale faster than $O(\nu^{-1/3})$.

Lemma 5.3. *If $u(y) = \sum_{n=1}^{\infty} a_n\sin(3^n y)$ is a Weierstrass function, $a_n\in\mathbb{R}$, $3^{-n\alpha} \leqslant |a_n| \leqslant C_0 3^{-n\alpha}$ for some constants $\alpha\in(0,1)$, $C_0 > 1$ and $1 \leqslant |a_n|/|a_{n+1}| \leqslant 3$, then there exist positive constants ε and C such that for every $\nu > 0$ and every integer $k \neq 0$ satisfying $\nu|k|^{-1} \leqslant 1/2$, (5.1) holds for $\widetilde{\lambda}_{\nu,k} = \nu^{\frac{\alpha}{\alpha+2}}|k|^{\frac{2}{\alpha+2}}$.*

Proof. For every $\delta \in (0,1)$, there exists $m \in \mathbb{Z}$, $m > 0$, such that $3^{-m}\pi \leqslant \delta < 3^{-m+1}\pi$. As $|a_m| \geqslant 3^{-m\alpha}$, by Lemma 5.2, we have $\omega_1(\delta,u) \geqslant \omega_1(3^{-m}\pi,u) \geqslant C^{-1}3^{-3m}|a_m|^2 \geqslant C^{-1}3^{-3m}|3^{-m\alpha}|^2 = C^{-1}3^{-(3+2\alpha)m} \geqslant C^{-1}(\delta/(3\pi))^{3+2\alpha} \geqslant C^{-1}\delta^{3+2\alpha}$. Now the result follows from Theorem 5.1. □

Lemma 5.4. *If $u(y) = \sum_{n=1}^{\infty} a_n \sin(3^n y)$ is a Weierstrass function, $a_n \in \mathbb{R}$, $n^{-\alpha} \leqslant |a_n| \leqslant C_0 n^{-\alpha}$ for some constants $\alpha \in (1,2)$, $C_0 > 1$ and $1 \leqslant |a_n|/|a_{n+1}| \leqslant 3$, then there exist positive constants ε and C such that for every $\nu > 0$ and every integer $k \neq 0$ satisfying $\nu|k|^{-1} \leqslant 1/2$, (5.1) holds for $\widetilde{\lambda}_{\nu,k} = |k|(\ln(|k|/\nu))^{-\alpha}$.*

Proof. For every $\delta \in (0,1)$, there exists $m \in \mathbb{Z}$, $m > 0$, such that $3^{-m}\pi \leqslant \delta < 3^{-m+1}\pi$, and thus $m \leqslant \log_3(\pi/\delta) + 1 \leqslant C(1 - \ln\delta)$. As $|a_m| \geqslant m^{-\alpha}$, by Lemma 5.2, we have $\omega_1(\delta,u) \geqslant \omega_1(3^{-m}\pi,u) \geqslant C^{-1}3^{-3m}|a_m|^2 \geqslant C^{-1}3^{-3m}m^{-2\alpha} \geqslant C^{-1}\delta^3(1-\ln\delta)^{-2\alpha}$. Since $\widetilde{\lambda}_{\nu,k} = |k|(\ln(|k|/\nu))^{-\alpha}$, taking

$$\delta = (\nu/\widetilde{\lambda}_{\nu,k})^{1/2} = (\nu/|k|)^{\frac{1}{2}}(\ln(|k|/\nu))^{\frac{\alpha}{2}} \in (0,1),$$

we have

$$\begin{aligned}\delta\omega_1(\delta, ku/\nu) &= \delta(|k|/\nu)^2\omega_1(\delta,u) = \delta(\delta^{-2}(\ln(|k|/\nu))^{\alpha})^2\omega_1(\delta,u)\\ &\geqslant C^{-1}\delta(\delta^{-2}(\ln(|k|/\nu))^{\alpha})^2\delta^3(1-\ln\delta)^{-2\alpha}\\ &= C^{-1}(\ln(|k|/\nu))^{2\alpha}(1-\ln\delta)^{-2\alpha}.\end{aligned}$$

We also have $\delta \geqslant C^{-1}(\nu/|k|)^{\frac{1}{2}}$, $\ln\delta \geqslant (1/2)\ln(\nu/|k|) - C$ and $1 - \ln\delta \leqslant C - (1/2)\ln(\nu/|k|) = C + (1/2)\ln(|k|/\nu) \leqslant C\ln(|k|/\nu)$, which imply that $\delta\omega_1(\delta, ku/\nu) \geqslant C_1$ for an absolute constant $C_1 > 0$.

By Lemma 4.3, for $\delta = (\nu/\widetilde{\lambda}_{\nu,k})^{1/2} > 0$, we have

$$\Psi_1(ku/\nu) \geqslant (\varphi^{-1}(\delta\omega_1(\delta, ku/\nu))/\delta)^2,$$

and

$$\begin{aligned}\nu\Psi_1(ku/\nu) &\geqslant \nu(\varphi^{-1}(\delta\omega_1(\delta,ku/\nu))/\delta)^2 \geqslant \nu(\varphi^{-1}(C_1)/\delta)^2\\ &= \nu(\varphi^{-1}(C_1))^2/((\nu/\widetilde{\lambda}_{\nu,k})^{1/2})^2 = (\varphi^{-1}(C_1))^2\widetilde{\lambda}_{\nu,k}.\end{aligned}$$

Thus,

$$\begin{aligned}\|S_\nu(t)P_k\|_{L^2\to L^2} &= \|\mathrm{e}^{-tL_{k,\nu}}\|_{L^2\to L^2} \leqslant \|R_\nu(t)P_k\|_{L^2\to L^2} = \|\mathrm{e}^{-tR_{k,\nu}}\|_{L^2\to L^2}\\ &\leqslant \mathrm{e}^{-t\nu\Psi_1(ku/\nu)+\pi/2} \leqslant \mathrm{e}^{-\varepsilon\widetilde{\lambda}_{\nu,k}t+\pi/2}, \quad \forall\, t \geqslant 0,\end{aligned}$$

where $\varepsilon = (\varphi^{-1}(C_1))^2 > 0$ is a constant. This completes the proof. □

Acknowledgements This work was supported by China Postdoctoral Science Foundation (Grant No. 2018M 630016). The author thanks Professor Zhifei Zhang for many helpful suggestions.

References

1 Beck M, Wayne C E. Metastability and rapid convergence to quasi-stationary bar states for the two-dimensional Navier-Stokes equations. Proc Roy Soc Edinburgh Sect A, 2013, 143: 905–927

2 Bedrossian J, Coti Zelati M. Enhanced dissipation, hypoellipticity, and anomalous small noise inviscid limits in shear flows. Arch Ration Mech Anal, 2017, 224: 1161–1204

3 Bedrossian J, Germain P, Masmoudi N. On the stability threshold for the 3D Couette flow in Sobolev regularity. Ann of Math (2), 2017, 185: 541–608

4 Bedrossian J, He S. Suppression of blow-up in Patlak-Keller-Segel via shear flows. SIAM J Math Anal, 2017, 49: 4722–4766

5 Constantin P, Kiselev A, Ryzhik L, et al. Diffusion and mixing in fluid flow. Ann of Math (2), 2008, 168: 643–674

6 Coti Zelati M, Delgadino M G, Elgindi T M. On the relation between enhanced dissipation time-scales and mixing rates. Comm Pure Appl Math, 2020, 73: 1205–1244

7 Engel K J, Nagel R. One-parameter Semigroups for Linear Evolution Equations. Graduate Texts in Mathematics, 194. New York: Springer-Verlag, 2000

8 Gallagher I, Gallay T, Nier F. Spectral asymptotics for large skew-symmetric perturbations of the harmonic oscillator. Int Math Res Not IMRN, 2009, 12: 2147–2199

9 Grenier E, Nguyen T, Rousset F, et al. Linear inviscid damping and enhanced viscous dissipation of shear flows by using the conjugate operator method. J Funct Anal, 2020, 278: 108339

10 Helffer B, Sjöstrand J. From resolvent bounds to semigroup bounds. arXiv:1001.4171, 2010

11 Ibrahim S, Maekawa Y, Masmoudi N. On pseudospectral bound for non-selfadjoint operators and its application to stability of Kolmogorov flows. Ann PDE, 2019, 5: 14

12 Kato T. Perturbation Theory for Linear Operators. Grundlehren der mathematischen Wissenschaften, vol. 132. Berlin: Springer, 1966

13 Kiselev A, Zlatoš A. Quenching of combustion by shear flows. Duke Math J, 2006, 132: 49–72

14 Li T, Wei D, Zhang Z. Pseudospectral bound and transition threshold for the 3D Kolmogorov flow. Comm Pure Appl Math, 2020, 73: 465–557

15 Villani C. Hypocoercivity. Memoirs of the American Mathematical Society, vol. 202. Providence: Amer Math Soc, 2009

16 Wei D, Zhang Z. Transition threshold for the 3D Couette flow in Sobolev space. Comm Pure Appl Math, 2021, in press

17 Wei D, Zhang Z, Zhao W. Linear inviscid damping and enhanced dissipation for the Kolmogorov flow. Adv Math, 2020, 362: 106963