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Difference of composition operators over the half-plane

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Abstract To overcome the unboundedness of the half-plane, we use Khinchine's inequality and atom decomposition techniques to provide joint Carleson measure characterizations when the difference of composition operators is bounded or compact from standard weighted Bergman spaces to Lebesgue spaces over the half-plane for all index choices. For applications, we obtain direct analytic characterizations of the bounded and compact differences of composition operators on such spaces. This paper concludes with a joint Carleson measure characterization when the difference of composition operators is Hilbert-Schmidt.

Keywords weighted Bergman space, joint Carleson measure, composition operator, Khinchine's inequality, atom decomposition, Hilbert-Schmidt

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1 Introduction

Let Π^+ be the upper half of the complex plane, i.e., $\Pi^+ := \{z \in \mathbb{C} : \Im z > 0\}$. Let μ be a positive Borel measure on Π^+. For $0 < p < \infty$, let $L^p(\mu)$ denote the Lebesgue space on Π^+ with the measure μ, i.e., $L^p(\mu)$ comprises all measurable complex functions f defined on Π^+ for which the "norm"

$$\|f\|_{L^p} := \left\{ \int_{\Pi^+} |f|^p d\mu \right\}^{\frac{1}{p}}$$

is finite. When $0 < p < 1$, the space $L^p(\mu)$ is a complete metric space under the translation-invariant metric $(f, g) \mapsto \|f - g\|_{L^p}^p$. When $1 \leqslant p < \infty$, the space $L^p(\mu)$ is a Banach space. In particular, $L^2(\mu)$ is a Hilbert space. For $\alpha > -1$, let

$$dA_\alpha(z) := c_\alpha(\Im z)^\alpha dA(z),$$

where $c_\alpha = \frac{2^\alpha(\alpha+1)}{\pi}$ is a constant and dA is the Lebesgue area measure on Π^+. For $0 < p < \infty$, we denote the standard weighted Bergman space by $A^p_\alpha(\Pi^+)$, which comprises holomorphic functions of $L^p(dA_\alpha)$. It is known that each space $A^p_\alpha(\Pi^+)$ is a closed subspace of $L^p(dA_\alpha)$. For convenience, we use $\|f\|_{A^p_\alpha}$ to represent the norm of $f \in A^p_\alpha(\Pi^+)$.

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Let $H(\Pi^+)$ and $S(\Pi^+)$ be the sets of all holomorphic functions and holomorphic self-maps on Π^+, respectively. The composition operator C_φ is defined by

$$C_\varphi f = f \circ \varphi, \quad f \in H(\Pi^+).$$

Extensive study of the theory of composition operators has been conducted during the past four decades in various settings. For various aspects of the theory of composition operators acting on holomorphic function spaces, see [11, 21].

One of the most important problems in the study of composition operators is to characterize compact differences of such operators (see [4, 5, 7, 14, 15, 18, 19, 23]). In particular, in a study [18] on weighted Bergman spaces over the unit disk, the compact difference of two composition operators is characterized by a particular cancellation property of the inducing maps at every "bad" boundary point, which prevents each composition operator in the difference from being compact. For general linear combinations of composition operators, see [1, 3, 7, 8, 15, 16, 23].

It is known that in contrast to the unit disk case, some composition operators are not bounded on $A^p_\alpha(\Pi^+)$, and no composition operator on $A^p_\alpha(\Pi^+)$ is compact (see [12, 22]). Recently, the notion of the joint Carleson measure was introduced by Koo and Wang [14] for their study of differences of composition operators over the ball. Choe et al. [6] used the joint Carleson measure to provide a characterization of bounded and compact differences of composition operators on $A^p_\alpha(\Pi^+)$ and showed that there are several instances of distinct composition operators with compact differences.

Continuing along this line, we use Khinchine's inequality and atom decomposition techniques, which are quite different from the methods used in [6, 20], to provide analogous joint Carleson measure characterizations when the difference of composition operators is bounded or compact from standard weighted Bergman spaces $A^p_\alpha(\Pi^+)$ to Lebesgue spaces $L^q(\mu)$ for all index choices, including the much more delicate case in which $0 < q < p < \infty$.

The rest of this paper is organized as follows. Section 2 presents some necessary basic prerequisites and technical lemmas. Section 3 is devoted to characterizations of boundedness and compactness of differences of composition operators (see Theorem 3.1 and 3.2). For applications, we obtain direct analytic characterizations of the bounded and compact differences of composition operators between such spaces (see Theorems 3.3 and 3.4). In addition, we show that there are symbols φ and ψ inducing bounded or compact difference $C_\varphi - C_\psi$ acting from $A^p_\alpha(\Pi^+)$ to $A^q_\beta(\Pi^+)$ for any $0 < p, q < \infty$. Section 4 presents some joint Carleson measure characterizations when the difference of composition operators is Hilbert-Schmidt from $A^2_\alpha(\Pi^+)$ to $L^2(\mu)$ (see Theorem 4.1).

2 Preliminaries

In this section, we recall some basic facts about $A^p_\alpha(\Pi^+)$ and prove some technical lemmas that we will need in the future.

Throughout the paper, we use the same letter C to denote various positive constants that might change at each occurrence. Variables indicating the dependency of constants C will often be specified in parentheses. We use the notation $X \lesssim Y$ or $Y \gtrsim X$ for non-negative quantities X and Y so that $X \leqslant CY$ for some inessential constant $C > 0$. Similarly, we use the notation $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

The pseudo-hyperbolic distance $\rho : \Pi^+ \times \Pi^+ \to [0, 1)$ is

$$\rho(z, w) := \left| \frac{z - w}{z - \overline{w}} \right|.$$

For $z \in \Pi^+$ and $0 < \delta < 1$, let $E_\delta(z)$ denote the pseudo-hyperbolic disk centered at z with radius δ. An elementary calculation shows that $E_\delta(z)$ is actually a Euclidean disk centered at $x + \mathrm{i}\frac{1+\delta^2}{1-\delta^2}y$ with radius $\frac{2\delta}{1-\delta^2}y$, where $\mathrm{i}^2 = -1$, $x = \Re z$, and $y = \Im z$. Furthermore, for $\alpha > -1$, $a, z, w \in \Pi^+$, and $0 < \delta < 1$, the following inequalities hold:

$$\frac{1-\rho(z,w)}{1+\rho(z,w)} \leqslant \frac{\Im z}{\Im w} \leqslant \frac{1+\rho(z,w)}{1-\rho(z,w)}, \tag{2.1}$$

$$\frac{1-\rho(z,w)}{1+\rho(z,w)} \leqslant \left|\frac{z-\overline{a}}{w-\overline{a}}\right| \leqslant \frac{1+\rho(z,w)}{1-\rho(z,w)}, \tag{2.2}$$

$$\frac{1-\rho(z,w)}{1+\rho(z,w)} \leqslant \left|\frac{z-\overline{w}}{2\Im z}\right| \leqslant \frac{1+\rho(z,w)}{1-\rho(z,w)} \tag{2.3}$$

and

$$A_\alpha[E_\delta(z)] \approx (\Im z)^{\alpha+2}. \tag{2.4}$$

The above facts can be found in [10].

We will use the submean value type inequality

$$|f(z)|^p \leqslant \frac{C}{(\Im z)^{\alpha+2}} \int_{E_\delta(z)} |f(w)|^p dA_\alpha(w), \quad z \in \Pi^+ \tag{2.5}$$

for all $f \in H(\Pi^+)$ and some constant $C = C(\alpha,\delta)$ (see [10, Lemma 3.6]). In particular,

$$|f(z)|^p \leqslant \frac{C}{(\Im z)^{\alpha+2}} \|f\|^p_{A^p_\alpha}, \quad z \in \Pi^+. \tag{2.6}$$

For a positive Borel measure μ on Π^+, we have the following submean value property with respect to the pseudo-hyperbolic disk.

Proposition 2.1. *Assume that $\alpha > -1$ and $0 < \delta < 1$, and μ is a positive Borel measure on Π^+. Then,*

$$\mu[E_\delta(z)] \leqslant \frac{C}{(\Im z)^{\alpha+2}} \int_{E_\delta(z)} \mu[E_\delta(w)] dA_\alpha(w), \quad z \in \Pi^+,$$

where $C = C(\alpha,\delta) > 0$ is a constant.

Proof. Let $z \in \Pi^+$ and $0 < \delta < 1$. By Fubini's theorem and (2.1),

$$\begin{aligned}
\int_{E_\delta(z)} \mu[E_\delta(w)] dA_\alpha(w) &= \int_{E_\delta(z)} \int_{E_\delta(w)} d\mu(u) dA_\alpha(w) \\
&= \int_{\Pi^+} \int_{E_\delta(z)} \chi_{E_\delta(u)}(w) dA_\alpha(w) d\mu(u) \\
&\geqslant \int_{E_\delta(z)} \int_{E_\delta(z)\cap E_\delta(u)} dA_\alpha(w) d\mu(u) \\
&\approx (\Im z)^\alpha \int_{E_\delta(z)} A[E_\delta(z)\cap E_\delta(u)] d\mu(u) \\
&\geqslant (\Im z)^\alpha \mu[E_\delta(z)] \inf_{u\in E_\delta(z)} A[E_\delta(z)\cap E_\delta(u)] \\
&\geqslant C(\Im z)^{\alpha+2} \mu[E_\delta(z)],
\end{aligned}$$

where $C = C(\alpha,\delta) > 0$ is a constant. The last inequality holds because both $E_\delta(z)$ and $E_\delta(u)$ are the Euclidean disks and $\Im u \approx \Im z$ when $u \in E_\delta(z)$, and the above infimum is attained when

$$u_0 := \Re u + \mathrm{i}\frac{1+\delta^2}{1-\delta^2}\Im u$$

is on the boundary of $E_\delta(z)$. □

Given $\alpha > -1$, it follows from (2.6) that each point evaluation is a continuous linear function on $A^2_\alpha(\Pi^+)$. Thus, for each $z \in \Pi^+$, there exists a unique reproducing kernel $K^{(\alpha)}_z \in A^2_\alpha(\Pi^+)$, i.e.,

$$f(z) = \int_{\Pi^+} f(w)\overline{K^{(\alpha)}_z(w)} dA_\alpha(w)$$

for $f \in A^2_\alpha(\Pi^+)$. The explicit formula of $K_z^{(\alpha)}$ is

$$K_z^{(\alpha)}(w) = \left(\frac{\mathrm{i}}{w-\bar{z}}\right)^{\alpha+2}.$$

Let $\widehat{\Pi}^+ = \overline{\Pi^+} \cup \{\infty\}$. We consider $\lim_{z\to\partial\widehat{\Pi}^+} g(z) = 0$, if $g(z) \to 0$ as $\Im z \to 0^+$ and $g(z) \to 0$ as $|z| \to \infty$. This is equivalent when $\forall\, \epsilon > 0$, and there is a compact set $K \subset \Pi^+$ such that $\sup_{z\in\Pi^+\setminus K} |g(z)| < \epsilon$. Let

$$\tau_z(w) := \frac{1}{w-\bar{z}}, \quad z, w \in \Pi^+,$$

throughout this paper. The following lemma is cited from [6, Lemma 2.5].

Lemma 2.2. *Let $\alpha > -1, s > 0$, $z \in \Pi^+$, and $0 < p < \infty$. If $ps > \alpha + 2$, then*

$$\tau_z^s \in A^p_\alpha(\Pi^+), \quad \|\tau_z^s\|^p_{A^p_\alpha} \approx \frac{1}{(\Im z)^{ps-\alpha-2}},$$

and $\frac{\tau_z^s}{\|\tau_z^s\|_{A^p_\alpha}} \to 0$ as $z \to \partial\widehat{\Pi}^+$ uniformly on compact subsets of Π^+.

For $\alpha > -1$, $0 < p < \infty$, and a positive Borel measure μ on Π^+, μ is an (α, p, q)-Carleson measure if the embedding $A^p_\alpha(\Pi^+) \subset L^q(\mu)$ is continuous. In addition, if the embedding $A^p_\alpha(\Pi^+) \subset L^q(\mu)$ is compact, then μ is evaluated as a compact (α, p, q)-Carleson measure. For $\varphi \in S(\Pi^+)$, we define the pullback measure $\mu \circ \varphi^{-1}$ on Π^+:

$$(\mu \circ \varphi^{-1})[E] = \mu[\varphi^{-1}(E)]$$

for every Borel subset E of Π^+. Then, the identity

$$\int_{\Pi^+} (f \circ \varphi) d\mu = \int_{\Pi^+} f d(\mu \circ \varphi^{-1})$$

is valid for any Borel function $f > 0$ (see [13, p. 163]).

Lemma 2.3 is widely known, which can be proved through the following equation. For $s > 0$ and $\alpha > -1$, define T_s by

$$T_s f(z) = \int_{\Pi^+} \frac{(\Im w)^{s-\alpha-2}}{|z-\overline{w}|^s} f(w) dA_\alpha(w), \quad z \in \Pi^+, \tag{2.7}$$

$\forall\, f \in L^p(dA_\alpha)$.

Lemma 2.3. *Suppose that $s > 0$, $\alpha > -1$, and $1 < p < \infty$. If $s > 1 + \frac{\alpha+1}{p}$, then T_s is bounded on $L^p(dA_\alpha)$.*

Proof. For $1 < p < \infty$, let q be the conjugate exponent of p. Let

$$(A, B) = \left(-\frac{s-1}{q}, 0\right) \quad \text{and} \quad (C, D) = \left(-\frac{\alpha+1}{p}, \frac{s-\alpha-2}{p}\right).$$

Since $s > 1 + \frac{\alpha+1}{p}$ and $\alpha > -1$, $(A, B) \cap (C, D) \neq \emptyset$. Let $h(w) = (\Im w)^t$, where $t \in (A, B) \cap (C, D)$. Since $t > -\frac{s-1}{q}$, $s - 2 + qt > -1$. Thus, by Lemma 2.2,

$$\begin{aligned}\int_{\Pi^+} \frac{(\Im w)^{s-\alpha-2}}{|z-\overline{w}|^s} (h(w))^q dA_\alpha(w) &\approx \int_{\Pi^+} \frac{(\Im w)^{s-2+qt}}{|z-\overline{w}|^s} dA(w) \\ &\approx \int_{\Pi^+} \left|\frac{1}{(z-\overline{w})^{\frac{s}{q}}}\right|^q dA_{s-2+qt}(w) \\ &\approx (h(z))^q. \end{aligned} \tag{2.8}$$

In addition, since $t > -\frac{\alpha+1}{p}$, we have $pt + \alpha > -1$. Thus, by Lemma 2.2,

$$\int_{\Pi^+} \frac{(\Im w)^{s-\alpha-2}}{|z-\overline{w}|^s} (h(z))^p dA_\alpha(z) \approx (\Im w)^{s-\alpha-2} \int_{\Pi^+} \frac{(\Im z)^{pt+\alpha}}{|z-\overline{w}|^s} dA(z)$$

$$\approx (\Im w)^{s-\alpha-2}\int_{\Pi^+}\left|\frac{1}{(z-\overline{w})^{\frac{s}{p}}}\right|^p dA_{pt+\alpha}(z)$$
$$\approx (h(w))^p. \tag{2.9}$$

Now, from (2.8) and (2.9), we determine that T_s is bounded on $L^p(dA_\alpha)$ by Schur's test [25, Theorem 3.6]. □

Furthermore, we recall some terminologies. When we consider $0<\delta<1$ and $E_\delta(z_n)$ are pairwise disjoint, a sequence $\{z_n\}\subset\Pi^+$ is called δ-separated. In addition, we say that $\{z_n\}\subset\Pi^+$ is separated if it is δ-separated for some δ. A sequence $\{z_n\}\subset\Pi^+$ is called a δ-lattice if it is $\frac{\delta}{2}$-separated and $\Pi^+=\bigcup_{n=1}^\infty E_\delta(z_n)$. A δ-lattice can be explicitly constructed by using almost the same argument as that in [25].

The following lemma is cited from [10, Lemma 4.2].

Lemma 2.4. *Assume $s>0$ and $0<\delta<1$ with $(s+1)\delta<1$. If $\{z_n\}$ is δ-separated, then there exists a positive integer $N=N(s,\delta)$ such that no more than N of the balls $E_{s\delta}(z_n)$ contain a common point.*

The following lemmas are cited from [6, Lemma 3.2] and [6, Lemma 7.2], respectively.

Lemma 2.5. *Let $\alpha>-1, 0<p<\infty$ and $0<\delta'<\delta<1$. There is a constant $C=C(\alpha,p,\delta,\delta')>0$ such that*

$$|f(z)-f(w)|^p\leqslant C\frac{\rho^p(z,w)}{A_\alpha[E_\delta(z)]}\int_{E_\delta(z)}|f|^p dA_\alpha$$

for all $z,w\in\Pi^+$ with $w\in E_{\delta'}(z)$ and functions f holomorphic on $E_\delta(z)$.

Lemma 2.6. *Given that s is real and $0<\delta<1$, there is a constant $C=C(s,\delta)>0$ such that*

$$\left|\left(\frac{z-\overline{a}}{w-\overline{a}}\right)^s-1\right|\leqslant C\rho(z,w)$$

for $a,z,w\in\Pi^+$ with $\rho(z,w)<\delta$.

Using the above lemmas, we now prove the next lemma.

Lemma 2.7. *Assume that $0<p<\infty$, $\alpha>-1$ and $s>\max\{1,\frac{1}{p}\}+\frac{\alpha+1}{p}$. Let $\{a_k\}\subset\Pi^+$ be an η-lattice with $0<\eta<\frac{1}{4}$. Define $T: A^p_\alpha(\Pi^+)\to H(\Pi^+)$ by*

$$Tf(z)=\sum_{k=1}^\infty\frac{\overline{(\mathrm{i}^s)}(A_{s-2}[E_\eta(a_k)])f(a_k)}{(\overline{a}_k-z)^s}.$$

Then, the following inequality holds:

$$|f(z)-Tf(z)|\lesssim\sum_{k=1}^\infty\frac{\eta(\Im a_k)^{s-\frac{\alpha+2}{p}}}{|z-\overline{a}_k|^s}\left(\int_{E_{2\eta}(a_k)}|f|^p dA_\alpha\right)^{\frac{1}{p}}$$

for any $f\in A^p_\alpha(\Pi^+)$.

Proof. Let $0<\eta<\frac{1}{4}$, $s>\max\{1,\frac{1}{p}\}+\frac{\alpha+1}{p}$, and $\{a_k\}\subset\Pi^+$ be an η-lattice. Let $\beta=s-2$. Then $\beta>-1$. Performing the change-of-variable

$$\gamma(z)=\mathrm{i}\frac{1+z}{1-z},$$

which maps $\mathbb{D}$ (the unit disk in the complex plane $\mathbb{C}$) to Π^+, we obtain

$$f(z)=\int_{\Pi^+}f(w)\overline{\left(\frac{\mathrm{i}}{w-\overline{z}}\right)^s}dA_\beta(w).$$

Thus, we get

$$|f(z)-Tf(z)|\lesssim\sum_{k=1}^\infty\int_{E_\eta(a_k)}\left|\frac{f(w)}{(z-\overline{w})^s}-\frac{f(a_k)}{(z-\overline{a}_k)^s}\right|dA_\beta(w)$$

$$\leqslant \sum_{k=1}^{\infty} \frac{1}{|z-\overline{a}_k|^s} \int_{E_\eta(a_k)} |f(w)-f(a_k)| dA_\beta(w)$$
$$+ \sum_{k=1}^{\infty} \frac{1}{|z-\overline{a}_k|^s} \int_{E_\eta(a_k)} \left| \left(\frac{z-\overline{a}_k}{z-\overline{w}} \right)^s - 1 \right| |f(w)| dA_\beta(w)$$
$$=: I + II.$$

For $w \in E_\eta(a_k)$, by Lemma 2.5, there exists a constant $C > 0$ such that

$$I \leqslant \sum_{k=1}^{\infty} \frac{1}{|z-\overline{a}_k|^s} (A_\beta[E_\eta(a_k)]) \left(\frac{C\rho^p(w,a_k)}{A_\alpha[E_{2\eta}(a_k)]} \int_{E_{2\eta}(a_k)} |f|^p dA_\alpha \right)^{\frac{1}{p}}$$
$$\lesssim \sum_{k=1}^{\infty} \frac{\eta(\Im a_k)^{s-\frac{\alpha+2}{p}}}{|z-\overline{a}_k|^s} \left(\int_{E_{2\eta}(a_k)} |f|^p dA_\alpha \right)^{\frac{1}{p}}.$$

By (2.5) and Lemma 2.6,

$$II \lesssim \sum_{k=1}^{\infty} \frac{\eta}{|z-\overline{a}_k|^s} \int_{E_\eta(a_k)} |f(w)| dA_\beta(w)$$
$$\lesssim \sum_{k=1}^{\infty} \frac{\eta(\Im a_k)^{s-\frac{\alpha+2}{p}}}{|z-\overline{a}_k|^s} \left(\int_{E_{2\eta}(a_k)} |f|^p dA_\alpha \right)^{\frac{1}{p}}.$$

Thus, we get

$$|f(z) - Tf(z)| \lesssim \sum_{k=1}^{\infty} \frac{\eta(\Im a_k)^{s-\frac{\alpha+2}{p}}}{|z-\overline{a}_k|^s} \left(\int_{E_{2\eta}(a_k)} |f|^p dA_\alpha \right)^{\frac{1}{p}}.$$

This completes the proof. □

Based on Lemmas 2.3, 2.4 and 2.7, we have the following atom decomposition theorem on $A^p_\alpha(\Pi^+)$, whose proof is similar to the argument of [24, Theorem 2.30] that is provided below.

Theorem 2.8. *Suppose that $0 < p < \infty, \alpha > -1$, and $s > \max\{1, \frac{1}{p}\} + \frac{\alpha+1}{p}$. Then, there exists a constant $0 < \lambda < 1$ such that for any δ-lattice $\{w_n\} \subset \Pi^+$, where $0 < \delta < \lambda$, the space $A^p_\alpha(\Pi^+)$ comprises functions of the form*

$$f = \sum_{n=1}^{\infty} c_n \frac{\tau^s_{w_n}}{\|\tau^s_{w_n}\|_{A^p_\alpha}} \in A^p_\alpha(\Pi^+), \tag{2.10}$$

where $\{c_n\} \in l^p$, and

$$\|f\|_{A^p_\alpha} \approx \inf\{\|\{c_n\}\|_{l^p} : \{c_n\} \text{ satisfies } (2.10)\}.$$

Proof. Let $s > \max\{1, \frac{1}{p}\} + \frac{\alpha+1}{p}$, and $0 < \delta < 1$. Consider a function f defined by (2.10), where $\{w_n\}$ is a δ-lattice and $\{c_n\} \in l^p$. If $0 < p \leqslant 1$, then $\|f\|_{A^p_\alpha} \leqslant \|\{c_n\}\|_{l^p}$. When $1 < p < \infty$, let

$$F(z) = \sum_{n=1}^{\infty} |c_n| (A_\alpha[E_\delta(w_n)])^{-1/p} \chi_{E_\delta(w_n)}(z).$$

It clearly shows that $\|F\|_{L^p} \lesssim \|\{c_n\}\|_{l^p}$. Then, by (2.1), (2.2) and Lemma 2.2,

$$T_s(F)(z) = \sum_{n=1}^{\infty} |c_n| (A_\alpha[E_\delta(w_n)])^{-1/p} \int_{E_\delta(w_n)} \frac{(\Im w)^{s-\alpha-2}}{|z-\overline{w}|^s} dA_\alpha(w) \gtrsim |f(z)|.$$

Since $s > 1 + \frac{\alpha+1}{p}$, by Lemma 2.3, we conclude that $f \in A^p_\alpha(\Pi^+)$ and $\|f\|_{A^p_\alpha} \lesssim \|\{c_n\}\|_{l^p}$.

To show that every function $f \in A^p_\alpha(\Pi^+)$ admits a representation of the form given in (2.10), we fix a δ-lattice $\{w_n\}$, where $0 < \delta < 1/4$. Furthermore, by Lemmas 2.4 and 2.7, and the first part of this proof, there exists a constant $C > 0$ such that

$$\int_{\Pi^+} |f(z) - Tf(z)|^p dA_\alpha(z) < C\delta^p \int_{\Pi^+} |f(z)| dA_\alpha(z).$$

If δ is sufficiently small to ensure that $C\delta^p < 1$, then the operator $I-T$ on $A^p_\alpha(\Pi^+)$ has norm less than 1, where I is the identity operator. It follows from standard functional analysis that the operator T is invertible on $A^p_\alpha(\Pi^+)$. Therefore, every $f \in A^p_\alpha(\Pi^+)$ admits the following representation:

$$f(z) = \sum_{n=1} c_n \frac{\tau^s_{w_n}(z)}{\|\tau^s_{w_n}\|_{A^p_\alpha}},$$

where

$$c_n = \overline{(\mathrm{i}^s)}(A_{s-2}[E_\delta(w_n)])\|\tau^s_{w_n}\|_{A^p_\alpha} g(w_n)$$

and $g = T^{-1}f$. Then, by (2.1), (2.4), (2.5), Lemmas 2.2 and 2.4, $\|\{c_n\}\|_{l^p} \lesssim \|f\|_{A^p_\alpha}$. The proof is completed when we substitute $\lambda = \delta$. □

The complicated part of our principal results also requires Khinchine's inequality. Therefore, the Rademacher functions $r_n : [0,1] \to [-1,1]$ are defined as

$$r_n(t) := \operatorname{sgn}(\sin(2^n \pi t)).$$

Khinchine's inequality says that for $0 < p < \infty$, there are constants $0 < A_p \leqslant B_p < \infty$ such that

$$A_p\left(\sum_{n=1}^m |c_n|^2\right)^{\frac{p}{2}} \leqslant \int_0^1 \left|\sum_{n=1}^m c_n r_n(t)\right|^p dt \leqslant B_p\left(\sum_{n=1}^m |c_n|^2\right)^{\frac{p}{2}} \tag{2.11}$$

for all natural numbers m and all complex numbers $c_1, \ldots, c_m$.

In addition, we define some notations that will be used in the sequel for brevity. For $z \in \Pi^+$, $0 < \delta < 1$, $\alpha > -1$, and a positive Borel measure μ on Π^+, let

$$H_{\alpha,\mu,\delta}(z) = \frac{\mu[E_\delta(z)]}{(\Im z)^{\frac{q(\alpha+2)}{p}}} \quad \text{and} \quad G_{\alpha,\mu,\delta}(z) = \frac{\mu[E_\delta(z)]}{(\Im z)^{\alpha+2}}. \tag{2.12}$$

By definition, a positive Borel measure μ is locally finite when $\mu(K) < \infty$ for any compact set $K \subset \Pi^+$.

Remark 2.9. If $\sup_{z\in\Pi^+} H_{\alpha,\mu,\delta}(z) < \infty$, then μ is locally finite. In particular, for any compact set $K \subset \Pi^+$, we have

$$\sup_{z\in K} H_{\alpha,\mu,\delta}(z) \leqslant \sup_{z\in\Pi^+} H_{\alpha,\mu,\delta}(z) < \infty.$$

Note that $\sup_{z\in K} \Im z \lesssim 1$. Then $\sup_{z\in K} \mu[E_\delta(z)] \lesssim 1$. Since K can be covered by finitely many $E_\delta(z)$, μ is locally finite.

To characterize an (α,p,q)-Carleson measure for $0 < q < p < \infty$, we use an idea of Luecking [17]. First, we need the following lemma.

Lemma 2.10. *Assume that $0 < q < p < \infty$, and μ is a positive Borel measure on Π^+. If μ is an (α,p,q)-Carleson measure, then there exists a constant $0 < \lambda < \frac{1}{4}$ such that for any δ-lattice $\{w_n\} \subset \Pi^+$, where $0 < \delta < \lambda$, the sequence $\{H_{\alpha,\mu,2\delta}(w_n)\}_n$ is in $l^{\frac{p}{p-q}}$.*

Proof. Let $s > \max\{1, \frac{1}{p}\} + \frac{\alpha+1}{p}$. Considering a sequence $\{c_n\} \in l^p$ and $t \in [0,1]$, by the definition of the Rademacher functions r_n, we have

$$\|\{c_n r_n(t)\}\|_{l^p} = \|\{c_n\}\|_{l^p}.$$

By Theorem 2.8, there exists a constant $0 < \lambda < 1$ such that for any δ-lattice $\{w_n\} \subset \Pi^+$, where $0 < \delta < \lambda$,

$$f_t := \sum_{n=1}^\infty c_n r_n(t) \frac{\tau^s_{w_n}}{\|\tau^s_{w_n}\|_{A^p_\alpha}} \in A^p_\alpha(\Pi^+) \quad \text{and} \quad \|f_t\|_{A^p_\alpha} \lesssim \|\{c_n\}\|_{l^p}.$$

Generally, we can assume that $0<\lambda<\frac{1}{4}$. Since μ is an (α,p,q)-Carleson measure, we deduce that

$$\int_{\Pi^+}\left|\sum_{n=1}^{\infty}c_nr_n(t)\frac{\tau^s_{w_n}(z)}{\|\tau^s_{w_n}\|_{A^p_\alpha}}\right|^q d\mu(z)\lesssim\|f_t\|^q_{A^p_\alpha}\lesssim\|\{c_n\}\|^q_{l^p}. \tag{2.13}$$

Then, by Khinchine's inequality and Fubini's theorem,

$$\begin{aligned}&\int_{\Pi^+}\left[\sum_{n=1}^{\infty}|c_n|^2\frac{|\tau^s_{w_n}(z)|^2}{\|\tau^s_{w_n}\|^2_{A^p_\alpha}}\right]^{\frac{q}{2}}d\mu(z)\\&\quad\leqslant\frac{1}{A_q}\int_{\Pi^+}\int_0^1\left|\sum_{n=1}^{\infty}c_nr_n(t)\frac{\tau^s_{w_n}(z)}{\|\tau^s_{w_n}\|_{A^p_\alpha}}\right|^q dtd\mu(z)\\&\quad\leqslant\frac{1}{A_q}\int_0^1\int_{\Pi^+}\left|\sum_{n=1}^{\infty}c_nr_n(t)\frac{\tau^s_{w_n}(z)}{\|\tau^s_{w_n}\|_{A^p_\alpha}}\right|^q d\mu(z)dt\\&\quad\lesssim\|\{c_n\}\|^q_{l^p}.\end{aligned} \tag{2.14}$$

By (2.3),

$$\begin{aligned}&\sum_{n=1}^{\infty}|c_n|^qH_{\alpha,\mu,2\delta}(w_n)\\&\quad=\sum_{n=1}^{\infty}|c_n|^q\int_{E_{2\delta}(w_n)}\frac{1}{(\Im w_n)^{\frac{q(\alpha+2)}{p}}}d\mu(z)\\&\quad\approx\int_{\Pi^+}\sum_{n=1}^{\infty}\chi_{E_{2\delta}(w_n)}(z)|c_n|^q\left[\frac{(\Im w_n)^s}{|z-\overline{w_n}|^s}\frac{1}{(\Im w_n)^{\frac{(\alpha+2)}{p}}}\right]^qd\mu(z)\\&\quad=:M.\end{aligned}$$

By Lemma 2.4, there exists a positive integer N that does not belong to more than N sets $E_{2\delta}(w_{n_i})$ with some $n_1<n_2<\cdots<n_N$ for any $z\in\Pi^+$. Thus,

$$\begin{aligned}&\sum_{n=1}^{\infty}\chi_{E_{2\delta}(w_n)}(z)|c_n|^q\left(\frac{(\Im w_n)^s}{|z-\overline{w_n}|^s}\frac{1}{(\Im w_n)^{\frac{(\alpha+2)}{p}}}\right)^q\\&\quad=\sum_{i=1}^{N}\chi_{E_{2\delta}(w_{n_i})}(z)|c_{n_i}|^q\left(\frac{(\Im w_{n_i})^s}{|z-\overline{w_{n_i}}|^s}\frac{1}{(\Im w_{n_i})^{\frac{(\alpha+2)}{p}}}\right)^q\\&\quad\leqslant\sum_{i=1}^{N}\left[|c_{n_i}|^2\left(\frac{(\Im w_{n_i})^s}{|z-\overline{w_{n_i}}|^s}\frac{1}{(\Im w_{n_i})^{\frac{(\alpha+2)}{p}}}\right)^2\right]^{\frac{q}{2}}\\&\quad\lesssim\left(\sum_{n=1}^{\infty}|c_n|^2\frac{|\tau^s_{w_n}(z)|^2}{\|\tau^s_{w_n}\|^2_{A^p_\alpha}}\right)^{\frac{q}{2}}.\end{aligned} \tag{2.15}$$

Then, by (2.14),

$$M\lesssim\int_{\Pi^+}\left(\sum_{n=1}^{\infty}|c_n|^2\frac{|\tau^s_{w_n}(z)|^2}{\|\tau^s_{w_n}\|^2_{A^p_\alpha}}\right)^{\frac{q}{2}}d\mu(z)\lesssim\|\{c_n\}\|^q_{l^p},$$

which implies that

$$\sum_{n=1}^{\infty}|c_n|^qH_{\alpha,\mu,2\delta}(w_n)\lesssim\|\{c_n\}\|^q_{l^p}.$$

For any $\{b_n\}\in l^{\frac{p}{q}}$, $\{(b_n)^{\frac{1}{q}}\}\in l^p$. Thus, for any $\{b_n\}\in l^{\frac{p}{q}}$,

$$\sum_{n=1}^{\infty}|b_n|H_{\alpha,\mu,2\delta}(w_n)\lesssim\|\{b_n\}\|_{l^{\frac{p}{q}}}.$$

Thus, we deduce that

$$\{H_{\alpha,\mu,2\delta}(w_n)\}_n \in (l^{\frac{p}{q}})^* = l^{\frac{p}{p-q}}.$$

This completes the proof. □

The (α,p,q)-Carleson measures for $A^p_\alpha(\Pi^+)$ can be characterized in the following manners.

Theorem 2.11. *Suppose that $\alpha > -1$ and μ is a positive Borel measure on Π^+. Then the following statements hold*:

(1) *If $0 < p \leqslant q < \infty$, then μ is an (α,p,q)-Carleson measure if and only if $\sup_{z\in\Pi^+} H_{\alpha,\mu,\delta}(z) < \infty$ for some* (*or all*) *$0 < \delta < 1$.*

(2) *If $0 < p \leqslant q < \infty$, then μ is a compact (α,p,q)-Carleson measure if and only if μ is locally finite and $\lim_{z\to\partial\widehat{\Pi}^+} H_{\alpha,\mu,\delta}(z) = 0$ for some* (*or all*) *$0 < \delta < 1$.*

(3) *If $0 < q < p < \infty$, then the following statements are equivalent*:

(a) *μ is an (α,p,q)-Carleson measure*;

(b) *μ is a compact (α,p,q)-Carleson measure*;

(c) *there exists $0 < \lambda < \frac{1}{4}$ such that $\{H_{\alpha,\mu,2\delta}(w_n)\}_n \in l^{\frac{p}{p-q}}$ for all $0 < \delta < \lambda$ and any δ-lattice $\{w_n\} \subset \Pi^+$*;

(d) *there exists $0 < \delta < \frac{1}{4}$ such that $\{H_{\alpha,\mu,2\delta}(w_n)\}_n \in l^{\frac{p}{p-q}}$ for some δ-lattice $\{w_n\} \subset \Pi^+$*;

(e) *there exists $0 < \delta < \frac{1}{4}$ such that $G_{\alpha,\mu,\delta} \in L^{\frac{p}{p-q}}(dA_\alpha)$,*

where $H_{\alpha,\mu,\delta}$ and $G_{\alpha,\mu,\delta}$ are defined in (2.12)*. Moreover, if μ is an (α,p,q)-Carleson measure, then μ is locally finite.*

Proof. Since (1) is much easier than (2), whose proof is omitted to the reader, we only provide proofs for (2) and (3). Suppose $0 < p \leqslant q < \infty$. To prove sufficiency, assume

$$\lim_{z\to\partial\widehat{\Pi}^+} H_{\alpha,\mu,\delta}(z) = 0$$

for some δ, and μ is locally finite. Then, $\forall\,\epsilon > 0$, there exists a compact set $K \subset \Pi^+$ such that

$$\sup_{z\in\Pi^+\setminus K} H_{\alpha,\mu,\delta}(z) < \epsilon.$$

Let $\{f_n\}$ be a sequence in $A^p_\alpha(\Pi^+)$ that converges to 0 uniformly on compact subsets of Π^+, and $\|f_n\|_{A^p_\alpha} \leqslant M$ for some positive constant M. By (2.1), (2.5) and Fubini's theorem,

$$\begin{aligned}\int_{\Pi^+} |f_n(z)|^q d\mu(z) &\lesssim \int_{\Pi^+} \left[\frac{1}{(\Im z)^{\alpha+2}} \int_{E_\delta(z)} |f_n(w)|^q dA_\alpha(w)\right] d\mu(z)\\ &= \int_{\Pi^+} \left[\int_{\Pi^+} \frac{\chi_{E_\delta(w)}(z)}{(\Im z)^{\alpha+2}} d\mu(z)\right] |f_n(w)|^q dA_\alpha(w)\\ &\approx \int_{\Pi^+} G_{\alpha,\mu,\delta}(w)|f_n(w)|^q dA_\alpha(w). \end{aligned}\tag{2.16}$$

Again,

$$\begin{aligned}\int_{\Pi^+} G_{\alpha,\mu,\delta}(w)|f_n(w)|^q dA_\alpha(w) &= \int_K G_{\alpha,\mu,\delta}(w)|f_n(w)|^q dA_\alpha(w)\\ &\quad + \int_{\Pi^+\setminus K} G_{\alpha,\mu,\delta}(w)|f_n(w)|^q dA_\alpha(w)\\ &=: I(f_n) + II(f_n). \end{aligned}\tag{2.17}$$

Since μ is locally finite, we present an argument in one line to see that $I(f_n) \to 0$ as $n \to \infty$. It follows from (2.6) that

$$II(f_n) \leqslant \int_{\Pi^+\setminus K} H_{\alpha,\mu,\delta}(w)(\Im w)^{(\frac{(q-p)(\alpha+2)}{p})}|f_n(w)|^{q-p}|f_n(w)|^p dA_\alpha(w)$$

$$\lesssim \epsilon\|f_n\|_{A^p_\alpha}^{q-p}\int_{\Pi^+\setminus K}|f_n(w)|^p dA_\alpha(w)\leqslant \epsilon\|f_n\|_{A^p_\alpha}^q\leqslant \epsilon M^q.$$

Thus,

$$\limsup_{n\to\infty}\int_{\Pi^+}|f_n(z)|^q d\mu(z)\lesssim \epsilon M^q.$$

Due to the arbitrariness of ϵ, we determine that $f_n\to 0$ in $L^q(\mu)$, as desired.

For necessity, assume that μ is a compact (α,p,q)-Carleson measure. Then, μ is locally finite from (1). Let $f_z=\frac{\tau_z^s}{\|\tau_z^s\|_{A^p_\alpha}}$, where $s>\frac{\alpha+2}{p}$, $z\in\Pi^+$. For any $0<\delta<1$, from (2.3) and Lemma 2.2,

$$\begin{aligned}\lim_{z\to\partial\widehat{\Pi}^+}H_{\alpha,\mu,\delta}(z)&\approx\lim_{z\to\partial\widehat{\Pi}^+}\int_{E_\delta(z)}|f_z(w)|^q d\mu(w)\\&\leqslant\lim_{z\to\partial\widehat{\Pi}^+}\int_{\Pi^+}|f_z(w)|^q d\mu(w)\\&=0,\end{aligned}$$

as desired.

For the proof of (3), suppose $0<q<p<\infty$. (a) $\Leftrightarrow$ (b) is proved in [9, Theorem 5.4] for the unweighted harmonic Bergman space on Π^+ when $1<q<p<\infty$. However, it still holds for a weighted holomorphic Bergman space on Π^+ and $0<q<p<\infty$. Since (c) $\Rightarrow$ (d) is trivial and (a) $\Rightarrow$ (c) follows from Lemma 2.10, it sufficiently proves (d) $\Rightarrow$ (e) $\Rightarrow$ (a).

(d) $\Rightarrow$ (e). Suppose that there exists a constant $0<\delta<\frac{1}{4}$ such that $\{H_{\alpha,\mu,2\delta}(w_n)\}_n\in l^{\frac{p}{p-q}}$, where $\{w_n\}\subset\Pi^+$ is a δ-lattice. Let $0<\delta'\leqslant\frac{\delta}{\delta+2}$. Then, we have $E_{\delta'}(z)\subset E_{2\delta}(w_n)$ and $\Im z\approx\Im w_n$ for $z\in E_\delta(w_n)$ from (2.1). By (2.4) and Lemma 2.4,

$$\begin{aligned}\int_{\Pi^+}[G_{\alpha,\mu,\delta'}(z)]^{\frac{p}{p-q}}dA_\alpha(z)&\leqslant\sum_{n=1}^\infty\int_{E_{2\delta}(w_n)}[G_{\alpha,\mu,\delta'}(z)]^{\frac{p}{p-q}}dA_\alpha(z)\\&\lesssim\sum_{n=1}^\infty[G_{\alpha,\mu,2\delta}(w_n)]^{\frac{p}{p-q}}(\Im w_n)^{\alpha+2}\\&=\sum_{n=1}^\infty[H_{\alpha,\mu,2\delta}(w_n)]^{\frac{p}{p-q}}<\infty,\end{aligned}\tag{2.18}$$

as desired.

In the above case, for any $z\in\Pi^+$, z belongs to at most N of the sets $E_\delta(w_n)$ from Lemma 2.4. Therefore, we assume $z\in E_\delta(w_{n_j})$, where $j=1,2,\dots,N$. Since $E_{\delta'}(z)\subset E_{2\delta}(w_{n_j})$ and $\Im z\approx\Im w_{n_j}$,

$$H_{\alpha,\mu,\delta'}(z)\lesssim H_{\alpha,\mu,2\delta}(w_{n_j})\leqslant\|\{H_{\alpha,\mu,2\delta}(w_n)\}_n\|_{l^{\frac{p}{p-q}}}.$$

Thus, $\sup_{z\in\Pi^+}H_{\alpha,\mu,\delta'}(z)<\infty$. Hence, μ is locally finite from Remark 2.9.

(e) $\Rightarrow$ (a). Suppose that there exists $0<\delta<\frac{1}{4}$ such that $G_{\alpha,\mu,\delta}\in L^{\frac{p}{p-q}}(dA_\alpha)$. For any $f\in A^p_\alpha(\Pi^+)$, by an argument similar to (2.16),

$$\int_{\Pi^+}|f(z)|^q d\mu(z)\lesssim\int_{\Pi^+}G_{\alpha,\mu,\delta}(w)|f(w)|^q dA_\alpha(w).$$

By Hölder's inequality,

$$\int_{\Pi^+}|f(z)|^q d\mu(z)\lesssim\left(\int_{\Pi^+}[G_{\alpha,\mu,\delta}(w)]^{\frac{p}{p-q}}dA_\alpha(w)\right)^{\frac{p-q}{p}}\|f\|_{A^p_\alpha}^q,$$

as desired. □

The following lemma provides an estimate of the difference of our test functions in terms of the pseudo-hyperbolic distance.

Lemma 2.12. *Given $s > 0$ and $0 < \delta < 1$, there exists a constant $C = C(\delta, s) > 0$ such that*

$$|\tau_a^s(w) - \tau_a^s(z)| \geqslant C|\tau_a^s(w)|\rho(z,w)$$

for all $a, z \in \Pi^+$ and $w \in E_\delta(a)$.

Proof. Let $s > 0$ and $0 < \delta < 1$. From (2.2),

$$\left|1 - \frac{\tau_a(z)}{\tau_a(w)}\right| = \rho(z,w)\left|\frac{z - \overline{w}}{z - \overline{a}}\right| \approx \rho(z,w) \tag{2.19}$$

for $a, z \in \Pi^+$ and $w \in E_\delta(a)$. Thus, by the triangle inequality,

$$\left|\frac{\tau_a(z)}{\tau_a(w)}\right|^s \leqslant M$$

for some $M \geqslant 1$ depending only on δ and s. If $\Re(\frac{\tau_a(z)}{\tau_a(w)})^s \geqslant \frac{1}{2}$, let

$$K = \left\{\xi \in \mathbb{C} : |\xi| \leqslant M \text{ and } \Re\xi \geqslant \frac{1}{4}\right\}.$$

Then, K is a nonempty compact subset of $\mathbb{C}$. Let

$$h(\xi) = (\xi)^{\frac{1}{s}} \quad \text{and} \quad \xi_0 := \left(\frac{\tau_a(z)}{\tau_a(w)}\right)^s.$$

Thus, by the mean value property,

$$\begin{aligned}\left|1 - \frac{\tau_a(z)}{\tau_a(w)}\right| &= |h(1) - h(\xi_0)| \leqslant |1 - \xi_0| \sup_{\xi \in K} |h'(\xi)| \\ &= \frac{1}{s}|1 - \xi_0| \max\{4^{1-\frac{1}{s}}, M^{\frac{1}{s}-1}\},\end{aligned}$$

i.e.,

$$\left|1 - \left(\frac{\tau_a(z)}{\tau_a(w)}\right)^s\right| \gtrsim \left|1 - \frac{\tau_a(z)}{\tau_a(w)}\right|.$$

Thus,

$$|\tau_a^s(w) - \tau_a^s(z)| \gtrsim |\tau_a^s(w)|\left|1 - \frac{\tau_a(z)}{\tau_a(w)}\right| \approx |\tau_a^s(w)|\rho(z,w).$$

Conversely, if $\Re(\frac{\tau_a(z)}{\tau_a(w)})^s < \frac{1}{2}$, then

$$\left|1 - \left(\frac{\tau_a(z)}{\tau_a(w)}\right)^s\right| \geqslant \left|1 - \Re\left(\frac{\tau_a(z)}{\tau_a(w)}\right)^s\right| > \frac{1}{2}.$$

Thus,

$$\left|1 - \frac{\tau_a(z)}{\tau_a(w)}\right| \leqslant 1 + \left|\frac{\tau_a(z)}{\tau_a(w)}\right| \leqslant 2(1 + M^{\frac{1}{s}})\left|1 - \left(\frac{\tau_a(z)}{\tau_a(w)}\right)^s\right|.$$

Therefore, there exists a constant $C > 0$ depending on δ and s such that

$$\begin{aligned}|\tau_a^s(w) - \tau_a^s(z)| &= |\tau_a^s(w)|\left|1 - \left(\frac{\tau_a(z)}{\tau_a(w)}\right)^s\right| \\ &\geqslant C|\tau_a^s(w)|\left|1 - \frac{\tau_a(z)}{\tau_a(w)}\right| \\ &\approx |\tau_a^s(w)|\rho(z,w).\end{aligned}$$

This completes the proof. □

For $\alpha > -1$, let

$$H_z(w) = \frac{(\Im z)^{\alpha+2}}{(w-\overline{z})^{2(\alpha+2)}}, \quad z, w \in \Pi^+$$

throughout this paper and

$$G_{\alpha,\mu,\delta}(z) = \frac{\mu[E_\delta(z)]}{(\Im z)^{\alpha+2}}.$$

Lemma 2.13. *Suppose that $\alpha > -1$, $0 < q < p < \infty$, $0 < \delta < 1$, and μ is a positive Borel measure on Π^+. Then, $G_{\alpha,\mu,\delta} \in L^{\frac{p}{p-q}}(dA_\alpha)$ if and only if the function*

$$F(z) = \int_{\Pi^+} |H_z(w)| d\mu(w), \quad z \in \Pi^+$$

is in $L^{\frac{p}{p-q}}(dA_\alpha)$.

Proof. For sufficiency, suppose that $0 < \delta < 1$ and $F \in L^{\frac{p}{p-q}}(dA_\alpha)$. Since

$$G_{\alpha,\mu,\delta}(z) \approx \int_{E_\delta(z)} |H_z(w)| d\mu(w) \leqslant \int_{\Pi^+} |H_z(w)| d\mu(w) = F(z), \quad z \in \Pi^+,$$

$G_{\alpha,\mu,\delta} \in L^{\frac{p}{p-q}}(dA_\alpha)$.

For necessity, suppose $G_{\alpha,\mu,\delta} \in L^{\frac{p}{p-q}}(dA_\alpha)$. Since $2(\alpha+2) > \alpha+2$, $H_z \in A^1_\alpha(\Pi^+)$ from Lemma 2.2. It follows from (2.5) that

$$|H_z(w)| \lesssim \left(\frac{\Im z}{\Im w}\right)^{\alpha+2} \int_{E_\delta(w)} \frac{1}{|u-\overline{z}|^{2(\alpha+2)}} dA_\alpha(u), \quad w \in \Pi^+.$$

Thus, from (2.1) and Fubini's theorem,

$$\begin{aligned}
F(z) &= \int_{\Pi^+} |H_z(w)| d\mu(w) \\
&\lesssim \int_{\Pi^+} \left[\left(\frac{\Im z}{\Im w}\right)^{\alpha+2} \int_{E_\delta(w)} \frac{1}{|u-\overline{z}|^{2(\alpha+2)}} dA_\alpha(u)\right] d\mu(w) \\
&\leqslant \int_{\Pi^+} \left[\frac{1}{(\Im w)^{\alpha+2}} \int_{E_\delta(w)} \frac{1}{|u-\overline{z}|^{\alpha+2}} dA_\alpha(u)\right] d\mu(w) \\
&= \int_{\Pi^+} \left[\int_{\Pi^+} \frac{\chi_{E_\delta(u)}(w)}{(\Im w)^{\alpha+2}} d\mu(w)\right] \frac{1}{|u-\overline{z}|^{\alpha+2}} dA_\alpha(u) \\
&\approx \int_{\Pi^+} \frac{\mu[E_\delta(u)]}{(\Im u)^{\alpha+2}} \frac{1}{|u-\overline{z}|^{\alpha+2}} dA_\alpha(u) \\
&= T_{\alpha+2}(G_{\alpha,\mu,\delta})(z),
\end{aligned}$$

where $T_{\alpha+2}$ is defined in (2.7). Note that

$$\alpha + 2 - \left(1 + \frac{\alpha+1}{p/(p-q)}\right) = (\alpha+1)\frac{2p-q}{p} > 0. \tag{2.20}$$

Furthermore, by Lemma 2.3, the operator $T_{\alpha+2}$ is bounded on $L^{\frac{p}{p-q}}(dA_\alpha)$. Thus, $F \in L^{\frac{p}{p-q}}(dA_\alpha)$. □

3 Boundedness and compactness of difference

In this section, we characterize bounded and compact differences of composition operators by means of joint measures. We define some notations used in the rest of this paper. For $\varphi, \psi \in S(\Pi^+)$ and $0 < \delta < 1$, we let

$$\sigma(z) = \sigma_{\varphi,\psi}(z) := \rho(\varphi(z), \psi(z))$$

and

$$\Omega_\delta := \{z \in \Pi^+ : \sigma(z) < \delta\}. \tag{3.1}$$

Given $\alpha > -1$ and $0 < q < \infty$, we define the joint pullback measure $\omega_{\mu,q}$ as

$$\omega_{\mu,q}(E) = \int_{\varphi^{-1}(E)} \sigma^q d\mu + \int_{\psi^{-1}(E)} \sigma^q d\mu$$

for any Borel set $E \subset \Pi^+$. Then, $\omega_{\mu,q}$ is actually the sum of two pullback measures, $(\sigma^q d\mu) \circ \varphi^{-1}$ and $(\sigma^q d\mu) \circ \psi^{-1}$.

Our principal result regarding the bounded difference of composition operators is the following characterization.

Theorem 3.1. *Suppose $0 < p, q < \infty$, and $\alpha > -1$. Let $\varphi, \psi \in S(\Pi^+)$ and μ be a positive Borel measure on Π^+. Then, $C_\varphi - C_\psi$ is bounded from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$ if and only if $\omega_{\mu,q}$ is an (α, p, q)-Carleson measure.*

Proof. For sufficiency, assume that $\omega_{\mu,q}$ is an (α, p, q)-Carleson measure. Let $0 < \delta < 1$. For $f \in A^p_\alpha(\Pi^+)$,

$$\begin{aligned} \int_{\Pi^+} |(C_\varphi - C_\psi)f|^q d\mu &= \int_{\Omega_{\frac{\delta}{2}}} |(C_\varphi - C_\psi)f|^q d\mu + \int_{\Pi^+\setminus\Omega_{\frac{\delta}{2}}} |(C_\varphi - C_\psi)f|^q d\mu \\ &\lesssim \int_{\Omega_{\frac{\delta}{2}}} |(C_\varphi - C_\psi)f|^q d\mu + \int_{\Pi^+\setminus\Omega_{\frac{\delta}{2}}} |f(\varphi)|^q + |f(\psi)|^q d\mu \\ &=: I(f) + II(f). \end{aligned} \tag{3.2}$$

The second term of the above equation is easily handled. In particular, for $z \in \Pi^+ \setminus \Omega_{\frac{\delta}{2}}$, $\frac{1}{2} \leqslant \frac{\sigma(z)}{\delta}$. Thus,

$$\begin{aligned} II(f) &\leqslant \frac{2^q}{\delta^q} \int_{\Pi^+\setminus\Omega_{\frac{\delta}{2}}} (|f(\varphi)|^q + |f(\psi)|^q)\sigma^q d\mu \\ &\lesssim \int_{\Pi^+} (|f(\varphi)|^q + |f(\psi)|^q)\sigma^q d\mu \\ &= \int_{\Pi^+} |f|^q d\omega_{\mu,q}. \end{aligned} \tag{3.3}$$

Since $\omega_{\mu,q}$ is an (α, p, q)-Carleson measure, $II(f) \lesssim \|f\|^q_{A^p_\alpha}$.

Furthermore, we estimate the first term of (3.2) by Lemma 2.5 and Fubini's theorem,

$$\begin{aligned} I(f) &\lesssim \int_{\Omega_{\frac{\delta}{2}}} \left[\frac{\sigma^q(z)}{A_\alpha[E_\delta(\varphi(z))]} \int_{E_\delta(\varphi(z))} |f(w)|^q dA_\alpha(w)\right] d\mu(z) \\ &= \int_{\Pi^+} \left[\int_{\Omega_{\frac{\delta}{2}} \cap \varphi^{-1}[E_\delta(w)]} \frac{\sigma^q(z)}{A_\alpha[E_\delta(\varphi(z))]} d\mu(z)\right] |f(w)|^q dA_\alpha(w). \end{aligned} \tag{3.4}$$

From (2.4) and (2.1), we have

$$A_\alpha[E_\delta(\varphi(z))] \approx (\Im\varphi(z))^{\alpha+2} \approx (\Im w)^{\alpha+2}$$

for all $z \in \varphi^{-1}[E_\delta(w)]$. The same estimate holds when the roles of φ and ψ are interchanged. Thus, from (3.4),

$$I(f) \lesssim \int_{\Pi^+} G_{\alpha,\omega_{\mu,q},\delta}(w)|f(w)|^q dA_\alpha(w). \tag{3.5}$$

If $0 < q < p < \infty$, by Hölder's inequality,

$$\begin{aligned}&\int_{\Pi^+} G_{\alpha,\omega_{\mu,q},\delta}(w)|f(w)|^q dA_\alpha(w)\\&\quad\leqslant \bigg(\int_{\Pi^+} [G_{\alpha,\omega_{\mu,q},\delta}(w)]^{\frac{p}{p-q}} dA_\alpha(w)\bigg)^{\frac{p-q}{p}} \|f\|_{A^p_\alpha}^q. \end{aligned} \tag{3.6}$$

If $0 < p \leqslant q < \infty$, by (2.6), then

$$\begin{aligned}&\int_{\Pi^+} G_{\alpha,\omega_{\mu,q},\delta}(w)|f(w)|^q dA_\alpha(w)\\&\quad\leqslant \Big(\sup_{w\in\Pi^+} H_{\alpha,\omega_{\mu,q},\delta}(w)\Big)\int_{\Pi^+} (\Im w)^{(\frac{(q-p)(\alpha+2)}{p})}|f(w)|^{q-p}|f(w)|^p dA_\alpha(w)\\&\quad\lesssim \Big(\sup_{w\in\Pi^+} H_{\alpha,\omega_{\mu,q},\delta}(w)\Big)\|f\|_{A^p_\alpha}^{q-p}\int_{\Pi^+} |f(w)|^p dA_\alpha(w)\\&\quad= \Big(\sup_{w\in\Pi^+} H_{\alpha,\omega_{\mu,q},\delta}(w)\Big)\|f\|_{A^p_\alpha}^q. \end{aligned} \tag{3.7}$$

Therefore, by Theorem 2.11, (3.6) and (3.7), it is always the case that

$$\int_{\Pi^+} |(C_\varphi - C_\psi)f|^q d\mu \lesssim \|f\|_{A^p_\alpha}^q.$$

This proves the sufficiency.

For necessity, assume that $C_\varphi - C_\psi$ is bounded from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$. If $0 < q < p < \infty$, let $s > \max\{1, \frac{1}{p}\} + \frac{\alpha+1}{p}$. Taking $\{c_k\} \in l^p$, $t \in [0,1]$, and the Rademacher functions r_k by the atom decomposition theorem (see Theorem 2.8), there exists a constant $0 < \lambda < 1$ such that

$$f_t := \sum_{k=1}^\infty c_k r_k(t)\frac{\tau^s_{w_k}}{\|\tau^s_{w_k}\|_{A^p_\alpha}} \in A^p_\alpha(\Pi^+) \quad \text{and} \quad \|f_t\|_{A^p_\alpha} \lesssim \|\{c_k\}\|_{l^p},$$

where $\{w_k\} \subset \Pi^+$ is a δ-lattice and $0 < \delta < \lambda$. Without loss of generality, we can assume $0 < \lambda < \frac{1}{4}$. Since $C_\varphi - C_\psi$ is bounded from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$,

$$\int_{\Pi^+} \bigg|\sum_{k=1}^\infty c_k r_k(t)\frac{(C_\varphi - C_\psi)\tau^s_{w_k}(z)}{\|\tau^s_{w_k}\|_{A^p_\alpha}}\bigg|^q d\mu(z) \lesssim \|f_t\|_{A^p_\alpha}^q \lesssim \|\{c_k\}\|_{l^p}^q.$$

Then, by an argument similar to that of (2.14),

$$\int_{\Pi^+} \bigg[\sum_{k=1}^\infty |c_k|^2\frac{|(C_\varphi - C_\psi)\tau^s_{w_k}(z)|^2}{\|\tau^s_{w_k}\|_{A^p_\alpha}^2}\bigg]^{\frac{q}{2}} d\mu(z) \lesssim \|\{c_k\}\|_{l^p}^q. \tag{3.8}$$

By (3.8) and Lemma 2.12,

$$\begin{aligned}\sum_{k=1}^\infty |c_k|^q H_{\alpha,v,2\delta}(w_k) &= \sum_{k=1}^\infty |c_k|^q \int_{\varphi^{-1}(E_{2\delta}(w_k))} \frac{\sigma^q(z)}{(\Im w_k)^{\frac{q(\alpha+2)}{p}}} d\mu(z)\\&= \int_{\Pi^+} \sum_{k=1}^\infty |c_k|^q \chi_{\varphi^{-1}(E_{2\delta}(w_k))}(z)\frac{\sigma^q(z)}{(\Im w_k)^{\frac{q(\alpha+2)}{p}}} d\mu(z)\\&\lesssim \int_{\Pi^+} \sum_{k=1}^\infty |c_k|^q \chi_{\varphi^{-1}(E_{2\delta}(w_k))}(z)\bigg|\frac{(C_\varphi - C_\psi)\tau^s_{w_k}(z)}{\|\tau^s_{w_k}\|_{A^p_\alpha}}\bigg|^q d\mu(z)\\&=: M.\end{aligned}$$

By Lemma 2.4, there exists a positive integer N such that for any $z \in \Pi^+$, $\varphi(z)$ does not belong to more than N sets $E_{2\delta}(w_n)$. By an argument similar to that of (2.15),

$$\begin{aligned} M &\lesssim \int_{\Pi^+} \left[\sum_{k=1}^{\infty} |c_k|^2 \frac{|(C_\varphi - C_\psi)\tau_{w_k}^s(z)|^2}{\|\tau_{w_k}^s\|_{A_\alpha^p}^2} \right]^{\frac{q}{2}} d\mu(z) \\ &\lesssim \int_{\Pi^+} \left[\sum_{k=1}^{\infty} |c_k|^2 \frac{|(C_\varphi - C_\psi)\tau_{w_k}^s(z)|^2}{\|\tau_{w_k}^s\|_{A_\alpha^p}^2} \right]^{\frac{q}{2}} d\mu(z). \end{aligned}$$

By this and (3.8),

$$\sum_{k=1}^{\infty} |c_k|^q H_{\alpha,v,2\delta}(w_k) \lesssim \|\{c_k\}\|_{l^p}^q.$$

The same estimate holds when we replace φ by ψ. Thus, we deduce

$$\sum_{k=1}^{\infty} |c_k|^q H_{\alpha,\omega_{\mu,q},2\delta}(w_k) \lesssim \|\{c_k\}\|_{l^p}^q.$$

Thus, for any $\{b_k\} \in l^{\frac{p}{q}}$,

$$\sum_{k=1}^{\infty} |b_k| H_{\alpha,\omega_{\mu,q},2\delta}(w_k) \lesssim \|\{b_k\}\|_{l^{\frac{p}{q}}},$$

which implies that

$$\{H_{\alpha,\omega_{\mu,q},2\delta}(w_n)\}_n \in (l^{\frac{p}{q}})^* = l^{\frac{p}{p-q}}.$$

Then, by Theorem 2.11, $\omega_{\mu,q}$ is an (α,p,q)-Carleson measure.

If $0 < p \leqslant q < \infty$, let $s > \frac{\alpha+2}{p}$. For any $0 < \delta < 1$, by (2.2) and Lemma 2.12,

$$|(C_\varphi - C_\psi)\tau_w^s(z)|^q \gtrsim \sigma^q(z)|\tau_w^s(\varphi(z))|^q \approx \frac{\sigma^q(z)}{(\Im w)^{sq}},$$

where $z \in \varphi^{-1}[E_\delta(w)]$. Thus,

$$\begin{aligned} \int_{\Pi^+} |(C_\varphi - C_\psi)\tau_w^s(z)|^q d\mu(z) &\geqslant \int_{\varphi^{-1}(E_\delta(w))} |(C_\varphi - C_\psi)\tau_w^s(z)|^q d\mu(z) \\ &\approx \frac{1}{(\Im w)^{sq}} \int_{\varphi^{-1}(E_\delta(w))} \sigma^q(z) d\mu(z). \end{aligned} \tag{3.9}$$

The same estimate holds when the roles of φ and ψ are interchanged. Then, we deduce

$$\frac{\|(C_\varphi - C_\psi)\tau_w^s\|_{L^q}^q}{\|\tau_w^s\|_{A_\alpha^p}^q} \gtrsim H_{\alpha,\omega_{\mu,q},\delta}(w). \tag{3.10}$$

Since $C_\varphi - C_\psi$ is bounded from $A_\alpha^p(\Pi^+)$ to $L^q(\mu)$,

$$\frac{\|(C_\varphi - C_\psi)\tau_w^s\|_{L^q}^q}{\|\tau_w^s\|_{A_\alpha^p}^q} \lesssim 1.$$

Thus, $\omega_{\mu,q}$ is an (α,p,q)-Carleson measure due to Theorem 2.11. □

Our principal result regarding the compact difference of composition operators is the following characterization.

Theorem 3.2. *Suppose $0 < p, q < \infty$, and $\alpha > -1$. Let $\varphi, \psi \in S(\Pi^+)$, and μ be a positive Borel measure on Π^+. Then, $C_\varphi - C_\psi$ is compact from $A_\alpha^p(\Pi^+)$ to $L^q(\mu)$ if and only if $\omega_{\mu,q}$ is a compact (α,p,q)-Carleson measure. Moreover, if $0 < q < p < \infty$, then $C_\varphi - C_\psi$ is compact from $A_\alpha^p(\Pi^+)$ to $L^q(\mu)$ if and only if it is bounded.*

Proof. We use the same notation as in the proof of Theorem 3.1. Note that $C_\varphi - C_\psi$ is compact from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$ if and only if $(C_\varphi - C_\psi)f_n \to 0$ in $L^q(\mu)$ for any bounded sequence $\{f_n\}$ in $A^p_\alpha(\Pi^+)$ satisfying $f_n \to 0$ uniformly on compact subsets of Π^+. This can be proved by modifying the argument in [11, Proposition 3.11].

For sufficiency, assume $\omega_{\mu,q}$ is a compact (α,p,q)-Carleson measure. Then $\omega_{\mu,q}$ is locally finite from Theorem 2.11. Let f_n be a sequence in $A^p_\alpha(\Pi^+)$ that converges to 0 uniformly on compact subsets of Π^+, and $\|f_n\|_{A^p_\alpha} \leqslant M$ for some positive constants M. Then, $\lim_{n\to\infty} II(f_n) = 0$ by (3.3).

If $0 < q < p < \infty$, by Theorem 2.11, there exists $0 < \delta < 1$ such that $G_{\alpha,\omega_{\mu,q},\delta} \in L^{\frac{p}{p-q}}(dA_\alpha)$. For $\epsilon > 0$, let

$$K_\epsilon = \left\{ z \in \Pi^+ : |z - \epsilon \mathrm{i}| \leqslant \frac{1}{\epsilon} \text{ and } \Im z \geqslant \epsilon \right\}$$

and

$$Q_\epsilon = \Pi^+ \setminus K_\epsilon.$$

By (3.5) and Hölder's inequality,

$$\begin{aligned} I(f_n) &\lesssim \int_{\Pi^+} G_{\alpha,\omega_{\mu,q},\delta}(w)|f_n(w)|^q dA_\alpha(w) \\ &\leqslant \int_{K_\epsilon} G_{\alpha,\omega_{\mu,q},\delta}(w)|f_n(w)|^q dA_\alpha(w) \\ &\quad + \int_{\Pi^+} \chi_{Q_\epsilon}(w) G_{\alpha,\omega_{\mu,q},\delta}(w)|f_n(w)|^q dA_\alpha(w) \\ &\leqslant \int_{K_\epsilon} G_{\alpha,\omega_{\mu,q},\delta}(w)|f_n(w)|^q dA_\alpha(w) \\ &\quad + \left(\int_{\Pi^+} [\chi_{Q_\epsilon}(w) G_{\alpha,\omega_{\mu,q},\delta}(w)]^{\frac{p}{p-q}} dA_\alpha(w) \right)^{\frac{p-q}{p}} M^q. \end{aligned}$$

Since $\omega_{\mu,q}$ is locally finite and f_n converges to 0 uniformly on compact subsets of each K_ϵ,

$$\limsup_{n\to\infty} \int_{K_\epsilon} G_{\alpha,\omega_{\mu,q},\delta}(w)|f_n(w)|^q dA_\alpha(w) = 0.$$

Then,

$$\limsup_{n\to\infty} I(f_n) \lesssim \left(\int_{\Pi^+} [\chi_{Q_\epsilon}(w) G_{\alpha,\omega_{\mu,q},\delta}(w)]^{\frac{p}{p-q}} dA_\alpha(w) \right)^{\frac{p-q}{p}}.$$

Note that

$$\chi_{Q_\epsilon}(w) G_{\alpha,\omega_{\mu,q},\delta}(w) \to 0 \quad \text{as} \quad \epsilon \to 0^+$$

for every $w \in \Pi^+$, since

$$\chi_{Q_\epsilon}(w) G_{\alpha,\omega_{\mu,q},\delta}(w) \leqslant G_{\alpha,\omega_{\mu,q},\delta}(w), \quad w \in \Pi^+ \quad \text{and} \quad G_{\alpha,\omega_{\mu,q},\delta} \in L^{\frac{p}{p-q}}(dA_\alpha).$$

Owing to the dominated convergence theorem,

$$\limsup_{n\to\infty} I(f_n) = 0.$$

Thus, $C_\varphi - C_\psi$ is compact from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$.

If $0 < p \leqslant q < \infty$, then repeating the argument in (3.5) and (2.17) and using Theorem 2.11 yields that $\lim_{n\to\infty} I(f_n) = 0$. Thus, $C_\varphi - C_\psi$ is compact from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$, as desired.

For necessity, suppose $C_\varphi - C_\psi$ is compact from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$. Then $\omega_{\mu,q}$ is locally finite from Theorems 2.11 and 3.1. If $0 < q < p < \infty$, then $C_\varphi - C_\psi$ is bounded from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$. From Theorem 3.1, $\omega_{\mu,q}$ is an (α,p,q)-Carleson measure. By Theorem 2.11, this is equivalent to $\omega_{\mu,q}$ being a compact (α,p,q)-Carleson measure. If $0 < p \leqslant q < \infty$, we deduce from (3.10), Lemma 2.2 and Theorem 2.11 that $\omega_{\mu,q}$ is a compact (α,p,q)-Carleson measure, as desired.

Note that if $0 < q < p < \infty$, by Theorems 2.11 and 3.1, $C_\varphi - C_\psi$ is bounded from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$ if and only if $\omega_{\mu,q}$ is an (α, p, q)-Carleson measure if and only if $\omega_{\mu,q}$ is a compact (α, p, q)-Carleson measure if and only if $C_\varphi - C_\psi$ is compact from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$. □

With the theorems above, we obtain the following direct analytic characterizations of the bounded and compact difference of composition operators between such spaces. Therefore, we set some notations. For $\varphi, \psi \in S(\Pi^+)$, $z \in \Pi^+$, let

$$H_{\mu,\varphi,\psi}(z) = \int_{\Pi^+} |H_z^{\frac{1}{p}}(\varphi(w)) - H_z^{\frac{1}{p}}(\psi(w))|^q d\mu(w)$$

and

$$G_{\mu,\varphi,\psi}(z) = \int_{\Pi^+} |H_z^{\frac{1}{q}}(\varphi(w)) - H_z^{\frac{1}{q}}(\psi(w))|^q d\mu(w).$$

Theorem 3.3. *Suppose that $0 < p \leqslant q < \infty, \alpha > -1$, and μ is a positive Borel measure on Π^+. Let $\varphi, \psi \in S(\Pi^+)$. Then, the following statements hold*:

(1) *$C_\varphi - C_\psi$ is bounded from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$ if and only if* $\sup_{z\in\Pi^+} H_{\mu,\varphi,\psi}(z) < \infty$;

(2) *$C_\varphi - C_\psi$ is compact from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$ if and only if $\omega_{\mu,q}$ is locally finite and* $\lim_{z\to\partial\widehat{\Pi}^+} H_{\mu,\varphi,\psi}(z) = 0$.

Proof. Sufficiency. Taking $s = \frac{2(\alpha+2)}{p}$ in (3.10) yields that

$$H_{\alpha,\omega_{\mu,q},\delta}(z) \lesssim H_{\mu,\varphi,\psi}(z), \quad z \in \Pi^+.$$

By Theorems 2.11 and 3.1 (resp. Theorem 3.2), $C_\varphi - C_\psi$ is bounded (resp. compact) from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$ if $\sup_{z\in\Pi^+} H_{\mu,\varphi,\psi}(z) < \infty$ (resp. $\omega_{\mu,q}$ is locally finite and $\lim_{z\to\partial\widehat{\Pi}^+} H_{\mu,\varphi,\psi}(z) = 0$).

Necessity. With

$$s = \frac{2(\alpha+2)}{p} \quad \text{and} \quad g_z := \frac{\tau_z^s}{\|\tau_z^s\|_{A^p_\alpha}}$$

for $z \in \Pi^+$, $g_z \in A^p_\alpha(\Pi^+)$ and $g_z \to 0$ uniformly on any compact subsets of Π^+ as $z \to \partial\widehat{\Pi}^+$ by Lemma 2.2. Note that

$$H_{\mu,\varphi,\psi}(z) = \int_{\Pi^+} |(C_\varphi - C_\psi)g_z(w)|^q d\mu(w).$$

If $C_\varphi - C_\psi$ is bounded from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$, then $\sup_{z\in\Pi^+} H_{\mu,\varphi,\psi}(z) < \infty$. If $C_\varphi - C_\psi$ is compact from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$, then

$$\lim_{z\to\partial\widehat{\Pi}^+} H_{\mu,\varphi,\psi}(z) = 0.$$

By Theorems 2.11 and 3.2, $\omega_{\mu,q}$ is locally finite. □

Theorem 3.4. *Suppose that $0 < q < p < \infty, \alpha > -1$, and μ is a positive Borel measure on Π^+. Let $\varphi, \psi \in S(\Pi^+)$. Then, the following statements are equivalent*:

(1) *$C_\varphi - C_\psi$ is bounded from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$*;

(2) *$C_\varphi - C_\psi$ is compact from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$*;

(3) $G_{\mu,\varphi,\psi} \in L^{\frac{p}{p-q}}(dA_\alpha)$.

Proof. It is sufficient to prove (1) ⇔ (3). (1) ⇒ (3). Suppose that $C_\varphi - C_\psi$ is bounded from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$. Then, by Theorems 2.11 and 3.1, there exists $0 < \delta < 1/4$ such that

$$G_{\alpha,\omega_{\mu,q},\delta} \in L^{\frac{p}{p-q}}(dA_\alpha), \quad z \in \Pi^+.$$

Taking $s = \frac{2(\alpha+2)}{q}$ and $f_z = (\Im z)^{\frac{s}{2}} \tau_z^s$, we have

$$G_{\mu,\varphi,\psi}(z) = \int_{\Pi^+} |(C_\varphi - C_\psi) f_z|^q d\mu(w)$$

$$
\begin{aligned}
&= \int_{\Omega_{\frac{\delta}{2}}} |(C_\varphi - C_\psi) f_z|^q d\mu(w) + \int_{\Pi^+ \setminus \Omega_{\frac{\delta}{2}}} |(C_\varphi - C_\psi) f_z|^q d\mu(w) \\
&\lesssim \int_{\Omega_{\frac{\delta}{2}}} |f_z(\varphi) - f_z(\psi)|^q d\mu(w) + \int_{\Pi^+ \setminus \Omega_{\frac{\delta}{2}}} |f_z(\varphi)|^q + |f_z(\psi)|^q d\mu(w) \\
&=: I(f_z) + II(f_z).
\end{aligned}
$$

By an argument similar to that of (3.4),

$$
\begin{aligned}
I(f_z) &\lesssim \int_{\Pi^+} G_{\alpha,\omega_{\mu,q},\delta}(w) \frac{(\Im z)^{\alpha+2}}{|w-\overline{z}|^{2(\alpha+2)}} dA_\alpha(w) \\
&\leqslant \int_{\Pi^+} G_{\alpha,\omega_{\mu,q},\delta}(w) \frac{1}{|w-\overline{z}|^{(\alpha+2)}} dA_\alpha(w) \\
&= T_{\alpha+2}(G_{\alpha,\omega_{\mu,q},\delta})(z),
\end{aligned}
$$

where $T_{\alpha+2}$ is defined in (2.7). By an argument similar to that of (3.3),

$$
II(f_z) \lesssim \int_{\Pi^+} [|H_z(\varphi(w))| + |H_z(\psi(w))|] \sigma^q(w) d\mu(w).
$$

Then, by (2.20) and Lemma 2.3, the operator $T_{\alpha+2}$ is bounded on $L^{\frac{p}{p-q}}(dA_\alpha)$. By Lemma 2.13 and Theorem 3.1, the function

$$
F(z) = \int_{\Pi^+} [|H_z(\varphi(w))| + |H_z(\psi(w))|] \sigma^q(w) d\mu(w)
$$

is in $L^{\frac{p}{p-q}}(dA_\alpha)$. Therefore, $G_{\mu,\varphi,\psi} \in L^{\frac{p}{p-q}}(dA_\alpha)$.

(3)$\Rightarrow$ (1). Suppose that the function $G_{\mu,\varphi,\psi}$ is in $L^{\frac{p}{p-q}}(dA_\alpha)$. For any $0<\delta<1$, $w \in \varphi^{-1}(E_\delta(z))$, and $z \in \Pi^+$, by Lemma 2.12,

$$
\int_{\Pi^+} |H_z^{\frac{1}{q}}(\varphi(w)) - H_z^{\frac{1}{q}}(\psi(w))|^q d\mu(w) \gtrsim \frac{1}{(\Im z)^{\alpha+2}} \int_{\varphi^{-1}(E_\delta(z))} \sigma^q(w) d\mu(w).
$$

The same estimate holds when the roles of φ and ψ are interchanged. Therefore, we deduce

$$
G_{\mu,\varphi,\psi}(z) \gtrsim G_{\alpha,\omega_{\mu,q},\delta}(z), \quad z \in \Pi^+.
$$

Therefore, by Theorems 2.11 and 3.1, $C_\varphi - C_\psi$ is bounded from $A^p_\alpha(\Pi^+)$ to $L^q(\mu)$. □

We close this section with the following example, which shows that there exist symbols φ and ψ inducing bounded or compact difference $C_\varphi - C_\psi$ acting from $A^p_\alpha(\Pi^+)$ to $A^q_\beta(\Pi^+)$ for any $0<p,q<\infty$ by using Theorems 2.11, 3.1 and 3.2. The example below is cited from [6, Example 7.8]. Based on that proof, a straightforward calculation yields the conclusions.

Example 3.5. For $0<p,q<\infty$ and $s>0$, let

$$
\varphi(z) = 2\pi\mathrm{i} + \log(z+e\mathrm{i}) \quad \text{and} \quad \psi(z) = \varphi(z) + \frac{\pi}{(z+e\mathrm{i})^s (\log(z+e\mathrm{i}))^{\frac{1}{q}}}.
$$

Then, the following claims hold for $\alpha, \beta > -1$:

(1) neither C_φ nor C_ψ is bounded from $A^p_\alpha(\Pi^+)$ to $A^q_\beta(\Pi^+)$;

(2) if $0<p\leqslant q<\infty$, then $C_\varphi - C_\psi$ is bounded/compact from $A^p_\alpha(\Pi^+)$ to $A^q_\beta(\Pi^+)$ if and only if $s \geqslant \frac{(\beta+2)}{q}$;

(3) if $0<q<p<\infty$, then $C_\varphi - C_\psi$ is bounded/compact from $A^p_\alpha(\Pi^+)$ to $A^q_\beta(\Pi^+)$ if and only if $s > \frac{(\beta+2)}{q}$.

4 Hilbert-Schmidt difference

In this section, we characterize the Hilbert-Schmidt character of differences of composition operators from $A^2_\alpha(\Pi^+)$ to $L^2(\mu)$ by means of the joint Carleson measure. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be separable Hilbert spaces. A compact linear operator T from X to Y is Hilbert-Schmidt if

$$\|T\|^2_{HS(X,Y)} := \sum_{j=1}^{\infty} \|Te_j\|^2_Y < \infty$$

for any orthonormal basis $\{e_n\}$ of X. For $\varphi, \psi \in S(\Pi^+)$, $C_\varphi - C_\psi$ is Hilbert-Schmidt from $A^2_\alpha(\Pi^+)$ to $L^2(\mu)$ if and only if

$$\|C_\varphi - C_\psi\|^2_{HS(A^2_\alpha(\Pi^+),L^2(\mu))} = \sum_{j=1}^{\infty} \|(C_\varphi - C_\psi)e_j\|^2_{L^2} < \infty.$$

From [2, Proposition 3.1],

$$\begin{aligned}\|C_\varphi - C_\psi\|^2_{HS(A^2_\alpha(\Pi^+),L^2(\mu))} &= \sum_{j=1}^{\infty} \|(C_\varphi - C_\psi)e_j\|^2_{L^2} \\ &= \int_{\Pi^+}\int_{\Pi^+} |K^{(\alpha)}_{\varphi(z)}(w) - K^{(\alpha)}_{\psi(z)}(w)|^2 dA_\alpha(w)d\mu(z) \\ &= \int_{\Pi^+}\int_{\Pi^+} |K^{(\alpha)}_{w}(\varphi(z)) - K^{(\alpha)}_{w}(\psi(z))|^2 d\mu(z)dA_\alpha(w). \end{aligned} \tag{4.1}$$

Recall

$$\omega_{\mu,q}(E) = \int_{\varphi^{-1}(E)} \sigma^q d\mu + \int_{\psi^{-1}(E)} \sigma^q d\mu.$$

Our principal result on the Hilbert-Schmidt difference of composition operators is the following.

Theorem 4.1. *Assume that $\alpha > -1$, $\varphi, \psi \in S(\Pi^+)$, and μ is a positive Borel measure on Π^+. Then, compact operator $C_\varphi - C_\psi$ is Hilbert-Schmidt from $A^2_\alpha(\Pi^+)$ to $L^2(\mu)$ if and only if*

$$F(w) = \frac{\omega_{\mu,2}[E_{2\delta}(w)]}{(\Im w)^{2(\alpha+2)}} \in L^1(dA_\alpha)$$

if and only if

$$\left\{\frac{\omega_{\mu,2}[E_{2\delta}(a_n)]}{(\Im a_n)^{\alpha+2}}\right\}_n \in l^1,$$

where $\{a_n\} \subset \Pi^+$ is a δ-lattice and $0 < \delta < \frac{1}{4}$. Moreover,

$$\|C_\varphi - C_\psi\|^2_{HS(A^2_\alpha(\Pi^+),L^2(\mu))} \approx \int_{\Pi^+} \frac{\omega_{\mu,2}[E_{2\delta}(w)]}{(\Im w)^{2(\alpha+2)}} dA_\alpha(w) \approx \sum_{n=1}^{\infty} \frac{\omega_{\mu,2}[E_{2\delta}(a_n)]}{(\Im a_n)^{\alpha+2}}.$$

Proof. Let $0 < \delta < \frac{1}{4}$. Then, by (2.3), (4.1) and Lemma 2.12,

$$\begin{aligned}&\|C_\varphi - C_\psi\|^2_{HS(A^2_\alpha(\Pi^+),L^2(\mu))} \\ &\quad \geqslant \int_{\Pi^+}\int_{\varphi^{-1}(E_{2\delta}(w))} |K^{(\alpha)}_{w}(\varphi(z)) - K^{(\alpha)}_{w}(\psi(z))|^2 d\mu(z)dA_\alpha(w) \\ &\quad \geqslant \int_{\Pi^+}\int_{\varphi^{-1}(E_{2\delta}(w))} \frac{\sigma^2(z)}{(\Im w)^{2(\alpha+2)}} d\mu(z)dA_\alpha(w).\end{aligned}$$

The same estimate holds when the roles of φ and ψ are interchanged. Thus,

$$\|C_\varphi - C_\psi\|^2_{HS(A^2_\alpha(\Pi^+),L^2(\mu))} \gtrsim \int_{\Pi^+} \frac{\omega_{\mu,2}[E_{2\delta}(w)]}{(\Im w)^{2(\alpha+2)}} dA_\alpha(w). \tag{4.2}$$

For any δ-lattice $\{a_n\} \subset \Pi^+$, by Proposition 2.1 and Lemma 2.4,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\omega_{\mu,2}[E_{2\delta}(a_n)]}{(\Im a_n)^{\alpha+2}} &\lesssim \sum_{n=1}^{\infty} \frac{1}{(\Im a_n)^{2(\alpha+2)}} \int_{E_{2\delta}(a_n)} \omega_{\mu,2}[E_\delta(z)] dA_\alpha(z) \\ &\approx \sum_{n=1}^{\infty} \int_{E_{2\delta}(a_n)} \frac{\omega_{\mu,2}[E_{2\delta}(z)]}{(\Im z)^{2(\alpha+2)}} dA_\alpha(z) \\ &\lesssim \int_{\Pi^+} \frac{\omega_{\mu,2}[E_{2\delta}(z)]}{(\Im z)^{2(\alpha+2)}} dA_\alpha(z). \end{aligned} \tag{4.3}$$

Let $0 < \delta' \leqslant \frac{\delta}{\delta+2}$. From (3.1),

$$\begin{aligned}&\int_{\Pi^+} |K_w^{(\alpha)}(\varphi(z)) - K_w^{(\alpha)}(\psi(z))|^2 d\mu(z) \\ &= \int_{\Omega_{\frac{\delta'}{2}}} |K_w^{(\alpha)}(\varphi(z)) - K_w^{(\alpha)}(\psi(z))|^2 d\mu(z) \\ &\quad + \int_{\Pi^+ \setminus \Omega_{\frac{\delta'}{2}}} |K_w^{(\alpha)}(\varphi(z)) - K_w^{(\alpha)}(\psi(z))|^2 d\mu(z) \\ &\lesssim \int_{\Omega_{\frac{\delta'}{2}}} |K_w^{(\alpha)}(\varphi(z)) - K_w^{(\alpha)}(\psi(z))|^2 d\mu(z) \\ &\quad + \int_{\Pi^+ \setminus \Omega_{\frac{\delta'}{2}}} |K_w^{(\alpha)}(\varphi(z))|^2 + |K_w^{(\alpha)}(\psi(z))|^2 d\mu(z) \\ &=: I(K_w^{(\alpha)}) + II(K_w^{(\alpha)}). \end{aligned} \tag{4.4}$$

By (2.2) and an argument similar to (3.4),

$$I(K_w^{(\alpha)}) \lesssim \int_{\Pi^+} \frac{\omega_{\mu,2}[E_{\delta'}(u)]}{(\Im u)^{\alpha+2}} |K_w^{(\alpha)}(u)|^2 dA_\alpha(u).$$

Since $E_{\delta'}(u) \subset E_{2\delta}(a_n)$, $\Im u \approx \Im a_n$, and $|K_w^{(\alpha)}(a_n)| \approx |K_w^{(\alpha)}(u)|$ for $u \in E_\delta(a_n)$ from (2.1) and (2.2),

$$\begin{aligned}&\int_{\Pi^+} \frac{\omega_{\mu,2}[E_{\delta'}(u)]}{(\Im u)^{\alpha+2}} |K_w^{(\alpha)}(u)|^2 dA_\alpha(u) \\ &\leqslant \sum_{n=1}^{\infty} \int_{E_{2\delta}(a_n)} \frac{\omega_{\mu,2}[E_{\delta'}(u)]}{(\Im u)^{\alpha+2}} |K_w^{\alpha}(u)|^2 dA_\alpha(u) \\ &\lesssim \sum_{n=1}^{\infty} |K_w^{(\alpha)}(a_n)|^2 \omega_{\mu,2}[E_{2\delta}(a_n)] \\ &= \sum_{n=1}^{\infty} |K_{a_n}^{(\alpha)}(w)|^2 \omega_{\mu,2}[E_{2\delta}(a_n)]. \end{aligned}$$

Thus,

$$I(K_w^{(\alpha)}) \lesssim \sum_{n=1}^{\infty} |K_{a_n}^{(\alpha)}(w)|^2 \omega_{\mu,2}[E_{2\delta}(a_n)]. \tag{4.5}$$

By (2.2) and an argument similar to that of (3.3),

$$II(K_w^{(\alpha)}) \lesssim \int_{\Pi^+} |K_w^{(\alpha)}(u)|^2 d\omega_{\mu,2}(u).$$

From (2.2),

$$\int_{\Pi^+} |K_w^{(\alpha)}(u)|^2 d\omega_{\mu,2}(u) \leqslant \sum_{n=1}^{\infty} \int_{E_{2\delta}(a_n)} |K_w^{(\alpha)}(u)|^2 d\omega_{\mu,2}(u)$$

$$\approx \sum_{n=1}^{\infty} |K_w^{(\alpha)}(a_n)|^2 \omega_{\mu,2}[E_{2\delta}(a_n)]$$
$$= \sum_{n=1}^{\infty} |K_{a_n}^{(\alpha)}(w)|^2 \omega_{\mu,2}[E_{2\delta}(a_n)].$$

Thus,

$$II(K_w^{(\alpha)}) \lesssim \sum_{n=1}^{\infty} |K_{a_n}^{(\alpha)}(w)|^2 \omega_{\mu,2}[E_{2\delta}(a_n)]. \tag{4.6}$$

By (4.4)–(4.6) and Lemma 2.2,

$$\|C_\varphi - C_\psi\|^2_{HS(A^2_\alpha(\Pi^+),L^2(\mu))} \lesssim \sum_{n=1}^{\infty} \omega_{\mu,2}[E_{2\delta}(a_n)] \int_{\Pi^+} |K_{a_n}^{(\alpha)}(w)|^2 dA_\alpha(w)$$
$$\approx \sum_{n=1}^{\infty} \frac{\omega_{\mu,2}[E_{2\delta}(a_n)]}{(\Im a_n)^{\alpha+2}}. \tag{4.7}$$

From (4.2), (4.3) and (4.7),

$$\|C_\varphi - C_\psi\|^2_{HS(A^2_\alpha(\Pi^+),L^2(\mu))} \approx \int_{\Pi^+} \frac{\omega_{\mu,2}[E_{2\delta}(w)]}{(\Im w)^{2(\alpha+2)}} dA_\alpha(w) \approx \sum_{n=1}^{\infty} \frac{\omega_{\mu,2}[E_{2\delta}(a_n)]}{(\Im a_n)^{\alpha+2}},$$

as desired. □

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