SCIENCE CHINA
Mathematics

• ARTICLES •

July 2020 Vol. 63 No. 7: 1285–1306
https://doi.org/10.1007/s11425-018-9427-6

Global well-posedness for the 2D incompressible magneto-micropolar fluid system with partial viscosity

Hongxia Lin[1,2] & Zhaoyin Xiang[2,*]

[1]*Geomathematics Key Laboratory of Sichuan Province, Chengdu University of Technology, Chengdu* 610059*, China;*
[2]*School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu* 611731*, China*

Email: linhongxia18@126.com, zxiang@uestc.edu.cn

Received June 28, 2018; accepted November 14, 2018; published online August 7, 2019

Abstract In this paper, we consider an initial-boundary value problem for the 2D incompressible magneto-micropolar fluid equations with zero magnetic diffusion and zero spin viscosity in the horizontally infinite flat layer with Navier-type boundary conditions. We establish the global well-posedness of strong solutions around the equilibrium $(0, e_1, 0)$.

Keywords global well-posedness, 2D magneto-micropolar fluid equations, zero magnetic diffusion, zero spin viscosity

MSC(2010) 35J35, 65M70

Citation: Lin H X, Xiang Z Y. Global well-posedness for the 2D incompressible magneto-micropolar fluid system with partial viscosity. Sci China Math, 2020, 63: 1285–1306, https://doi.org/10.1007/s11425-018-9427-6

1 Introduction

The magneto-micropolar fluid equations can be used to describe the motion of aggregates of small solid ferromagnetic particles relative to viscous magnetic fluids under the action of magnetic fields, such as salt water, ester, fluorocarbon, etc., which is of great importance in practical and mathematical applications [2]. Let U be the fluid velocity, B be the magnetic field, W be the micro-rotational velocity representing the angular velocity of the rotation of the particles of the fluid, and P be the scalar pressure. The incompressible magneto-micropolar fluid equations take the following form:

$$\begin{cases} \partial_t U + (U\cdot\nabla)U - 2\chi\nabla\times W = (\mu+\chi)\Delta U - \nabla P + (B\cdot\nabla)B, \\ \partial_t B + (U\cdot\nabla)B = \nu\Delta B + (B\cdot\nabla)U, \\ \partial_t W + (U\cdot\nabla)W + 4\chi W - 2\chi\nabla\times U - \kappa\nabla\mathrm{div}W = \gamma\Delta W, \\ \nabla\cdot U = \nabla\cdot B = 0. \end{cases} \tag{1.1}$$

* Corresponding author

Here, the nonnegative parameters μ, χ, ν, γ and κ are associated with the properties of the materials: μ is the kinematic viscosity, χ is the vortex viscosity, $\frac{1}{\nu}$ is the magnetic Reynolds number, and γ and κ are the spin viscosities. Because of their mathematical and physical importance, there is a great amount of literature on the mathematical theory of magneto-micropolar fluid flows. Before going into our mathematical analysis, we recall progress on System (1.1) and the related models.

If the magnetic field is neglected, i.e., $B=0$, System (1.1) is transformed to the following micropolar fluid equations proposed by Eringen [16]:

$$\begin{cases} \partial_t U + (U\cdot\nabla)U - 2\chi\nabla\times W = (\mu+\chi)\Delta U - \nabla P, \\ \partial_t W + (U\cdot\nabla)W + 4\chi W - 2\chi\nabla\times U - \kappa\nabla \mathrm{div} W = \gamma\Delta W, \\ \nabla\cdot U = 0. \end{cases} \tag{1.2}$$

Physically, the micropolar fluid system (1.2) represents fluids consisting of rigid randomly oriented particles suspended in a viscous medium. The global existence of weak solutions and strong solutions to the Cauchy problem of System (1.2) with full viscosities has been obtained by Galdi and Rionero [17]. In the remarkable paper [13], Dong and Zhang found a new quantity for the 2D system (1.2) with zero angular viscosity, i.e., $\gamma=0$, and proved the global existence and uniqueness of smooth solutions. Then Xue [35] proved the global well-posedness of the system with rough initial data and showed the vanishing microrotation viscosity limit in the case of zero kinematic viscosity or zero angular viscosity, while Dong et al. [12] recently studied the global regularity and large time behavior of solutions to the 2D micropolar equations with only angular viscosity dissipation (i.e., $\mu+\chi=0$ in (1.2)). For the more related works on System (1.2), we refer to [7, 10, 11] and the references therein.

On the other hand, when the micro-rotation effects are neglected, i.e., $W=0$, Equations (1.1) will become the usual magnetohydrodynamic (MHD) equations by taking $\chi=0$:

$$\begin{cases} \partial_t U + (U\cdot\nabla)U = (\mu+\chi)\Delta U - \nabla P + (B\cdot\nabla)B, \\ \partial_t B + (U\cdot\nabla)B = \nu\Delta B + (B\cdot\nabla)U, \\ \nabla\cdot U = \nabla\cdot B = 0. \end{cases} \tag{1.3}$$

It is well known that the 2D MHD equations (1.3) with full viscosities have a unique and global classical solution (see [15, 29]). There is also some important recent progress in the global existence of the partial viscous MHD system posed on the whole space. For example, Cao et al. [4], Cao and Wu [5], Cao et al. [6] and Du and Zhou [14] established the global regularity of the 2D MHD equations with mixed partial dissipation for any smooth initial data, and Jiu et al. [18] proved a weaker version of the small data global existence result for the zero kinematic viscosity case (i.e., $\mu+\chi=0$ in (1.3)). In the latter case, the global existence of solutions was proved by Zhou and Zhu [40] for small initial data on periodic domain. However, in the case without magnetic diffusion (i.e., $\nu=0$), the question of whether the smooth solution of the 2D system (1.3) develops singularity in finite time has been a long-standing open problem. In the recent remarkable paper [20], Lin et al. proved the global existence of the smooth solution of System (1.3) with $\nu=0$ around the trivial solution $((1,0)^\top,0)$, which motivated a series of results in this framework [23, 24, 37–39] (see also [1, 3, 9, 19, 34] for the 3D case and [21, 22, 26] for the 3D toy model case). We also remark that the initial-boundary value problem of System (1.3) with $\nu=0$ was investigated by [25, 30].

For the full magneto-micropolar fluid equations (1.1) with full viscosities, the existence, uniqueness and regularity of weak solutions were established by the standard energy method (see [28, 33]). Wang and Wang [31] established a blow-up criterion for the two-dimensional Cauchy problem of System (1.1) without magnetic diffusion and spin viscosities (i.e., $\nu=0$ and $\gamma=0$ in (1.1)). Then Cheng and Liu [8] further investigated the global existence and uniqueness of the classical solution for the case of vertical kinematic viscosity, horizontal magnetic diffusion and horizontal vortex viscosity, and Yamazaki [36] studied the global regularity of the 2D system (1.1) with zero spin viscosity $\gamma=0$. Recently, Wei et al. [32] also showed the global existence and optimal convergence rates of solutions for 3D compressible magneto-micropolar fluid equations.

Motivated by the above works, we are concerned with the initial-boundary value problem of the 2D magneto-micropolar fluid equations with zero magnetic diffusion and zero spin viscosity, i.e., $\nu = \gamma = \kappa = 0$ in (1.1). Precisely, our governing equations are expressed as follows:

$$\begin{cases} \partial_t U + (U \cdot \nabla)U - 2\chi \nabla \times W = (\mu + \chi)\Delta U - \nabla P + (B \cdot \nabla)B, \\ \partial_t B + (U \cdot \nabla)B = (B \cdot \nabla)U, \\ \partial_t W + (U \cdot \nabla)W + 4\chi W - 2\chi \nabla \times U = 0, \\ \nabla \cdot U = \nabla \cdot B = 0, \end{cases} \tag{1.4}$$

where $U = (U_1, U_2), B = (B_1, B_2)$, $\nabla \times W := (\partial_2 W, -\partial_1 W)$ and $\nabla \times U := \partial_1 U_2 - \partial_2 U_1$. Our main purpose is to establish the global regularity for System (1.4) around the equilibrium $(0, e_1, 0)$. Thus, we will set $U := u + 0$, $B := b + e_1$, $W_3 := w + 0$, and $\mu = \chi = \frac{1}{2}$ for simplicity. Then System (1.4) can be reformulated as

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nabla \times w = \Delta u - \nabla P + (b \cdot \nabla)b + \partial_1 b, \\ \partial_t b + (u \cdot \nabla)b = (b \cdot \nabla)u + \partial_1 u, \\ \partial_t w + (u \cdot \nabla)w + 2w - \nabla \times u = 0, \\ \nabla \cdot u = \nabla \cdot b = 0 \end{cases} \tag{1.5}$$

in $\Omega \times (0, +\infty)$ with the initial data

$$u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x), \quad w(x, 0) = w_0(x) \tag{1.6}$$

in Ω, where $u_0 = (u_{01}, u_{02})$ and $b_0 = (b_{01}, b_{02})$. Similar to [25,30], we assume that the domain Ω occupied by the fluid is a horizontally infinite flat layer

$$\Omega := \{x = (x_1, x_2) \,|\, -\infty < x_1 < +\infty,\, 0 < x_2 < 1\}. \tag{1.7}$$

On the boundary, we impose the usual Navier-slip condition for the fluid $u = (u_1, u_2)$:

$$u \cdot n = 0 \quad \text{and} \quad \nabla \times u = 0 \quad \text{on} \quad \partial\Omega,$$

and assume that the container is perfectly conducting for the magnetic field $b = (b_1, b_2)$, i.e.,

$$b \cdot n = 0 \quad \text{on} \quad \partial\Omega.$$

Summarily, we have the following boundary conditions for u, b and w:

$$u_2 = \partial_2 u_1 = 0 \quad \text{and} \quad b_2 = 0 \quad \text{on} \quad \partial\Omega. \tag{1.8}$$

This kind of boundary condition is also useful in the study of the viscous surface wave equation (see, for example, [27]).

Our result asserts that the 2D incompressible magneto-micropolar fluid equations (1.5)–(1.8) possess a unique global classical solution for suitable small initial data. More precisely, we have the following theorem.

Theorem 1.1. *Suppose that* $(u_0, b_0, w_0) \in H^2(\Omega)$ *satisfies* $\operatorname{div} u_0 = 0$ *and* $\operatorname{div} b_0 = 0$ *in* Ω, *and* $u_{02} = \partial_2 u_{01} = 0$, $b_{02} = \partial_2 b_{01} = 0$, $w_0 = 0$ *on* $\partial\Omega$. *Let*

$$\|(u_0, b_0, w_0)\|^2_{H^2(\Omega)} \leqslant \varepsilon_0^2$$

for some suitable small $\varepsilon_0 \in (0, 1)$. *Then System* (1.5)–(1.8) *admits a unique global solution* $(u, b, w) \in C([0, +\infty); H^2(\Omega))$ *such that* $\partial_2 b_1 = 0$ *and* $w = 0$ *on* $\partial\Omega$ *and*

$$\begin{aligned} &(\|(u, b, w)(t)\|^2_{H^2} + \|(u_t, w_t)(t)\|^2_{L^2}) \\ &\quad + \int_0^t (\|(\nabla u, w)(s)\|^2_{H^2} + \|\partial_1 b(s)\|^2_{H^1} + \|u_t(s)\|^2_{H^1} + \|(b_t, w_t)(s)\|^2_{L^2}) ds \leqslant C\varepsilon_0^2 \end{aligned} \tag{1.9}$$

for any $t > 0$ *and some uniform constant* $C > 0$.

To prove Theorem 1.1, there are two main difficulties. The first one is due to the partial dissipation of the magnetic b, which is similar to that of the MHD system (1.3) with zero magnetic diffusion and can be observed by a spectral analysis for the linearized system of (1.5), while the second one arises from the zero spin viscosity of the micro-rotational velocity w. These difficulties will be overcome by a series of careful energy estimates. Roughly speaking, the dissipation for b in (1.9) is of the form $\|\partial_1 b(s)\|_{H^1}$ rather than $\|\nabla b(s)\|_{H^1}$. This will result in several trouble terms of particular forms: $\int(\partial_2 b_1)^2\partial_1 u_1$ (see Lemma 3.3) and $\int(\partial_2^2 b_1)^2\partial_1 u_1$ (see Lemma 3.4), which cannot be absorbed by the dissipation from a direct interpolation. Motivated by [22, 25], we will replace $\partial_1 u_1$ by $\partial_t b_1 + u\cdot\nabla b_1 - b\cdot\nabla u_1$, which will provide us more dissipation. On the other hand, the dissipation for w will resort to the damping mechanism in (1.5). Here, the key observation is that the rotation terms in $(1.5)_1$ and $(1.5)_3$ can be absorbed by the dissipation if we take the rotation for $(1.5)_1$ and the Laplacian for $(1.5)_3$, respectively (see Lemma 3.4), which will be a failed recursive argument if we use the Stokes estimates as [25]. Finally, the trading time derivative and spacial derivative is also new to our current work.

Remark 1.2. Since the magnetic equation and the micro-rotational velocity equation are hyperbolic and characteristic, no boundary condition needs to be imposed for the magnetic field component b_1 or the micro-rotational velocity w. This will result in an essential difficulty if we want to establish the global existence of System (1.1) by using the basic energy method. Fortunately, the boundary condition $\partial_2 b_1 = 0$ and $w = 0$ on $\partial\Omega$ can be propagated by the equations $(1.5)_2$ and $(1.5)_3$ if $\partial_2 b_{01} = 0$ and $w_0 = 0$ on $\partial\Omega$, respectively.

Remark 1.3. The global well-posedness result in Theorem 1.1 also holds for the whole space case $\Omega = \mathbb{R}^2$, due to the fact that all boundary terms actually vanish in the process of energy estimates. In this case, indeed, the time derivatives can be removed from the energy functional $A(t)$ and the dissipation energy $B(t)$ defined in Section 3.

The rest of this paper is organized as follows. In Section 2, we present some preliminary materials, including the interpolation inequalities involving the derivatives in one direction and the local well-posedness of System (1.1). The global uniform *a priori* estimates for System (1.1), which will yield the proof of Theorem 1.1 by a continuity argument, will be established in Section 3.

Notations. Throughout this paper, we denote

$$\int f := \iint_\Omega f(x_1,x_2)dx_1dx_2,\quad \|g\|_{L^p} := \|g\|_{L^p(\Omega)},\quad \|(f,g,h)\|_{L^2}^2 := \|f\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\Omega)}^2 + \|h\|_{L^2(\Omega)}^2.$$

For simplicity, we also set $\partial_t := \frac{\partial}{\partial t}$, $\partial_i := \frac{\partial}{\partial x_i}$, and $\partial_i\partial_j = \frac{\partial^2}{\partial x_i\partial x_j}$ for $i,\ j = 1,2$. Finally, we will use C to denote some absolute positive constant, which may be different on different lines.

2 Preliminaries

In this section, we first give several basic interpolation inequalities and then state the Stokes estimates. A local well-posedness will also be presented at the end of this section.

We begin with the basic inequalities involving the derivatives in one direction.

Lemma 2.1 (See [25]). *Let Ω be the strip domain defined by* (1.7). *Assume $v = (v_1, v_2) \in H^2(\Omega)$ with* $\operatorname{div} v = 0$ *and $v_2|_{\partial\Omega} = 0$. Then we have*

$$\|v_2\|_{L^\infty(\Omega)} \leqslant C\|\partial_1 v\|_{H^1(\Omega)}, \tag{2.1}$$

$$\|v\|_{L^\infty(\Omega)} \leqslant C\|v\|_{H^1(\Omega)}^{\frac12}\|\partial_1 v\|_{H^1(\Omega)}^{\frac12}, \tag{2.2}$$

$$\|\nabla(v\cdot\nabla v)\|_{L^2(\Omega)} \leqslant C\|v\|_{H^2(\Omega)}\|\partial_1 v\|_{H^1(\Omega)}. \tag{2.3}$$

Next, we present the local well-posedness of System (1.5)–(1.8).

Theorem 2.2. *Suppose that $(u_0, b_0, w_0) \in H^2(\Omega)$ satisfies* $\operatorname{div} u_0 = 0$ *and* $\operatorname{div} b_0 = 0$ *in Ω, and $u_{02} = \partial_2 u_{01} = 0$, $b_{02} = \partial_2 b_{01} = 0$, $w_0 = 0$ on $\partial\Omega$. Then there exists $T > 0$ such that System* (1.5)–(1.8) *admits a unique solution $(u, b, w) \in C([0,T]; H^2(\Omega))$ with $\partial_2 b_1 = 0$ and $w = 0$ on $\partial\Omega$.*

Theorem 2.2 can be proved by firstly constructing an iteration scheme for System (1.5)–(1.8), which provides a series of approximate solutions to (1.5)–(1.8), and then deriving uniform bounds for the approximate solutions to pass the limit. This procedure is more or less standard and thus we omit the details.

3 A priori estimates

In this section, we will focus on the uniform *a priori* estimates of solutions to System (1.5)–(1.8), which are the key ingredients of our global well-posedness analysis. For simplicity, we introduce a new energy functional

$$A(t)=\|(u,b,w)(t)\|_{H^2}^2+\|u_t(t)\|_{L^2}^2$$

and the dissipation energy

$$B(t)=\|(\nabla u,w)(t)\|_{H^2}^2+\|\partial_1 b(t)\|_{H^1}^2+\|u_t(t)\|_{H^1}^2+\|(b_t,w_t)(t)\|_{L^2}^2.$$

Then we have the following *a priori* estimates.

Proposition 3.1. *Let* (u,b,w) *be the solution of System* (1.5)–(1.8) *with the initial data* (u_0,b_0,w_0) *satisfying* $\operatorname{div} u_0=\operatorname{div} b_0=0$ *in* Ω *and* $u_{02}=\partial_2 u_{01}=b_{02}=\partial_2 b_{01}=w_0=0$ *on* $\partial\Omega$. *If*

$$\sup_{0\leqslant t\leqslant T}\|(u,b,w)(t)\|_{H^2}^2\leqslant c_0^2 \tag{3.1}$$

for some $c_0\in(0,1)$, *then there exists a uniform constant* C_0 *such that*

$$A(t)+\int_0^t B(s)ds\leqslant C_0\|(u_0,b_0,w_0)\|_{H^2}^2 \tag{3.2}$$

for any $t\in[0,T]$ *provided that* c_0 *is suitably small.*

We split the proof of Proposition 3.1 into a series of lemmas and begin with the basic L^2 estimate of (u,b,w).

Lemma 3.2 (L^2 estimate of (u,b,w))**.** *Let* (u,b,w) *be the solution of System* (1.5)–(1.8) *with the initial data satisfying the conditions in Proposition* 3.1. *Then we have*

$$\|(u,b,w)(t)\|_{L^2}^2+\frac{1}{2}\int_0^t\|(\nabla u,w)(s)\|_{L^2}^2ds\leqslant\|(u_0,b_0,w_0)\|_{L^2}^2 \tag{3.3}$$

for any $t\in[0,T]$.

Proof. Lemma 3.2 can be proved by a direct energy method and thus we omit the details for simplicity. □

Lemma 3.3 (L^2 estimate of $(\nabla u,\nabla b,\nabla w)$)**.** *Let* (u,b,w) *be the solution of System* (1.5)–(1.8) *with the initial data satisfying the conditions in Proposition* 3.1. *Then we have*

$$\begin{aligned}&\left(\|(\nabla u,\nabla b,\nabla w)(t)\|_{L^2}^2-4\int b_1(\partial_2 b_1)^2\right)+\frac{1}{2}\int_0^t\|(\Delta u,\nabla w)(s)\|_{L^2}^2ds\\&\quad\leqslant C\int_0^t(\|(u,b)(s)\|_{H^2}+\|b(s)\|_{H^2}^2)B(s)ds+\left(\|(\nabla u_0,\nabla b_0,\nabla w_0)\|_{L^2}^2-4\int b_{01}(\partial_2 b_{01})^2\right)\end{aligned} \tag{3.4}$$

for any $t\in[0,T]$.

Proof. Taking the L^2 inner product of the first three equations in (1.5) with $-\Delta u$, $-\Delta b$ and $-\Delta w$, respectively, we obtain

$$\frac{1}{2}\frac{d}{dt}\|(\nabla u,\nabla b,\nabla w)\|_{L^2}^2+\|(\Delta u,2\nabla w)\|_{L^2}^2$$

$$
\begin{aligned}
&= \int (u\cdot\nabla)u\cdot\Delta u - \int (b\cdot\nabla)b\cdot\Delta u + \int (u\cdot\nabla)b\cdot\Delta b - \int (b\cdot\nabla)u\cdot\Delta b \\
&\quad + \int (u\cdot\nabla)w\cdot\Delta w - \int \nabla\times w\cdot\Delta u - \int \nabla\times u\Delta w. \qquad (3.5)
\end{aligned}
$$

Here, we used the fact that

$$
\int \nabla P\cdot\Delta u = \int \partial_1 P\Delta u_1 + \partial_2 P\Delta u_2 = -\int P\Delta\partial_1 u_1 - \int P\Delta\partial_2 u_2 = -\int p\,\mathrm{div} u = 0
$$

and

$$
\begin{aligned}
\int \partial_1 b\cdot\Delta u &= -\int b_1\partial_1^3 u_1 - \int b_1\partial_2^2\partial_1 u_1 - \int b_2\partial_1^3 u_2 - \int b_2\partial_2^2\partial_1 u_2 \\
&= -\int \partial_1^2 b_1\partial_1 u_1 - \int \partial_2^2 b_1\partial_1 u_1 - \int \partial_1^2 b_2\partial_1 u_2 - \int \partial_2^2 b_2\partial_1 u_2 = -\int \Delta b\cdot\partial_1 u
\end{aligned}
$$

due to $\Delta u_2 = \partial_1^2 u_2 - \partial_1\partial_2 u_1 = 0$ and $\partial_2 u_1 = \partial_2 b_1 = b_2 = u_2 = 0$ on $\partial\Omega$. Then we rewrite (3.5) as

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla u,\nabla b,\nabla w)\|_{L^2}^2 + \|(\Delta u, 2\nabla w)\|_{L^2}^2 \\
&\quad = \int ((u\cdot\nabla)u - (b\cdot\nabla)b)\cdot\Delta u + \int (u\cdot\nabla)b\cdot\partial_2^2 b - \int (b\cdot\nabla)u\cdot\partial_2^2 b \\
&\qquad + \int [(u\cdot\nabla)b - (b\cdot\nabla)u]\cdot\partial_1^2 b + \int (u\cdot\nabla)w\Delta w - \int (\nabla\times w\cdot\Delta u + \nabla\times u\Delta w) \\
&\quad =: J_{11} + J_{12} + \cdots + J_{16}. \qquad (3.6)
\end{aligned}
$$

We now estimate $J_{11},\dots,J_{16}$ one by one. For J_{11}, we use Hölder's inequality, (2.1), Sobolev's inequality and Young's inequality to have

$$
\begin{aligned}
J_{11} &= \int ((u\cdot\nabla)u - b_1\partial_1 b - b_2\partial_2 b)\cdot\Delta u \\
&\leqslant (\|u\|_{L^\infty}\|\nabla u\|_{L^2} + \|b_1\|_{L^\infty}\|\partial_1 b\|_{L^2} + \|b_2\|_{L^\infty}\|\partial_2 b\|_{L^2})\|\Delta u\|_{L^2} \\
&\leqslant C(\|u\|_{H^2}\|\nabla u\|_{L^2} + \|b_1\|_{H^2}\|\partial_1 b\|_{L^2} + \|\partial_1 b\|_{H^1}\|b\|_{H^1})\|\Delta u\|_{L^2} \\
&\leqslant C(\|u\|_{H^2} + \|b\|_{H^2})(\|\partial_1 b\|_{H^1}^2 + \|\nabla u\|_{H^1}^2) \\
&\leqslant C(\|u\|_{H^2} + \|b\|_{H^2})B(t). \qquad (3.7)
\end{aligned}
$$

For J_{12}, since $\partial_2 b_1 = 0$ and $u\cdot\nabla b_2 = 0$ on $\partial\Omega$, the integration by parts together with Hölder's inequality and Young's inequality gives that

$$
\begin{aligned}
J_{12} &= -\int \partial_2(u\cdot\nabla b_1)\partial_2 b_1 - \int \partial_2(u\cdot\nabla b_2)\partial_2 b_2 \\
&= -\int (\partial_2 u\cdot\nabla b_1)\partial_2 b_1 - \int (u\cdot\nabla\partial_2 b_1)\partial_2 b_1 - \int (\partial_2 u\cdot\nabla b_2)\partial_2 b_2 - \int (u\cdot\nabla\partial_2 b_2)\partial_2 b_2 \\
&= -\int \partial_2 u_1\partial_1 b_1\partial_2 b_1 - \int \partial_2 u_2(\partial_2 b_1)^2 + \int (\partial_2 u\cdot\nabla b_2)\partial_1 b_1 \\
&\leqslant \|\partial_2 u_1\|_{L^\infty}\|\partial_1 b_1\|_{L^2}\|\partial_2 b_1\|_{L^2} + \|\partial_2 u\|_{L^\infty}\|\nabla b_2\|_{L^2}\|\partial_1 b_1\|_{L^2} + \int \partial_1 u_1(\partial_2 b_1)^2 \\
&\leqslant C\|\nabla u\|_{H^2}\|\partial_1 b\|_{L^2}\|b\|_{H^1} + \int \partial_1 u_1(\partial_2 b_1)^2 \\
&\leqslant C\|b\|_{H^1}B(t) + \int \partial_1 u_1(\partial_2 b_1)^2, \qquad (3.8)
\end{aligned}
$$

where we used $\int (u\cdot\nabla\partial_2 b_2)\partial_2 b_2 = 0$ and $\int (u\cdot\nabla\partial_2 b_1)\partial_2 b_1 = 0$ in the third equality.

To estimate J_{13}, we first rewrite it as

$$
J_{13} = -\int b_1\partial_1 u_1\partial_2^2 b_1 - \int b_2\partial_2 u_1\partial_2^2 b_1 + \int \partial_2(b\cdot\nabla u_2)\partial_2 b_2
$$

$$= -\int b_1\partial_1 u_1\partial_2^2 b_1 - \int b_2\partial_2 u_1\partial_2^2 b_1 + \int (\partial_2 b\cdot\nabla u_2 + b\cdot\nabla\partial_2 u_2)\partial_2 b_2$$
$$=: J_{131} + J_{132} + J_{133}. \tag{3.9}$$

Clearly, it follows from the integration by parts that

$$\begin{aligned} J_{131} &= \int \partial_2 b_1\partial_1 u_1\partial_2 b_1 + \int b_1\partial_1\partial_2 u_1\partial_2 b_1 \\ &= \int \partial_1 u_1(\partial_2 b_1)^2 - \int \partial_1 b_1\partial_2 u_1\partial_2 b_1 - \int b_1\partial_2 u_1\partial_2\partial_1 b_1 \\ &\leqslant \int \partial_1 u_1(\partial_2 b_1)^2 + \|\partial_1 b_1\|_{L^2}\|\partial_2 u_1\|_{L^\infty}\|\partial_2 b_1\|_{L^2} + \|b_1\|_{L^\infty}\|\partial_2 u_1\|_{L^2}\|\partial_2\partial_1 b_1\|_{L^2} \\ &\leqslant \int \partial_1 u_1(\partial_2 b_1)^2 + C\|\partial_1 b_1\|_{H^1}\|\nabla u\|_{H^2}\|b\|_{H^2} \\ &\leqslant \int \partial_1 u_1(\partial_2 b_1)^2 + C\|b\|_{H^2}B(t). \end{aligned} \tag{3.10}$$

Similarly, we have

$$\begin{aligned} J_{132} + J_{133} &\leqslant \|b_2\|_{L^\infty}\|\partial_2 u_1\|_{L^2}\|\partial_2^2 b_1\|_{L^2} + (\|\partial_2 b\|_{L^2}\|\nabla u_2\|_{L^\infty} + \|b\|_{L^\infty}\|\nabla\partial_2 u_2\|_{L^2})\|\partial_2 b_2\|_{L^2} \\ &\leqslant C\|\partial_1 b\|_{H^1}\|\partial_2 u_1\|_{L^2}\|\partial_2^2 b_1\|_{L^2} + C\|b\|_{H^2}\|\nabla u\|_{H^2}\|\partial_1 b_1\|_{L^2} \\ &\leqslant C\|b\|_{H^2}(\|\partial_1 b\|_{H^1}^2 + \|\nabla u\|_{H^2}^2) \\ &\leqslant C\|b\|_{H^2}B(t). \end{aligned} \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9), we obtain

$$J_{13} \leqslant C\|b\|_{H^2}B(t) + \int \partial_1 u_1(\partial_2 b_1)^2. \tag{3.12}$$

Similarly, for J_{14} and J_{15}, applying Hölder's inequality and Sobolev's inequality, we deduce

$$\begin{aligned} J_{14} + J_{15} &= -\int (\partial_1 u\cdot\nabla)b\cdot\partial_1 b - \int (b\cdot\nabla)u\cdot\partial_1^2 b + \int (u\cdot\nabla)w\Delta w \\ &\leqslant \|\partial_1 u\|_{L^4}\|\nabla b\|_{L^2}\|\partial_1 b\|_{L^4} + \|b\|_{L^\infty}\|\nabla u\|_{L^2}\|\partial_1^2 b\|_{L^2} + \|u\|_{L^\infty}\|\nabla w\|_{L^2}\|\Delta w\|_{L^2} \\ &\leqslant C\|\partial_1 u\|_{H^1}\|\nabla b\|_{L^2}\|\partial_1 b\|_{H^1} + C\|b\|_{H^2}\|\nabla u\|_{L^2}\|\partial_1^2 b\|_{L^2} + C\|u\|_{H^2}\|\nabla w\|_{L^2}\|\Delta w\|_{L^2} \\ &\leqslant C\|b\|_{H^2}(\|\partial_1 b\|_{H^1}^2 + \|\nabla u\|_{H^1}^2) + C\|u\|_{H^2}\|w\|_{H^2}^2 \\ &\leqslant C(\|b\|_{H^2} + \|u\|_{H^2})B(t), \end{aligned} \tag{3.13}$$

where we have used the integration by parts for the first term of J_{14}.

Finally, for the linear term J_{16}, it follows from the integration by parts that

$$\begin{aligned} J_{16} &= -\int \partial_2 w\Delta u_1 + \int \partial_1 w\Delta u_2 - \int \partial_1 u_2\Delta w + \int \partial_2 u_1\Delta w \\ &= -\int \partial_2 w\Delta u_1 + \int \partial_1 w\Delta u_2 - \int \Delta\partial_1 u_2\, w + \int \Delta\partial_2 u_1\, w \\ &= -2\int \partial_2 w\Delta u_1 + 2\int \partial_1 w\Delta u_2. \end{aligned}$$

Thus, by Hölder's inequality and Young's inequality, we obtain

$$J_{16} \leqslant 2\|\nabla w\|_{L^2}\|\Delta u\|_{L^2} \leqslant \frac{4}{3}\|\nabla w\|_{L^2}^2 + \frac{3}{4}\|\Delta u\|_{L^2}^2. \tag{3.14}$$

Substituting all the estimates of $J_{11},\dots,J_{16}$ into (3.6), we conclude that

$$\frac{d}{dt}\|(\nabla u,\nabla b,\nabla w)\|_{L^2}^2 + \frac{1}{2}\|(\Delta u,\nabla w)\|_{L^2}^2 \leqslant C(\|u\|_{H^2} + \|b\|_{H^2})B(t) + 4\int (\partial_2 b_1)^2\partial_1 u_1. \tag{3.15}$$

It remains to estimate the integral term in (3.15). From $(1.5)_2$, we have $\partial_1 u_1 = \partial_t b_1 + u \cdot \nabla b_1 - b \cdot \nabla u_1$ and thus

$$\begin{aligned}\int (\partial_2 b_1)^2 \partial_1 u_1 &= \int (\partial_2 b_1)^2 (\partial_t b_1 + u \cdot \nabla b_1 - b \cdot \nabla u_1)\\ &= \frac{d}{dt}\int (\partial_2 b_1)^2 b_1 - 2\int b_1 \partial_2 b_1 \partial_2 \partial_t b_1 + \int (\partial_2 b_1)^2 (u \cdot \nabla b_1 - b \cdot \nabla u_1)\\ &= \frac{d}{dt}\int (\partial_2 b_1)^2 b_1 - 2\int b_1 \partial_2 b_1 \partial_2 (\partial_1 u_1 - u \cdot \nabla b_1 + b \cdot \nabla u_1) + \int (\partial_2 b_1)^2 (u \cdot \nabla b_1 - b \cdot \nabla u_1),\end{aligned}$$

which implies that

$$\begin{aligned}&-\frac{d}{dt}\int (\partial_2 b_1)^2 b_1 + \int \partial_1 u_1 (\partial_2 b_1)^2\\ &\quad = -2\int b_1 \partial_2 b_1 \partial_2 \partial_1 u_1 + 2\int b_1 \partial_2 b_1 \partial_2 (u \cdot \nabla b_1) - 2\int b_1 \partial_2 b_1 \partial_2 (b \cdot \nabla u_1)\\ &\qquad + \int (\partial_2 b_1)^2 (u \cdot \nabla b_1 - b \cdot \nabla u_1). \qquad (3.16)\end{aligned}$$

Since

$$\begin{aligned}2\int b_1 \partial_2 b_1 \partial_2 (u \cdot \nabla b_1) &= 2\int b_1 \partial_2 b_1 (\partial_2 u \cdot \nabla b_1) + 2\int b_1 \partial_2 b_1 (u \cdot \nabla \partial_2 b_1)\\ &= 2\int b_1 \partial_2 b_1 (\partial_2 u \cdot \nabla b_1) + \int b_1 u \cdot \nabla (\partial_2 b_1)^2\\ &= 2\int b_1 \partial_2 b_1 (\partial_2 u \cdot \nabla b_1) - \int (\partial_2 b_1)^2 u \cdot \nabla b_1,\end{aligned}$$

we can deduce from (3.16) that

$$\begin{aligned}&-\frac{d}{dt}\int (\partial_2 b_1)^2 b_1 + \int \partial_1 u_1 (\partial_2 b_1)^2\\ &= -2\int b_1 \partial_2 b_1 \partial_2 \partial_1 u_1 + 2\int b_1 \partial_2 b_1 (\partial_2 u \cdot \nabla b_1) - \int (2 b_1 \partial_2 b_1 \partial_2 (b \cdot \nabla u_1) + (\partial_2 b_1)^2 b \cdot \nabla u_1)\\ &=: J_{17} + J_{18} + J_{19}. \qquad (3.17)\end{aligned}$$

For J_{17}, the integration by parts together with Hölder's inequality, Sobolev's embedding and Young's inequality gives that

$$\begin{aligned}J_{17} &= 2\int (\partial_1 b_1 \partial_2 b_1 + b_1 \partial_1 \partial_2 b_1) \partial_2 u_1\\ &\leqslant 2(\|\partial_1 b_1\|_{L^4} \|\partial_2 b_1\|_{L^2} + \|b_1\|_{L^4} \|\partial_1 \partial_2 b_1\|_{L^2}) \|\partial_2 u_1\|_{L^4}\\ &\leqslant C \|\partial_1 b_1\|_{H^1} \|b_1\|_{H^1} \|\partial_2 u_1\|_{H^1}\\ &\leqslant C \|b\|_{H^1} B(t). \qquad (3.18)\end{aligned}$$

For J_{18}, noticing that

$$\int b_1 (\partial_2 b_1)^2 \partial_2 u_2 = -\int (\partial_2 b_1)^2 \partial_2 u_2 b_1 - \int \partial_2^2 b_1 \partial_2 u_2 b_1^2 - \int \partial_2 b_1 \partial_2^2 u_2 b_1^2$$

by the integration by parts, i.e.,

$$2\int b_1 (\partial_2 b_1)^2 \partial_2 u_2 = -\int \partial_2^2 b_1 \partial_2 u_2 b_1^2 - \int \partial_2 b_1 \partial_2^2 u_2 b_1^2,$$

we can deduce that

$$J_{18} = 2\int b_1 \partial_2 b_1 \partial_2 u_1 \partial_1 b_1 + 2\int b_1 (\partial_2 b_1)^2 \partial_2 u_2$$

$$
\begin{aligned}
&= 2\int b_1\partial_2 b_1\partial_2 u_1\partial_1 b_1 - \int \partial_2^2 b_1\partial_2 u_2 b_1^2 - \int \partial_2 b_1\partial_2^2 u_2 b_1^2\\
&\leqslant 2\|b_1\|_{L^\infty}\|\partial_2 b_1\|_{L^4}\|\partial_2 u_1\|_{L^2}\|\partial_1 b_1\|_{L^4} + (\|\partial_2^2 b_1\|_{L^2}\|\partial_2 u_2\|_{L^2} + \|\partial_2 b_1\|_{L^2}\|\partial_2^2 u_2\|_{L^2})\|b_1\|_{L^\infty}^2\\
&\leqslant C\|b_1\|_{H^2}^2\|\partial_2 u_1\|_{L^2}\|\partial_1 b_1\|_{H^1} + C\|\partial_2 b_1\|_{H^1}\|\partial_2 u_2\|_{H^1}\|b_1\|_{H^1}\|\partial_1 b_1\|_{H^1}\\
&\leqslant C\|b\|_{H^2}^2 B(t).
\end{aligned}
\tag{3.19}
$$

Here, we used (2.2) in the second inequality. Similarly, we have

$$
\begin{aligned}
J_{19} &= -3\int b_1(\partial_2 b_1)^2\partial_1 u_1 - 2\int b_1\partial_2 b_1\partial_2 b_2\partial_2 u_1 - 2\int b_1^2\partial_2 b_1\partial_2\partial_1 u_1\\
&\quad - 2\int b_1\partial_2 b_1 b_2\partial_2^2 u_1 - \int b_2(\partial_2 b_1)^2\partial_2 u_1\\
&= -\frac{3}{2}\bigg(\int \partial_2^2 b_1\partial_2 u_2 b_1^2 + \int \partial_2 b_1\partial_2^2 u_2 b_1^2\bigg) - 2\int b_1\partial_2 b_1\partial_2 b_2\partial_2 u_1\\
&\quad - 2\int b_1^2\partial_2 b_1\partial_2\partial_1 u_1 - 2\int b_1\partial_2 b_1 b_2\partial_2^2 u_1 - \int b_2(\partial_2 b_1)^2\partial_2 u_1\\
&= -\frac{3}{2}\int \partial_2^2 b_1\partial_2 u_2 b_1^2 + \frac{1}{2}\int \partial_2 b_1\partial_2^2 u_2 b_1^2 + 2\int b_1\partial_2 b_1\partial_1 b_1\partial_2 u_1 - 2\int b_1\partial_2 b_1 b_2\partial_2^2 u_1 - \int b_2(\partial_2 b_1)^2\partial_2 u_1
\end{aligned}
$$

and thus

$$
\begin{aligned}
J_{19} &\leqslant \frac{3}{2}(\|\partial_2^2 b_1\|_{L^2}\|\partial_2 u_2\|_{L^2} + \|\partial_2 b_1\|_{L^2}\|\partial_2^2 u_2\|_{L^2})\|b_1\|_{L^\infty}^2 + 2(\|\partial_2 b_1\|_{L^4}\|\partial_1 b_1\|_{L^2}\|\partial_2 u_1\|_{L^4}\\
&\quad + \|\partial_2 b_1\|_{L^2}\|b_2\|_{L^\infty}\|\partial_2^2 u_1\|_{L^2})\|b_1\|_{L^\infty} + \|b_2\|_{L^\infty}\|\partial_2 b_1\|_{L^4}^2\|\partial_2 u_1\|_{L^2}\\
&\leqslant C\|\partial_2 b_1\|_{H^1}\|\partial_2 u_2\|_{H^1}\|b_1\|_{H^1}\|\partial_1 b_1\|_{H^1} + C\|b_1\|_{H^2}^2\|\partial_1 b\|_{H^1}\|\partial_2 u_1\|_{H^1}\\
&\leqslant C\|b\|_{H^2}^2 B(t).
\end{aligned}
\tag{3.20}
$$

Substituting (3.18)–(3.20) into (3.17), we obtain

$$
-\frac{d}{dt}\int(\partial_2 b_1)^2 b_1 + \int(\partial_2 b_1)^2\partial_1 u_1 \leqslant C\|b\|_{H^2}^2 B(t),
$$

which together with (3.15) leads to

$$
\frac{d}{dt}\|(\nabla u,\nabla b,\nabla w)\|_{L^2}^2 - 4\frac{d}{dt}\int(\partial_2 b_1)^2 b_1 + \frac{1}{2}\|(\Delta u,\nabla w)\|_{L^2}^2 \leqslant C(\|u\|_{H^2} + \|b\|_{H^2} + \|b\|_{H^2}^2)B(t).
$$

This completes the proof of Lemma 3.3. □

Lemma 3.4 (L^2 estimate of $(\Delta u,\Delta b,\Delta w)$). *Let* (u,b,w) *be the solution of System* (1.5)–(1.8) *with the initial data satisfying the conditions in Proposition* 3.1. *Then we have*

$$
\begin{aligned}
&\bigg(\|(\nabla(\nabla\times u),\nabla(\nabla\times b),\Delta w)(t)\|_{L^2}^2 - 6\int b_1(\partial_2^2 b_1)^2 + 21\int b_1^2(\partial_2^2 b_1)^2\bigg)\\
&\quad + \frac{1}{2}\int_0^t \|(\Delta(\nabla\times u),\Delta w)(s)\|_{L^2}^2 ds\\
&\quad\leqslant C\int_0^t(\|(u,b,w)(s)\|_{H^2} + \|b(s)\|_{H^2}^2 + \|b(s)\|_{H^2}^3)B(s)ds\\
&\qquad + \bigg(\|(\nabla(\nabla\times u_0),\nabla(\nabla\times b_0),\Delta w_0)\|_{L^2}^2 - 6\int b_{01}(\partial_2^2 b_{01})^2 + 21\int b_{01}^2(\partial_2^2 b_{01})^2\bigg)
\end{aligned}
\tag{3.21}
$$

for any $t\in[0,T]$.

Proof. Applying $\nabla\times$ to the first two equations and Δ to the third equation in (1.5), respectively, and setting $h := \nabla\times u$ and $j := \nabla\times b$, we have

$$
\begin{cases}
\partial_t h + (u\cdot\nabla)h + \Delta w = \Delta h + (b\cdot\nabla)j + \partial_1 j,\\
\partial_t j + (u\cdot\nabla)j = (b\cdot\nabla)h + 2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - 2\partial_1 u_1(\partial_1 b_2 + \partial_2 b_1) + \partial_1 h,\\
\partial_t\Delta w + \Delta((u\cdot\nabla)w) + 2\Delta w - \Delta h = 0
\end{cases}
\tag{3.22}
$$

in $\Omega\times(0,T)$. Due to $u_2=\partial_2 u_1=0$ and $b_2=\partial_2 b_1=0$ on $\partial\Omega$, we also have

$$h=0\quad\text{and}\quad j=0\quad\text{on}\quad\partial\Omega.$$

Then we can take the L^2 inner product of the three equations in (3.22) with $-\Delta h$, $-\Delta j$ and Δw, respectively, and deduce from the integration by parts that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla h,\nabla j,\Delta w)\|_{L^2}^2+\|(\Delta h,2\Delta w)\|_{L^2}^2\\
&\quad=2\int\Delta h\Delta w+\int(u\cdot\nabla)h\,\Delta h+\int(u\cdot\nabla)j\,\Delta j-\int(b\cdot\nabla)j\,\Delta h-\int(b\cdot\nabla)h\,\Delta j\\
&\qquad-2\int\partial_1 b_1(\partial_1 u_2+\partial_2 u_1)\Delta j+2\int\partial_1 u_1(\partial_1 b_2+\partial_2 b_1)\Delta j-\int\Delta((u\cdot\nabla)w)\Delta w\\
&\quad=:J_{21}+J_{22}+\cdots+J_{28}.
\end{aligned}\tag{3.23}$$

Here, we have used the fact that

$$\int\partial_1 j\Delta h+\int\partial_1 h\Delta j=-\int j\Delta\partial_1 h+\int\partial_1 h\Delta j=\int\nabla j\cdot\nabla\partial_1 h-\int\nabla\partial_1 h\cdot\nabla j=0.$$

We now estimate $J_{21},J_{22},\ldots,J_{28}$ one by one. Clearly, for J_{21}, we have

$$J_{21}\leqslant 2\|\Delta w\|_{L^2}\|\Delta h\|_{L^2}\leqslant\frac{4}{3}\|\Delta w\|_{L^2}^2+\frac{3}{4}\|\Delta h\|_{L^2}^2.\tag{3.24}$$

For J_{22}, it follows from Hölder's inequality and Sobolev's embedding that

$$J_{22}\leqslant\|u\|_{L^4}\|\nabla h\|_{L^4}\|\Delta h\|_{L^2}\leqslant C\|u\|_{H^1}\|\nabla h\|_{H^1}^2\leqslant C\|u\|_{H^1}\|\nabla u\|_{H^2}^2\leqslant C\|u\|_{H^2}B(t).\tag{3.25}$$

For J_{23}, since

$$\int(u\cdot\nabla)\partial_1 j\,\partial_1 j+\int(u\cdot\nabla)\partial_2 j\,\partial_2 j=0,$$

we can first use the integration by parts to rewrite it as

$$\begin{aligned}
J_{23}&=-\int(\partial_1 u\cdot\nabla)j\,\partial_1 j-\int(\partial_2 u\cdot\nabla)j\,\partial_2 j\\
&=-\int(\partial_1 u\cdot\nabla)j\,\partial_1 j-\int\partial_2 u_1\partial_1 j\partial_2 j-\int\partial_2 u_2((\partial_1^2 b_1)^2+2\partial_1^2 b_1\partial_2^2 b_1+(\partial_2^2 b_1)^2)
\end{aligned}$$

by $(u\cdot\nabla)j=0$ on $\partial\Omega$ and $\partial_2 j=-\Delta b_1$. Then we have

$$\begin{aligned}
J_{23}&\leqslant 2\|\nabla u\|_{L^\infty}(\|\nabla j\|_{L^2}\|\partial_1 j\|_{L^2}+\|\partial_1^2 b_1\|_{L^2}^2+2\|\partial_1^2 b_1\|_{L^2}\|\partial_2^2 b_1\|_{L^2})-\int\partial_2 u_2(\partial_2^2 b_1)^2\\
&\leqslant C\|\nabla u\|_{H^2}\|\partial_1\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}+\int\partial_1 u_1(\partial_2^2 b_1)^2\\
&\leqslant C\|b\|_{H^2}B(t)+\int\partial_1 u_1(\partial_2^2 b_1)^2
\end{aligned}\tag{3.26}$$

by Hölder's inequality and Sobolev's embedding.

Similarly, due to $(b\cdot\nabla)j=(b\cdot\nabla)h=0$ on $\partial\Omega$ and $\partial_2 j=-\Delta b_1$ again, we also have

$$\begin{aligned}
J_{24}+J_{25}&=\int(\partial_i b\cdot\nabla)j\partial_i h+\int(\partial_i b\cdot\nabla)h\partial_i j+\int(b\cdot\nabla)(\partial_i j\partial_i h)\\
&=\int(\partial_i b\cdot\nabla)j\partial_i h+\int(\partial_i b\cdot\nabla)h\partial_i j\\
&=\int\partial_i b_1\partial_1 j\partial_i h+\int\partial_1 b_2\partial_2 j\partial_1 h+2\int\partial_2 b_2\partial_2 j\partial_2 h+\int(\partial_1 b\cdot\nabla)h\partial_1 j+\int\partial_2 b_1\partial_1 h\partial_2 j\\
&=\int\partial_i b_1\partial_1 j\partial_i h+\int\partial_1 b_2\partial_2 j\partial_1 h-2\int\partial_1 b_1\partial_2 j\partial_2 h+\int(\partial_1 b\cdot\nabla)h\partial_1 j
\end{aligned}$$

$$-\int \partial_2 b_1 \partial_1 h \partial_1^2 b_1 - \int \partial_2 b_1 \partial_1 h \partial_2^2 b_1.$$

Noticing that

$$\int \partial_2 b_1 \partial_1 h \partial_2^2 b_1 = \frac{1}{2}\int \partial_1 h \partial_2 (\partial_2 b_1)^2 = \frac{1}{2}\int \partial_2 h \partial_1 (\partial_2 b_1)^2 = \int \partial_2 h \partial_2 b_1 \partial_2 \partial_1 b_1,$$

we can obtain from Hölder's inequality and Sobolev's embedding that

$$\begin{aligned} J_{24} + J_{25} &\leqslant \|\nabla b\|_{L^4}\|\partial_1 j\|_{L^2}\|\nabla h\|_{L^4} + 4\|\partial_1 b\|_{L^4}\|\nabla h\|_{L^4}\|\nabla j\|_{L^2} + 2\|\nabla b\|_{L^4}\|\nabla h\|_{L^4}\|\partial_1 \nabla b\|_{L^2} \\ &\leqslant C\|\nabla b\|_{H^1}\|\partial_1 b\|_{H^1}\|\nabla h\|_{H^1} \\ &\leqslant C\|b\|_{H^2}\|\partial_1 b\|_{H^1}\|\nabla u\|_{H^2} \\ &\leqslant C\|b\|_{H^2} B(t). \end{aligned} \tag{3.27}$$

For J_{26}, a similar argument leads to

$$\begin{aligned} J_{26} &= 2\int (\nabla \partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) + \partial_1 b_1(\nabla \partial_1 u_2 + \nabla \partial_2 u_1)) \cdot \nabla j \\ &\leqslant 4(\|\nabla \partial_1 b_1\|_{L^2}\|\nabla u\|_{L^\infty} + \|\partial_1 b_1\|_{L^4}\|\nabla^2 u\|_{L^4})\|\nabla j\|_{L^2} \\ &\leqslant C\|\partial_1 b\|_{H^1}\|\nabla u\|_{H^2}\|\nabla^2 b\|_{L^2} \\ &\leqslant C\|b\|_{H^2} B(t). \end{aligned} \tag{3.28}$$

The estimates for J_{27} are more subtle. We first rewrite it as

$$\begin{aligned} J_{27} = &-2\int (\partial_1^2 u_1(\partial_1 b_2 + \partial_2 b_1) + \partial_1 u_1(\partial_1^2 b_2 + \partial_1 \partial_2 b_1))\partial_1 j \\ &- 2\int (\partial_1 \partial_2 u_1(\partial_1 b_2 + \partial_2 b_1) + \partial_1 u_1(\partial_1 \partial_2 b_2 + \partial_2^2 b_1))\partial_2 j \\ = &-2\int (\partial_1^2 u_1(\partial_1 b_2 + \partial_2 b_1) + \partial_1 u_1(\partial_1^2 b_2 + \partial_1 \partial_2 b_1))\partial_1 j - 2\int (\partial_1 \partial_2 u_1 \partial_1 b_2 + \partial_1 u_1 \partial_1 \partial_2 b_1)\partial_2 j \\ &+ 2\int (\partial_1 \partial_2 u_1 \partial_2 b_1 + \partial_1 u_1 \partial_2^2 b_1)\partial_1^2 b_1 + 2\int \partial_1 \partial_2 u_1 \partial_2 b_1 \partial_2^2 b_1 + 2\int \partial_1 u_1 (\partial_2^2 b_1)^2. \end{aligned}$$

Then Hölder's inequality and Sobolev's embedding give that

$$\begin{aligned} J_{27} &\leqslant 2(\|\partial_1^2 u_1\|_{L^4}\|\nabla b\|_{L^4} + \|\partial_1 u_1\|_{L^\infty}\|\nabla^2 b\|_{L^2})\|\partial_1 j\|_{L^2} + 2(\|\partial_1 \partial_2 u_1\|_{L^4}\|\partial_1 b_2\|_{L^4} \\ &\quad + \|\partial_1 u_1\|_{L^\infty}\|\partial_1 \partial_2 b_2\|_{L^2})\|\partial_2 j\|_{L^2} + 2(\|\partial_1 \partial_2 u_1\|_{L^4}\|\partial_2 b_1\|_{L^4} + \|\partial_1 u_1\|_{L^\infty}\|\partial_2^2 b_1\|_{L^2})\|\partial_1^2 b_1\|_{L^2} \\ &\quad + 2\|\partial_2^2 u_1\|_{L^4}\|\partial_2 b_1\|_{L^4}\|\partial_1 \partial_2 b_1\|_{L^2} + 2\int \partial_1 u_1 (\partial_2^2 b_1)^2 \\ &\leqslant C\|\nabla u\|_{H^2}(\|\nabla b\|_{H^1}\|\partial_1 j\|_{L^2} + \|\partial_1 b\|_{H^1}\|\partial_2 j\|_{L^2} + \|\nabla b\|_{H^1}\|\partial_1 b\|_{H^1}) + 2\int \partial_1 u_1 (\partial_2^2 b_1)^2 \\ &\leqslant C\|\nabla u\|_{H^2}\|b\|_{H^2}\|\partial_1 b\|_{H^1} + 2\int \partial_1 u_1 (\partial_2^2 b_1)^2 \\ &\leqslant C\|b\|_{H^2} B(t) + 2\int \partial_1 u_1 (\partial_2^2 b_1)^2. \end{aligned} \tag{3.29}$$

Here, we used the fact that

$$\int \partial_1 \partial_2 u_1 \partial_2 b_1 \partial_2^2 b_1 = \int \partial_2^2 u_1 \partial_2 b_1 \partial_1 \partial_2 b_1$$

in the first inequality.

Finally, for J_{28}, a direct calculation yields that

$$J_{28} = \int (\Delta u \cdot \nabla) w \Delta w + 2\int (\nabla u \cdot \nabla) \nabla w \Delta w$$

$$
\begin{aligned}
&\leqslant (\|\Delta u\|_{L^4}\|\nabla w\|_{L^4} + 2\|\nabla u\|_{L^\infty}\|\nabla^2 w\|_2)\|\Delta w\|_2 \\
&\leqslant C\|\nabla u\|_{H^2}\|w\|_{H^2}^2 \\
&\leqslant C\|w\|_{H^2}B(t).
\end{aligned} \tag{3.30}
$$

Summarily, substituting (3.24)–(3.30) into (3.23), we can conclude that

$$
\frac{d}{dt}\|(\nabla h, \nabla j, \Delta w)\|_{L^2}^2 + \frac{1}{2}\|(\Delta h, \Delta w)\|_{L^2}^2 \leqslant C\|(u,b,w)\|_{H^2}B(t) + 6\int(\partial_2^2 b_1)^2\partial_1 u_1. \tag{3.31}
$$

It remains to estimate the last integral $\int(\partial_2^2 b_1)^2\partial_1 u_1$. To this end, we use the magnetic equation of (1.5) to obtain

$$
\int(\partial_2^2 b_1)^2\partial_1 u_1 = \int(\partial_2^2 b_1)^2\partial_t b_1 + \int(\partial_2^2 b_1)^2 u\cdot\nabla b_1 - \int(\partial_2^2 b_1)^2 b\cdot\nabla u_1. \tag{3.32}
$$

The first term on the right-hand side of (3.32) can be rewritten as

$$
\begin{aligned}
\int(\partial_2^2 b_1)^2\partial_t b_1 &= \frac{d}{dt}\int b_1(\partial_2^2 b_1)^2 - 2\int b_1\partial_2^2 b_1\partial_2^2 b_{1t} \\
&= \frac{d}{dt}\int b_1(\partial_2^2 b_1)^2 - 2\int b_1\partial_2^2 b_1\partial_2^2(\partial_1 u_1 - u\cdot\nabla b_1 + b\cdot\nabla u_1).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&-\frac{d}{dt}\int b_1(\partial_2^2 b_1)^2 + \int\partial_1 u_1(\partial_2^2 b_1)^2 \\
&= -2\int b_1\partial_2^2 b_1\partial_2^2\partial_1 u_1 + 2\int b_1\partial_2^2 b_1\partial_2^2(u\cdot\nabla b_1) - 2\int b_1\partial_2^2 b_1\partial_2^2(b\cdot\nabla u_1) \\
&\quad + \int(\partial_2^2 b_1)^2 u\cdot\nabla b_1 - \int(\partial_2^2 b_1)^2 b\cdot\nabla u_1 \\
&= 2\int\partial_1 b_1\partial_2^2 b_1\partial_2^2 u_1 - 2\int\partial_2 b_1\partial_2\partial_1 b_1\partial_2^2 u_1 - 2\int b_1\partial_2\partial_1 b_1\partial_2^3 u_1 + 2\int b_1\partial_2^2 b_1\partial_2^2 u\cdot\nabla b_1 \\
&\quad + 4\int b_1\partial_2^2 b_1\partial_2 u\cdot\partial_2\nabla b_1 - 2\int b_1\partial_2^2 b_1\partial_2^2(b\cdot\nabla u_1) - \int(\partial_2^2 b_1)^2 b\cdot\nabla u_1 \\
&=: J_{31} + J_{32} + \cdots + J_{37}.
\end{aligned} \tag{3.33}
$$

Here, we used the fact that $2\int b_1\partial_2^2 b_1 u\cdot\nabla\partial_2^2 b_1 = -\int(\partial_2^2 b_1)^2 u\cdot\nabla b_1$ in the second equality. By using Hölder's inequality, Sobolev's embedding and Young's inequality, we obtain

$$
\begin{aligned}
J_{31} + J_{32} + J_{33} &\leqslant 2\|\partial_1 b_1\|_{L^4}\|\partial_2^2 b_1\|_{L^2}\|\partial_2^2 u_1\|_{L^4} + 2\|\partial_2 b_1\|_{L^4}\|\partial_2\partial_1 b_1\|_{L^2}\|\partial_2^2 u_1\|_{L^4} \\
&\quad + 2\|b_1\|_{L^\infty}\|\partial_2\partial_1 b_1\|_{L^2}\|\partial_2^3 u_1\|_{L^2} \\
&\leqslant C\|\partial_1 b_1\|_{H^1}\|b\|_{H^2}\|\partial_2^2 u_1\|_{H^1} \\
&\leqslant C\|b\|_{H^2}B(t).
\end{aligned} \tag{3.34}
$$

For J_{34}, we have

$$
\begin{aligned}
J_{34} &= 2\int b_1\partial_2^2 b_1\partial_2^2 u_1\partial_1 b_1 - 2\int b_1\partial_2^2 b_1\partial_2\partial_1 u_1\partial_2 b_1 \\
&= 2\int b_1\partial_2^2 b_1\partial_2^2 u_1\partial_1 b_1 + 2\int(\partial_2 b_1)^3\partial_2\partial_1 u_1 + 2\int b_1(\partial_2 b_1)^2\partial_2^2\partial_1 u_1 \\
&= 2\int b_1\partial_2^2 b_1\partial_2^2 u_1\partial_1 b_1 - 6\int(\partial_2 b_1)^2\partial_1\partial_2 b_1\partial_2 u_1 - 2\int\partial_1 b_1(\partial_2 b_1)^2\partial_2^2 u_1 - 4\int b_1\partial_1\partial_2 b_1\partial_2 b_1\partial_2^2 u_1 \\
&\leqslant C\|b_1\|_{L^\infty}\|\partial_2^2 b_1\|_{L^2}\|\partial_2^2 u_1\|_{L^4}\|\partial_1 b_1\|_{L^4} + C\|\partial_2 b_1\|_{L^4}^2\|\partial_2\partial_1 b_1\|_{L^2}\|\partial_2 u_1\|_{L^\infty} \\
&\quad + C\|\partial_1 b_1\|_4\|\partial_2 b_1\|_{L^4}^2\|\partial_2^2 u_1\|_{L^4} + C\|b_1\|_{L^\infty}\|\partial_2\partial_1 b_1\|_{L^2}\|\partial_2 b_1\|_{L^4}\|\partial_2^2 u_1\|_{L^4}
\end{aligned}
$$

$$\leqslant C\|b\|_{H^2}\|\partial_1 b\|_{H^1}\|\nabla u\|_{H^2}$$
$$\leqslant C\|b\|_{H^2}^2 B(t). \tag{3.35}$$

Finally, for J_{35}, J_{36} and J_{37}, we first rewrite them as

$$\begin{aligned}
J_{35}+J_{36}+J_{37} &= 4\int b_1\partial_2^2 b_1\partial_2 u_1\partial_2\partial_1 b_1 - 2\int b_1\partial_2^2 b_1\partial_2^2 b_2\partial_2 u_1 - 4\int b_1\partial_2^2 b_1\partial_2 b_1\partial_2\partial_1 u_1\\
&\quad - 4\int b_1\partial_2^2 b_1\partial_2 b_2\partial_2^2 u_1 - 2\int b_1^2\partial_2^2 b_1\partial_1\partial_2^2 u_1 - 2\int b_1 b_2\partial_2^2 b_1\partial_2^3 u_1\\
&\quad - \int b_2(\partial_2^2 b_1)^2\partial_2 u_1 - 7\int b_1(\partial_2^2 b_1)^2\partial_1 u_1\\
&= 6\int b_1\partial_2^2 b_1\partial_2\partial_1 b_1\partial_2 u_1 - 6\int(\partial_2 b_1)^2\partial_1\partial_2 b_1\partial_2 u_1 - 2\int \partial_1 b_1(\partial_2 b_1)^2\partial_2^2 u_1\\
&\quad - 4\int b_1\partial_2 b_1\partial_1\partial_2 b_1\partial_2^2 u_1 + 4\int b_1\partial_2^2 b_1\partial_1 b_1\partial_2^2 u_1 - 2\int b_1^2\partial_2^2 b_1\partial_1\partial_2^2 u_1\\
&\quad - 2\int b_1 b_2\partial_2^2 b_1\partial_2^3 u_1 - \int b_2(\partial_2^2 b_1)^2\partial_2 u_1 - 7\int b_1(\partial_2^2 b_1)^2\partial_1 u_1.
\end{aligned} \tag{3.36}$$

On the right-hand side of (3.36), all the integrals except for the last one can be bounded by

$$\begin{aligned}
&\|b_1\|_{L^\infty}\|\partial_2^2 b_1\|_{L^2}\|\partial_2\partial_1 b_1\|_{L^2}\|\partial_2 u_1\|_{L^\infty} + \|\partial_2 b_1\|_{L^4}^2\|\partial_1\partial_2 b_1\|_{L^2}\|\partial_2 u_1\|_{L^\infty}\\
&\quad + \|\partial_1 b_1\|_{L^4}\|\partial_2 b_1\|_{L^4}^2\|\partial_2^2 u_1\|_{L^4} + \|b_1\|_{L^\infty}\|\partial_2 b_1\|_{L^4}\|\partial_1\partial_2 b_1\|_{L^2}\|\partial_2^2 u_1\|_{L^4}\\
&\quad + \|b_1\|_{L^\infty}\|\partial_2^2 b_1\|_{L^2}\|\partial_1 b_1\|_{L^4}\|\partial_2^2 u_1\|_{L^4} + \|b\|_{L^\infty}^2\|\partial_2^2 b_1\|_{L^2}\|\nabla\partial_2^2 u_1\|_{L^2}\\
&\quad + \|b_2\|_{L^\infty}\|\partial_2^2 b_1\|_{L^2}^2\|\partial_2 u_1\|_{L^\infty}
\end{aligned}$$

and thus by

$$\|b\|_{H^2}^2\|\partial_1 b\|_{H^1}\|\nabla u\|_{H^2}.$$

Here we used (2.1) and (2.2). Then we have

$$J_{35}+J_{36}+J_{37} \leqslant C\|b\|_{H^2}^2 B(t) - 7\int b_1(\partial_2^2 b_1)^2\partial_1 u_1. \tag{3.37}$$

Substituting (3.34), (3.35) and (3.37) into (3.33), we deduce that

$$-\frac{d}{dt}\int b_1(\partial_2^2 b_1)^2 + \int(\partial_2^2 b_1)^2\partial_1 u_1 \leqslant C(\|b\|_{H^2}+\|b\|_{H^2}^2)B(t) - 7\int b_1(\partial_2^2 b_1)^2\partial_1 u_1. \tag{3.38}$$

We will take a similar procedure to establish the estimate of the last integral $\int b_1(\partial_2^2 b_1)^2\partial_1 u_1$ to that of $\int(\partial_2^2 b_1)^2\partial_1 u_1$. The key observation that such an inference will be closed is that we have the higher order nonlinearity in $\int b_1(\partial_2^2 b_1)^2\partial_1 u_1$ than that in $\int(\partial_2^2 b_1)^2\partial_1 u_1$. Precisely, by the magnetic equation of (1.5), we see

$$\begin{aligned}
\int b_1(\partial_2^2 b_1)^2\partial_1 u_1 &= \int b_1(\partial_2^2 b_1)^2\partial_t b_1 + \int b_1(\partial_2^2 b_1)^2(u\cdot\nabla b_1 - b\cdot\nabla u_1)\\
&= \frac{1}{2}\frac{d}{dt}\int b_1^2(\partial_2^2 b_1)^2 - \int b_1^2\partial_2^2 b_1\partial_2^2 b_{1t} + \int b_1(\partial_2^2 b_1)^2(u\cdot\nabla b_1 - b\cdot\nabla u_1).
\end{aligned}$$

Then using the magnetic equation of (1.5) again, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int b_1^2(\partial_2^2 b_1)^2 - \int b_1(\partial_2^2 b_1)^2\partial_1 u_1\\
&\quad = \int b_1^2\partial_2^2 b_1\partial_2^2 b_{1t} - \int b_1(\partial_2^2 b_1)^2(u\cdot\nabla b_1 - b\cdot\nabla u_1)\\
&\quad = \int b_1^2\partial_2^2 b_1\partial_2^2(b\cdot\nabla u_1 - u\cdot\nabla b_1 + \partial_1 u_1) - \int b_1(\partial_2^2 b_1)^2(u\cdot\nabla b_1 - b\cdot\nabla u_1)
\end{aligned}$$

$$
\begin{aligned}
&= \int b_1^2 \partial_2^2 b_1 (\partial_2^2 b \cdot \nabla u_1 + 2\partial_2 b \cdot \nabla \partial_2 u_1 + b \cdot \nabla \partial_2^2 u_1) \\
&\quad - \int b_1^2 \partial_2^2 b_1 (\partial_2^2 u \cdot \nabla b_1 + 2\partial_2 u \cdot \nabla \partial_2 b_1) + \int b_1^2 \partial_2^2 b_1 \partial_2^2 \partial_1 u_1 + \int b_1 (\partial_2^2 b_1)^2 b \cdot \nabla u_1.
\end{aligned}
$$

Here in the last equality we have used the fact that

$$
- \int b_1^2 \partial_2^2 b_1 u \cdot \nabla \partial_2^2 b_1 = \int b_1 (\partial_2^2 b_1)^2 u \cdot \nabla b_1.
$$

Thus it follows from Hölder's inequality, Sobolev's embedding and Young's inequality that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int b_1^2 (\partial_2^2 b_1)^2 - \int b_1 (\partial_2^2 b_1)^2 \partial_1 u_1 \\
&\quad \leqslant C\|b\|_{L^\infty}^2 \|b\|_{H^2} (\|b\|_{H^2} \|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_4 \|\nabla b\|_{L^4} + \|b\|_{L^\infty} \|\nabla^3 u_1\|_{L^2} + \|\nabla^3 u\|_{L^2}) \\
&\quad \leqslant C\|b\|_{H^2}^2 \|\partial_1 b\|_{H^1} (\|b\|_{H^2} \|\nabla u\|_{H^2} + \|\nabla u\|_{H^2}) \\
&\quad \leqslant C(\|b\|_{H^2}^2 + \|b\|_{H^2}^3) B(t).
\end{aligned} \tag{3.39}
$$

Combining (3.38) and (3.39), we obtain

$$
\frac{d}{dt}\bigg(- \int b_1 (\partial_2^2 b_1)^2 + \frac{7}{2}\int b_1^2 (\partial_2^2 b_1)^2 \bigg) + \int (\partial_2^2 b_1)^2 \partial_1 u_1 \leqslant C(\|b\|_{H^2} + \|b\|_{H^2}^2 + \|b\|_{H^2}^3) B(t), \tag{3.40}
$$

which together with (3.31) yields that

$$
\begin{aligned}
&\frac{d}{dt}\bigg(\|(\nabla h, \nabla j, \Delta w)\|_{L^2}^2 - 6\int b_1 (\partial_2^2 b_1)^2 + 21 \int b_1^2 (\partial_2^2 b_1)^2 \bigg) + \frac{1}{2}\|(\Delta h, \Delta w)\|_{L^2}^2 \\
&\quad \leqslant C(\|(u, b, w)\|_{H^2} + \|b\|_{H^2}^2 + \|b\|_{H^2}^3) B(t).
\end{aligned}
$$

This completes the proof of Lemma 3.4. □

Lemma 3.5 (Dissipation of $\partial_1 b$). *Let (u, b, w) be the solution of System* (1.5)–(1.8) *with the initial data satisfying the conditions in Proposition* 3.1. *Then we have*

$$
\begin{aligned}
&\int \partial_1 b \cdot u + \frac{1}{2}\int_0^t \|\partial_1 b(s)\|_{L^2}^2 ds - \int_0^t \|(\Delta u, \partial_1 u, \nabla w)(s)\|_{L^2}^2 ds \\
&\quad \leqslant C\int_0^t \|(u, b)(s)\|_{H^2} B(s) ds + \int \partial_1 b_0 \cdot u_0
\end{aligned} \tag{3.41}
$$

for any $t \in [0, T]$.

Proof. Multiplying $(1.5)_1$ and $(1.5)_2$ by $\partial_1 b$ and $\partial_1 u$, respectively, and integrating the resulted equations over Ω, we get

$$
\begin{aligned}
&\int \partial_t u \cdot \partial_1 b + \int (u \cdot \nabla u) \cdot \partial_1 b - \int \Delta u \cdot \partial_1 b = \int (b \cdot \nabla b) \cdot \partial_1 b + \|\partial_1 b\|_{L^2}^2 + \int \nabla \times w \cdot \partial_1 b, \\
&\int \partial_t b \cdot \partial_1 u + \int (u \cdot \nabla b) \cdot \partial_1 u = \int (b \cdot \nabla u) \cdot \partial_1 u + \|\partial_1 u\|_{L^2}^2.
\end{aligned}
$$

Since $\int u_t \cdot \partial_1 b = -\int \partial_1 u_t \cdot b$, we can subtract the second equation from the first one to deduce that

$$
\begin{aligned}
\frac{d}{dt}\int \partial_1 u \cdot b + \|\partial_1 b\|_{L^2}^2 - \|\partial_1 u\|_{L^2}^2 &= \int (u \cdot \nabla u) \cdot \partial_1 b - \int (b \cdot \nabla b) \cdot \partial_1 b - \int (u \cdot \nabla b) \cdot \partial_1 u \\
&\quad + \int (b \cdot \nabla u) \cdot \partial_1 u - \int \Delta u \cdot \partial_1 b - \int \nabla \times w \cdot \partial_1 b.
\end{aligned} \tag{3.42}
$$

For the nonlinear terms on the right-hand side of (3.42), we have

$$
\int (u \cdot \nabla u) \cdot \partial_1 b - \int (b \cdot \nabla b) \cdot \partial_1 b - \int (u \cdot \nabla b) \cdot \partial_1 u + \int (b \cdot \nabla u) \cdot \partial_1 u
$$

$$
\begin{aligned}
&= \int (u\cdot\nabla u)\cdot\partial_1 b - \int b_1|\partial_1 b|^2 - \int b_2\partial_2 b\cdot\partial_1 b - \int u_1\partial_1 b\cdot\partial_1 u - \int u_2\partial_2 b\cdot\partial_1 u + \int (b\cdot\nabla u)\cdot\partial_1 u \\
&\leqslant \|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\partial_1 b\|_{L^2} + \|b_1\|_{L^\infty}\|\partial_1 b\|_{L^2}^2 + \|b_2\|_{L^\infty}\|\partial_2 b\|_{L^2}\|\partial_1 b\|_{L^2} \\
&\quad + \|u_2\|_{L^\infty}\|\partial_2 b\|_{L^2}\|\partial_1 u\|_{L^2} + \|b\|_{L^\infty}\|\nabla u\|_{L^2}\|\partial_1 u\|_{L^2} \\
&\leqslant \|u\|_{H^2}\|\nabla u\|_{L^2}\|\partial_1 b\|_{L^2} + \|b\|_{H^2}\|\partial_1 b\|_{H^1}^2 + \|b\|_{H^1}\|\partial_1 u\|_{H^1}^2 + \|b\|_{H^2}\|\nabla u\|_{L^2}^2 \\
&\leqslant C(\|u\|_{H^2} + \|b\|_{H^2})(\|\nabla u\|_{H^1}^2 + \|\partial_1 b\|_{H^1}^2) \\
&\leqslant C(\|u\|_{H^2} + \|b\|_{H^2})B(t),
\end{aligned} \tag{3.43}
$$

while for the linear terms, it is easy to see that

$$
\begin{aligned}
-\int \Delta u\cdot\partial_1 b - \int \nabla\times w\cdot\partial_1 b &\leqslant (\|\Delta u\|_{L^2} + \|\nabla w\|_{L^2})\|\partial_1 b\|_{L^2} \\
&\leqslant (\|\Delta u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) + \frac{1}{2}\|\partial_1 b\|_{L^2}^2.
\end{aligned} \tag{3.44}
$$

Inserting the estimates (3.43) and (3.44) into (3.42) leads to (3.41). This completes the proof of Lemma 3.5. □

Lemma 3.6 (Dissipation of (u_t, b_t, w_t)). *Let* (u, b, w) *be the solution of System* (1.5)–(1.8) *with the initial data satisfying the conditions in Proposition* 3.1. *Then we have*

$$
\begin{aligned}
&\|(\nabla u, 2w)(t)\|_{L^2}^2 + \int_0^t \|(u_t, b_t, w_t)(s)\|_{L^2}^2 ds - 4\int_0^t \|(\nabla u, \partial_1 b, \nabla w)(s)\|_{L^2}^2 ds \\
&\quad \leqslant C\int_0^t \|(u,b)(s)\|_{H^2}^2 B(s)dx + \|(\nabla u_0, 2w_0)\|_{L^2}^2
\end{aligned} \tag{3.45}
$$

for any $t\in[0,T]$.

Proof. Taking the L^2 inner product of the first three equations in (1.5) with u_t, b_t and w_t, respectively, we can show from the integration by parts that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla u, 2w)\|_{L^2}^2 + \|(u_t, b_t, w_t)\|_{L^2}^2 \\
&\quad = -\int (u\cdot\nabla)u\cdot u_t + \int (b\cdot\nabla)b\cdot u_t - \int (u\cdot\nabla)b\cdot b_t + \int (b\cdot\nabla)u\cdot b_t - \int (u\cdot\nabla)w w_t \\
&\qquad + \int \partial_1 b\cdot u_t + \int \partial_1 u\cdot b_t + \int \nabla\times w\cdot u_t + \int \nabla\times u\, w_t
\end{aligned} \tag{3.46}
$$

due to $\partial_t u_2 = \partial_2 u_1 = 0$ and $w_t = w = 0$ on $\partial\Omega$. We first use Hölder's inequality, Sobolev's embedding and Young's inequality to bound the nonlinear terms on the right-hand side of (3.46) as

$$
\begin{aligned}
&-\int (u\cdot\nabla)u\cdot u_t + \int (b\cdot\nabla)b\cdot u_t - \int (u\cdot\nabla)b\cdot b_t + \int (b\cdot\nabla)u\cdot b_t - \int (u\cdot\nabla)w w_t \\
&\quad = -\int ((u\cdot\nabla)u - b_1\partial_1 b - b_2\partial_2 b)\cdot u_t - \int (u_1\partial_1 b + u_2\partial_2 b - (b\cdot\nabla)u)\cdot b_t - \int (u\cdot\nabla)w w_t \\
&\quad \leqslant (\|u\|_{L^\infty}\|\nabla u\|_{L^2} + \|b_1\|_{L^\infty}\|\partial_1 b\|_{L^2} + \|b_2\|_{L^\infty}\|\partial_2 b\|_{L^2})\|u_t\|_{L^2} \\
&\qquad + (\|u_1\|_{L^\infty}\|\partial_1 b\|_{L^2} + \|u_2\|_{L^\infty}\|\partial_2 b\|_{L^2} + \|b\|_{L^\infty}\|\nabla u\|_{L^2})\|b_t\|_{L^2} + \|u\|_{L^\infty}\|\nabla w\|_{L^2}\|w_t\|_{L^2} \\
&\quad \leqslant C(\|u\|_{H^2}\|\nabla u\|_{L^2} + \|b\|_{H^2}\|\partial_1 b\|_{H^1})\|u_t\|_{L^2} + C(\|u\|_{H^2}\|\partial_1 b\|_{L^2} + \|b\|_{H^2}\|\nabla u\|_{H^1})\|b_t\|_{L^2} \\
&\qquad + C\|u\|_{H^2}\|\nabla w\|_{L^2}\|w_t\|_{L^2} \\
&\quad \leqslant \frac{1}{4}\|(u_t, b_t, w_t)\|_{L^2}^2 + C(\|u\|_{H^2}^2 + \|b\|_{H^2}^2)(\|\nabla u\|_{H^1}^2 + \|\partial_1 b\|_{H^1}^2 + \|\nabla w\|_{L^2}^2) \\
&\quad \leqslant \frac{1}{4}\|(u_t, b_t, w_t)\|_{L^2}^2 + C(\|u\|_{H^2}^2 + \|b\|_{H^2}^2)B(t).
\end{aligned} \tag{3.47}
$$

For the linear terms on the right-hand side of (3.46), on the other hand, it is clear that

$$
\int \partial_1 b\cdot u_t + \int \partial_1 u\cdot b_t + \int \nabla\times w\cdot u_t + \int \nabla\times u\, w_t
$$

$$\leqslant \frac{1}{4}\|(u_t, b_t, w_t)\|_{L^2}^2 + 2(\|\partial_1 b\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2). \tag{3.48}$$

Combining (3.46)–(3.48), we can deduce the desired estimate (3.45). This completes the proof of Lemma 3.6. □

Lemma 3.7 (Dissipation of ∇u_t). *Let (u, b, w) be the solution of System* (1.5)–(1.8) *with the initial data satisfying the conditions in Proposition* 3.1. *Then we have*

$$\|u_t(t)\|_{L^2}^2 + \int_0^t \|\nabla u_t(s)\|_{L^2}^2 ds - 2\int_0^t \|(b_t, w_t)(s)\|_{L^2}^2 ds \leqslant C\int_0^t \|(u, b)(s)\|_{H^2} B(s) ds + \|u_t(0)\|_{L^2}^2 \tag{3.49}$$

for any $t \in [0, T]$.

Proof. Applying ∂_t to $(1.5)_1$ and taking the L^2 inner product of the resulted equation with u_t, we can use the integration by parts to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 &= -\int ((u\cdot\nabla)u)_t \cdot u_t + \int ((b\cdot\nabla)b)_t u_t + \int \partial_1 b_t \cdot u_t + \int \nabla\times w_t \cdot u_t\\ &= \int (u_t\cdot\nabla)u\cdot u_t - \int (b_t\cdot\nabla)u_t\cdot b - \int (b\cdot\nabla)u_t\cdot b_t - \int b_t\cdot\partial_1 u_t + \int w_t \nabla\times u_t.\end{aligned}$$

It then follows from Hölder's inequality, Sobolev's embedding and Young's inequality that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2\\ &\quad\leqslant \|\nabla u\|_{L^2}\|u_t\|_{L^4}^2 + 2\|b\|_{L^\infty}\|b_t\|_{L^2}\|\nabla u_t\|_{L^2} + \|b_t\|_{L^2}\|\partial_1 u_t\|_{L^2} + \|w_t\|_{L^2}\|\nabla u_t\|_{L^2}\\ &\quad\leqslant C\|\nabla u\|_{L^2}(\|u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{1}{2}} + \|u_t\|_{L^2})^2 + C\|b\|_{H^2}\|b_t\|_{L^2}\|\nabla u_t\|_{L^2} + (\|b_t\|_{L^2} + \|w_t\|_{L^2})\|\nabla u_t\|_{L^2}\\ &\quad\leqslant \frac{1}{2}\|\nabla u_t\|_{L^2}^2 + C(\|\nabla u\|_{L^2} + \|b\|_{H^2})(\|u_t\|_{H^1}^2 + \|b_t\|_{L^2}^2) + (\|b_t\|_{L^2}^2 + \|w_t\|_{L^2}^2)\\ &\quad\leqslant \frac{1}{2}\|\nabla u_t\|_{L^2}^2 + C(\|u\|_{H^1} + \|b\|_{H^2})B(t) + (\|b_t\|_{L^2}^2 + \|w_t\|_{L^2}^2),\end{aligned}$$

from which we can deduce the desired estimate (3.49). This completes the proof of Lemma 3.7. □

Lemma 3.8 (Dissipation of $\nabla\partial_1 b$). *Let (u, b, w) be the solution of System* (1.5)–(1.8) *with the initial data satisfying the conditions in Proposition* 3.1. *Then we have*

$$\begin{aligned}&\left(\|\Delta b(t)\|_{L^2}^2 - 6\int b_1(\partial_2^2 b_1)^2 + 21\int b_1^2(\partial_2^2 b_1)^2\right)\\ &\quad + \left(\int_0^t \|\nabla\partial_1 b(s)\|_{L^2}^2 ds - 2\int_0^t \|\nabla u_t(s)\|_{L^2}^2 ds - 8\int_0^t \|\nabla^2 w(s)\|_{L^2}^2 ds\right)\\ &\qquad \leqslant C\int_0^t (\|(u, b)(s)\|_{H^2} + \|b(s)\|_{H^2}^2 + \|b(s)\|_{H^2}^3) B(s) ds\\ &\qquad\quad + \left(\|\Delta b_0\|_{L^2}^2 - 6\int b_{01}(\partial_2^2 b_{01})^2 + 21\int b_{01}^2(\partial_2^2 b_{01})^2\right)\end{aligned} \tag{3.50}$$

for any $t \in [0, T]$.

Proof. Firstly, we apply Δ to $(1.5)_2$ and take the L^2 inner product of the resulted equation with Δb to get

$$\frac{1}{2}\frac{d}{dt}\|\Delta b\|_{L^2}^2 = \int \Delta b\cdot\Delta(\partial_1 u) + \int \Delta(b\cdot\nabla u - u\cdot\nabla b)\cdot\Delta b. \tag{3.51}$$

To obtain the dissipation of $\nabla\partial_1 b$, we first rewrite the first term on the right-hand side of (3.51). Since $\int \Delta b\cdot\nabla\partial_1 P = 0$ due to $\Delta b_2 = 0$ on $\partial\Omega$, we see from the velocity equation of (1.5) that

$$\int \Delta b\cdot\Delta(\partial_1 u) = -\int \Delta b\cdot\partial_1^2 b + \int \Delta b\cdot\partial_1 u_t - \int \Delta b\cdot\partial_1(\nabla\times w) + \int \Delta b\cdot\partial_1(u\cdot\nabla u - b\cdot\nabla b). \tag{3.52}$$

The dissipation will arise from the first integral on the right-hand side of (3.52). Indeed, the integration by parts gives that

$$\begin{aligned}\int \Delta b\cdot\partial_1^2 b &= \int \partial_1^2 b_1\partial_1^2 b_1 + \int \partial_2^2 b_1\partial_1^2 b_1 + \int \Delta b_2\partial_1^2 b_2\\&= \int \partial_1^2 b_1\partial_1^2 b_1 - \int \partial_2 b_1\partial_2\partial_1^2 b_1 - \int \partial_1\Delta b_2\partial_1 b_2\\&= \int \partial_1^2 b_1\partial_1^2 b_1 + \int \partial_1\partial_2 b_1\partial_2\partial_1 b_1 + \int \partial_1\nabla b_2\cdot\partial_1\nabla b_2\\&= \|\nabla\partial_1 b\|_{L^2}^2. \end{aligned}\tag{3.53}$$

For the linear terms on the right-hand side of (3.52), we have

$$\begin{aligned}\int \Delta b\cdot\partial_1 u_t &= \int \partial_1^2 b_1\partial_1 u_{1t} + \int \partial_2^2 b_1\partial_1 u_{1t} + \int \Delta b_2\partial_1 u_{2t}\\&= \int \partial_1^2 b_1\partial_1 u_{1t} - \int \partial_2 b_1\partial_2\partial_1 u_{1t} - \int \nabla b_2\cdot\partial_1\nabla u_{2t}\\&= \int \partial_1^2 b_1\partial_1 u_{1t} + \int \partial_1\partial_2 b_1\partial_2 u_{1t} + \int \partial_1\nabla b_2\cdot\nabla u_{2t}\\&\leqslant \|\partial_1\nabla b\|_{L^2}\|\nabla u_t\|_{L^2} \leqslant \frac{1}{4}\|\nabla\partial_1 b\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2\end{aligned}\tag{3.54}$$

and

$$\begin{aligned}-\int \Delta b\cdot\partial_1(\nabla\times w) &= -\int \partial_1^2 b\cdot\partial_1(\nabla\times w) - \int \partial_2^2 b_1\partial_1\partial_2 w + \int \partial_2^2 b_2\partial_1^2 w\\&= -\int \partial_1^2 b\cdot\partial_1(\nabla\times w) - \int \partial_1\partial_2 b_1\partial_2^2 w - \int \partial_2\partial_1 b_1\partial_1^2 w\\&\leqslant \|\partial_1\nabla b\|_{L^2}(\|\nabla\times\partial_1 w\|_{L^2} + \|\partial_1^2 w\|_{L^2} + \|\partial_2^2 w\|_{L^2})\\&\leqslant \frac{1}{4}\|\nabla\partial_1 b\|_{L^2}^2 + 4\|\nabla^2 w\|_{L^2}^2.\end{aligned}\tag{3.55}$$

Inserting (3.52)–(3.55) into (3.51), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Delta b\|_{L^2}^2 + \frac{1}{2}\|\partial_1\nabla b\|_{L^2}^2 - \|\nabla u_t\|_{L^2}^2 - 4(\|\partial_1^2 w\|_{L^2}^2 + \|\partial_2^2 w\|_{L^2}^2)\\&\quad\leqslant \int \partial_1(u\cdot\nabla u)\cdot\Delta b - \int \partial_1(b\cdot\nabla b)\cdot\Delta b + \int \Delta(b\cdot\nabla u)\cdot\Delta b - \int \Delta(u\cdot\nabla b)\cdot\Delta b\\&\quad=: J_{41} + J_{42} + J_{43} + J_{44}.\end{aligned}\tag{3.56}$$

We now estimate J_{41}, J_{42}, J_{43} and J_{44} one by one. For J_{41}, we can use the integration by parts and (2.3) to deduce that

$$\begin{aligned}J_{41} &= -\int (u_1\partial_1 u_1)\Delta\partial_1 b_1 - \int (u_1\partial_1 u_2)\Delta\partial_1 b_2 - \int (u_2\partial_2 u)\cdot\Delta\partial_1 b\\&= \int \nabla(u_1\partial_1 u_1)\cdot\nabla\partial_1 b_1 + \int \nabla(u_1\partial_1 u_2)\cdot\nabla\partial_1 b_2 + \int \nabla(u_2\partial_2 u)\cdot\nabla\partial_1 b\\&\leqslant \|\nabla(u\cdot\nabla u)\|_{L^2}(\|\nabla\partial_1 b_1\|_{L^2} + \|\nabla\partial_1 b_2\|_{L^2} + \|\nabla\partial_1 b\|_{L^2})\\&\leqslant C\|u\|_{H^2}\|\nabla u\|_{H^1}\|\partial_1 b\|_{H^1}\\&\leqslant C\|u\|_{H^2}B(t).\end{aligned}\tag{3.57}$$

Similarly, for J_{42}, we have

$$J_{42} = \int b_1\partial_1 b_1\Delta\partial_1 b_1 + \int b_1\partial_1 b_2\Delta\partial_1 b_2 + \int b_2\partial_2 b\Delta\partial_1 b$$

$$
\begin{aligned}
&= -\int \nabla(b_1\partial_1 b_1)\cdot\nabla\partial_1 b_1 - \int \nabla(b_1\partial_1 b_2)\cdot\nabla\partial_1 b_2 - \int \nabla(b_2\partial_2 b)\cdot\nabla\partial_1 b\\
&\leqslant C\|\nabla(b\cdot\nabla b)\|_{L^2}\|\nabla\partial_1 b\|_{L^2}\\
&\leqslant C\|b\|_{H^2}\|\partial_1 b\|_{H^1}\|\nabla\partial_1 b_1\|_{L^2}\\
&\leqslant C\|b\|_{H^2}B(t).
\end{aligned}\tag{3.58}
$$

The estimates for J_{43} and J_{44} are more subtle. For J_{43}, we first split it into three terms

$$
\begin{aligned}
J_{43} &= \int (\Delta b\cdot\nabla u + 2(\nabla b\cdot\nabla)\nabla u + b\cdot\nabla\Delta u)\cdot\partial_1^2 b + \int (\Delta b\cdot\nabla u_2 + 2(\nabla b\cdot\nabla)\nabla u_2 + b\cdot\nabla\Delta u_2)\partial_2^2 b_2\\
&\quad + \int (\Delta b\cdot\nabla u_1 + 2(\nabla b\cdot\nabla)\nabla u_1 + b\cdot\nabla\Delta u_1)\partial_2^2 b_1\\
&=: J_{431} + J_{432} + J_{433}.
\end{aligned}\tag{3.59}
$$

By Hölder's inequality, Sobolev's embedding and Young's inequality, we have

$$
\begin{aligned}
J_{431} &\leqslant (\|\Delta b\|_{L^2}\|\nabla u\|_{L^\infty} + 2\|\nabla b\|_{L^4}\|\nabla^2 u\|_{L^4} + \|b\|_{L^\infty}\|\nabla^3 u\|_{L^2})\|\partial_1^2 b\|_{L^2}\\
&\leqslant C\|b\|_{H^2}\|\nabla u\|_{H^2}\|\partial_1 b\|_{H^1}\\
&\leqslant C\|b\|_{H^2}(\|\nabla u\|_{H^2}^2 + \|\partial_1 b\|_{H^1}^2)\\
&\leqslant C\|b\|_{H^2}B(t)
\end{aligned}\tag{3.60}
$$

and

$$
\begin{aligned}
J_{432} &\leqslant (\|\Delta b\|_{L^2}\|\nabla u_2\|_{L^\infty} + 2\|\nabla b\|_{L^4}\|\nabla^2 u_2\|_{L^4} + \|b\|_{L^\infty}\|\nabla^3 u_2\|_{L^2})\|\partial_2^2 b_2\|_{L^2}\\
&\leqslant C\|b\|_{H^2}\|\nabla u_2\|_{H^2}\|\partial_2\partial_1 b\|_{L^2}\\
&\leqslant C\|b\|_{H^2}(\|\nabla u\|_{H^2}^2 + \|\partial_1 b\|_{H^1}^2)\\
&\leqslant C\|b\|_{H^2}B(t).
\end{aligned}\tag{3.61}
$$

To estimate J_{433}, we first use the integration by parts to rewrite it as

$$
\begin{aligned}
J_{433} &= \int \partial_1^2 b_1\partial_1 u_1\partial_2^2 b_1 + \int \partial_1 u_1(\partial_2^2 b_1)^2 + \int \partial_1^2 b_2\partial_2 u_1\partial_2^2 b_1 + \int \partial_2^2 b_2\partial_2 u_1\partial_2^2 b_1\\
&\quad + 2\int (\partial_1 b\cdot\nabla)\partial_1 u_1\partial_2^2 b_1 + 2\int \partial_2 b_1\partial_2\partial_1 u_1\partial_2^2 b_1 + 2\int \partial_2 b_2\partial_2^2 u_1\partial_2^2 b_1\\
&\quad + \int b_1\partial_1\Delta u_1\partial_2^2 b_1 + \int b_2\partial_2\Delta u_1\partial_2^2 b_1\\
&= \int \partial_1^2 b_1\partial_1 u_1\partial_2^2 b_1 + \int (\partial_2^2 b_1)^2\partial_1 u_1 + \int \partial_1^2 b_2\partial_2 u_1\partial_2^2 b_1 - \int \partial_2\partial_1 b_1\partial_2 u_1\partial_2^2 b_1\\
&\quad + 2\int (\partial_1 b\cdot\nabla)\partial_1 u_1\partial_2^2 b_1 - 2\int \partial_2^2 u_1\partial_2 b_1\partial_1\partial_2 b_1 - 2\int \partial_1 b_1\partial_2^2 u_1\partial_2^2 b_1\\
&\quad - \int \partial_1 b_1\Delta u_1\partial_2^2 b_1 + \int \partial_2 b_1\Delta u_1\partial_2\partial_1 b_1 + \int b_1\partial_2\Delta u_1\partial_2\partial_1 b_1 + \int b_2\partial_2\Delta u_1\partial_2^2 b_1.
\end{aligned}\tag{3.62}
$$

Then it is clear that except for the second term on the right-hand side of (3.62) the other integrals can be bounded by

$$
\begin{aligned}
&\|\partial_1 b\|_{H^1}\|\nabla u_1\|_{L^\infty}\|\partial_2^2 b_1\|_{L^2} + \|\partial_1 b\|_{L^4}\|\nabla^2 u_1\|_{L^4}\|\partial_2^2 b_1\|_{L^2} + \|\nabla^2 u_1\|_{L^4}\|\partial_2 b_1\|_{L^4}\|\partial_1\partial_2 b_1\|_{L^2}\\
&\quad + \|b_1\|_{L^\infty}\|\partial_2\Delta u_1\|_{L^2}\|\partial_2\partial_1 b_1\|_{L^2} + \|b_2\|_{L^\infty}\|\partial_2\Delta u_1\|_{L^2}\|\partial_2^2 b_1\|_{L^2}
\end{aligned}
$$

and thus by

$$\|b\|_{H^2}\|\partial_1 b\|_{H^1}\|\nabla u\|_{H^2}.$$

This means that

$$J_{433} \leqslant C\|b\|_{H^2}B(t) + \int (\partial_2^2 b_1)^2\partial_1 u_1,$$

which together with (3.59)–(3.61) yields that

$$J_{43} \leqslant C\|b\|_{H^2}B(t) + \int (\partial_2^2 b_1)^2 \partial_1 u_1. \tag{3.63}$$

Similarly, for J_{44}, we rewrite it as

$$\begin{aligned} J_{44} &= -\int (\Delta u \cdot \nabla b + 2(\nabla u \cdot \nabla) \cdot \nabla b) \cdot \partial_1^2 b - \int (\Delta u \cdot \nabla b_1 + 2(\nabla u \cdot \nabla) \cdot \nabla b_1)\partial_2^2 b_1 \\ &\quad - \int (\Delta u \cdot \nabla b_2 + 2(\nabla u \cdot \nabla) \cdot \nabla b_2)\partial_2^2 b_2 \\ &=: J_{441} + J_{442} + J_{443}. \end{aligned} \tag{3.64}$$

Clearly, we have

$$\begin{aligned} J_{441} + J_{443} &\leqslant (\|\Delta u\|_{L^4}\|\nabla b\|_{L^4} + 2\|\nabla u\|_{L^\infty}\|\nabla^2 b\|_{L^2})(\|\partial_1^2 b\|_{L^2} + \|\partial_2^2 b_2\|_{L^2}) \\ &\leqslant C(\|\Delta u\|_{H^1}\|\nabla b\|_{H^1} + \|\nabla u\|_{H^2}\|b\|_{H^2})(\|\partial_1^2 b\|_{L^2} + \|\partial_2\partial_1 b_1\|_{L^2}) \\ &\leqslant C\|\nabla u\|_{H^2}\|b\|_{H^2}\|\partial_1 b\|_{H^1} \\ &\leqslant C\|b\|_{H^2}B(t). \end{aligned} \tag{3.65}$$

We now deal with J_{442}:

$$\begin{aligned} J_{442} &= -\int \Delta u_1 \partial_1 b_1 \partial_2^2 b_1 - \int \partial_1^2 u_2 \partial_2 b_1 \partial_2^2 b_1 + \int \partial_2\partial_1 u_1 \partial_2 b_1 \partial_2^2 b_1 \\ &\quad - 2\int \partial_1 u \cdot \nabla \partial_1 b_1 \partial_2^2 b_1 - 2\int \partial_2 u_1 \partial_1\partial_2 b_1 \partial_2^2 b_1 - 2\int \partial_2 u_2 \partial_2^2 b_1 \partial_2^2 b_1 \\ &= -\int \Delta u_1 \partial_1 b_1 \partial_2^2 b_1 - \int \partial_2\partial_1 u_2 \partial_2 b_1 \partial_1\partial_2 b_1 + \int \partial_2^2 u_1 \partial_2 b_1 \partial_1 \partial_2 b_1 \\ &\quad - 2\int \partial_1 u \cdot \nabla \partial_1 b_1 \partial_2^2 b_1 - 2\int \partial_2 u_1 \partial_1 \partial_2 b_1 \partial_2^2 b_1 + 2\int (\partial_2^2 b_1)^2 \partial_1 u_1. \end{aligned} \tag{3.66}$$

On the right-hand side of (3.64), all the integrals except for the last one can be bounded by

$$\|\Delta u_1\|_{L^4}\|\partial_1 b_1\|_{L^4}\|\partial_2^2 b_1\|_{L^2} + \|\nabla^2 u\|_{L^4}\|\partial_2 b_1\|_{L^4}\|\partial_1\partial_2 b_1\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla\partial_1 b_1\|_{L^2}\|\partial_2^2 b_1\|_{L^2}$$

and thus by

$$\|b\|_{H^2}\|\partial_1 b\|_{H^1}\|\nabla u\|_{H^2}.$$

This means that

$$J_{442} \leqslant C\|b\|_{H^2}B(t) + 2\int (\partial_2^2 b_1)^2 \partial_1 u_1,$$

which together with (3.64)–(3.66) gives that

$$J_{44} \leqslant C\|b\|_{H^2}B(t) + 2\int (\partial_2^2 b_1)^2 \partial_1 u_1. \tag{3.67}$$

Substituting (3.57), (3.58), (3.63) and (3.67) into (3.56), we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta b\|_{L^2}^2 + \frac{1}{2}\|\nabla\partial_1 b\|_{L^2}^2 - \|\nabla u_t\|_{L^2}^2 - 4\|\nabla^2 w\|_{L^2}^2 \leqslant C(\|u\|_{H^2} + \|b\|_{H^2})B(t) + 3\int (\partial_2^2 b_1)^2 \partial_1 u_1,$$

which together with (3.40) gives that

$$\begin{aligned} &\frac{d}{dt}\left(\|\Delta b\|_{L^2}^2 - 6\int b_1(\partial_2^2 b_1)^2 + 21\int b_1^2(\partial_2^2 b_1)^2\right) + (\|\nabla\partial_1 b\|_{L^2}^2 - 2\|\nabla u_t\|_{L^2}^2 - 8\|\nabla^2 w\|_{L^2}^2) \\ &\quad \leqslant C(\|u\|_{H^2} + \|b\|_{H^2} + \|b\|_{H^2}^2 + \|b\|_{H^2}^3)B(t). \end{aligned}$$

This completes the proof of Lemma 3.8. □

Proof of Proposition 3.1. Firstly, we remark that

$$\|\Delta u\|_{L^2} = \|\nabla^2 u\|_{L^2} = \|\nabla(\nabla\times u)\|_{L^2} \quad \text{and} \quad \|\nabla^3 u\|_{L^2} = \|\Delta(\nabla\times u)\|_{L^2}$$

and that

$$\|\nabla^2 b\|_{L^2} = \|\nabla(\nabla\times b)\|_{L^2} \quad \text{and} \quad \|\Delta w\|_{L^2} = \|\nabla^2 w\|_{L^2}$$

by the integration by parts. Then we can combine (3.3), (3.4) and (3.21) to deduce that

$$\begin{aligned}
&\left(2\|(u,b,w)(t)\|_{H^2}^2 - 8\int b_1(\partial_2 b_1)^2 - 12\int b_1(\partial_2^2 b_1)^2 + 42\int b_1^2(\partial_2^2 b_1)^2\right) + \int_0^t \|(\nabla u, w)(s)\|_{H^2}^2 ds \\
&\quad \leqslant C\int_0^t (\|(u,b,w)(s)\|_{H^2} + \|b(s)\|_{H^2}^2 + \|b(s)\|_{H^2}^3) B(s) ds \\
&\qquad + \left(2\|(u_0,b_0,w_0)\|_{H^2}^2 - 8\int b_{01}(\partial_2 b_{01})^2 - 12\int b_{01}(\partial_2^2 b_{01})^2 + 42\int b_{01}^2(\partial_2^2 b_{01})^2\right) \qquad (3.68)
\end{aligned}$$

for any $t\in[0,T]$. Thus multiplying (3.41), (3.45), (3.49) and (3.50) by $\frac{1}{2}$, $\frac{1}{32}$, $\frac{1}{128}$ and $\frac{1}{512}$, respectively, and adding the resulted inequalities into (3.68), we obtain

$$\begin{aligned}
&\left(\|(u,b,w)(t)\|_{H^2}^2 + \frac{1}{128}\|u_t(t)\|_{L^2}^2 - 64\|b\|_{L^\infty}\|b\|_{H^2}^2\right) \\
&\quad + \frac{1}{512}\int_0^t (\|(\nabla u, w)(s)\|_{H^2}^2 + \|\partial_1 b(s)\|_{H^1}^2 + \|(u_t,b_t,w_t)(s)\|_{L^2}^2 + \|\nabla u_t(s)\|_{L^2}^2) ds \\
&\quad \leqslant C\int_0^t (\|(u,b,w)(s)\|_{H^2} + \|(u,b)(s)\|_{H^2}^2 + \|b(s)\|_{H^2}^3) B(s) ds \\
&\qquad + \left(3\|(u_0,b_0,w_0)\|_{H^2}^2 + \frac{1}{128}\|u_t(0)\|_{L^2}^2 + 64\|b_0\|_{L^\infty}\|b_0\|_{H^2}^2\right)
\end{aligned}$$

for any $t\in[0,T]$, which together with the assumption (3.1) yields that

$$\begin{aligned}
&\|(u,b,w)(t)\|_{H^2}^2 + \|\partial_t u(t)\|_{L^2}^2 + \int_0^t (\|(\nabla u, w)(s)\|_{H^2}^2 + \|\partial_1 b(s)\|_{H^1}^2 + \|u_t\|_{H^1} + \|(b_t,w_t)(s)\|_{L^2}^2) ds \\
&\quad \leqslant C(\|(u_0,b_0,w_0)\|_{H^2}^2 + \|\partial_t u(0)\|_{L^2}^2) + Cc_0\int_0^t B(s) ds \\
&\quad \leqslant C\|(u_0,b_0,w_0)\|_{H^2}^2 + Cc_0\int_0^t B(s) ds
\end{aligned}$$

for some c_0 suitably small. Here, we also used the fact that

$$\|\partial_t u(0)\|_{L^2} \leqslant C(\|(u_0,b_0)\|_{H^2} + \|w_0\|_{H^1})$$

in the last inequality, which follows from the compatibility conditions. Therefore, we have

$$A(t) + \int_0^t B(s) ds \leqslant C\|(u_0,b_0,w_0)\|_{H^2}^2 + Cc_0\int_0^t B(s) ds.$$

By choosing some c_0 suitably small, we conclude that

$$A(t) + \int_0^t B(s) ds \leqslant C\|(u_0,b_0,w_0)\|_{H^2}^2.$$

This completes the proof of Proposition 3.1. □

Proof of Theorem 1.1. We first take ε_0 suitably small such that $C_0\varepsilon_0^2 \leqslant \frac{1}{2}c_0^2$. Let

$$T^* = \sup\{t \mid \|(u,b,w)(t)\|_{H^2}^2 \leqslant c_0^2\}.$$

If $T^* < +\infty$, we will see from the uniform *a priori* estimates (3.2) in Proposition 3.1 that

$$\sup_{0\leqslant t\leqslant T^*} \|(u,b,w)(t)\|_{H^2}^2 \leqslant C_0\varepsilon_0^2 \leqslant \frac{1}{2}c_0^2,$$

which contradicts the definition of T^*. Hence, we conclude $T^* = +\infty$.

The uniform estimate (1.9) follows from Proposition 3.1. This completes the proof of Theorem 1.1. □

Acknowledgements The first author was supported by National Natural Science Foundation of China (Grant No. 11701049), the China Postdoctoral Science Foundation (Grant No. 2017M622989) and the Opening Fund of Geomathematics Key Laboratory of Sichuan Province (Grant No. scsxdz201707). The second author was supported by National Natural Science Foundation of China (Grant Nos. 11571063 and 11771045). The authors are very grateful to the referees for their detailed comments and helpful suggestions, which greatly improved the manuscript, and to Professor Lili Du for suggesting this problem.

References

1 Abidi H, Zhang P. On the global solution of a 3-D MHD system with initial data near equilibrium. Comm Pure Appl Math, 2017, 70: 1509–1561
2 Berkovski B, Bashtovoy V. Magnetic Fluids and Applications Handbook. New York: Begell House, 1996
3 Cai Y, Lei Z. Global well-posedness of the incompressible magnetohydrodynamics. Arch Ration Mech Anal, 2018, 228: 969–993
4 Cao C, Regmi D, Wu J. The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion. J Differential Equations, 2013, 254: 2661–2681
5 Cao C, Wu J. Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion. Adv Math, 2011, 226: 1803–1822
6 Cao C, Wu J, Yuan B. The 2D incompressible magnetohydrodynamics equations with only magnetic diffusion. SIAM J Math Anal, 2014, 46: 588–602
7 Chen Q, Miao C. Global well-posedness for the micropolar fluid system in critical Besov spaces. J Differential Equations, 2012, 252: 2698–2724
8 Cheng J, Liu Y. Global regularity of the 2D magnetic micropolar fluid flows with mixed partial viscosity. Comput Math Appl, 2015, 70: 66–72
9 Deng W, Zhang P. Large time behavior of solutions to 3-D MHD system with initial data near equilibrium. Arch Ration Mech Anal, 2018, 230: 1017–1102
10 Dong B, Chen Z. Regularity criteria of weak solutions to the three-dimensional micropolar flows. J Math Phys, 2009, 50: 103525
11 Dong B, Chen Z. Asymptotic profiles of solutions to the 2D viscous incompressible micropolar fluid flows. Discrete Contin Dyn Syst, 2009, 23: 765–784
12 Dong B, Li J, Wu J. Global well-posedness and large-time decay for the 2D micropolar equations. J Differential Equations, 2017, 262: 3488–3523
13 Dong B, Zhang Z. Global regularity of the 2D micropolar fluid flows with zero angular viscosity. J Differential Equations, 2010, 249: 200–213
14 Du L, Zhou D. Global well-posedness of two-dimensional magnetohydrodynamic flows with partial dissipation and magnetic diffusion. SIAM J Math Anal, 2015, 47: 1562–1589
15 Duvaut G, Lions J L. Inéquations en thermoélasticité et magnétohydrodynamique. Arch Ration Mech Anal, 1972, 46: 241–279
16 Eringen A C. Theory of micropolar fluids. J Math Mech, 1966, 16: 1–18
17 Galdi G, Rionero S. A note on the existence and uniqueness of solutions of micropolar fluid equations. Internat J Engrg Sci, 1977, 14: 105–108
18 Jiu Q, Niu D, Wu J, et al. The 2D magnetohydrodynamic equations with magnetic diffusion. Nonlinearity, 2015, 28: 3935–3955
19 Lei Z. On axially symmetric incompressible magnetohydrodynamics in three dimensions. J Differential Equations, 2015, 259: 3202–3215
20 Lin F, Xu L, Zhang P. Global small solutions to 2D incompressible MHD system. J Differential Equations, 2015, 259: 5440–5485
21 Lin F, Zhang P. Global small solutions to MHD type system (I): 3D case. Comm Pure Appl Math, 2014, 67: 531–580
22 Lin F, Zhang T. Global small solutions to a complex fluid model in 3D. Arch Ration Mech Anal, 2015, 216: 905–920

23 Pan R, Zhou Y, Zhu Y. Global classical solutions of three dimensional viscous MHD system without magnetic diffusion on periodic boxes. Arch Ration Mech Anal, 2018, 227: 637–662

24 Ren X, Wu J, Xiang Z, et al. Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion. J Funct Anal, 2015, 267: 5440–5485

25 Ren X, Xiang Z, Zhang Z. Global well-posedness for the 2D MHD equations without magnetic diffusion in a strip domain. Nonlinearity, 2016, 29: 1257–1291

26 Ren X X, Xiang Z Y, Zhang Z F. Global existence and decay of smooth solutions for the 3-D MHD-type equations without magnetic diffusion. Sci China Math, 2016, 59: 1949–1974

27 Ren X X, Xiang Z Y, Zhang Z F. Low regularity well-posedness for the viscous surface wave equation. Sci China Math, 2019, 62: 1887–1924

28 Rojas-Medar M A, Boldrini J L. Magneto-micropolar fluid motion: Existence of weak solutions. Rev Mat Complut, 1998, 11: 443–460

29 Sermange M, Temam R. Some mathematical questions related to the MHD equations. Comm Pure Appl Math, 1983, 36: 635–664

30 Tan Z, Wang Y. Global well-posedness of an initial-boundary value problem for viscous non-resistive MHD systems. SIAM J Math Anal, 2018, 50: 1432–1470

31 Wang Y, Wang Y. Blow-up criterion for two-dimensional magneto-micropolar fluid equations with partial viscosity. Math Methods Appl Sci, 2011, 34: 2125–2135

32 Wei R, Guo B, Li Y. Global existence and optimal convergence rates of solutions for 3D compressible magneto-micropolar fluid equations. J Differential Equations, 2017, 263: 2457–2480

33 Xiang Z, Yang H. On the regularity criteria for the 3D magneto-micropolar fluids in terms of one directional derivative. Bound Value Probl, 2012, 2012: 139

34 Xu L, Zhang P. Global small solutions to three-dimensional incompressible magnetohydrodynamical system. SIAM J Math Anal, 2015, 47: 26–65

35 Xue L. Well posedness and zero microrotation viscosity limit of the 2D micropolar fluid equations. Math Methods Appl Sci, 2011, 34: 1760–1777

36 Yamazaki K. Global regularity of the two-dimensional magneto-micropolar fluid system with zero angular viscosity. Discrete Contin Dyn Syst, 2015, 35: 2193–2207

37 Zhai Z, Zhang T. Global existence and uniqueness theorem to 2-D incompressible non-resistive MHD system with non-equilibrium background magnetic field. J Differential Equations, 2016, 261: 3519–3550

38 Zhang T. An elementary proof of the global existence and uniqueness theorem to 2-D incompressible non-resistive MHD system. ArXiv:1404.5681, 2014

39 Zhang T. Global solutions to the 2D viscous, non-resistive MHD system with large background magnetic field. J Differential Equations, 2016, 260: 5450–5480

40 Zhou Y, Zhu Y. Global classical solutions of 2D MHD system with only magnetic diffusion on periodic domain. J Math Phys, 2018, 59: 081505