SCIENCE CHINA
Mathematics

• ARTICLES •

May 2020 Vol. 63 No. 5: 873–886
https://doi.org/10.1007/s11425-018-9405-4

Bifurcation of limit cycles of the nongeneric quadratic reversible system with discontinuous perturbations

Jihua Yang

School of Mathematics and Computer Science, Ningxia Normal University, Guyuan 756000, China
Email: jihua1113@163.com

Received February 7, 2018; accepted September 29, 2018; published online July 16, 2019

Abstract By using the Picard-Fuchs equation and the property of the Chebyshev space to the discontinuous differential system, we obtain an upper bound of the number of limit cycles for the nongeneric quadratic reversible system when it is perturbed inside all discontinuous polynomials with degree n.

Keywords quadratic reversible system, Melnikov function, Picard-Fuchs equation, Chebyshev space

MSC(2010) 34C05, 34C07

Citation: Yang J H. Bifurcation of limit cycles of the nongeneric quadratic reversible system with discontinuous perturbations. Sci China Math, 2020, 63: 873–886, https://doi.org/10.1007/s11425-018-9405-4

1 Introduction

Stimulated by discontinuous phenomena in the real world, such as biology [9], nonlinear oscillations [18], impact and friction mechanics [1], great interest has appeared for studying the number of limit cycles and their relative positions of discontinuous differential systems. Similar to the smooth differential system, one of the main problems in the qualitative theory of non-smooth differential systems is the study of their limit cycles, and many methodologies have been developed, such as the Abelian integral method (or the first order Melnikov function) [11, 12, 19, 20], and the averaging method [2, 3, 10, 13–15]. This problem can be seen as an extension of the infinitesimal Hilbert's 16th problem to the discontinuous world.

The list of quadratic centers at (0,0), almost all the orbits of which are cubic, looks as follows [8, 21]:

The Hamiltonian system Q_3^H: $\dot{z} = -\mathrm{i}z - z^2 + 2|z|^2 + (b + \mathrm{i}c)\bar{z}^2$.

The Hamiltonian triangle: $\dot{z} = -\mathrm{i}z + \bar{z}^2$.

The reversible system: $\dot{z} = -\mathrm{i}z + (2b + 1)z^2 + 2|z|^2 + b\bar{z}^2$, $b \neq -1$.

The generic Lotka-Volterra system: $\dot{z} = -\mathrm{i}z + (1 - c\mathrm{i})z^2 + c\mathrm{i}\bar{z}^2$, $c = \pm\frac{1}{\sqrt{3}}$.

Under the perturbations of continuous polynomials of degree n, Horozov and Iliev [7] proved that the number of limit cycles for Q_3^H and the Hamiltonian triangle does not exceed $5n + 15$, and Zhao et al. [21] proved that the number of limit cycles for reversible and generic Lotka-Volterra systems does not exceed $7n$.

Let $z = x + \mathrm{i}y$ and by a linear transformation, the reversible system can be written [21] as

$$\begin{cases} \dot{x} = xy, \\ \dot{y} = \dfrac{3}{2}y^2 + ax^2 - 2(a+1)x + a + 2, \end{cases} \tag{1.1}$$

where $a \in \mathbb{R}$. When $a = -2$, System (1.1) corresponds to the nongeneric case of the reversible system (1.1):

$$\begin{cases} \dot{x} = xy, \\ \dot{y} = \dfrac{3}{2}y^2 - 2x^2 + 2x, \end{cases} \tag{1.2}$$

whose first integral is

$$H(x, y) = x^{-3}\left(\frac{1}{2}y^2 - 2x^2 + x\right) = h, \quad h \in (-1, 0) \tag{1.3}$$

with the integrating factor $\mu(x, y) = x^{-4}$.

In the present paper, by using the Picard-Fuchs equation and the property of the Chebyshev space, we investigate the number of limit cycles of System (1.2) under discontinuous polynomial perturbations of degree n. System (1.2) has a center (1,0) and $h = -1$ corresponds to the center (1,0) (see Figure 1). The perturbed system of (1.2) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} xy + \varepsilon f^+(x, y) \\ \frac{3}{2}y^2 - 2x^2 + 2x + \varepsilon g^+(x, y) \end{pmatrix}, & y > 0, \\ \begin{pmatrix} xy + \varepsilon f^-(x, y) \\ \frac{3}{2}y^2 - 2x^2 + 2x + \varepsilon g^-(x, y) \end{pmatrix}, & y < 0, \end{cases} \tag{1.4}$$

where $0 < |\varepsilon| \ll 1$,

$$f^{\pm}(x, y) = \sum_{i+j=0}^{n} a^{\pm}_{i,j} x^i y^j, \quad g^{\pm}(x, y) = \sum_{i+j=0}^{n} b^{\pm}_{i,j} x^i y^j, \quad i, j \in \mathbb{N}.$$

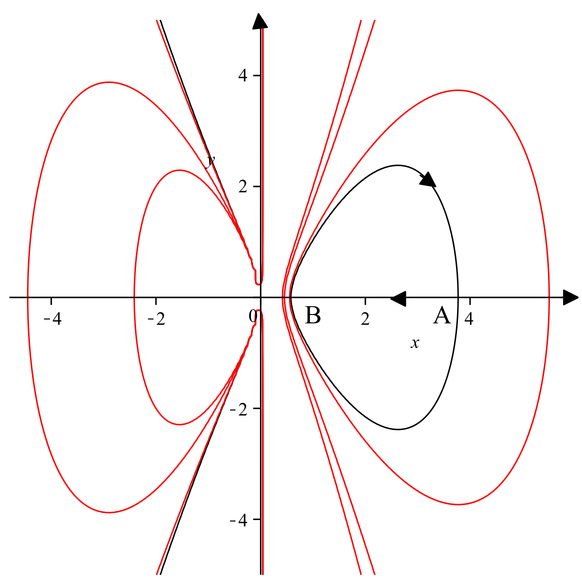

Figure 1 (Color online) The phase portrait of System (1.2)

From [12, Theorem 1.1] and [5, Theorem 1.1], by linear transformations, we know that the first order Melnikov function $M(h)$ of System (1.4) is

$$
\begin{aligned}
M(h) = &\int_{\Gamma_h^+} x^{-4}[g^+(x,y)dx - f^+(x,y)dy] \\
&+ \int_{\Gamma_h^-} x^{-4}[g^-(x,y)dx - f^-(x,y)dy], \quad h \in (-1,0),
\end{aligned} \tag{1.5}
$$

where

$$
\begin{aligned}
\Gamma_h^+ &= \{(x,y) \,|\, H(x,y) = h, h \in (-1,0), y > 0\}, \\
\Gamma_h^- &= \{(x,y) \,|\, H(x,y) = h, h \in (-1,0), y < 0\},
\end{aligned}
$$

and its number of zeros gives an upper bound of the number of limit cycles of System (1.4) bifurcating from the period annulus.

Our main results are the following two theorems.

Theorem 1.1. *Suppose that $h \in (-1,0)$.*

(i) *If $n = 2, 3$, then the number of limit cycles of System* (1.4) *bifurcating from the period annulus is not more than* 40 (*counting multiplicity*).

(ii) *If $4 \leqslant n \leqslant 7$, then the number of limit cycles of System* (1.4) *bifurcating from the period annulus is not more than $24n - 56$* (*counting multiplicity*).

(iii) *If $n \geqslant 8$, then the number of limit cycles of System* (1.4) *bifurcating from the period annulus is not more than $22n - 64$* (*counting multiplicity*).

Theorem 1.2. *Suppose that $h \in (-1,0)$, $a_{i,j}^+ = a_{i,j}^-$ and $b_{i,j}^+ = b_{i,j}^-$.*

(i) *If $n = 2, 3$, then the number of limit cycles of System* (1.4) *bifurcating from the period annulus is not more than* 4 (*counting multiplicity*).

(ii) *If $n \geqslant 4$, then the number of limit cycles of System* (1.4) *bifurcating from the period annulus is not more than $3n - 8$* (*counting multiplicity*).

Remark 1.3. (i) By using the Picard-Fuchs equation, we greatly simplified the computation of the first order Melnikov function. Then we can estimate the number of zeros of the first order Melnikov function which controls the number of limit cycles of the corresponding perturbed system benefited from the property of the Chebyshev space.

(ii) The perturbation as in (1.4) can be found in many practical applications, such as in the slender rocking block model and nonlinear compliant oscillator (see [6, 16, 17] and the references therein).

(iii) If $h \in (-1,0)$, $a_{i,j}^+ = a_{i,j}^-$ and $b_{i,j}^+ = b_{i,j}^-$, then Zhao et al. [21] obtained that the number of limit cycles of System (1.4) bifurcating from the period annulus is not more than $3n-4$ for $n \geqslant 4$; 8 for $n = 3$; 5 for $n = 2$ (counting multiplicity).

The rest of the paper is organized as follows: In Section 2, we obtain the algebraic structure of the first order Melnikov function $M(h)$ and the Picard-Fuchs equations satisfied by the generators of $M(h)$ are also obtained. Finally, we prove Theorems 1.1 and 1.2 in Section 3.

2 The algebraic structure of $M(h)$ and the Picard-Fuchs equation

In this section, we obtain the algebraic structure of the first order Melnikov function $M(h)$. For $h \in (-1,0)$, we denote

$$
I_{i,j}(h) = \int_{\Gamma_h^+} x^{i-4} y^j dx, \quad J_{i,j}(h) = \int_{\Gamma_h^-} x^{i-4} y^j dx.
$$

We first prove the following results.

Lemma 2.1. *Suppose that $h \in (-1,0)$, $i = -1, 0, 1, \dots$ and $j = 0, 1, 2, \dots$*

(i) The following equalities hold:

$$\begin{cases} I_{-1,1}(h) = \dfrac{1}{7}[hI_{1,1}(h) + 8I_{0,1}(h)], \\ I_{0,0}(h) = \dfrac{1}{3}[hI_{2,0}(h) + 4I_{1,0}(h)], \\ I_{-1,2}(h) = \dfrac{4}{3}(h+1)I_{2,0}(h), \\ I_{1,0}(h) = I_{2,0}(h), \\ I_{-1,3}(h) = 12[I_{1,1}(h) - I_{0,1}(h)], \end{cases} \tag{2.1}$$

$$\begin{cases} I_{-1,4}(h) = 4[I_{1,2}(h) - I_{0,2}(h)], \\ I_{0,3}(h) = 4[I_{2,1}(h) - I_{1,1}(h)], \\ I_{1,2}(h) = \dfrac{1}{h}[2I_{0,2}(h) - 3I_{-1,2}(h)], \\ I_{2,1}(h) = \dfrac{1}{h}[4I_{1,1}(h) - 5I_{0,1}(h)], \\ I_{3,0}(h) = \dfrac{1}{h}\left[\dfrac{1}{2}I_{0,2}(h) - 2I_{2,0}(h) + I_{1,0}(h)\right]. \end{cases} \tag{2.2}$$

(ii) If $4 \leqslant n \leqslant 7$, then

$$\begin{cases} I_{i,2j+1}(h) = \dfrac{1}{h^{n-3}}[\bar{\alpha}(h)I_{0,1}(h) + \bar{\beta}(h)I_{1,1}(h)], & i+2j+1 = n, \\ I_{i,2j}(h) = \dfrac{1}{h^{n-3}}[\bar{\gamma}(h)I_{2,0}(h) + \bar{\delta}(h)I_{0,2}(h)], & i+2j = n, \end{cases}$$

where $\bar{\alpha}(h)$, $\bar{\beta}(h)$, $\bar{\gamma}(h)$ and $\bar{\delta}(h)$ are polynomials of h with $\deg\bar{\alpha}(h), \deg\bar{\delta}(h) \leqslant n-4$ and $\deg\bar{\beta}(h), \deg\bar{\gamma}(h) \leqslant n-3$.

(iii) If $n \geqslant 8$, then

$$\begin{cases} I_{i,2j+1}(h) = \dfrac{1}{h^{n-3}}[\bar{\alpha}(h)I_{0,1}(h) + \bar{\beta}(h)I_{1,1}(h)], & i+2j+1 = n, \\ I_{i,2j}(h) = \dfrac{1}{h^{n-3}}\bar{\gamma}(h)I_{2,0}(h), & i+2j = n, \end{cases}$$

where $\bar{\alpha}(h)$, $\bar{\beta}(h)$ and $\bar{\gamma}(h)$ are polynomials of h with $\deg\bar{\alpha}(h) \leqslant n-5$ and $\deg\bar{\beta}(h), \deg\bar{\gamma}(h) \leqslant n-4$.

Proof. Let D be the interior of $\Gamma_h^+ \cup \overrightarrow{AB}$ (see the black line in Figure 1). Using Green's formula, we have for $j \geqslant 0$,

$$\begin{aligned} \int_{\Gamma_h^+} x^i y^j \, dy &= \oint_{\Gamma_h^+ \cup \overrightarrow{AB}} x^i y^j \, dy - \int_{\overrightarrow{AB}} x^i y^j \, dy \\ &= \oint_{\Gamma_h^+ \cup \overrightarrow{AB}} x^i y^j \, dy = -i \iint_D x^{i-1} y^j \, dxdy, \\ \int_{\Gamma_h^+} x^{i-1} y^{j+1} dx &= \oint_{\Gamma_h^+ \cup \overrightarrow{AB}} x^{i-1} y^{j+1} dx = (j+1) \iint_D x^{i-1} y^j \, dxdy. \end{aligned}$$

Hence,

$$\int_{\Gamma_h^+} x^i y^j \, dy = -\frac{i}{j+1} \int_{\Gamma_h^+} x^{i-1} y^{j+1} dx, \quad j \geqslant 0. \tag{2.3}$$

In a similar way, we have

$$\int_{\Gamma_h^-} x^i y^j \, dy = -\frac{i}{j+1} \int_{\Gamma_h^-} x^{i-1} y^{j+1} dx, \quad j \geqslant 0. \tag{2.4}$$

By a straightforward calculation and noting that (2.3) and (2.4), we obtain

$$
\begin{aligned}
M(h) &= \int_{\Gamma_h^+} x^{-4}(g^+(x,y)dx - f^+(x,y)dy) \\
&\quad + \int_{\Gamma_h^-} x^{-4}(g^-(x,y)dx - f^-(x,y)dy) \\
&= \int_{\Gamma_h^+} \sum_{i+j=0}^{n} b_{i,j}^+ x^{i-4} y^j\,dx - \int_{\Gamma_h^+} \sum_{i+j=0}^{n} a_{i,j}^+ x^{i-4} y^j\,dy \\
&\quad + \int_{\Gamma_h^-} \sum_{i+j=0}^{n} b_{i,j}^- x^{i-4} y^j\,dx - \int_{\Gamma_h^-} \sum_{i+j=0}^{n} a_{i,j}^- x^{i-4} y^j\,dy \\
&= \sum_{i+j=0}^{n} b_{i,j}^+ \int_{\Gamma_h^+} x^{i-4} y^j\,dx + \sum_{i+j=0}^{n} \frac{i-4}{j+1} a_{i,j}^+ \int_{\Gamma_h^+} x^{i-5} y^{j+1} dx \\
&\quad + \sum_{i+j=0}^{n} b_{i,j}^- \int_{\Gamma_h^-} x^{i-4} y^j\,dx + \sum_{i+j=0}^{n} \frac{i-4}{j+1} a_{i,j}^- \int_{\Gamma_h^-} x^{i-5} y^{j+1} dx \\
&= \sum_{i+j=0, i\geqslant -1, j\geqslant 0}^{n} \tilde{a}_{i,j} I_{i,j}(h) + \sum_{i+j=0, i\geqslant -1, j\geqslant 0}^{n} \tilde{b}_{i,j} J_{i,j}(h) \\
&=: \sum_{i+j=0, i\geqslant -1, j\geqslant 0}^{n} \rho_{i,j} I_{i,j}(h),
\end{aligned} \tag{2.5}
$$

where in the last equality we have used that $J_{i,j}(h) = (-1)^{j+1} I_{i,j}(h)$.

Differentiating (1.3) with respect to x, we obtain

$$
x^{-3} y \frac{\partial y}{\partial x} - \frac{3}{2} x^{-4} y^2 + 2x^{-2} - 2x^{-3} = 0. \tag{2.6}
$$

Multiplying (2.6) by $x^i y^{j-2} dx$, integrating over Γ_h^+ and noting that (2.3), we have

$$
(2i + 3j - 6) I_{i,j} = 4j(I_{i+2,j-2} - I_{i+1,j-2}). \tag{2.7}
$$

Similarly, multiplying (1.3) by $x^{i-4} y^j dx$ and integrating over Γ_h^+ yields

$$
hI_{i,j} = \frac{1}{2} I_{i-3,j+2} - 2I_{i-1,j} + I_{i-2,j}. \tag{2.8}
$$

Eliminating $I_{i-3,j+2}$ by (2.7) and (2.8) gives

$$
(2i + 3j - 6)hI_{i,j} = (2i + j - 10) I_{i-2,j} - 4(i + j - 4) I_{i-1,j}. \tag{2.9}
$$

From (2.7) we have

$$
I_{1,0} = I_{2,0}, \quad I_{-1,3} = 12(I_{1,1} - I_{0,1}). \tag{2.10}
$$

From (2.8) we obtain

$$
hI_{2,0} = \frac{1}{2} I_{-1,2} - 2I_{1,0} + I_{0,0}. \tag{2.11}
$$

Taking $(i,j) = (2,0), (1,1)$ in (2.9) we have

$$
I_{0,0} = \frac{1}{3}(hI_{2,0} + 4I_{1,0}), \quad I_{-1,1} = \frac{1}{7}(hI_{1,1} + 8I_{0,1}). \tag{2.12}
$$

Hence,

$$
I_{0,0} = \frac{1}{3}(h + 4) I_{2,0}. \tag{2.13}
$$

From (2.10)–(2.12) we get

$$I_{-1,2} = \frac{4}{3}(h+1)I_{2,0}. \tag{2.14}$$

(2.10) and (2.12)–(2.14) imply (2.1) holds. In a similar way, applying the equalities (2.7) and (2.9), we can obtain (2.2). Hence, the conclusion (i) holds. By some straightforward calculations according to (2.7) and (2.9), we can get the conclusion (ii).

(iii) Now we prove the conclusion (iii) by induction on n. Without loss of generality, we only show the case $i+2j+1=n$. With the help of Maple, from (2.7) and (2.9) and noting the conclusions (i) and (ii), we obtain

$$\begin{cases} I_{-1,9} = -\dfrac{768}{46189h^5}[(200h^3+3000h^2+2024h+512)I_{0,1} \\ \qquad +(663h^4+326h^3+239h^2+64h)I_{1,1}], \\ I_{0,8} = -\dfrac{2048}{315h^5}(h+1)^4 I_{2,0}, \\ I_{1,7} = -\dfrac{64}{7293h^5}[(385h^3+1385h^2+1480h+512)I_{0,1} \\ \qquad +(139h^3+171h^2+64h)I_{1,1}], \\ I_{2,6} = -\dfrac{128}{35h^5}(h+1)^3 I_{2,0}, \\ I_{3,5} = -\dfrac{16}{3003h^5}[(480h^2+1000h+512)I_{0,1}+(39h^3+111h^2+64h)I_{1,1}], \\ I_{4,4} = -\dfrac{32}{105h^5}(h+1)^2(h+8)I_{2,0}, \\ I_{5,3} = -\dfrac{4}{1001h^5}[(77h^2+584h+512)I_{0,1}+(59h^2+64h)I_{1,1}], \\ I_{6,2} = \dfrac{4}{15h^5}(h+1)(3h+8)I_{2,0}, \\ I_{7,1} = -\dfrac{1}{231h^5}[(232h+512)I_{0,1}+(15h^2+64h)I_{1,1}], \\ I_{8,0} = -\dfrac{1}{5h^5}(h^2+12h+16)I_{2,0}, \end{cases}$$

which imply that the conclusion holds for $n=8$. Now assume that (iii) holds for $i+2l+1 \leqslant k-1$ $(k \geqslant 9)$. For $i+2l+1=k$, if k is an even number, then taking

$$(i, 2l+1) = (-1, k+1)$$

in (2.7) and

$$(i, 2l+1) = (1, k-1), (3, k-3), \ldots, (k-3, 3), (k-1, 1)$$

in (2.9), respectively, we have

$$\boldsymbol{A}\begin{pmatrix} I_{-1,k+1} \\ I_{1,k-1} \\ I_{3,k-3} \\ \vdots \\ I_{k-3,3} \\ I_{k-1,1} \end{pmatrix} = \frac{1}{h}\begin{pmatrix} \frac{4(k+1)}{5-3k}hI_{0,k-1} \\ \frac{1}{3k-7}\left[(k-9)I_{-1,k-1}-4(k-4)I_{0,k-1}\right] \\ \frac{1}{3k-9}\left[(k-7)I_{1,k-3}-4(k-4)I_{2,k-3}\right] \\ \vdots \\ \frac{1}{2k-3}(2k-13)I_{k-5,3}-4(k-4)I_{k-4,3} \\ \frac{1}{2k-5}(2k-11)I_{k-3,1}-4(k-4)I_{k-2,1} \end{pmatrix}, \tag{2.15}$$

where

$$\boldsymbol{A}=\begin{pmatrix}1 & \frac{4(k+1)}{5-3k} & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & 0 & \cdots & 0 & 1\end{pmatrix}$$

is a $\frac{k+2}{2}\times\frac{k+2}{2}$ matrix and $\det\boldsymbol{A}=1$. If k is an odd number taking $(i,2l+1)=(0,k)$ in (2.7) and $(i,2l+1)=(2,k-2),(4,k-4),\dots,(k-3,3),(k-1,1)$ in (2.9), respectively, we have

$$\boldsymbol{B}\begin{pmatrix}I_{0,k}\\ I_{2,k-2}\\ I_{4,k-4}\\ \vdots\\ I_{k-3,3}\\ I_{k-1,1}\end{pmatrix}=\frac{1}{h}\begin{pmatrix}\frac{4k}{6-3k}hI_{1,k-2}\\ \frac{1}{3k-8}\big[(k-8)I_{0,k-2}-4(k-4)I_{1,k-2}\big]\\ \frac{1}{3k-10}\big[(k-6)I_{2,k-4}-4(k-4)I_{3,k-4}\big]\\ \vdots\\ \frac{1}{2k-3}\big(2k-13)I_{k-5,3}-4(k-4)I_{k-4,3}\\ \frac{1}{2k-5}\big(2k-11)I_{k-3,1}-4(k-4)I_{k-2,1}\end{pmatrix}, \tag{2.16}$$

where

$$\boldsymbol{B}=\begin{pmatrix}1 & \frac{4k}{6-3k} & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & 0 & \cdots & 0 & 1\end{pmatrix}$$

is a $\frac{k+1}{2}\times\frac{k+1}{2}$ matrix and $\det\boldsymbol{B}=1$. Hence, we can get that $I_{i,2l+1}$ can be expressed by $I_{0,1}$ and $I_{1,1}$ for $i+2l+1=k$ by the induction hypothesis.

From (2.15) and (2.16), we have for $(i,2l+1)=(-1,k+1)$ or $(i,2l+1)=(0,k)$,

$$\begin{aligned}I_{-1,k+1}(h)&=\frac{1}{h^{k-3}}[h\alpha^{(k-1)}(h)I_{0,1}+h\beta^{(k-1)}(h)I_{1,1}]\\ &=:\frac{1}{h^{k-3}}[\alpha^{(k)}(h)I_{0,1}+\beta^{(k)}(h)I_{1,1}],\quad k\text{ even},\\ I_{0,k}(h)&=\frac{1}{h^{k-3}}[h\alpha^{(k-1)}(h)I_{0,1}+h\beta^{(k-1)}(h)I_{1,1}]\\ &=:\frac{1}{h^{k-3}}[\alpha^{(k)}(h)I_{0,1}+\beta^{(k)}(h)I_{1,1}],\quad k\text{ odd},\end{aligned}$$

where $\alpha^{(k-1)}(h)$ and $\beta^{(k-1)}(h)$ are polynomials in h. By the induction hypothesis we obtain that

$$\deg\alpha^{(k-1)}(h)\leqslant k-6,\quad \deg\beta^{(k-1)}(h)\leqslant k-5.$$

Therefore,

$$\deg\alpha^{(k)}(h)\leqslant k-5,\quad \deg\beta^{(k)}(h)\leqslant k-4.$$

In a similar way, we can prove the cases for $(i,2l+1)=(1,k-1),(3,k-3),\dots,(k-3,3),(k-1,1)$ or $(i,2l+1)=(2,k-2),(4,k-4),\dots,(k-3,3),(k-1,1)$. This completes the proof. □

Lemma 2.2. *Suppose that* $h\in(-1,0)$.

(i) *If* $n=2,3$, *then*

$$M(h)=\alpha(h)I_{0,1}(h)+\beta(h)I_{1,1}(h)+\gamma(h)I_{2,0}(h)+\delta(h)I_{0,2}(h), \tag{2.17}$$

where $\alpha(h)$ *is a constant, and* $\beta(h)$, $\gamma(h)$ *and* $\delta(h)$ *are polynomials of* h *with* $\deg\beta(h),\deg\gamma(h),\deg\delta(h)\leqslant 1$.

(ii) *If* $4\leqslant n\leqslant 7$, *then*

$$M(h)=\frac{1}{h^{n-3}}[\alpha(h)I_{0,1}(h)+\beta(h)I_{1,1}(h)+\gamma(h)I_{2,0}(h)+\delta(h)I_{0,2}(h)],$$

where $\alpha(h)$, $\beta(h)$, $\gamma(h)$ *and* $\delta(h)$ *are polynomials of* h *with* $\deg\alpha(h),\deg\delta(h)\leqslant n-4$ *and* $\deg\beta(h),\deg\gamma(h)\leqslant n-3$.

(iii) *If* $n\geqslant 8$, *then*

$$M(h)=\frac{1}{h^{n-3}}[\alpha(h)I_{0,1}(h)+\beta(h)I_{1,1}(h)+\gamma(h)I_{2,0}(h)+\delta(h)I_{0,2}(h)],$$

where $\alpha(h)$, $\beta(h)$, $\gamma(h)$ *and* $\delta(h)$ *are polynomials of* h *with* $\deg\alpha(h)\leqslant n-5$, $\deg\beta(h),\deg\gamma(h)\leqslant n-4$ *and* $\deg\delta(h)\leqslant 3$.

Lemma 2.3. (i) *The vector function* $(I_{0,1},I_{1,1})^{\mathrm{T}}$ *satisfies the following Picard-Fuchs equation*:

$$\begin{pmatrix} I_{0,1}\\ I_{1,1}\end{pmatrix}=\begin{pmatrix} \frac{4}{5}h+\frac{16}{15} & \frac{4}{15}h\\ \frac{4}{3} & \frac{4}{3}h\end{pmatrix}\begin{pmatrix} I'_{0,1}\\ I'_{1,1}\end{pmatrix}. \tag{2.18}$$

(ii) *The vector function* $(I_{2,0},I_{0,2})^{\mathrm{T}}$ *satisfies the following Picard-Fuchs equation*:

$$\begin{pmatrix} I_{2,0}\\ I_{0,2}\end{pmatrix}=\begin{pmatrix} 2h+2 & 0\\ 4h+4 & h\end{pmatrix}\begin{pmatrix} I'_{2,0}\\ I'_{0,2}\end{pmatrix}. \tag{2.19}$$

Proof. From (1.3) we get

$$\frac{\partial y}{\partial h}=\frac{x^3}{y},$$

which implies

$$I'_{i,j}=j\int_{\Gamma_h^+}x^{i-1}y^{j-2}dx. \tag{2.20}$$

Hence,

$$I_{i,j}=\frac{1}{j+2}I'_{i-3,j+2}. \tag{2.21}$$

Multiplying the both sides of (2.20) by h, we have

$$hI'_{i,j}=\frac{j}{2(j+2)}I'_{i-3,j+2}-2I'_{i-1,j}+I'_{i-2,j}. \tag{2.22}$$

From (2.3) and (2.20) we have for $j\geqslant 1$,

$$\begin{aligned} I_{i,j}&=\int_{\Gamma_h^+}x^{i-4}y^j dx=-\frac{j}{i-3}\int_{\Gamma_h^+}x^{i-3}y^{j-1}dy\\ &=-\frac{j}{i-3}\int_{\Gamma_h^+}x^{i-3}y^{j-1}\frac{3hx^2+4x-1}{y}dx\\ &=-\frac{1}{i-3}(3hI'_{i,j}+4I'_{i-1,j}-I'_{i-2,j}). \end{aligned} \tag{2.23}$$

(2.21)–(2.23) imply

$$I_{i,j} = -\frac{4}{2i+j-6}(hI'_{i,j} + I'_{i-1,j}), \quad j \geqslant 1. \tag{2.24}$$

From (2.21) and noting (2.14) we obtain

$$I_{2,0} = \frac{1}{2}I'_{-1,2} = \frac{2}{3}I_{2,0} + \frac{2}{3}(h+1)I'_{2,0}.$$

Hence,

$$I_{2,0} = 2(h+1)I'_{2,0}. \tag{2.25}$$

From (2.24) we have

$$I_{0,1} = \frac{4}{5}(hI'_{0,1} + I'_{-1,1}), \quad I_{1,1} = \frac{4}{3}(hI'_{1,1} + I'_{0,1}), \quad I_{0,2} = hI'_{0,2} + I'_{-1,2}, \tag{2.26}$$

and noting (2.12) and (2.14) we obtain the conclusions (i) and (ii). This completes the proof. □

Lemma 2.4. *For $h \in (-1, 0)$,*

$$I_{2,0}(h) = c_1\sqrt{h+1}, \quad I_{0,2}(h) = 2c_1\sqrt{h+1} - c_1 h \ln\frac{1-\sqrt{h+1}}{1+\sqrt{h+1}},$$

where c_1 is a constant.

Proof. From (2.19) we have $I_{2,0}(h) = c_1\sqrt{h+1}$, where c_1 is a constant. Therefore, we have for $h \in (-1, 0)$,

$$I_{0,2}(h) = c_2 h + 2c_1\sqrt{h+1} - c_1 h \ln\frac{1-\sqrt{h+1}}{1+\sqrt{h+1}},$$

where c_2 is a constant. Since $I_{0,2}(-1) = 0$, we have $c_2 = 0$. Hence, $I_{0,2}(h) = 2c_1\sqrt{h+1} - c_1 h \ln\frac{1-\sqrt{h+1}}{1+\sqrt{h+1}}$. This completes the proof. □

Taking $(i, j) = (4, 1), (3, 1)$ in (2.9) respectively and bearing in mind (2.2), we get

$$I_{3,1}(h) = -\frac{1}{h}I_{1,1}(h), \quad I_{4,1}(h) = -\frac{1}{5h}[I_{2,1}(h) + 4I_{3,1}(h)].$$

Hence, $I_{0,1}(h) = h^2 I_{4,1}(h)$. Using Green's formula, we have

$$I_{4,1}(h) = \int_{\Gamma_h^+} ydx = \oint_{\Gamma_h^+ \cup \overrightarrow{AB}} ydx = \iint_D dxdy \neq 0,$$

where D is the interior of $\Gamma_h^+ \cup \overrightarrow{AB}$ (see Figure 1). Thus, $I_{0,1}(h) \neq 0$ for $h \in (-1, 0)$. Noting that $\frac{\partial y}{\partial h} = x^3 y^{-1}$ and $dx = xydt$, we have

$$I'_{0,1}(h) = \int_{\Gamma_h^+} x^{-4}\frac{\partial y}{\partial h}dx = \int_0^{t_0} dt \neq 0,$$

where t_0 is the time from the left end point to the right end point of Γ_h^+. So we can get the following lemma.

Lemma 2.5. *Let $\omega_1(h) = \frac{I_{1,1}(h)}{I_{0,1}(h)}$ and $\omega_2(h) = \frac{I'_{1,1}(h)}{I'_{0,1}(h)}$ for $h \in (-1, 0)$. Then $\omega_1(h)$ and $\omega_2(h)$ satisfy the following Riccati equations*:

$$G(h)\omega'_1(h) = \frac{1}{4}h\omega_1^2(h) - \frac{1}{2}(h-2)\omega_1(h) - \frac{5}{4} \tag{2.27}$$

and

$$G(h)\omega'_2(h) = -\frac{1}{4}h\omega_2^2(h) - \frac{1}{2}h\omega_2(h) - \frac{1}{4}, \tag{2.28}$$

respectively, where $G(h)=h(h+1)$.

Proof. From (2.18), we have

$$G(h)\begin{pmatrix} I'_{0,1}(h) \\ I'_{1,1}(h) \end{pmatrix} = \begin{pmatrix} \frac{5}{4}h & -\frac{1}{4}h \\ -\frac{5}{4} & \frac{3}{4}h+1 \end{pmatrix} \begin{pmatrix} I_{0,1}(h) \\ I_{1,1}(h) \end{pmatrix}$$

and

$$G(h)\begin{pmatrix} I''_{0,1}(h) \\ I''_{1,1}(h) \end{pmatrix} = \begin{pmatrix} \frac{1}{4}h & -\frac{1}{4}h \\ -\frac{1}{4} & -\frac{1}{4}h \end{pmatrix} \begin{pmatrix} I'_{0,1}(h) \\ I'_{1,1}(h) \end{pmatrix},$$

where $G(h)=h(h+1)$. Noting that $G(h)\neq 0$ for $h\in(-1,0)$ and

$$\omega_1'(h)=\frac{I'_{1,1}(h)}{I_{0,1}(h)}-\omega_1(h)\frac{I'_{0,1}(h)}{I_{0,1}(h)},\quad \omega_2'(h)=\frac{I''_{1,1}(h)}{I'_{0,1}(h)}-\omega_2(h)\frac{I''_{0,1}(h)}{I'_{0,1}(h)},$$

we obtain (2.27) and (2.28). This completes the proof. □

3 Proof of the main results

In order to prove Theorem 1.1, we first introduce some helpful results in the literature. Let V be a finite-dimensional vector space of functions, real-analytic on an open interval $\mathbb{I}$.

Definition 3.1 (See [4]). *We say that S is a Chebyshev space, provided that each non-zero function in S has at most* $\dim(S)-1$ *zeros, counted with multiplicity.*

Proposition 3.2 (See [4]). *The solution space S of a second order linear analytic differential equation*

$$x''+a_1(t)x'+a_2(t)x=0$$

on an open interval $\mathbb{I}$ is a Chebyshev space if and only if there exists a nowhere vanishing solution $x_0(t)\in S$ $(x_0(t)\neq 0,\ \forall\, t\in\mathbb{I})$.

Proposition 3.3 (See [4]). *Suppose the solution space of the homogeneous equation*

$$x''+a_1(t)x'+a_2(t)x=0$$

is a Chebyshev space and let $R(t)$ be an analytic function on $\mathbb{I}$ having l zeros (counted with multiplicity). Then every solution $x(t)$ of the non-homogeneous equation

$$x''+a_1(t)x'+a_2(t)x=R(t)$$

has at most $l+2$ zeros on $\mathbb{I}$.

In the following we denote by $\#\{\varphi(h)=0,h\in(a,b)\}$ the number of isolated zeros of $\varphi(h)$ on (a,b) taking into account the multiplicity, and we also denote by $\Theta_k(h)$ the polynomial of degree at most k.

Lemma 3.4. *Suppose that $h\in(-1,0)$.*

(i) *If $n=2,3$, then there exist polynomials $P_2^1(h)$, $P_1^1(h)$ and $P_0^1(h)$ of h with degree respectively 4, 3 and 2 such that $L^1(h)\Phi(h)=0$.*

(ii) *If $4\leqslant n\leqslant 7$, then there exist polynomials $P_2^2(h)$, $P_1^2(h)$ and $P_0^2(h)$ of h with degree respectively $2n-4$, $2n-5$ and $2n-6$ such that $L^2(h)\Phi(h)=0$.*

(iii) *If $n\geqslant 8$, then there exist polynomials $P_2^3(h)$, $P_1^3(h)$ and $P_0^3(h)$ of h with degree respectively $2n-6$, $2n-7$ and $2n-8$ such that $L^3(h)\Phi(h)=0$, where*

$$\Phi(h)=\alpha(h)I_{0,1}(h)+\beta(h)I_{1,1}(h)$$

and

$$L^i(h) = P_2^i(h)\frac{d^2}{dh^2} + P_1^i(h)\frac{d}{dh} + P_0^i(h), \quad i = 1, 2, 3. \tag{3.1}$$

Proof. Without loss of generality, we only prove (iii). (i) and (ii) can be shown similarly. By (2.18), we have

$$V'(h) = (E - B)^{-1}(Bh + C)V''(h),$$

where $V(h) = (I_{0,1}(h), I_{1,1}(h))^{\mathrm{T}}$, and

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{4}{5} & \frac{4}{15} \\ 0 & \frac{4}{3} \end{pmatrix}, \quad C = \begin{pmatrix} \frac{16}{15} & 0 \\ \frac{4}{3} & 0 \end{pmatrix}.$$

Hence,

$$\begin{aligned} \Phi(h) &= \tau(h)V(h) = \tau(h)(Bh + C)V'(h) \\ &= \tau(h)(Bh + C)(E - B)^{-1}(Bh + C)V''(h) \\ &=: \Theta_{n-3}(h)I''_{0,1}(h) + \Theta_{n-2}(h)I''_{1,1}(h), \end{aligned}$$

where $\tau(h) = (\alpha(h), \beta(h))$, $\Theta_{n-3}(h)$ denotes a polynomial in h of degree at most $n-3$, etc. For $\Phi'(h)$, we have

$$\begin{aligned} \Phi'(h) &= \tau'(h)V(h) + \tau(h)V'(h) \\ &= [\tau'(h)(Bh + C) + \tau(h)](E - B)^{-1}(Bh + C)V''(h) \\ &=: \Theta_{n-4}(h)I''_{0,1}(h) + \Theta_{n-3}(h)I''_{1,1}(h). \end{aligned}$$

In a similar way, we have

$$\Phi''(h) = \Theta_{n-5}(h)I''_{0,1}(h) + \Theta_{n-4}(h)I''_{1,1}(h).$$

Next, suppose that

$$P_2(h) = \sum_{k=0}^{2n-6} p_{2,k}h^k, \quad P_1(h) = \sum_{m=0}^{2n-7} p_{1,m}h^m, \quad P_0(h) = \sum_{l=0}^{2n-8} p_{0,l}h^l \tag{3.2}$$

are polynomials of h with coefficients $p_{2,k}$, $p_{1,m}$ and $p_{0,l}$ to be determined such that $L(h)\Phi(h) = 0$ for

$$0 \leqslant k \leqslant 2n - 6, \quad 0 \leqslant m \leqslant 2n - 7, \quad 0 \leqslant l \leqslant 2n - 8. \tag{3.3}$$

By a straightforward computation, we have

$$\begin{aligned} L(h)\Phi(h) &= P_2(h)\Phi''(h) + P_1(h)\Phi'(h) + P_0(h)\Phi(h) \\ &= [P_2(h)\Theta_{n-5}(h) + P_1(h)\Theta_{n-4}(h) + P_0(h)\Theta_{n-3}(h)]I''_{0,1}(h) \\ &\quad + [P_2(h)\Theta_{n-4}(h) + P_1(h)\Theta_{n-3}(h) + P_0(h)\Theta_{n-2}(h)]I''_{1,1}(h) \\ &=: X(h)I''_{0,1}(h) + Y(h)I''_{1,1}(h), \end{aligned}$$

where $X(h)$ and $Y(h)$ are polynomials of h with $\deg X(h) \leqslant 3n - 11$ and $\deg Y(h) \leqslant 3n - 10$.

Let

$$X(h) = \sum_{i=0}^{3n-11} x_i h^i, \quad Y(h) = \sum_{j=0}^{3n-10} y_j h^j,$$

where x_i and y_j are expressed by $p_{2,k}$, $p_{1,m}$ and $p_{0,l}$ in (3.2) linearly, k, m and l satisfy (3.3). Let

$$x_i = 0, \quad y_j = 0, \quad 0 \leqslant i \leqslant 3n - 11, \quad 0 \leqslant j \leqslant 3n - 10. \tag{3.4}$$

Then System (3.4) is homogenous linear equations with $6n-19$ equations and about $6n-18$ variables of $p_{2,k}$, $p_{1,m}$ and $p_{0,l}$ for k, m and l satisfying (3.3). It follows from the theory of linear algebra that there exist $p_{2,k}$, $p_{1,m}$ and $p_{0,l}$ such that (3.4) holds, which yields $L(h)\Phi(h)=0$. This completes the proof. □

Lemma 3.5. *Let $\Phi(h)=\alpha(h)I_{0,1}(h)+\beta(h)I_{1,1}(h)$.*

(i) *If $n=2,3$, then $\Phi(h)$ has at most 4 zeros on $(-1,0)$, taking into account the multiplicity.*

(ii) *If $4\leqslant n\leqslant 7$, then $\Phi(h)$ has at most $3n-8$ zeros on $(-1,0)$, taking into account the multiplicity.*

(iii) *If $n\geqslant 8$, then $\Phi(h)$ has at most $3n-11$ zeros on $(-1,0)$, taking into account the multiplicity.*

Proof. We only prove (iii). (i) and (ii) can be proved in a similar way. Let $\chi_1(h)=\alpha(h)+\beta(h)\omega_1(h)$. So $\Phi(h)=I_{0,1}(h)\chi_1(h)$ which implies

$$\#\{\Phi(h)=0, h\in(-1,0)\}=\#\{\chi_1(h)=0, h\in(-1,0)\}.$$

By (2.27) we know that $\chi_1(h)$ satisfies

$$G(h)\beta(h)\chi_1'(h)=\frac{1}{4}h\chi_1(h)^2+F_1(h)\chi_1(h)+F_0(h) \tag{3.5}$$

with $\deg F_0(h)\leqslant 2n-8$. Recall that the inequality (4.8) in [22] is

$$\nu\leqslant\sigma+\lambda+1,$$

where ν, σ and λ correspond here to $\#\{\chi_1(h)=0, h\in(-1,0)\}$, $\#\{F_0(h)=0, h\in(-1,0)\}$ and $\#\{\beta(h)=0, h\in(-1,0)\}$, respectively. Hence, we have for $h\in(-1,0)$,

$$\#\{\chi_1(h)=0\}\leqslant\#\{\beta(h)=0\}+\#\{F_0(h)=0\}+1\leqslant 3n-11.$$

Hence,

$$\#\{\Phi(h)=0, h\in(-1,0)\}=\#\{\chi_1(h)=0, h\in(-1,0)\}\leqslant 3n-11.$$

This completes the proof. □

Proof of Theorem 1.1. We only prove (iii). (i) and (ii) can be proved similarly.

Let $M_1(h)=h^{n-3}M(h)$. Then $M_1(h)$ has the same zeros as $M(h)$ on $(-1,0)$. For the sake of clearness, we split the proof into three steps.

(1) For $h\in(-1,0)$, $L^3(h)M_1(h)=R(h)$, where $L^3(h)$ is defined by (3.1),

$$R(h)=\Theta_{2n-4}(h)\ln\frac{1-\sqrt{h+1}}{1+\sqrt{h+1}}+\Theta_{3n-9}(h)\frac{1}{h(h+1)^{\frac{3}{2}}}. \tag{3.6}$$

In fact, from Lemma 2.4, we have

$$\begin{aligned}\Psi(h)&:=\gamma(h)I_{2,0}(h)+\delta(h)I_{0,2}(h)\\&=c_1[\gamma(h)+2\delta(h)]\sqrt{h+1}-c_1h\delta(h)\ln\frac{1-\sqrt{h+1}}{1+\sqrt{h+1}}\\&:=\Theta_{n-4}(h)\sqrt{h+1}+h\Theta_3(h)\ln\frac{1-\sqrt{h+1}}{1+\sqrt{h+1}},\\\Psi'(h)&=\Theta_{n-4}(h)\frac{1}{\sqrt{h+1}}+\Theta_3(h)\ln\frac{1-\sqrt{h+1}}{1+\sqrt{h+1}},\\\Psi''(h)&=\Theta_{n-3}(h)\frac{1}{h(h+1)^{\frac{3}{2}}}+\Theta_2(h)\ln\frac{1-\sqrt{h+1}}{1+\sqrt{h+1}}.\end{aligned} \tag{3.7}$$

From Lemma 3.4(iii), we have

$$L^3(h)M_1(h)=L^3(h)\Psi(h)=P_2^3(h)\Psi''(h)+P_1^3(h)\Psi'(h)+P_0^3(h)\Psi(h). \tag{3.8}$$

Substituting (3.7) into (3.8) gives (3.6).

(2) Zeros of $R(h)$ for $h \in (-1,0)$.

Denote that $U = \{h \in (-1,0) \,|\, \Theta_{2n-4}(h) = 0\}$. For $h \in (-1,0)\backslash U$, by detailed computations, we get

$$\left(\frac{R(h)}{\Theta_{2n-4}(h)}\right)' = \frac{\Theta_{5n-12}(h)}{\Theta_{2n-4}^2(h)h^2(h+1)^{\frac{5}{2}}}. \tag{3.9}$$

Since $h^2(h+1)^{\frac{5}{2}} \neq 0$ for $h \in (-1,0)$, we have

$$\#\{R(h) = 0, h \in (-1,0)\} \leqslant 7n - 15. \tag{3.10}$$

(3) Zeros of $M(h)$ for $h \in (-1,0)$.

By Lemma 3.5, we have $\Phi(h)$ has at most $3n-11$ zeros on $(-1,0)$. We assume that

$$P_2^3(\tilde{h}_i) = 0, \quad \Phi(\bar{h}_j) = 0, \quad \tilde{h}_i, \bar{h}_j \in (-1,0), \quad 1 \leqslant i \leqslant 2n-6, \quad 1 \leqslant j \leqslant 3n-11.$$

Denote $\tilde{h}_i$ and $\bar{h}_j$ by h_m^*, and reorder them such that $h_m^* < h_{m+1}^*$ for $m = 1, 2, \ldots, 5n-17$. Let

$$\Delta_s = (h_s^*, h_{s+1}^*), \quad s = 0, 1, \ldots, 5n-17,$$

where $h_0^* = -1$, $h_{5n-16}^* = 0$. Then $P_2^3(h) \neq 0$ and $\Phi(h) \neq 0$ for $h \in \Delta_s$ and $L^3(h)\Phi(h) = 0$. By Proposition 3.2, the solution space of

$$L^3(h) = P_2^3(h)\left(\frac{d^2}{dh^2} + \frac{P_1^3(h)}{P_2(h)}\frac{d}{dh} + \frac{P_0^3(h)}{P_2(h)}\right) = 0$$

is a Chebyshev space on Δ_s. By Proposition 3.3, $M_1(h)$ has at most $2 + l_s$ zeros for $h \in \Delta_s$, where l_s is the number of zeros of $R(h)$ on Δ_s. Therefore, we obtain for $h \in (-1,0)$,

$$\begin{aligned}
\#\{M(h) = 0\} &= \#\{M_1(h) = 0\} \\
&\leqslant \#\{R(h) = 0\} + 2 \cdot \text{the number of the intervals of } \Delta_s \\
&\quad + \text{the number of the end points of } \Delta_s \\
&\leqslant 22n - 64.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.2. If $a_{i,j}^+ = a_{i,j}^-$ and $b_{i,j}^+ = b_{i,j}^-$, i.e., the system (1.4) is smooth. Since Γ_h is symmetric with respect to the x-axis for $h \in (-1,0)$, $A_{i,2l}(h) = \oint_{\Gamma_h} x^{i-4}y^{2l}dx = 0$, $l = 0, 1, 2, \ldots$, where

$$\Gamma_h = \Gamma_h^+ \cup \Gamma_h^-, \quad A_{i,j}(h) = I_{i,j}(h) + J_{i,j}(h).$$

Hence, from Lemma 2.2 we have

$$M(h) = \begin{cases} \dfrac{1}{h^{n-3}}[\tilde{\alpha}(h)A_{0,1}(h) + \tilde{\beta}(h)A_{1,1}(h)], & n = 2, 3, \\ \dfrac{1}{h^{n-3}}[\alpha(h)A_{0,1}(h) + \beta(h)A_{1,1}(h)], & n \geqslant 4, \end{cases}$$

where $\tilde{\alpha}(h)$ is a constant, and $\tilde{\beta}(h)$, $\alpha(h)$ and $\beta(h)$ are polynomials of h with $\deg \tilde{\beta}(h) \leqslant 1$, $\deg \alpha(h) \leqslant n-4$ and $\deg \beta(h) \leqslant n-3$. By the same proof of Lemma 3.5, we have

$$\#\{M(h) = 0, h \in (-1,0)\} \leqslant \begin{cases} 4, & n = 2, 3, \\ 3n - 8, & n \geqslant 4. \end{cases}$$

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 11701306, 11671040 and 11601250), Higher Educational Science Program of Ningxia (Grant No. NGY201789), Construction of First-class Disciplines of Higher Education of Ningxia (pedagogy) (Grant No. NXYLXK2017B11) and Key Program of Ningxia Normal University (Grant No. NXSFZD1708).

References

1 Bernardo M, Budd C, Champneys A, et al. Piecewise-smooth Dynamical Systems, Theory and Applications. London: Springer-Verlag, 2008

2 Cen X, Li S, Zhao Y. On the number of limit cycles for a class of discontinuous quadratic differetnial systems. J Math Anal Appl, 2017, 449: 314–342

3 Dong G, Liu C. Note on limit cycles for m-piecewise discontinuous polynomial Liénard differential equations. Z Angew Math Phys, 2017, 68: 97

4 Gavrilov L, Iliev I. Quadratic perturbations of quadratic codimension-four centers. J Math Anal Appl, 2009, 357: 69–76

5 Han M, Sheng L. Bifurcation of limit cycles in piecewise smooth systems via Melnikov function. J Appl Anal Comput, 2015, 5: 809–815

6 Hogan S. Heteroclinic bifurcations in damped rigid block motion. Proc R Soc Lond Ser A Math Phys Eng Sci, 1992, 439: 155–162

7 Horozov E, Iliev I. Linear estimate for the number of zeros of Abelian integrals with cubic Hamiltonians. Nonlinearity, 1998, 11: 1521–1537

8 Iliev I. Perturbations of quadratic centers. Bull Sci Math, 1998, 122: 107–161

9 Krivan V. On the Gause predator-prey model with a refuge: A fresh look at the history. J Theoret Biol, 2011, 274: 67–73

10 Li S, Liu C. A linear estimate of the number of limit cycles for some planar piecewise smooth quadratic differential system. J Math Anal Appl, 2015, 428: 1354–1367

11 Liang F, Han M, Romanovski V. Bifurcation of limit cycles by perturbing a piecewise linear Hamiltonian system with a homoclinic loop. Nonlinear Anal, 2012, 75: 4355–4374

12 Liu X, Han M. Bifurcation of limit cycles by perturbing piecewise Hamiltonian systems. Internat J Bifur Chaos Appl Sci Engrg, 2010, 20: 1379–1390

13 Llibre J, Mereu A. Limit cycles for discontinuous quadratic differential systems. J Math Anal Appl, 2014, 413: 763–775

14 Llibre J, Mereu A, Novaes D. Averaging theory for discontinuous piecewise differential systems. J Differential Equations, 2015, 258: 4007–4032

15 Llibre J, Teixeira M. Limit cycles for m-piecewise discontinuous polynomial Liénard differential equations. Z Angew Math Phys, 2015, 66: 51–66

16 Peng Z, Lang Z, Billings S, et al. Analysis of bilinear oscillators under harmonic loading using nonlinear output frequency response functions. Internat J Mech Sci, 2007, 49: 1213–1225

17 Shen J, Du Z. Heteroclinic bifurcation in a class of planar piecewise smooth systems with multiple zones. Z Angew Math Phys, 2016, 67: 42

18 Teixeira M. Perturbation theory for non-smooth systems. In: Encyclopedia of Complexity and Systems Science. New York: Springer, 2009, 6697–6709

19 Xiong Y. Limit cycle bifurcations by perturbing a piecewise Hamiltonian system with a double homoclinic loop. Internat J Bifur Chaos Appl Sci Engrg, 2016, 26: 1650103

20 Yang J, Zhao L. Limit cycle bifurcations for piecewise smooth Hamiltonian systems with a generalized eye-figure loop. Internat J Bifur Chaos Appl Sci Engrg, 2016, 26: 1650204

21 Zhao Y, Li W, Li C, et al. Linear estimate of the number of zeros of Abelian integrals for quadratic centers having almost all their orbits formed by cubics. Sci China Ser A, 2002, 45: 964–974

22 Zhao Y, Zhang Z. Linear estimate of the number of zeros of Abelian integrals for a kind of quartic Hamiltonians. J Differential Equations, 1999, 155: 73–88