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Bounded type Siegel disks of finite type maps with few singular values

Dedicated to the Memory of Professor Lei Tan

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Abstract Let U be an open subset of the Riemann sphere $\widehat{\mathbb{C}}$. We give sufficient conditions for which a finite type map $f : U \to \widehat{\mathbb{C}}$ with at most three singular values has a Siegel disk compactly contained in U and whose boundary is a quasicircle containing a unique critical point. The main tool is quasiconformal surgery à la Douady-Ghys-Herman-Świątek. We also give sufficient conditions for which, instead, Δ has not compact closure in U. The main tool is the Schwarzian derivative and area inequalities à la Graczyk-Świątek.

Keywords Siegel disks, quasicircles, quasiconformal surgery, Schwarzian derivative

MSC(2010) 37F30, 37F40, 37F50

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1 Introduction

1.1 Singular values and finite type maps

The origin of the notion of singular values is hard to trace back. It seems to date from Hurwitz or before (see [4, 18, 19]).

Definition 1.1. Let $f : \mathcal{S} \to \mathcal{S}'$ be an analytic map between Riemann surfaces. A *regular value* is a point $z \in \mathcal{S}'$ which has an *evenly covered* neighbourhood V, i.e., V is open and the restriction $f : f^{-1}(V) \to V$ is a covering. A point $z \in \mathcal{S}'$ which is not a regular value is called a *singular value*. An *asymptotic value* is a point $z \in \mathcal{S}'$ which is the limit of the image by f of a continuous path in $\mathcal{S}$ going to infinity[1)]. A *critical value* is the image by f of a critical point of f.

The following is well known; see for example Proposition 1 in [12][2)].

Proposition 1.2. *The set of singular values* $\operatorname{Sing} f$ *is the closure of the union of the set* $\mathrm{C}(f)$ *of critical values and of the set* $\mathrm{A}(f)$ *of asymptotic values*:

$$\operatorname{Sing} f = \overline{\mathrm{C}(f) \cup \mathrm{A}(f)}.$$

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1) In the sense of Alexandrov, i.e., leaving every compact set.

2) The notation $\operatorname{Sing} f$ is not used in [12]. Instead, he uses $\operatorname{Sing} f^{-1}$ to refer to $\mathrm{C}(f) \cup \mathrm{A}(f)$, not to its closure.

Remark 1.3. Some authors use the term singular values to denote $\mathrm{C}(f) \cup \mathrm{A}(f)$, not its closure. Of course if this union is finite then it equals its closure.

Definition 1.4 (See [6, 10, 11])**.** A *finite type map* is an analytic map $f : \mathrm{dom}\, f = \mathcal{S} \to \mathcal{S}'$, where $\mathcal{S}$ and $\mathcal{S}'$ are two Riemann surfaces, $\mathcal{S}'$ is compact, f is open, f has no removable isolated singularities, and the set of singular values of f is finite.

Being open for an analytic map is equivalent to not being constant on any connected component of the domain.

1.2 Statement

Theorem 1.5. *Let $\widehat{\mathbb{C}}$ be the Riemann sphere, U be an open subset, and $f : U \to \widehat{\mathbb{C}}$ be a finite type map such that*

- $\mathrm{Sing}\, f \subset \{a, b, c\}$ *for some* $a, b, c \in \widehat{\mathbb{C}}$,
- $a \in U$ *and a is a neutral fixed point of f, with eigenvalue* $\mathrm{e}^{2\pi \mathrm{i}\theta}$ *with* $\theta \in \mathbb{R}$,
- θ *is a bounded type irrational,*
- *either* $c \in \widehat{\mathbb{C}} - U$ *or* $f(c) = c$.

Let Δ be the Siegel disk of f at a. Consider the lift γ starting from a of any injective *path γ' going from a to b while avoiding $\{a, b, c\}$ in between. Then either*

(1) *γ ends on a non-critical point in U, in which case $U = \widehat{\mathbb{C}}$ and f is a homography*[3]*,*

(2) *or γ ends on a critical point, called the* main critical point *and then Δ is a quasidisk whose boundary going through the main critical point, and through no other critical point,*

(3) *or γ leaves every compact subset of U and then Δ has not compact closure in U.*

Kneser [20] proved that the set of homeomorphisms of $\widehat{\mathbb{C}}$ is connected. Using the group of homographies, the set of homeomorphisms of $\widehat{\mathbb{C}}$ that fix three given points is also connected. It implies that any two paths γ' as in the theorem above are homotopic relative to $\{a, b, c\}$. By homotopy lifting (Lemma 5.2 applied to $X = [0, 1]$ or $X = [0, 1)$) this implies that the three cases are exclusive even for different γ'. It also implies that in Case (2), the endpoint is independent of the choice of γ'. It is called the *main critical point*.

The three cases cover all possibilities because if the path does not leave every compact subset of U then it has to converge for otherwise its set of accumulation would be a continuum bigger than one point but then since γ' converges, f would map such a continuum to one point, which contradicts the fact that f is analytic and open.

In Subsection 3.2, we give some information on the quasiconformal constant of Δ (see Proposition 3.7).

1.3 About isolated removable singularities and restrictions of finite type maps

In the definition of finite type maps there is the requirement that no isolated singularity is removable. Our proof does not require this condition and would work with a modified definition of finite type maps that omits this requirement. *However, this would not have any advantage.*

Indeed, removing an isolated point z from the domain of f requires adding $f(z)$ to $\mathrm{Sing}\, f$. In particular if the restricted map satisfies the condition of our theorem, then the initial map also does. Moreover, the conclusion of the theorem on the initial map is more informative as on the restricted map (if the main critical point turns out to be removed, we pass from Case (2) to Case (1) and hence loose the quasicircle property for instance).

Remark 1.6. More generally, how much can we restrict a finite type map into finite type maps? Assume that we have a finite type map f, and want to take a restriction g that is still a finite type map, except that we do not require for f nor g that there is no isolated singularity. Note that $f(\mathrm{dom}\, f \cap \partial\, \mathrm{dom}\, g) \subset \mathrm{Sing}\, g$. Since $\mathrm{Sing}\, g$ is finite, this implies that $\mathrm{dom}\, f \cap \partial\, \mathrm{dom}\, g$ is discrete, and hence *we*

[3] This case does not require all the assumptions: as we will see in the proof, it is enough for instance to keep the first assumption ($\mathrm{Sing}\, f \subset \{a, b, c\}$), to assume that $a \in U$ is fixed and non-critical, and of course that γ ends on a non-critical point in U.

can only remove isolated points. If instead we use the full definition of finite type maps (i.e., requiring that there is no isolated singularity) it follows that *restricting the domain of a finite type map without restricting its range never gives a finite type map.* Note also that if the range is connected there is no way to restrict it, because in the definition of the finite type map, the range must be compact.

1.4 Applications

Our main result covers some families for which it was already known, but also many new cases. It does not cover some other families for which it is known by different methods. Let us detail this here.

Let us denote by (i) and (ii) the following cases:

(i) The boundary of the Siegel disk is a quasicircle containing a critical point.

(ii) The Siegel disk is not compactly contained in the domain of f.

If the rotation number has bounded type, then (i) and (ii) cover all the cases by a theorem of Graczyk and Świątek (see [13]). Recall that we call U the domain of f and that f must be a finite type map from U to $\widehat{\mathbb{C}}$ with $\operatorname{Sing} f \subset \{a, b, c\}$.

That a bounded type Siegel disk of period one of a quadratic polynomial satisfies (i) has been known to follow from work of Herman and Świątek from the Douady-Ghys surgery (see [8]). This result has been extended (to all periods) for all polynomials of degree $\geqslant 2$ by an unpublished work of Shishikura, and to all rational maps of degree $\geqslant 2$ by Zhang [26]. Our result applies only to period one Siegel disks of specific polynomials or rational maps and of course always yields (i) when the degree is at least two, so *it brings no novelty there.* For polynomials we take $c = \infty$ in the statement of our main theorem. The polynomial must be either unicritical or have at most two finite critical values (but can have possibly more critical points). If it has two finite critical values, one of them must be the Siegel disk center. Concerning rational maps, we should assume that a is a critical value, for otherwise the map would either be a homography or a bicritical rational map, and b and c would both have only one preimage and since c is fixed, the rational map would be conjugate to a polynomial. We do not enumerate here the rational maps to which our theorem applies, because it would be long and probably pointless given it is superseded by the work of Zhang, yet we give an example. Let us fix $a = 0$, $b = \infty$, $c = 1$ and let $f : z \mapsto P \circ \mu(z)$ where $P(z) = 3z^2 - 2z^3$ is the cubic polynomial whose finite critical points are 0 and 1 and are fixed (the other preimage of 0 is $3/2$ and the other preimage of 1 is $-1/2$), and $\mu(z)$ is a homography that satisfies $\mu(0) = 3/2$, $P'(3/2) \times \mu'(0) = \mathrm{e}^{2\pi \mathrm{i}\theta}$ and $\mu(1) =$ either 1 or $-1/2$. This gives two possible functions μ which yield two non-conjugate rational maps $f = P \circ \mu$, one for which the fixed critical value c is a critical point, the other for which it is not a critical point.

In the realm of transcendental maps, we take $c = \infty$ in the statement of our main theorem. We then recover the fact that the bounded type Siegel disks of period one in the exponential family $z \mapsto \exp(z) + \kappa$ are unbounded, i.e., satisfy (ii), which was first proved by Herman [17]. We also recover (i) in the case of the map $\mathrm{e}^{\mathrm{i}\theta} \sin z$, via a semi-conjugacy $z \mapsto z^2$ to the map $z \mapsto \mathrm{e}^{2\mathrm{i}\theta}(\sin \sqrt{z})^2$. Note that θ has bounded type if and only if 2θ has bounded type. This case was first treated by Zhang [25], and the boundary of its Siegel disk contains exactly *two* critical points. We also recover (i) for the maps studied by Chéritat [7]: in particular the horn maps of parabolic points of quadratic polynomials, when the center of the Siegel disk is one of the ends of the Écalle cylinder, and the uncountable family of entire maps with two critical values and no singular values, when the center of the Siegel disk is one of these values.

In fact the present article can be seen as generalization of [7], and stems from a remark that the second author made while reading it: in [7] the first author used specific properties of horn maps, the fact that they descend through extended Fatou coordinates, to get rigidity; the second author realized that this property was not needed, and introduced the homotopy arguments used here instead.

The set of finite type maps over $\widehat{\mathbb{C}}$ with only three singular values is quite big, much bigger than just the set of all horn maps of finite type maps: for example we can deform such maps by the following procedure, which can also be applied as soon as the domain of definition is not too big. More precisely let f be a finite type map whose domain U is such that there exists an injective map $\phi : U \to \widehat{\mathbb{C}}$ different from a homography. Fixing a and whose derivative equals one at the Siegel center of f, then if μ is any homography fixing a with derivative 1, then $\tilde{f} := f \circ \phi \circ \mu$ is still a finite type map. By choosing μ appropriately, we can preserve the conditions of the theorem, in particular the condition $\tilde{f}(c) \notin U$ or $\tilde{f}(c) = c$.

2 Proof of Case (1)

The assumption of Case (1) is that the path γ ends on a non-critical point of f.

Denote by b' the endpoint of γ. Denote by V the connected component containing a of the preimage by f of $\widehat{\mathbb{C}} - \{c\}$ and note that V contains b'. Denote

$$W = V - f^{-1}(\{a, b\}).$$

The restriction of f to W is a covering of $\widehat{\mathbb{C}} - \{a, b, c\}$. Choose a point z_0 on the curve γ close to a. Let $z_1 = f(z_0)$. Consider a small loop α winding once around a and based on z_0. The image of this loop by f is a small loop $\alpha' = f \circ \alpha$ winding once around a and based on z_1. Let β be concatenation of γ followed from z_0 to a point w_0 close to b', of a small loop around b' based on w_0 and then the first portion followed backwards. Let $\beta' = f \circ \beta$. The fundamental group G of $\widehat{\mathbb{C}} - \{a, b, c\}$ with basepoint z_1 is generated by α' and β'. The covering

$$f : W \to \widehat{\mathbb{C}} - \{a, b, c\}$$

is characterized by the image of the fundamental group of W based on z_0 by f, as a subgroup of G. This image contains the two generators α' and β' and thus the covering is trivial, in the sense that it is a homeomorphism. We saw that V contains a and b'. It cannot contain any other preimage of a or b for otherwise there would be a point near such a preimage that is mapped to the same point as a point near a or b', contradicting injectivity of f on W. Hence $V = W \cup \{a, b'\}$ and f is bijective from V to $\widehat{\mathbb{C}} - \{c\}$. Recall that f is analytic, and thus f is an isomorphism from V to $\widehat{\mathbb{C}} - \{c\}$. Now since V is a subset of the Riemann sphere that is isomorphic to $\mathbb{C}$, we necessarily have $V = \widehat{\mathbb{C}} - c'$ for some $c' \in \widehat{\mathbb{C}}$. The map f is an isomorphism from $\widehat{\mathbb{C}} - \{c'\}$ to $\widehat{\mathbb{C}} - \{c\}$ and thus a homography. This homography extends to $\widehat{\mathbb{C}}$ into a map sending c' to c. By hypothesis f has no isolated removable singularity, and hence $\operatorname{dom} f = \widehat{\mathbb{C}}$.

3 Proof of Case (2)

The assumption of Case (2) is that the path γ ends on a critical point of f.

We will use quasiconformal surgery. The initial idea is due to Ghys' works on degree two polynomials and starts from a Blaschke product, that induces a critical circle map of appropriate rotation numbers. Then the Blaschke product is modified inside the disk into a quasiconformal rotation. An invariant Beltrami form $\tilde{\mu}$ is naturally defined, and a properly normalized straightening of $\tilde{\mu}$ conjugates the Blaschke product to a map that shares sufficiently many properties with the degree two polynomial $z \mapsto \rho z + z^2$ for one to be able to prove it is equal to it.

Call b' the main critical point, i.e., the endpoint of γ.

3.1 A surgery

The following diagram summarizes the process (see the text for details):

$$\begin{array}{cccc} & \text{conformal map, Schwarz reflection} & \text{surgery} & \text{straightening} \\ f & \longrightarrow \beta & \longrightarrow \tilde{\beta} & \longrightarrow \tilde{f} = f. \\ \text{given, analytic} & \text{analytic, commutes with } 1/\bar{z} & \text{quasiregular, } = \beta \text{ outside } \mathbb{D} & \text{analytic, same cover as } f \end{array}$$

For a start, conjugating f by a homography we can assume that

$$a = 0, \quad b = 1 \quad \text{and} \quad c = \infty.$$

3.1.1 *A pre-model*

We construct here a pre-model β, following the construction done in [7]. This is illustrated in Figure 1.

Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$. Then $\mathbb{D} \cap \operatorname{Sing} f \subset \{0\}$. Let Δ' be the connected component containing 0 of $f^{-1}(\mathbb{D})$. The set Δ' is simply connected by Lemma 5.3 and since it contains 0 which is a non-critical preimage of 0, it follows that we are in the case $k = 1$ of the lemma. By Lemma 5.4, Δ' is a Jordan domain and f is injective on $\overline{\Delta'}$. By the Jordan-Schoenflies theorem, the set $\widehat{\mathbb{C}} - \overline{\Delta'}$ is homeomorphic to the unit disk too. There is thus a (non-unique) conformal map

$$\psi : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to \widehat{\mathbb{C}} - \overline{\Delta'}$$

fixing $\infty = c$:

$$\psi(c) = c.$$

By Caratheodory's theorem, ψ extends to the closures to a homeomorphism from $\widehat{\mathbb{C}} - \mathbb{D}$ to $\widehat{\mathbb{C}} - \Delta'$, that we denote

$$\overline{\psi} : \widehat{\mathbb{C}} - \mathbb{D} \to \widehat{\mathbb{C}} - \Delta'.$$

The critical point b' of f lies at the end of the pull-back starting from a of the segment $[a, b] = [0, 1]$. In particular,

$$b' \in \partial\Delta'.$$

Other critical points map to either 0, 1 or ∞, and hence by injectivity of f on $\overline{\Delta}'$, the point b' is the only critical point of f on $\overline{\Delta}'$.

Consider the map

$$\beta_{\text{half}} = f \circ \psi : U' \to \widehat{\mathbb{C}},$$

where $U' = \psi^{-1}(U) = \psi^{-1}(U \setminus \Delta')$. Since ψ is proper, the distance from $f \circ \psi(z)$ to $\mathbb{D}$ tends to 0 as $z \to \partial\mathbb{D}$. It follows from the Schwarz reflection theorem that there exists a holomorphic extension of $f \circ \psi$ to a map $\beta : \operatorname{dom}\beta \to \widehat{\mathbb{C}}$ defined on $\operatorname{dom}\beta = U' \cup \partial\mathbb{D} \cup s(U')$ where $s(1/z) = z$; in other words, β is the Schwarz reflection of β_{half}. Note also that if $c \in U$ and $f(c) = c$ then, since $c = \infty \notin \mathbb{D}$, we have $\beta(c) = \beta_{\text{half}}(c) = f(\psi(c)) = f(c) = c$:

$$\beta(c) = c. \tag{3.1}$$

The point

$$b'_\beta := \overline{\psi}^{-1}(b') \in \partial\mathbb{D}$$

is mapped to b by β and is a critical point of β of local degree $2d - 1$ where d is the local degree of the critical point b' of f. We call it the *main critical point* of β. Note that if $c \in \operatorname{dom} f$ then $\beta(c) = f(c) = c$.

Following a wide-spread convention, we will call a *critical circle map* any map from a circle to itself that is analytic, has at least one critical point, yet is a homeomorphism and last preserves orientation.

Assertion 3.1. The restriction of β to $\partial\mathbb{D}$ is a critical circle map.

Proof. The map β sends the circle $\partial\mathbb{D}$ to itself and is analytic by construction (this is one of the striking conclusions of the Schwarz reflection theorem). Since f is injective on $\partial\Delta'$ it follows that β is injective on the circle $\partial\mathbb{D}$. It is orientation preserving because f maps Δ' to $\mathbb{D}$; more precisely pick any non-critical point z of β in $\partial\mathbb{D}$; then $\psi^{-1}(z) \in \partial\Delta'$ is non-critical for f and f maps any nearby point outside Δ' to a point outside $\mathbb{D}$, and hence β maps any point close to z and outside $\mathbb{D}$ to a point outside $\mathbb{D}$, so is orientation preserving near this point, and hence everywhere since it is a homeomorphism. Since there is at least one critical point of β on $\partial\mathbb{D}$, it follows that the restriction of β to $\partial\mathbb{D}$ is a critical circle map. □

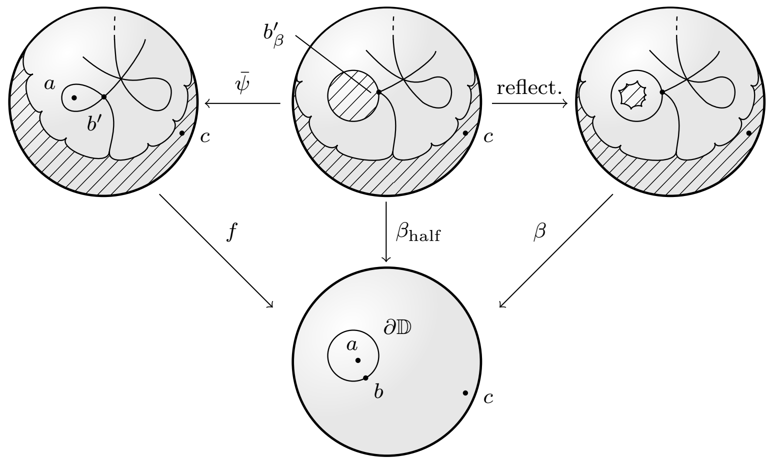

Figure 1 Sketch of the objects in the construction of the premodel β (see Subsection 3.1.1). The non-hatched part indicates domains of the maps from top to bottom. The graph inside the domains indicates a portion of the preimage of $(\partial\mathbb{D})$

3.1.2 *A model*

We turn to the construction of the model $\widetilde{\beta}$. This is illustrated in Figure 2. By the theory of the rotation number, there exists a unique $\tau \in \mathbb{R}/\mathbb{Z}$ such that $\beta \circ R_\tau$ has rotation number θ on $\partial\mathbb{D}$, where $R_\tau(z) = \mathrm{e}^{2\pi\mathrm{i}\tau} z$. Replacing β by $\beta \circ R_\tau$ amounts to replacing ψ by $\psi \circ R_\tau$, i.e., by another choice of conformal map from $\widehat{\mathbb{C}} - \overline{\mathbb{D}}$ to $\widehat{\mathbb{C}} - \overline{\Delta'}$ fixing ∞. So from now on we assume that

$$\beta \text{ has rotation number } \theta \text{ on the unit circle.}$$

We recall now one of the equivalent definitions of a quasisymmetric map from the circle to itself [14].

Definition 3.2. Let $k \geqslant 1$. Let $f : \partial\mathbb{D} \to \partial\mathbb{D}$ be a homeomorphism and $\tilde{f} : \mathbb{R} \to \mathbb{R}$ a continuous lift by the universal cover $\mathbb{R} \to \partial\mathbb{D}$: $x \mapsto \mathrm{e}^{2\pi\mathrm{i}x}$. The map f is termed k-quasisymmetric if it is orientation preserving and if $\forall\, x \in \mathbb{R}$ and $\forall\, h > 0$,

$$k^{-1} \leqslant \frac{|\tilde{f}(x+h) - \tilde{f}(x)|}{|\tilde{f}(x) - \tilde{f}(x-h)|} \leqslant k.$$

With this definition, a 1-quasisymmetric self map of $\partial\mathbb{D}$ is necessarily an isometry (rotation or reflection according to whether or not it preserves the orientation).

A theorem of Herman [15] and Świątek [24] (see Theorem 3.8 in the present article) ensures that a critical circle map with a rotation number of bounded type is quasisymmetrically conjugate to a rotation: there exists a quasisymmetric self map ϕ of $\partial\mathbb{D}$ such that

$$\forall\, z \in \partial\mathbb{D}, \quad \beta(z) = \phi^{-1} \circ R_\theta \circ \phi(z),$$

where $R_\theta(z) = \mathrm{e}^{2\pi\mathrm{i}\theta} z$. A theorem of Ahlfors and Beurling [3] ensures that a quasisymmetric circle map has an extension to a quasiconformal homeomorphism of the disk that we still denote by ϕ. We can assume moreover that $\phi(0) = 0$, replacing ϕ if necessary by its composition with an appropriate self-diffeomorphism of $\mathbb{D}$ equal to identity near $\partial\mathbb{D}$. We now modify the pre-model β by surgery into a model map $\tilde{\beta}$ as follows:

$$\forall\, z \in \widehat{\mathbb{C}} - \mathbb{D}, \quad \tilde{\beta}(z) = \beta(z),$$
$$\forall\, z \in \overline{\mathbb{D}}, \quad \tilde{\beta}(z) = \phi^{-1} \circ R_\theta \circ \phi.$$

These two definitions coincide on $\partial\mathbb{D}$. Since $a = 0$ we have $\tilde{\beta}(a) = a$. If $c \in \operatorname{dom} f$ then since $c = \infty$ we have $\tilde{\beta}(c) = \beta(c)$, $\tilde{\beta}(c) = c$. We now make use of two theorems.

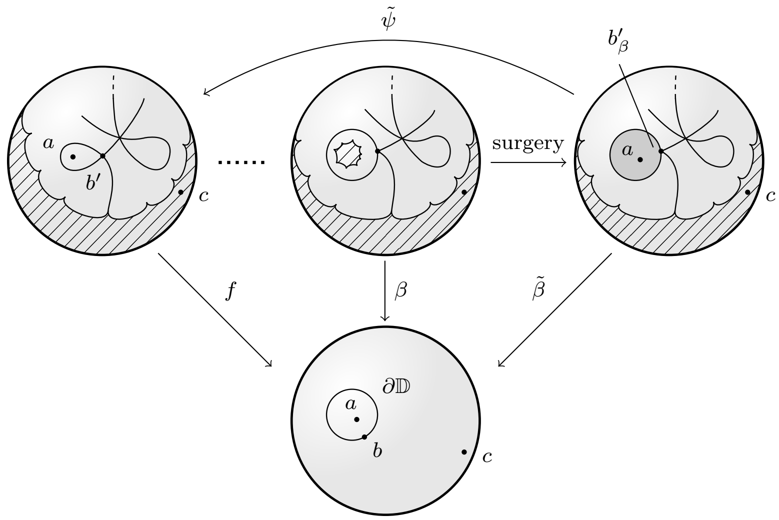

Figure 2 Sketch of the objects in the construction of the model $\tilde{\beta}$ (see Subsection 3.1.2). The set $\mathbb{D}$ is in dark grey to indicate where $\tilde{\beta}$ is not analytic. The map $\tilde{\psi}$ is a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$

Theorem 3.3 (Quasiconformal removability of analytic curves, see [21, Theorem 3.2 p. 202[4)]]). *If Γ is an analytic simple arc or a simple closed curve and $\phi : U \to V$ is a homeomorphism that is K-quasiconformal on $U - \Gamma$, then ϕ is K-quasiconformal on U.*

Theorem 3.4 (Rickman's lemma, see [23]). *Assume that $C \subset V \subset U \subset \widehat{\mathbb{C}}$ with U and V open and C closed rel. U. Assume that $\phi : V \to \widehat{\mathbb{C}}$ and $\Phi : U \to \widehat{\mathbb{C}}$ are homeomorphisms to their images, that ϕ is quasiconformal, that Φ is quasiconformal on $U - C$ and that $\Phi = \phi$ on C. Then*

(1) *Φ is quasiconformal,*

(2) *$\partial\Phi/\partial\bar{z} = \partial\phi/\partial\bar{z}$ almost everywhere on C.*

According to [5, Definition 1.34], a quasiregular map is a continuous map that is locally K-quasiconformal for a uniform K on the complement of a discrete set. From this and Theorem 3.4 above, we deduce that $\tilde{\beta}$ is quasiregular: the discrete set to remove from $\operatorname{dom}\tilde{\beta}$ is the set of critical points of β that do not lie in $\mathbb{D}$ (there is one such point b' on $\partial\mathbb{D}$, that we henceforth remove), and the only place where Theorem 3.4 is used is in neighborhoods W in $\mathbb{C}$ of points $z \in \partial\mathbb{D} - \{b'\}$, taking $\phi = \beta$, $\Phi = \tilde{\beta}$, $U = V = W$, $C = W - \mathbb{D}$.

By definition the relation $\tilde{\beta} = f \circ \overline{\psi}$ holds on $U' \cup \partial\mathbb{D}$. We want it to also hold on $\mathbb{D}$: it is convenient here, and will also be useful in some proof later.

Lemma 3.5. *There is a unique extension of $\overline{\psi}$ into a homeomorphism $\tilde{\psi}$ of $\widehat{\mathbb{C}}$, such that*

$$\tilde{\beta} = f \circ \tilde{\psi},$$

in the sense that the two hand sides have the same domain and are equal on it. Moreover $\tilde{\psi}$ is quasiconformal and $\tilde{\psi}(a) = a$.

Proof. Being a bijection, a homeomorphism $\tilde{\psi}$ satisfying the equation must satisfy $\tilde{\psi}(\mathbb{D}) = \Delta'$. The map f restricts to a bijection from $\overline{\Delta'}$ to $\overline{\mathbb{D}}$, and in this proof we call g its inverse. In particular $g(a) = a$. Note that $\overline{\psi}$ restricts to a bijection from $\partial\mathbb{D}$ to $\partial\Delta'$. Hence $\bar{\psi} = g \circ \tilde{\beta}$ holds on $\partial\mathbb{D}$. Also, we necessarily have $\tilde{\psi} = g \circ \tilde{\beta}$ on $\overline{\mathbb{D}}$. This implies $\tilde{\psi}(a) = a$. Conversely if we set $\tilde{\psi} = g \circ \tilde{\beta}$ on $\overline{\mathbb{D}}$, this matches with $\bar{\psi}$ on $\partial\mathbb{D}$, and hence we get a continuous bijection $\tilde{\psi}$ extending $\bar{\psi}$. Since the range $\widehat{\mathbb{C}}$ is compact, $\tilde{\psi}$ is a homeomorphism. By construction it is quasiconformal on the complement of $\partial\mathbb{D}$. Finally by Theorem 3.3, ψ is a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$. □

4) In [21], the definition of F is in Subsection 9.1 of Page 47 completed with the definition of K in Subsection 8.3 of Page 44, and M is the classical modulus of a (topological) quadrilateral in the complex plane.

As an immediate consequence, f and $\tilde{\beta}$ have the same set of critical values and the same set of asymptotic values and the same set of singular values:

$$\operatorname{Sing}\tilde{\beta} = \operatorname{Sing} f.$$

Note that by construction $\mathbb{D}$ is a rotation domain for $\tilde{\beta}$.

3.1.3 *Straightening*

Recall that ϕ is a quasiconformal map used to define $\tilde{\beta}$ in $\mathbb{D}$ (see the previous section). Let μ be the Beltrami form (ellipse field) on $\mathbb{D}$, defined as the pull-back by ϕ of the null form (for which all ellipses are circles). The restriction of $\tilde{\beta}$ to $\mathbb{D}$, which is equal to $\phi^{-1} \circ R_\theta \circ \phi$, preserves μ. By a standard procedure, one extends the Beltrami form μ into a unique $\tilde{\beta}$-invariant Beltrami form $\tilde{\mu}$ that vanishes outside $\bigcup_{n\in\mathbb{N}} \tilde{\beta}^{-n}(\mathbb{D})$. Let S be the unique straightening[5)] of $\tilde{\mu}$ that fixes a, b and c and let

$$\tilde{f} = S \circ \tilde{\beta} \circ S^{-1}.$$

If $c \in \operatorname{dom} f$ then since $f(c) = c \in \{a, b, c\}$ it is fixed by S and since $\tilde{\beta}(c) = c$ we get in particular,

$$\tilde{f}(c) = c.$$

The null Beltrami form is invariant by $\tilde{f}$, and hence $\tilde{f}$ is analytic. It is a finite type map with

$$\operatorname{Sing}\tilde{f} = \operatorname{Sing} f \subset \{a, b, c\}.$$

Proposition 3.6. *There exists a homography h with $h(a) = a$ and $h'(a) = 1$ and such that $f = \tilde{f} \circ h$. In particular* $\operatorname{dom}\tilde{f} = h(\operatorname{dom} f)$. *If moreover $c \notin U$ or $f(c) = c$ then $h = \mathrm{id}$, i.e., $\tilde{f} = f$.*

Proof. Consider the following commuting diagram:

$$\begin{array}{ccccc}
\widehat{\mathbb{C}} & \xleftarrow{\ \tilde{\psi}\ } & \widehat{\mathbb{C}} & \xrightarrow{\ S\ } & \widehat{\mathbb{C}} \\
& {\scriptstyle f}\searrow & \downarrow{\scriptstyle \tilde{\beta}} & & \downarrow{\scriptstyle \tilde{f}} \\
& & \widehat{\mathbb{C}} & \xrightarrow[\ S\ ]{} & \widehat{\mathbb{C}}.
\end{array}$$

The outer part reads

$$\begin{array}{ccc}
U & \xrightarrow{\ T\ } & \tilde{U} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\
\widehat{\mathbb{C}} & \xrightarrow[\ S\ ]{} & \widehat{\mathbb{C}}
\end{array}$$

with $T := S \circ \tilde{\psi}^{-1}$ and $\tilde{U} := \operatorname{dom}\tilde{f} = T(U)$. Note that T is in fact a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ and that the support of its Beltrami differential is contained in U.

Note that

$$\|\tilde{\mu}\|_\infty = \|\mu\|_\infty < 1.$$

For t a complex number with $|t| < 1/\|\tilde{\mu}\|_\infty$, let S_t be the unique straightening of $t\tilde{\mu}$ that fixes a, b and c: by a theorem of Ahlfors and Bers [2], the solution will depend continuously on t and better: for each $z \in \widehat{\mathbb{C}}$, the function $t \mapsto S_t(z)$ is analytic. It is a particular instance of holomorphic motion. The restriction to values of t in $[0, 1]$ gives an isotopy S_t rel $\{a, b, c\}$ from $S_0 = \mathrm{Id}_{\widehat{\mathbb{C}}}$ to $S_1 = S$.

By homotopy lifting (Lemma 5.2 applied to $g_t = S_{1-t} \circ f$) there exists a family $T_t : U \to \tilde{U}$ for $|t| < 1/\|\tilde{\mu}\|_\infty$ with $T_1 = T$ but T_0 not necessarily the identity, and satisfying $S_t \circ f = \tilde{f} \circ T_t$, i.e., the following diagram commutes:

5) S and ϕ may *not* coincide on $\mathbb{D}$

$$\begin{array}{ccc} U & \xrightarrow{T_t} & \tilde{U} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{f}} \\ \widehat{\mathbb{C}} & \xrightarrow[S_t]{} & \widehat{\mathbb{C}}. \end{array}$$

The family $T_t \circ T^{-1}$ is a holomorphic motion of $\tilde{U}$ that is constant on the set $\tilde{f}^{-1}(\{a,b,c\})$. Let us extend T_t into a map $\tilde{T}_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by setting $\tilde{T}_t(z) = T(z)$ for $z \in U$ and $\tilde{T}_t(z) = T(z)$ for $z \in \widehat{\mathbb{C}} - U$, i.e., on $\widehat{\mathbb{C}} - U$ it is constant with respect to t and equal to T. We still have that $\tilde{T}_t \circ T^{-1}$ is a holomorphic motion. In particular T_0 is quasiconformal. In fact, T_0 is holomorphic in U and since it is equal to the identity on $\widehat{\mathbb{C}} - U$, T_0 is holomorphic by the second claim of Rickman's lemma (see Theorem 3.4). Hence $T_0 = h$ for some homography h and since $S_0 = \text{id}$: $f = \tilde{f} \circ h$.

Let us prove that h fixes a and that $h'(a) = 1$. First, $\tilde{f}(a) = a \in \{a,b,c\}$ and hence the motion $T_t \circ T^{-1}$ is constant on a: $h(a) = T_0(a) = T(a) = S(\tilde{\psi}^{-1}(a)) = a$. Second, we have the relation $S_0 \circ f = \tilde{f} \circ h$. Now $S_0 = \text{id}$ and $f'(a) = \tilde{f}'(a)$: this follows from the topological invariance of the rotation number (for maps with a Siegel disk as f and $\tilde{f}$, this invariance follows from the invariance of the rotation number of circle maps). Hence $h'(a) = 1$ by the chain rule.

Now in the case that either $c \notin U$ or $f(c) = c$, let us prove that $h(c) = c$, from which it follows that $h = \text{id}$ and hence $\tilde{f} = f$. In the first case the holomorphic motion $T_t \circ T^{-1}$ is immobile on the complement of $\tilde{U}$, we get $T_0 = T$ on U, and hence $h(c) = T_0(c) = T(c) = S(\tilde{\psi}^{-1}(c)) = c$. In the other case, we have $\tilde{f}(c) = f(c) = c \in \{a,b,c\}$ and hence the motion is constant on c and we conclude similarly. □

3.2 Consequences

As a consequence, f has a Siegel disk Δ whose boundary is a quasicircle, and contains the critical point b' and its image b.

We have some information on the quasiconformal constant of $\partial\Delta$: let a_n be the entries of the continued fraction expansion of the bounded type number θ.

Proposition 3.7. *If d is the local degree of b' and $\sup a_n \leqslant M$ then Δ is a K-quasicircle with $K = K(d,M)$. Also, there exists an annulus separating Δ from $\partial\,\mathrm{dom}\, f$ and of modulus greater than or equal to $\varepsilon(d,M)$.*

Let us justify this proposition. We will use the following theorem that controls the quasi symmetry constant in the Herman-Świątek theorem.

Theorem 3.8 (See [15,16,24])**.** *Let $\mathcal{F}$ be a set of holomorphic maps defined in a neighbourhood of the unit circle and such that the following properties hold*:

(1) *there exists an open set A containing the unit circle C and contained in the domain of all $f \in \mathcal{F}$,*

(2) *$f(C) = C$ and the restriction of f to C has at least one critical point, yet is a homeomorphism and preserves the orientation,*

(3) *there exists M such that for every $f \in \mathcal{F}$, the rotation number of f has all its entries less than or equal to M,*

(4) *the class $\mathcal{F}$ is precompact on A for the Euclidean metric.*

Then there exists $k > 1$ such that all $f \in \mathcal{F}$ are k-quasi symmetrically conjugated to rotations.

It follows from [15] complemented by Herman [16]. Uniformity is stated in [16] only for Blaschke fractions, but the proof works exactly the same for families satisfying the above assumptions.

Lemma 3.9. *The set $\mathrm{dom}\,\beta$ contains an annulus A that contains $\partial\mathbb{D}$, is symmetric with respect to $\partial\mathbb{D}$, and whose modulus is greater than some positive universal constant that depends only on the local degree d of the main critical point b'. It follows that the set $\mathrm{dom}\, f - \overline{\Delta'}$ contains an annulus A' one of whose boundary components is $\partial\Delta'$, and whose modulus is half the modulus of A. Moreover β factors on A as $\beta\,|_A = g_d \circ \iota$ where g_d is a map independent of β, defined on an abstract Riemann surface homeomorphic to an annulus and taking values in $\widehat{\mathbb{C}}$, and ι is a conformal isomorphism. Moreover, $A \cap \beta^{-1}(a) = \{a\}$, $A \cap \beta^{-1}(b) = \{b'_\beta\}$ and $A \cap \beta^{-1}(c) = \emptyset$.*

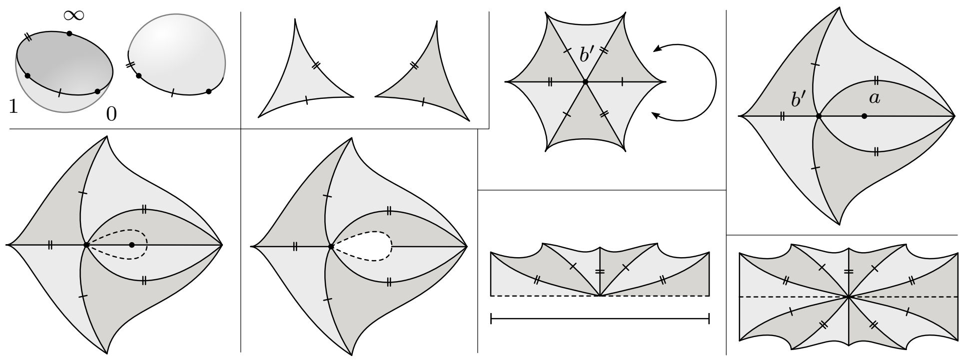

Figure 3 Illustrations for the proof of Lemma 3.9. In the first frame, the sphere $\widehat{\mathbb{C}}$ is cut along the real line, that passes through $a = 0$, $b = 1$, and $c = \infty$. We mark with respectively one, two and zero tick marks the segments $[0,1]$, $[1,\infty]$, $[\infty, 0]$. Each hemisphere has all its preimages by f complete and univalent. We can thus see $\operatorname{dom} f$, as a Riemann surface over $\widehat{\mathbb{C}}$, as a collection of a finite or countable number of copies of the two simply connected sets depicted in the second frame, that get glued together along their boundaries of the same colors. The conformal class of the two pieces, with their three marked boundary points, is unique, and the way two boundaries are glued is unique too: it has to be the identity in the chart represented in the first frame. Third frame: If d denotes the local degree of the critical point b', there must be exactly d distinct copies of each of the two pieces glued together around b' along their one tick and two ticks marked lines. Fourth frame: The fixed point a of f must be at the end of one of these one tick marked lines, and since a is not cricital, the two corresponding pieces are also glued along their boundaries without tick marks. In the fifth frame we added the dashed outline of Δ', the component of $f^{-1}(\mathbb{D})$ that contains a. This component is removed in the sixth frame and the set $\widehat{\mathbb{C}} - \overline{\Delta}'$ is conformally mapped to $\widehat{\mathbb{C}} - \overline{\mathbb{D}}$, represented on the seventh frame as half cylinder $\mathbb{H}/\mathbb{Z}$, unwrapped (one must identify the left and right boundaries). A reflection is performed on the last frame and gives us an annulus contained in $\operatorname{dom}\beta$ and whose conformal class depends only on d

Proof. Figure 3 illustrates the construction. Consider the upper and lower half planes $\mathbb{H}$ and $\mathbb{H}_-$ as subsets of $\widehat{\mathbb{C}}$. They contain none of the singular values. Since they are simply connected, it follows that for every connected component W of $f^{-1}(V)$ with $V = \mathbb{H}$ or $V = \mathbb{H}_-$, the map f is an analytic isomorphism from W to V. By studying the way the components W whose boundaries contain b' must be attached to each other along their boundaries, one can prove that f factors through a universal map on some topological disk $D \subset \operatorname{dom} f$ containing b' : $f_D = h_d \circ \iota_f$, $\iota_f : D \to D_0$. Let $A' = D - \Delta'$. The abstract Riemann surface D_0 is constructed using copies of $\mathbb{H}$ and $\mathbb{H}_-$, with the identity map as charts, that are glued along the three open segments that 0, 1 and ∞ cut $\mathbb{R}$ into and possibly glued at 0, 1 and ∞ too. The gluing is the identity map in these charts. The map h_d from a piece to $\widehat{\mathbb{C}}$ is also the identity map in these charts. If we remove from D_0 the component of $h_d^{-1}(\mathbb{D})$ that contains the preimage of a, we obtain the union A'_0 of an annulus and one of its boundary components, which is an analytic curve except at one corner. Then we can proceed to a reflection along this boundary minus the corner and complete it there, and we get an abstract Riemann surface A_0 that is homeomorphic to an annulus, and on which the map $h_d|_{A'_0}$ extends holomorphically into a map that we call g_d. Let A be the union of $\partial\mathbb{D}$, $A'' = \psi^{-1}(A')$ and the reflection of the latter with respect to $\partial\mathbb{D}$. By the Schwarz reflection theorem, the isomorphism $\iota_f \circ \psi$ from A'' to A'_0 reflects into an isomorphism $\iota : A \to A_0$. By definition of β it follows that $\beta|_{A''} = h_d \circ \iota_f \circ \psi$ and by analytic continuation we get $\beta|_A = g_d \circ \iota$. □

We need the following extension lemma, a variation of a theorem of Ahlfors and Beurling [3] (see also [9]).

Lemma 3.10 (See [3]). *Let ϕ be a k-quasisymmetric orientation preserving homeomorphism of $\partial\mathbb{D}$. Then ϕ has a continuous extension on $\overline{\mathbb{D}}$ to a homeomorphism that is K-quasiconformal on $\mathbb{D}$ and fixes 0, where K depends only on k.*

Proof. We list two remarks. First, the Ahlfor-Beurling theorem [3] is stated in a half plane $\mathbb{H}$, not in the unit disk. Second, the map they construct does not necessarily fix a point decided in advance, like $0 \in \mathbb{D}$ or $i \in \mathbb{H}$. However our claim is an easy consequence, using the exponential map. We justify it below for completeness.

Indeed, let $\tilde{\phi}$ be the lift of ϕ by $\mathbb{R} \to \partial\mathbb{D}$: $x \mapsto \mathrm{e}^{2\pi \mathrm{i} x}$. It commutes with T_1 because ϕ is an orientation preserving homeomorphism of $\partial\mathbb{D}$. By the definition of k-quasisymmetric that we chose earlier, $\tilde{\phi}$ satisfies the inequality

$$k^{-1} \leqslant \frac{|\tilde{\phi}(x+h) - \tilde{\phi}(x)|}{|\tilde{\phi}(x) - \tilde{\phi}(x-h)|} \leqslant k.$$

The following map $\hat{\phi} : \mathbb{H} \to \mathbb{H}$ is defined [3]:

$$\begin{aligned}\hat{\phi}(x + \mathrm{i}y) = &\frac{1}{2}\int_0^1 [\tilde{\phi}(x+ty) + \tilde{\phi}(x-ty)]dt \\ &+ \frac{\mathrm{i}r}{2}\int_0^1 [\tilde{\phi}(x+ty) - \tilde{\phi}(x-ty)]dt\end{aligned}$$

for some appropriate value of r that depends on k. Ahlfors and Beurling [3] proved that this gives a K-quasiconformal extension of $\tilde{\phi}$ with $K \leqslant k^2$. It is easy to see from the formula defining $\hat{\phi}$ that this extension commutes with T_1 too, so it pushes down by $z \mapsto \mathrm{e}^{2\pi \mathrm{i} z}$ to a homeomorphism of $\overline{\mathbb{D}}^* = \overline{\mathbb{D}} - \{0\}$ that is K-quasiconformal on $\mathbb{D}^*$ and has a continuous extension $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ fixing 0. By quasiconformal removability of 0 (see Theorem 3.3), f is K-quasiconformal on $\mathbb{D}$. □

We consider now the set $\mathcal{F}$ of all possible functions β obtained by our construction from maps f and satisfying the assumptions of Theorem 1.5, and with θ whose continued fraction entries are all less than or equal to M. Let us check that $\mathcal{F}$ satisfies the assumption of Theorem 3.8. Number (2) holds by construction (there is one and only one critical point of $\tilde{\beta}$ on $\partial\mathbb{D}$). Number (3) holds by hypothesis on θ. For Number (1), recall that we assume that either $c \notin \operatorname{dom} f$ or $f(c) = c$. In the first case, since $\operatorname{dom}\beta - \overline{\mathbb{D}} = \psi^{-1} \operatorname{dom} f$ and $\psi(c) = c$, it follows that $c \notin \operatorname{dom}\beta$. In the other case, where $f(c) = c$, note that $\beta(c) = c$ by Equation (3.1) but by the last sentence of Lemma 3.9, A contains no preimage of c by β, and hence A cannot contain c. In both cases, since $c = \infty$ we get

$$\infty \notin A.$$

It is known[6] that an annulus contained in $\mathbb{C} - \overline{\mathbb{D}}$, with one of whose boundary components being $\partial\mathbb{D}$, and not containing some other point $z \in \mathbb{C} - \overline{\mathbb{D}}$, has a modulus less than or equal to $\varepsilon(|z|)$ with $\varepsilon(R) \to 0$ as $R \to 1$. We can apply this to $A_{\text{half}} = A - \overline{\mathbb{D}}$ where A is given by Lemma 3.9. We have $\operatorname{mod} A_{\text{half}} = \frac{1}{2} \operatorname{mod} A = \operatorname{mod} A_{\text{half}} \leqslant \varepsilon(|c|)$. Since $\operatorname{mod} A$ depends only on d, it follows that $|z|$ cannot be too close to 1: A contains "$r < |z| < 1/r$" for some r that depends only on d. Hypothesis (1) follows. Hypothesis (4) follows from this and from the fact that β factors on A through a universal map as proved in Lemma 3.9, $\beta|_A = g_d \circ \iota$.

So we can apply Theorem 1.5 to our family $\mathcal{F}$: the conjugacies ϕ to a rotation of β on $\partial\mathbb{D}$ are k-quasisymmetric for a common k that depends on d and M. By Lemma 3.10, it follows that ϕ has a continuous extension on $\overline{\mathbb{D}}$ to a homeomorphism that is K-quasiconformal on $\mathbb{D}$ and fixes 0, where K depends only on k. It follows that $\|\tilde{\mu}\| \leqslant \frac{K-1}{K+1}$, and hence S is K-quasiconformal. This proves the first part of Proposition 3.7. For the second part, one can take the annulus $S(A_{\text{half}})$. A lower bound $\varepsilon(d, M)$ on its modulus follows from S being $K(d, M)$-qc and from the fact that the modulus of A_{half} depends only on d.

4 Proof of Case (3)

Here, we assume that γ leaves every compact subset of U.

Case 1. $c \notin U$.

By way of contradiction let us assume that the Siegel disk Δ is compactly contained in $U = \operatorname{dom} f$. By using [13] it follows that there is a critical point $p \in \partial\Delta \cap U$. Its image $f(p)$ is singular, and hence $f(p) \in \{a, b, c\}$. Now $f(\partial\Delta) \subset \partial\Delta$. Indeed f is proper on Δ since it is conjugate there to a rotation on

6) [1, Chapter III, Section A]: *Three extremal problems*, Problem I: Grötzsch.

a disk. Hence $f(p)$ belongs to the boundary of Δ relative to U. It follows that $f(p) \neq a$ and also that $f(p) \in U$. Since we assume $c \notin U$ this implies $f(p) \neq c$. Hence

$$f(p) = b.$$

Consider then an injective path α starting from a, contained in the Siegel disk and ending at a point z_0 close to p. Note that $f \circ \alpha$ is also injective, because f is injective on the Siegel disk. We endow $\widehat{\mathbb{C}}$ with a spherical metric and we may assume (shortening the path if necessary) that $f(z_0)$ is the closest point to b in the path. Locally, analytic maps are equivalent to $z \mapsto z^d$ for some $d \in \mathbb{N}$. Hence there exists a lift of the geodesic segment from $f(z_0)$ to b that starts from z_0 and ends on p. The concatenation of $f \circ \alpha$ and this segment is injective, goes from a to b, avoids c and the path $\gamma = \alpha \cdot \beta$ is its unique lift starting from a. We get a contradiction with the hypothesis.

Case 2. $f(c) = c$.

Assume by way of contradiction that Δ is compactly contained in $\operatorname{dom} f$.

Lemma 4.1. *It holds that $b \notin \partial\Delta$.*

Proof. Otherwise, consider $\varepsilon > 0$ small. By compactness, there is only finitely many preimages b_i' of b on $\partial\Delta$ and any point in Δ that maps in $B(b, \varepsilon)$ must be close to one of these preimages. By connectedness of Δ there is an injective path γ that starts from a and goes to $\partial B(b, \varepsilon)$ while staying in Δ. Complete it with a straight segment to b. It is still injective provided the first part is stopped as soon as it meets $\partial B(b, \varepsilon)$. On one hand, by what has been mentionned above, the preimage of the completed path will be contained in $\Delta \cup$ a finite union of small neighbourhoods of the b_i'. On the other hand the preimage of this segment must leave every compact subset of $\operatorname{dom} f$ by the paragraph following Theorem 1.5. This leads to a contradiction □

By assumption on $\operatorname{Sing} f$, a critical point is mapped to either a, b or c. Recall that[7] $f(\partial\Delta) \subset \partial\Delta$. Since neither a nor b is in $\partial\Delta$, we get the following corollary.

Corollary 4.2. *All critical points in $\partial\Delta$ map to c in one iteration of f.*

The main result of [13] tells us that $\partial\Delta$ contains at least one critical point of f. By compactness there is only finitely many of them on $\partial\Delta$, called $c_1', \ldots, c_m', m \geqslant 1$.

Since $c_1' \in \partial\Delta$ and $f(\partial\Delta) \subset \partial\Delta$, we get that $c \in \partial\Delta$. It follows that c cannot be critical for otherwise it would be a critical fixed point, and thus have an attracting basin which would be an open set disjoint from Δ and containing c, preventing $c \in \partial\Delta$. As a consequence for all integers k with $1 \leqslant k \leqslant m$, we have $c_k' \neq c$.

At this point we need to refine the result of [13]. Let S denote the Schwarzian derivative operator: $Sf = D(Nf) - (Nf)^2/2 = f'''/f' - (3/2)(f''/f')^2$, where $Df = f'$ and $Nf = (D^2 f)/Df = f''/f'$. One important fact in [13] was that, in the absence of critical points on $\partial\Delta$, the quantity Sf is bounded from above on Δ. This is not anymore true in our case, because Sf has double poles at all the critical points.

Let $h : \mathbb{D} \to \Delta$ be a conformal map fixing 0, so that

$$f \circ h = h \circ R_\theta \tag{4.1}$$

holds on $\mathbb{D}$. Since f has at least one critical point, its degree as a cover over $\widehat{\mathbb{C}} - \{a, b, c\}$ is at least two. Hence the fixed point a has at least one preimage a' different from a. A neighbourhood of a' will be disjoint from Δ. We will work in a coordinate system where $a = 0$ but where $a' = \infty$ (as opposed to the choice $c = \infty$ made in the previous section). In this case Δ is a bounded subset of $\mathbb{C}$. We will use the following area form:

$$dA = \rho(z)\ dx \wedge dy = \max\left(\{1\} \cup \left\{\frac{1}{|z - c_k'|^2};\ 1 \leqslant k \leqslant m\right\}\right) dx \wedge dy. \tag{4.2}$$

[7] For completeness, we justify this well-known fact: f being continuous we have $f(\overline{\Delta}) \subset \overline{\Delta}$. Hence if we had a point $z \in \partial\Delta$ such that $f(z) \notin \partial\Delta$ we would have $f(z) \in \Delta$. By definition of Δ, f is a bijection of Δ. Let $g : \Delta \to \Delta$ be its inverse, which is holomorphic too. Consider $z_n \in \Delta$ such that $z_n \to z$. Then $f(z_n) \to f(z)$ by continuity of f and hence $z_n = g(f(z_n)) \longrightarrow g(f(z))$ by continuity of g. Hence $z = g(f(z)) \in \Delta$ contradicting the assumption $z \in \partial\Delta$.

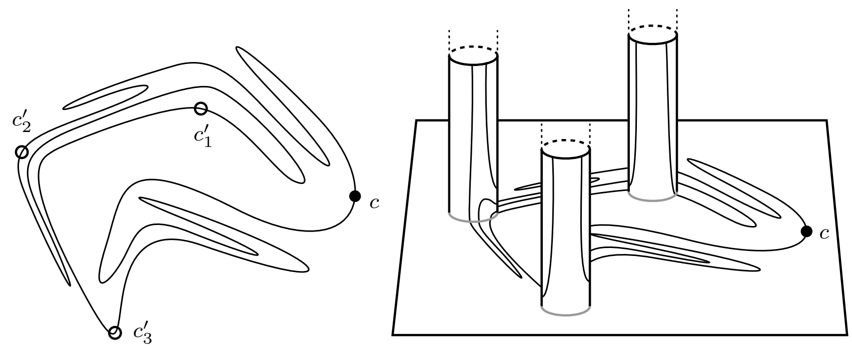

Figure 4 Illustration for the area form dA. Left: the continuous curve represents $\partial\Delta$; however one has to imagine that it is non-locally connected. The fixed point c is indicated, together with its critical preimages c_k'. Right: a representation of the same scene mapped in 3D space to a surface whose metric induces the area form dA; the three critical preimages of the fixed point c that were on $\partial\Delta$ have been blown up into infinite half cylinders

Note that we use a power two exponent no matter what the multiplicity of c_k' is. This is designed so that $|Sf(z)|/\rho(z)$ is bounded from above on $h(\mathbb{D})$:

$$\forall\, z \in h(\mathbb{D}), \quad |Sf(z)| \leqslant C\rho(z). \tag{4.3}$$

Indeed, the Schwarzian derivative near a critical point is a meromorphic function with a double pole, whose leading coefficient depends on the multiplicity of the critical point. Such an area form dA corresponds to a conformal metric $\sqrt{\rho(z)}|dz|$ that is isometric to an infinite cylinder near each c_k' (see Figure 4). We will use the coordinate $\log(z - c_k')/2\pi\mathrm{i}$.

Let $\lambda = \mathrm{e}^{2\pi\mathrm{i}\theta}$ so that

$$R_\theta(\zeta) = \lambda\zeta.$$

Lemma 4.3. *For* $\zeta \in \mathbb{D}$ *let*

$$H(\zeta) = Sh(\zeta)\zeta^2.$$

Then for all $K > 1$*, there exist* $Q_1 > 0$ *and* $r_0 < 1$ *such that* $\forall\, \zeta \in \mathbb{D}$ *with* $|\zeta| > r_0$ *and* $\forall\, i$ *with* $0 \leqslant i \leqslant K/(1-|\zeta|)$*,*

$$|H(\lambda^i\zeta) - H(\zeta)| \cdot (1-|\zeta|)^2 \leqslant Q_1 \operatorname{Area}_{dA} h(\operatorname{ring}(1-|\zeta|)),$$

where

$$\operatorname{ring}(\varepsilon) = \left\{\zeta \in \mathbb{D}\,;\, \frac{1}{2}\varepsilon < 1 - |\zeta| < \frac{3}{2}\varepsilon\right\}$$

and the area is measured using the area form dA *defined in Equation* (4.2).

Proof. We reproduce here the result of first computation done in the proof of [13, Lemma 2.1], which holds for any map satisfying Equation (4.1): for all $i \in \mathbb{N}$, and $\zeta \in \mathbb{D}$,

$$|H(\lambda^i\zeta) - H(\zeta)| = |Sf^i(h(\zeta))||\zeta h'(\zeta)|^2. \tag{4.4}$$

Here, Sf^i denotes the Schwarzian derivative of the i-th iterate of f. Denote $z_j = f^j(z)$. Similar to [13], we get from the chain rule for the Schwarzian that

$$|Sf^i(z)| \leqslant \sum_{j=0}^{i-1} |Sf(z_j)||(f^j)'(z)|^2. \tag{4.5}$$

We will estimate the following area: $\operatorname{Area}_{dA}\left[h(B(\zeta,(1-|\zeta|)/Q))\right]$ where the area is measured using dA defined in Equation (4.2) instead of the Euclidean metric (this is the main difference with [13]). We have

$$\operatorname{Area}_{dA}[h(B(\zeta,(1-|\zeta|)/Q))] = \int_{\zeta' \in B(\zeta,(1-|\zeta|)/Q)} \rho(h(\zeta'))|h'(\zeta')|^2 d\lambda(\zeta'),$$

where $d\lambda$ refers to the Lebesgue measure in $\mathbb{C}$. By Koebe's distortion theorems, there exists $\kappa > 1$ such that $\forall\,\zeta \in \mathbb{D}$, $\forall\,\zeta' \in B(\zeta, (1-|\zeta|)/2)$, we have $1/\kappa < |h'(\zeta')/h'(\zeta)| < \kappa$. Also, using the Schwarz-Pick metric (a.k.a. hyperbolic metric) of Δ and its comparison with the distance to $\partial\Delta$ there exists $\alpha > 1$ such that $\forall\,\zeta' \in B(\zeta, |1-\zeta|/2)$ and for all $z \in \mathbb{C}\setminus\Delta$: $d(h(\zeta'), z) \geqslant d(h(\zeta), z)/\alpha$ (indeed the hyperbolic distance in $\mathbb{D}$ from ζ to ζ' is bounded by a universal constant; the geodesic segment from ζ to ζ' is mapped by h to a curve in Δ whose hyperbolic length is shorter or equal; since $z \notin \Delta$, it follows that the element $u(w)|dw|$ of hyperbolic metric of Δ satisfies $u(w) \geqslant \frac{1}{4|w-z|}$ by Koebe's one quarter theorem; it follows that the curve has a length at least of $\frac{1}{4}|\log\frac{|h(\zeta')-z|}{|h(\zeta)-z|}|$). It follows that $\rho(h(\zeta')) \geqslant \rho(h(\zeta))/\alpha^2$. Hence $\forall\,Q \geqslant 2$,

$$\rho(h(\zeta))|h'(\zeta)|^2\pi(1-|\zeta|)^2/Q^2 \leqslant (\kappa\alpha)^2\,\mathrm{Area}_{dA}[h(B(\zeta,(1-|\zeta|)/Q))]. \tag{4.6}$$

Now given $K > 0$ we impose that $0 \leqslant i \leqslant K/(1-|\zeta|)$. As in [13], there exist then $r_0 < 1$ and $Q \geqslant 2$ that depend on K and on the upper bound Θ on the continued fraction entries of θ and such that if $|\zeta| > r_0$ then the balls $B(\lambda^j\zeta, (1-|\zeta|)/Q)$ are disjoint for j varying from 0 to i, and thus must be their images by h. Putting everything together we get

$$|H(\lambda^i\zeta) - H(\zeta)|(1-|\zeta|)^2 \leqslant Q_1\,\mathrm{Area}_{dA}[h(\mathrm{ring}')], \tag{4.7}$$

where $\mathrm{ring}' = \{\zeta' \in \mathbb{D};\, (1-\frac{1}{Q})(1-|\zeta|) < 1-|\zeta'| < (1+\frac{1}{Q})(1-|\zeta|)\}$, and Q_1 is some constant that depends on Q, i.e., only on K and on Θ. Note that we assume $Q \geqslant 2$ so the ring for Q is included in the ring for $Q = 2$ □

If we can prove that, for a given Q, this area tends to 0 as $|\zeta| \to 1$ (this claim is independend from Q), then the rest of the argument of [13] carries on and we get that $\partial\Delta$ is a finite union of quasiconformal arcs, hence locally connected. But then $\partial\Delta$ cannot contain a fixed point[8)], leading to a contradiction with $f(c) = c$.

According to the discussion above there only remains to prove the following lemma.

Lemma 4.4. *For the area form defined in Equation* (4.2),

$$\mathrm{Area}_{dA}\, h(\mathrm{ring}(\varepsilon)) \to 0$$

as $\varepsilon \to 1$.

Proof. Consider the sequence of annuli $A_n = h(\mathrm{ring}(3^{-n}))$. The A_n are disjoint. It is enough to prove $\mathrm{Area}\,A_n \to 0$: indeed $\forall\,\varepsilon < 1$, $\mathrm{ring}(\varepsilon)$ is contained in two consecutive such annuli union their common boundary which is a smooth curve.

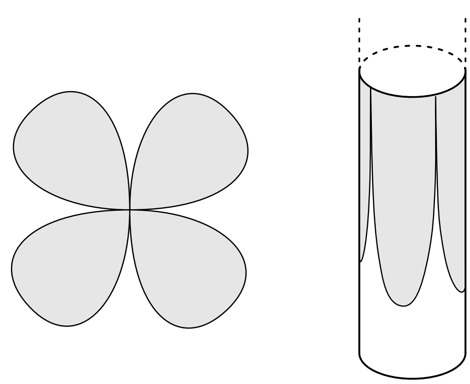

Figure 5 Left: preimage near a critical point of attracting petals of a parabolic fixed point. Right: their image in cylinder coordinates. In the latter, the width of the fjord between two petals decreases exponentially with respect to the height

8) This follows from [22, Lemma 18.7]. The lemma in this reference is stated for rational maps but holds in fact for any map as soon as the Siegel disk is compactly contained in the domain of the map.

In [13] the total area available was finite, and hence $\sum \operatorname{Area} A_n = \operatorname{Area} \Delta < +\infty$ whence $\operatorname{Area} A_n \to 0$. In our case there is an infinite amount of area available because of the cylindrical parts near the c'_k. We will prove that

$$\operatorname{Area}_{dA} \Delta < +\infty$$

still holds in our case. There are several cases according to the nature of the fixed point c.

- It cannot be attracting nor Siegel since $c \in \partial\Delta$.

- If c is parabolic, it leaves too little room between its attracting petals: indeed the Siegel disk must be disjoint from the attracting petals and therefore for each of the critical points c'_k, Δ is disjoint from the preimage of the petals near this critical point. These petals have an order of tangency > 1, which in cylinder coordinate (for which the area form becomes proportional to the Lebesgue measure) means that the space between two consecutive petals is decreasing exponentially with the distance as we proceed along the cylinder. Hence there is only a finite amount of area available to Δ (See Figure 5).

- If c is repelling, consider its linearizing coordinate ϕ defined in a neighbourhood V of c: $\phi \circ f(z) = f'(c) \times \phi(z)$. We may assume for convenience that $\phi(V)$ is a round disk. Consider the quotient map $\pi : \mathbb{C}^* \to \mathbb{C}^*/\Lambda$, where Λ is the group generated by the map $z \mapsto \lambda z$. The range is a torus. Consider the map $\Pi = \pi \circ \phi \circ f$, which is defined in a neighbourhood V'_k of c'_k. We take V'_k small enough so that $f(V'_k) \subset V$ and $V'_k \cap V = \emptyset$.

Assertion 4.5. The map Π is injective on $\Delta \cap \operatorname{dom} \Pi$.

Proof. Assume $\Pi(z) = \Pi(z')$. Consider the points $w = f(z)$ and $w' = f(z')$. From $\pi(\phi(w)) = \pi(\phi(w'))$ we deduce that one of w or w' is an iterated image of the other, say $w' = f^j(w)$, for some $j \geqslant 0$, with $f^n(w) \in V$ for all n with $0 \leqslant n \leqslant j$. So $f(z') = f(f^j(z))$ and because f is injective on Δ which contains both z' and $f^j(z)$, we get $z' = f^j(z)$. If $j > 1$ we get a contradiction because $z' \in V'_k$ and $f^j(z) \in V$ and these sets are disjoint. So $j = 0$, i.e., $z = z'$. □

As a corollary, the area of Δ in the cylinders is finite.

- If c is Cremer, consider a ball $B(c, \varepsilon)$ where f is injective. There is no more a linearizing coordinate ϕ. However, by a similar argument to the repelling case, for each k, the set $U_k := f(\Delta \cap B(c'_k, \varepsilon'))$ with ε' small enough has the property that for all $z \in U_k$, $f(z) \in B(c, \varepsilon)$ and the forward orbit $f^n(z)$, $n > 0$ is disjoint from U_k as long as it remains in $B(c, \varepsilon)$. Now slice the cylinder of c'_k into finite cylinders C_n of equal height. The set $f(C_n)$ is close to c where we do not have a cylinder. However, let us temporarily look at a neighbourhood of c in a cylinder coordinate $\log(z - c)/2\pi \mathrm{i}$. The set $f(C_n)$ is close to a sub-cylinder with straight boundaries and its height increases linearly with n. On $f(C_n)$ the map f is very close to a horizontal translation: the error term is exponentially small with respect to n. It implies that a point in $f(C_n)$ needs a number of iterates that is at least exponential in n to leave $B(c, \varepsilon)$. The disjointness implies that the area of $\Delta \cap C_n$ has to be exponentially small. □

5 Ramified coverings

This section is a collection of known facts about coverings and ramified coverings. For completeness, we provide a proof for each. Below, $\mathcal{S}$ and similar notation refer to arbitrary Riemann surfaces. Recall that $\mathrm{C}(f)$ denotes the set of critical values of f, $\mathrm{A}(f)$ the set of asymptotic values and that $\operatorname{Sing} f = \overline{\mathrm{C}(f) \cup \mathrm{A}(f)}$. Recall that a holomorphic map is open if and only if it is not constant on any of the connected components of its domain.

Lemma 5.1 (Path lifting)**.** *Let* $f : \operatorname{dom} f \to \mathcal{S}$ *be holomorphic, open with* $\operatorname{dom} f \subset \mathcal{S}'$, $\gamma : [0, 1] \to \mathcal{S}$ *be a path such that* $\gamma((0, 1)) \cap (\mathrm{C}(f) \cup \mathrm{A}(f)) = \emptyset$, *and* $z_0 \in \operatorname{dom} f$ *with* $f(z_0) = \gamma(0)$. *Then*

- *There exists a lift on* $[0, 1)$ *of* γ *starting from* z_0, *i.e.,* $\exists\, \tilde\gamma : [0, 1) \to \operatorname{dom} f$ *continuous with* $f \circ \tilde\gamma = \gamma$ *and* $\tilde\gamma(0) = z_0$.
- *If* z_0 *is not critical then the lift is unique.*
- *If* $\gamma(1) \notin \mathrm{A}(f)$ *then* $\tilde\gamma(s)$ *has a limit as* s *tends to* 1.

Proof. Let d be the local degree of f at z_0. According to basic complex analysis, near z_0 the map f is equivalent to $z \mapsto z^d$, i.e., there exist open neighborhoods U of z_0 and V of $f(z_0)$ and analytic isomorphisms $\phi : U \to \mathbb{D}$ and $\psi : V \to \mathbb{D}$ such that $\forall\, z \in \mathbb{D}$, $\psi \circ f \circ \phi^{-1}(z) = z^d$. By continuity of γ there exists $\varepsilon > 0$ such that $\gamma([0,\varepsilon]) \subset V$. By hypothesis $\gamma((0,\varepsilon]) \subset V - \{f(z_0)\}$. Since $f : U - \{z_0\} \to V - \{f(z_0)\}$ is a cover, it follows that there are exactly d lifts $\hat\gamma$ of the restriction of γ to $(0,\varepsilon]$ if we require $\hat\gamma(\varepsilon)$ to belong to U. Moreover, since $f : U \to V$ is equivalent to z^d as above, it follows that $\hat\gamma(t)$ necessarily tends to z_0 as $t \to 0$, so we can extend $\hat\gamma$ to a continuous function at 0 with value z_0.

Choose one such $\hat\gamma$, defined on $[0,\varepsilon]$ and consider the set $\mathcal{E}$ of continuous extensions $\tilde\gamma : [0,u) \to \operatorname{dom} f$ of $\hat\gamma$, with $\varepsilon \leqslant u \leqslant 1$, such that $\forall\, t \in [0,u)$, $f \circ \tilde\gamma(t) = \gamma(t)$. Let $\mathcal{U}$ be the set of possible u. Then $\varepsilon \in \mathcal{U}$ because $\hat\gamma \in \mathcal{E}$. By restriction, it is immediate that if $\varepsilon \leqslant u' \leqslant u \in \mathcal{U}$ then $u' \in \mathcal{U}$.

Next, we show that if $\tilde\gamma_1 : [0,u_1) \to \operatorname{dom} f$ and $\tilde\gamma_2 : [0,u_2) \to \operatorname{dom} f$ both belong to $\mathcal{E}$ then they coincide on $[0,u_3)$ with $u_3 = \min(u_1,u_2)$. It is enough to show that the set $\{t \in (0,u_3)\,;\, \tilde\gamma_1(t) = \tilde\gamma_2(t)\}$ is open, closed in $[0,u_3)$ and non-empty. It is closed by continuity; non-empty because it contains $(0,u)$; open because $\gamma((0,1))$ does not meet the set of critical values of f and hence f is locally injective near $\gamma_1(t) = \gamma_2(t)$ for all $t \in (0,u_3)$.

Let $u_\infty = \sup \mathcal{U}$. Then $u_\infty \in \mathcal{U}$ because by the previous paragraph we can take $\tilde\gamma : [0,u_\infty) \to \operatorname{dom} f$ to be the join of any sequence $\tilde\gamma_n : [0,u_n) \to \operatorname{dom} f$ with $u_n \in \mathcal{U}$ such that $u_n \to u_\infty$.

It remains to show that $u_\infty = 1$. Otherwise, consider the map $\tilde\gamma : [0,u) \to \operatorname{dom} f$ we just construct. Since $f \circ \tilde\gamma = \gamma$ on $[0,u)$, since γ has a limit at $t = u$ and since f is holomorphic and nowhere locally constant, it follows by the isolated zero theorem that $\tilde\gamma(t)$ either has a limit at $t = u$ or leaves every compact subset of $\operatorname{dom} f$. In the second case, $\gamma(u)$ is an asymptotic value, contradicting our assumption that $\gamma((0,1)) \cap \mathrm{A}(f) = \emptyset$. Hence $\tilde\gamma$ has a continuous extension at $t = u$. By the same arguments as in the first paragraph, we can then extend $\tilde\gamma$ further, leading to a contradiction with the definition of u_∞. Note that if $\gamma(1) \notin \mathrm{A}(f)$ then the same argument proves that $\tilde\gamma$ has an extension to $t = 1$ as a continuous lift of γ; this proves the third point of the lemma.

The second point stated in the lemma concerns uniqueness in the case where z_0 is not critical. If another lift $\tilde\gamma'$ exists with the same properties, then since $\tilde\gamma'_t$ is close to z_0 when t is small, the first paragraph above proves that $\tilde\gamma'$ and $\tilde\gamma$ are equal near 0. Uniqueness then follows from the third paragraph. □

Lemma 5.2 (Homotopy lifting). *The following commuting diagram illustrates the statement*:

$$\begin{array}{ccc} X & \overset{h_t}{\dashrightarrow} & \tilde{\mathcal{S}} \\ & {\scriptstyle g_t}\searrow & \downarrow{\scriptstyle f} \\ & & \mathcal{S}. \end{array}$$

Let X be a topological space. Let $f : \tilde{\mathcal{S}} \to \mathcal{S}$ be holomorphic and open. Let $g : [0,1] \times X \to \mathcal{S} : (t,x) \mapsto g_t(x)$ be continuous. Assume that $h_0 : X \to \tilde{\mathcal{S}}$ is continuous, that $f \circ h_0 = g_0$, that for every $x \in X$ with $g_0(x) \in \operatorname{Sing} f$, the function $t \mapsto g_t(x)$ is constant, and that for every $t \in (0,1]$, $g_t(x) \in \operatorname{Sing} f \Rightarrow g_0(x) \in \operatorname{Sing} f$. Then there exists a unique extension $h : [0,1] \times X \to \tilde{\mathcal{S}}$, $(t,x) \mapsto h_t(x)$ of h_0 that is continuous and lifts g, i.e., such that $\forall\, t$, $g_t = f \circ h_t$. It is constant with respect to t for all $x \in X$ such that $g_0(x) \in \operatorname{Sing} f$.

Proof. Existence: Since $\operatorname{Sing} f$ is closed, its complement in $\tilde{\mathcal{S}}$ is open. By the basic theory of singular values, f is a covering from $f^{-1}(\mathcal{S} - \operatorname{Sing} f)$ to $\mathcal{S} - \operatorname{Sing} f$. Hence by the theory of coverings, there is a unique continuous lift $t \mapsto h_t$ of $t \mapsto g_t$ starting from h_0, if we restrict to $\mathcal{S} - \operatorname{Sing} f$ in the range and to $f^{-1}(\mathcal{S} - \operatorname{Sing} f)$ in the domain. Let us extend h_t to every $x \in \operatorname{Sing} f$ by decreting that $h_t(x) = h_0(x)$. By the hypotheses, it automatically satisfies $f \circ h_t = g_t$ and is constant with respect to t for all x such that $g_0(x) \in \operatorname{Sing} f$, but it is less obvious that the map h we obtain is continuous.

We already know that h is continuous in the open subset $X - g_0^{-1}(\operatorname{Sing} f)$ of X. Consider $x_\infty \in X$ with $g_0(x_\infty) \in \operatorname{Sing} f$ and let $z_\infty = g_0(x_\infty)$ and $z'_\infty = h_0(x_\infty)$. Note that by construction, the function $t \mapsto h_t(x_\infty)$ is constant, equal to z'_∞. Let us show that the functions $t \mapsto h_t(x)$ converge uniformly to this constant function as $x \to x_\infty$. Since f is holomorphic and nowhere locally constant, for all neighborhood U of z'_∞, there is a connected open neighborhood V of z_∞ such that the connected component V' of $f^{-1}(V)$ containing z'_∞ is contained in U. Since g is continuous, $[0,1]$ compact and since $t \mapsto g_t(x_\infty)$ is constant equal to z_∞, it follows that there exists a neighborhood W of x_∞ such that $g_t(x)$ takes value in V whenever $x \in W$ and $t \in [0,1]$. By taking a smaller W we can also assume that $h_0(W) \subset V'$. It then follows that for all $x \in W$, $h_t(x) \in U$.

Uniqueness: Because f is holomorphic and nowhere locally constant, any time the function $t \mapsto g_t(x)$ is constant, its lifts $t \mapsto h_t(x)$ will be constant too. Hence h is determined when $g_0(x) \in \operatorname{Sing} f$. In addition, on $X - \operatorname{Sing} f$, uniqueness follows from the theory of coverings. □

Let us define a family of models ρ_k and ρ_k^* as follows. Let $\rho_0 : \mathbb{D} \to \mathbb{D}$, $z \mapsto 0$ and for $0 < k < +\infty$, $\rho_k : \mathbb{D} \to \mathbb{D}$, $z \mapsto z^k$. Let $\rho_\infty : H \to \mathbb{D}$, $z \mapsto \exp z$ where H is defined by "$\mathrm{Re}(z) < 0$". If $k < +\infty$ let ρ_k^* be the restriction to $\mathbb{D}^* = \mathbb{D} - \{0\}$. Note that all the models have the same range: $\mathbb{D}$. Not all are surjective, though.

Lemma 5.3. *Assume $f : \operatorname{dom} f \to \mathcal{S}'$ is holomorphic, W' is an open subset of $\mathcal{S}'$ isomorphic to $\mathbb{D}$, $a' \in W'$ and $W' \cap \operatorname{Sing} f \subset \{a'\}$. Let ψ be any isomorphism from W' to $\mathbb{D}$ mapping a' to 0. Then for all connected components W of $f^{-1}(W')$, the restriction $f : W \to W'$ is equivalent to one of the models $\rho = \rho_k$ or ρ_k^* above, and more precisely there exists an isomorphism $\phi : W \to \operatorname{dom} \rho$ such that $f = \psi^{-1} \circ \rho \circ \phi$ on W.*

Proof. The map f is a covering over $W' - \{a'\}$. If f is constant on W then this constant is a singular value, thus equal to a' so f is equivalent to ρ_0 via ψ and any choice of the isomorphism $\phi : W \to \mathbb{D}$. Otherwise, we use the classification of coverings. The fundamental group of $W' - \{a'\}$ is isomorphic to $\mathbb{Z}$. The group of deck transformations is a subgroup completely characterized by its index $k \in \mathbb{N}^* \cup \{\infty\}$. The restriction of f to $W - f^{-1}(a')$ is equivalent via ϕ and some *homeomorphism* ϕ to one of the ρ_k^* or to ρ_∞. This function ϕ must be analytic (this can be checked locally using $\psi \circ f = \rho_k^* \circ \phi$ as every other map in this diagram is analytic and locally invertible away from points corresponding to a'). If $W \cap f^{-1}(a') = \emptyset$ then we are done. Otherwise, consider a point $a \in W$ such that $f(a) = a'$ and let d be the local degree of f at a. Consider a simple loop around a in a small neighbourhood of a. Its image by f winds d times around a', and so does its image by $\psi \circ f$ around 0. Hence its image by ϕ, which is injective, is a simple loop whose image by ρ winds d times. It follows that $k = d$, in particular $k \neq +\infty$. Then, from the commutative diagram, it follows that ϕ extends continuously with $\phi(a) = a'$, and the extension is analytic by the erasable singularity theorem. Since the preimage of a small neighbourhood of 0 by ρ_k^* is connected, and f is continuous and $f^{-1}(a)$ is discrete, it follows that a is the only preimage of a' in W. □

Recall that a *Jordan curve* is the image of an injective loop in a topological surface. We use here the term *Jordan domain* to refer to open subsets W of a topological surface S for which there exists a homeomorphism from the closed unit disk of $\mathbb{C}$ to $\overline{W}$ mapping the open unit disk to W. If W is a Jordan domain then $\overline{W} - W$ is a Jordan curve, whence the name. With this terminology, the Jordan-Schoenflies theorem can be restated as follows: any Jordan curve in the plane bounds a Jordan domain.

Lemma 5.4. *Assume that we are in the case $k = 1$ of the previous lemma, i.e., $f : W \to W'$ is a bijection. Call a the unique preimage of a' in W. Assume also that W' is a Jordan domain as per the definition above. Last, assume that for all $v \in \operatorname{Sing} f \cap \partial W'$ there exists a path γ from a' to v within W' whose lift $\tilde{\gamma}$ starting from a has an endpoint in* $\operatorname{dom} f$. *Then W is a Jordan domain and the restriction of f to ∂W is a bijection to $\partial W'$.*

Proof. By homotopy lifting (see Lemma 5.2), the endpoint of $\tilde{\gamma}$ depends only on the endpoint of γ. It follows that $f^{-1} : W' \to W$ has a continuous extension g to $\overline{W'}$ taking values in $\operatorname{dom} f$, for otherwise one could construct a path γ whose lift does not converge in $\operatorname{dom} f$. The map $f \circ g$ is defined on $\overline{W'}$ and is the identity on W', and hence on $\overline{W'}$ by continuity. The map g must thus be injective. As an injective continuous map defined on a compact set, g is a homeomorphism to its image. Its image is a closed set hence must contain $\overline{W}$, and by continuity is contained in $\overline{W}$. □

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