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Twisted bounded-dilation group C^*-algebras as C^*-metric algebras

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Abstract We construct a class of C^*-metric algebras. We prove that for a discrete group Γ with a 2-cocycle σ, the closure of the seminorm $\|[M_\ell,\cdot]\|$ on $C_c(\Gamma,\sigma)$ is a Leibniz Lip-norm on the twisted reduced group C^*-algebra $C^*_r(\Gamma,\sigma)$ for the pointwise multiplication operator M_ℓ on $\ell^2(\Gamma)$, induced by a proper length function ℓ on Γ with the property of bounded θ-dilation. Moreover, the compact quantum metric space structures depend only on the cohomology class of 2-cocycles in the Lipschitz isometric sense.

Keywords discrete group, bounded θ-dilation, twisted reduced group C^*-algebra, C^*-metric algebra, Leibniz Lip-norm, compact quantum metric space

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1 Introduction

In [3], Connes recovered the geodesic distance on a compact Riemannian spin manifold M through a $*$-seminorm on a dense $*$-subalgebra of the unital C^*-algebra $C(M)$ of complex-valued continuous functions on M, which was constructed from a Dirac operator on M. More generally, Connes endowed the state space $\mathcal{S}(A)$ for a unital C^*-algebra A with a metric from a spectral triple $(A,\mathcal{H},D)$ using the formula

$$\rho_{L_D}(\mu,\nu)=\sup\{|\mu(a)-\nu(a)|:a\in A, L_D(a)=\|[D,a]\|\leqslant 1\}$$

for $\mu,\nu\in\mathcal{S}(A)$. A typical noncommutative example is the reduced group C^*-algebra of a discrete group endowed with a proper length function with a discrete and unbounded range. In 1998, Pavlović [14] and Rieffel [15] investigated the question of when the metric topology on the state space $\mathcal{S}(A)$ from ρ_{L_D} induces the weak $*$-topology. Later, Rieffel [16,18,19] discussed this question in a more general framework of order-unit spaces. More precisely, he defined a Lip-norm on an order-unit space as a seminorm whose induced metric topology on the state space is the weak $*$-topology as well as considering an order-unit space equipped with a Lip-norm as a compact quantum metric space [16,18].

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In [17], Rieffel provided compact quantum metric space structures on the noncommutative tori $C_r^*(\mathbb{Z}^d,\sigma)$, using length functions on $\mathbb{Z}^d$. As a follow-up, he posed the question of which other discrete groups with 2-cocycle σ could provide examples of compact quantum metric spaces. Later, Ozawa and Rieffel [13] showed that the reduced group C^*-algebras are compact quantum metric spaces for groups with a Haagerup-type condition (hyperbolic groups, in particular). Note that $\mathbb{Z}^d$ and hyperbolic groups are groups with a rapid decay property [4,6,13,21]. More generally, Antonescu and Christensen [1] constructed a large class of Lip-norms on a group Γ with the rapid decay property, and hence obtained compact quantum metric space structures on the reduced group C^*-algebra $C_r^*(\Gamma)$. Motivated by the σ-twisted rapid decay property in [12], Long and Wu [9] proved that twisted reduced group C^*-algebras for discrete groups with this property are compact quantum metric spaces. It is well known that a finitely generated discrete group is of polynomial growth if and only if it is nilpotent-by-finite [5,8,11,21]. In [2], Christ and Rieffel introduced three growth conditions for a group with a length function: strong polynomial growth, bounded doubling and polynomial growth, and proved that these growth conditions are equivalent for a proper length function on a finitely generated discrete group. They showed that for any proper length function on a discrete group Γ with the bounded doubling property, the topology on the state space $\mathcal{S}(C_r^*(\Gamma))$ from this metric coincides with the weak $*$-topology [2]. Moreover, they conjectured that this is also true for group C^*-algebras twisted by a 2-cocycle. The main goal of this paper is to prove this conjecture.

The rest of this paper is organized as follows. In Section 2, some basic concepts and results on the twisted reduced group C^*-algebras are introduced. In Section 3, for a discrete group with a proper length function, we introduce two seminorms on the twisted group algebras. One comes from the length function, and the other comes from the restriction of the left regular projective representation of the twisted group algebras. In addition, we prove that the latter depends only on the cohomology class of the 2-cocycle σ. In Section 4, we truncate the twisted convolution operators from the left regular projective representation by a family of cutoff functions, and control the truncated operators with respect to the operator norm and the seminorm $J_{D,\theta}$, respectively. In Section 5, the classical polynomial growth of the length function on a finitely generated group is generalized to the property of bounded dilation of the length function on a discrete group. For a discrete group equipped with a proper length function with the property of bounded dilation, we decompose the functions with finite support on this group into "three parts", and dominate them by the operator norm and the seminorm $J_{D,\theta}$, respectively. In Section 6, it is proved that for any 2-cocycle σ on a discrete group Γ endowed with a length function with the bounded θ-dilation property, the closure $\bar{L}_D$ of the seminorm L_D on the unital C^*-normed algebra $C_c(\Gamma,\sigma)$ is a Leibniz Lip-norm on the twisted reduced group C^*-algebra $C_r^*(\Gamma,\sigma)$, i.e., the pair $(C_r^*(\Gamma,\sigma),\bar{L}_D)$ is a C^*-metric algebra. It is further proved that the compact quantum metric space structures $(C_r^*(\Gamma,\sigma),\bar{L}_D)$ on $C_r^*(\Gamma,\sigma)$ depend only on the cohomology class of the 2-cocycle σ in the Lipschitz isometric sense.

2 Twisted reduced group C^*-algebras

Let Γ be a discrete group with the identity e, and let $\mathbb{T}$ be the unit circle in $\mathbb{C}$. A 2-*cocycle* on Γ is a function $\sigma:\Gamma\times\Gamma\mapsto\mathbb{T}$ satisfying

$$\sigma(s,t)\sigma(st,r)=\sigma(s,tr)\sigma(t,r)$$

for all $s,t,r\in\Gamma$. Two cocycles σ_1 and σ_2 are called *cohomologous* if there is a function $\rho:\Gamma\mapsto\mathbb{T}$ such that

$$\sigma_1(s,t)\sigma_2(s,t)^{-1}=\rho(s)\rho(t)\rho(st)^{-1}$$

for all $s,t\in\Gamma$. A 2-cocycle σ on Γ is called a *multiplier* if $\sigma(e,e)=1$.

Let Γ be a discrete group, and let σ be a 2-cocycle on Γ. Let $C_c(\Gamma,\sigma)$ denote the set of complex-valued functions on Γ with finite support. For $s\in\Gamma$, we define

$$\delta_s(t)=\begin{cases}1, & t=s,\\ 0, & t\neq s.\end{cases}$$

Then $\delta_s \in C_c(\Gamma,\sigma)$. $C_c(\Gamma,\sigma)$ becomes a $*$-algebra with the identity $\overline{\sigma(e,e)}\delta_e$ under the operations

$$\begin{aligned}
&(f+g)(s) = f(s)+g(s),\\
&(\alpha f)(s) = \alpha f(s),\\
&(f *_\sigma g)(s) = \sum_{t\in\Gamma} f(t)g(t^{-1}s)\sigma(t,t^{-1}s),\\
&f^*(s) = \overline{f(s^{-1})\sigma(s,s^{-1})},
\end{aligned}$$

for any $f,g \in C_c(\Gamma,\sigma)$, $s\in\Gamma$, and $\alpha\in\mathbb{C}$ (see [23]). When $\sigma\equiv 1$ is the trivial 2-cocycle, $C_c(\Gamma,\sigma)$ is the usual group algebra, and is denoted by $C_c(\Gamma)$ in this paper.

The left regular σ-projective representation of Γ on $\ell^2(\Gamma)$ is given by

$$\lambda_s(\delta_t) = \sigma(s,t)\delta_{st}$$

for any $s,t\in\Gamma$. This representation induces the left regular σ-projective representation λ of $C_c(\Gamma,\sigma)$ on $\ell^2(\Gamma)$ given by

$$\lambda(f) = \sum_{s\in\Gamma} f(s)\lambda_s$$

for any $f\in C_c(\Gamma,\sigma)$. More precisely, we have

$$\begin{aligned}
(\lambda(f)(\xi))(s) &= (f *_\sigma \xi)(s)\\
&= \sum_{t\in\Gamma} f(t)\sigma(t,t^{-1}s)\xi(t^{-1}s)\\
&= \sum_{r\in\Gamma} f(sr^{-1})\sigma(sr^{-1},r)\xi(r)
\end{aligned}$$

for any $s\in\Gamma$, $f\in C_c(\Gamma,\sigma)$, and $\xi\in\ell^2(\Gamma)$. It is here clear that λ is linear. For any $s,t,r\in\Gamma$, we have

$$\begin{aligned}
\lambda(\delta_s *_\sigma \delta_t)(\delta_r) &= \sigma(s,t)\sigma(st,r)\delta_{str}\\
&= \sigma(s,tr)\sigma(t,r)\delta_{str}\\
&= (\lambda(\delta_s)\lambda(\delta_t))(\delta_r),
\end{aligned}$$

i.e., $\lambda(\delta_s *_\sigma \delta_t) = \lambda(\delta_s)\lambda(\delta_t)$, and thus

$$\begin{aligned}
\lambda(f *_\sigma g) &= \sum_{s,t\in\Gamma} f(s)g(t)\lambda(\delta_s *_\sigma \delta_t)\\
&= \sum_{s,t\in\Gamma} f(s)g(t)\lambda(\delta_s)\lambda(\delta_t)\\
&= \lambda(f)\lambda(g)
\end{aligned}$$

for any $f,g\in C_c(\Gamma,\sigma)$. Moreover, for any $s,t\in\Gamma$, we have

$$(\lambda(\delta_s))^*(\delta_t) = \overline{\sigma(s,s^{-1}t)}\delta_{s^{-1}t} = \lambda({\delta_s}^*)(\delta_t).$$

For any $f\in C_c(\Gamma,\sigma)$, we have

$$\lambda(f^*) = \sum_{s\in\Gamma}\overline{f(s)}\lambda({\delta_s}^*) = \sum_{s\in\Gamma}\overline{f(s)}(\lambda(\delta_s))^* = (\lambda(f))^*.$$

Since $\lambda(\delta_s)$ is a unitary operator for all $s\in\Gamma$, we see that

$$\|\lambda(f)\| = \left\|\sum_{s\in\Gamma} f(s)\lambda(\delta_s)\right\| \leqslant \sum_{s\in\Gamma}|f(s)| = \|f\|_1$$

for any

$$f=\sum_{s\in\Gamma}f(s)\delta_s\in C_c(\Gamma,\sigma).$$

Thus, for any $f\in C_c(\Gamma,\sigma)$, we have $\lambda(f)\in B(\ell^2(\Gamma))$. We define

$$\|f\|=\|\lambda(f)\|,\quad f\in C_c(\Gamma,\sigma).$$

Then $\|\cdot\|$ is a C^*-norm on the $*$-algebra $C_c(\Gamma,\sigma)$. The *σ-twisted reduced group C^*-algebra* $C_r^*(\Gamma,\sigma)$ of Γ is the completion of $C_c(\Gamma,\sigma)$ for the norm $\|\cdot\|$. The map λ extends uniquely to a $*$-isomorphism, still denoted by λ, of $C_r^*(\Gamma,\sigma)$ into $B(\ell^2(\Gamma))$. In particular, if $\sigma\equiv 1$, $C_r^*(\Gamma,\sigma)$ is the reduced group C^*-algebra $C_r^*(\Gamma)$.

3 Two seminorms on twisted group algebras

Definition 3.1. A *length function* on a discrete group Γ is a function

$$\ell:\Gamma\mapsto[0,\infty)$$

such that

(i) $\ell(s)=0$ if and only if $s=e$, where e is the identity of Γ;

(ii) $\ell(s)=\ell(s^{-1})$ for all $s\in\Gamma$;

(iii) $\ell(s_1s_2)\leqslant\ell(s_1)+\ell(s_2)$ for all $s_1,s_2\in\Gamma$.

The length function ℓ is said to be *proper* if the set $\ell^{-1}([0,\beta])$ is finite for any real number $\beta>0$.

Let ℓ be a length function on a discrete group Γ, and let σ be a 2-cocycle on Γ. Let M_ℓ denote the (usually unbounded) operator on $\ell^2(\Gamma)$ of pointwise multiplication by the length function ℓ. We define the derivation Δ by the "Dirac" operator $D=M_\ell$ as

$$\Delta(\lambda(a))=[D,\lambda(a)]=[M_\ell,\lambda(a)]$$

for all $a\in C_r^*(\Gamma,\sigma)$. From this derivation, we can define a seminorm on the $*$-algebra $C_c(\Gamma,\sigma)$ as follows.

Definition 3.2. On the $*$-algebra $C_c(\Gamma,\sigma)$, we define

$$L_D(a)=\|\Delta(\lambda(a))\|=\|[D,\lambda(a)]\|$$

for any $a\in C_c(\Gamma,\sigma)$.

From the proof of Proposition 3.3 in [10], we have $L_D(\delta_s)=\|[D,\lambda_s]\|=\ell(s)$ for any $s\in\Gamma$, and thus $L_D(f)=\|[D,\lambda(f)]\|<\infty$ for any $f\in C_c(\Gamma,\sigma)$. It is clear that L_D is a seminorm on $C_c(\Gamma,\sigma)$.

For any $h\in\ell^\infty(\Gamma)$, let M_h denote the operator on $\ell^2(\Gamma)$ of pointwise multiplication by h. If E is a subset of Γ, let M_E denote M_{χ_E}, where χ_E is the characteristic function of E, and thus M_E is a projection on $\ell^2(\Gamma)$. For any $\alpha>0$, we set $B(\alpha)=\{r\in\Gamma:\ell(r)\leqslant\alpha\}$, and in this special case, we set $M_\alpha=M_{B(\alpha)}$.

For any $s,t\in\Gamma$ and $\xi\in\ell^2(\Gamma)$, we have

$$\begin{aligned}([D,\lambda_s](\xi))(t)&=((D\lambda_s-\lambda_sD)(\xi))(t)\\&=D(\lambda_s(\xi))(t)-\lambda_s(D(\xi))(t)\\&=\ell(t)(\lambda_s(\xi))(t)-(D(\xi))(s^{-1}t)\sigma(s,s^{-1}t)\\&=\ell(t)\xi(s^{-1}t)\sigma(s,s^{-1}t)-\ell(s^{-1}t)\xi(s^{-1}t)\sigma(s,s^{-1}t)\\&=(\ell(t)-\ell(s^{-1}t))\xi(s^{-1}t)\sigma(s,s^{-1}t).\end{aligned}$$

In particular, when $\xi=\delta_t$, we have

$$[D,\lambda_s](\delta_t)=(\ell(st)-\ell(t))\sigma(s,t)\delta_{st}$$

for all $s,t\in\Gamma$. Thus, for any $t\in\Gamma$, $f\in C_c(\Gamma,\sigma)$ and $\xi\in\ell^2(\Gamma)$, we have

$$\begin{aligned}[D,\lambda(f)](\xi)(t)&=\left[D,\sum_{s\in\Gamma}f(s)\lambda_s\right](\xi)(t)\\&=\sum_{s\in\Gamma}f(s)[D,\lambda_s](\xi)(t)\\&=\sum_{s\in\Gamma}f(s)(\ell(t)-\ell(s^{-1}t))\xi(s^{-1}t)\sigma(s,s^{-1}t)\\&=\sum_{r\in\Gamma}f(tr^{-1})(\ell(t)-\ell(r))\sigma(tr^{-1},r)\xi(r),\end{aligned}$$

and thus

$$[D,\lambda(f)](\delta_t)=\sum_{s\in\Gamma}f(s)[D,\lambda_s](\delta_t)=\sum_{s\in\Gamma}f(s)(\ell(st)-\ell(t))\sigma(s,t)\delta_{st}. \tag{3.1}$$

It is much more convenient to regard the operators $\lambda(f)$ and $[D,\lambda(f)]$ for any $f\in C_c(\Gamma,\sigma)$ as integral operators by means of the kernel functions. More concretely, the kernel function $\lambda(f)(r,s)$ for the operator $\lambda(f)$ is

$$\lambda(f)(r,s)=f(rs^{-1})\sigma(rs^{-1},s)$$

for any $r,s\in\Gamma$, and the kernel function $[D,\lambda(f)](r,s)$ for the operator $[D,\lambda(f)]$ is

$$[D,\lambda(f)](r,s)=(\ell(r)-\ell(s))\sigma(rs^{-1},s)f(rs^{-1})$$

for any $r,s\in\Gamma$. Thus, if $\ell(r)\neq\ell(s)$, then

$$f(rs^{-1})\sigma(rs^{-1},s)=(\ell(r)-\ell(s))^{-1}[D,\lambda(f)](r,s).$$

Proposition 3.3. *If ℓ is proper, then for any $f\in C_c(\Gamma,\sigma)$ and any $\alpha,\beta\in[0,\infty)$ with $\beta>\alpha\geqslant 0$, we have*

$$\|(I-M_\beta)\lambda(f)M_\alpha\|\leqslant(\beta-\alpha)^{-1}L_D(f).$$

Proof. For any $\xi\in\ell^2(\Gamma)$ and $r\in\Gamma$ with $\ell(r)>\beta$, we have

$$\begin{aligned}((I-M_\beta)\lambda(f)M_\alpha(\xi))(r)&=(1-\chi_{B(\beta)})(r)(\lambda(f)M_\alpha(\xi))(r)\\&=(\lambda(f)M_\alpha(\xi))(r)\\&=\sum_{s\in\Gamma}f(rs^{-1})\sigma(rs^{-1},s)(M_\alpha(\xi))(s)\\&=\sum_{s\in\Gamma}f(rs^{-1})\sigma(rs^{-1},s)\chi_{B(\alpha)}(s)\xi(s)\\&=\sum_{s\in B(\alpha)}(\ell(r)-\ell(s))^{-1}[D,\lambda(f)](r,s)\xi(s).\end{aligned}$$

For any $r,s\in\Gamma$ with $\ell(r)>\ell(s)$, we have

$$\begin{aligned}(\ell(r)-\ell(s))^{-1}&=\ell(r)^{-1}(1-\ell(s)/\ell(r))^{-1}\\&=\ell(r)^{-1}\sum_{k=0}^{\infty}\ell(s)^k\ell(r)^{-k};\end{aligned}$$

therefore,

$$((I-M_\beta)\lambda(f)M_\alpha(\xi))(r)=\sum_{s\in B(\alpha)}\ell(r)^{-1}\sum_{k=0}^{\infty}\ell(s)^k\ell(r)^{-k}[D,\lambda(f)](r,s)\xi(s)$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \sum_{s\in B(\alpha)} \ell(r)^{-1-k}[D,\lambda(f)](r,s)\ell(s)^k\xi(s)\\
&= \sum_{k=0}^{\infty} \ell(r)^{-1-k} \sum_{s\in\Gamma}[D,\lambda(f)](r,s)(D^kM_\alpha(\xi))(s)\\
&= \sum_{k=0}^{\infty} \ell(r)^{-1-k}([D,\lambda(f)]D^kM_\alpha(\xi))(r)\\
&= \left(\left(\sum_{k=0}^{\infty} D^{-1-k}(I-M_\beta)[D,\lambda(f)]D^kM_\alpha\right)(\xi)\right)(r).
\end{aligned}
$$

Thus, as operators from $M_\alpha(\ell^2(\Gamma))$ to $(I-M_\beta)(\ell^2(\Gamma))$, we have

$$(I-M_\beta)\lambda(f)M_\alpha = \sum_{k=0}^{\infty} D^{-1-k}(I-M_\beta)[D,\lambda(f)]D^kM_\alpha.$$

Since $\|D^{-1-k}(I-M_\beta)\| \leqslant \beta^{-1-k}$ and $\|[D,\lambda(f)]D^kM_\alpha\| \leqslant \alpha^k L_D(f)$, we have

$$\|(I-M_\beta)\lambda(f)M_\alpha\| \leqslant \beta^{-1}\sum_{k=0}^{\infty}\left(\frac{\alpha}{\beta}\right)^k L_D(f) = (\beta-\alpha)^{-1}L_D(f).$$

This completes the proof. □

From Proposition 3.3, we see that if ℓ is proper and $\theta > 1$, then for any $f \in C_c(\Gamma,\sigma)$ and any $\alpha > 0$, we have

$$(\theta-1)\alpha\|(I-M_{\theta\alpha})\lambda(f)M_\alpha\| \leqslant L_D(f),$$

which leads to the next definition.

Definition 3.4. If $\theta > 1$ and ℓ is proper, we define

$$J_{D,\theta}(f) = \sup\{(\theta-1)\alpha\|(I-M_{\theta\alpha})\lambda(f)M_\alpha\| : \alpha > 0\}$$

for any $f \in C_c(\Gamma,\sigma)$.

Proposition 3.5. *If ℓ is proper, then*

(i) *$J_{D,\theta}(f) \leqslant L_D(f)$ for any $f \in C_c(\Gamma,\sigma)$;*

(ii) *$J_{D,\theta}$ is a seminorm on $C_c(\Gamma,\sigma)$;*

(iii) *$J_{D,\theta}$ is a norm on W, where*

$$W = \{f \in C_c(\Gamma,\sigma) : \mathrm{tr}(\lambda(f)) = \langle\lambda(f)(\delta_e),\delta_e\rangle = 0\}.$$

Proof. (i) and (ii) are clear.

(iii) It is clear that W is a subspace of $C_c(\Gamma,\sigma)$. For any $f \in C_c(\Gamma,\sigma)$ and any $\alpha > 0$, we have

$$
\begin{aligned}
((I-M_{\theta\alpha})\lambda(f)M_\alpha)(\delta_e) &= (I-M_{\theta\alpha})(\lambda(f)(\delta_e))\\
&= (I-M_{\theta\alpha})\left(\sum_{r\in\Gamma} f(r)\sigma(r,e)\delta_r\right)\\
&= \sum_{\ell(r)>\theta\alpha} f(r)\sigma(r,e)\delta_r.
\end{aligned}
$$

If $f_0 \in W$ and $f_0 \neq 0$, then there is an $r_0 \in \Gamma$ such that $f_0(r_0) \neq 0$ and $r_0 \neq e$. Since $\ell(r_0) \neq 0$, we may choose an $\alpha_0 > 0$ such that $\theta\alpha_0 < \ell(r_0)$. Thus,

$$((I-M_{\theta\alpha_0})\lambda(f_0)M_{\alpha_0})(\delta_e) = \sum_{\ell(r)>\theta\alpha_0} f_0(r)\sigma(r,e)\delta_r \neq 0.$$

Eventually, $J_{D,\theta}(f_0) \neq 0$. □

Let σ_1 and σ_2 be two cohomologous 2-cocycles on Γ. Suppose ρ is the map such that

$$\sigma_1(s,t) = \rho(s)\rho(t)\overline{\rho(st)}\sigma_2(s,t)$$

for all $s, t \in \Gamma$. Then we can define a unitary operator U on $\ell^2(\Gamma)$ by

$$U : \xi = \sum_{s\in\Gamma} \xi(s)\delta_s \in \ell^2(\Gamma) \mapsto \sum_{s\in\Gamma} \xi(s)\rho(s)\delta_s \in \ell^2(\Gamma).$$

Let λ^{σ_1} be the left regular σ_1-projective representation of $C_c(\Gamma, \sigma_1)$ on $\ell^2(\Gamma)$, and let λ^{σ_2} be the left regular σ_2-projective representation of $C_c(\Gamma, \sigma_2)$ on $\ell^2(\Gamma)$. For any $\xi \in \ell^2(\Gamma)$ and $s, t \in \Gamma$, we have

$$\begin{aligned}(U\lambda_s^{\sigma_1}U^*(\xi))(t) &= \rho(t)(\lambda_s^{\sigma_1}U^*(\xi))(t) \\ &= \rho(t)\sigma_1(s, s^{-1}t)(U^*(\xi))(s^{-1}t) \\ &= \rho(t)\sigma_1(s, s^{-1}t)\overline{\rho(s^{-1}t)}\xi(s^{-1}t) \\ &= \rho(s)\sigma_2(s, s^{-1}t)\xi(s^{-1}t) \\ &= ((\rho(s)\lambda_s^{\sigma_2})(\xi))(t).\end{aligned}$$

Thus, we can define the map $\Phi : C_c(\Gamma, \sigma_1) \mapsto C_c(\Gamma, \sigma_2)$ by

$$\Phi(\lambda_s^{\sigma_1}) = U\lambda_s^{\sigma_1}U^* = \rho(s)\lambda_s^{\sigma_2} \tag{3.2}$$

for all $s \in \Gamma$. As the proof of Proposition 5.1 in [9], we can see that Φ is an isometrical $*$-isomorphism from $C_c(\Gamma, \sigma_1)$ onto $C_c(\Gamma, \sigma_2)$ induced by the unitary operator U. By density, Φ can be extended to a $*$-isomorphism between $C_r^*(\Gamma, \sigma_1)$ and $C_r^*(\Gamma, \sigma_2)$. Since U is a pointwise multiplication operator on $\ell^2(\Gamma)$ by ρ, we have

$$\begin{aligned}\|(I - M_{\theta\alpha})\Phi(a)M_\alpha\| &= \|(I - M_{\theta\alpha})U\lambda^{\sigma_1}(a)U^*M_\alpha\| \\ &= \|U(I - M_{\theta\alpha})\lambda^{\sigma_1}(a)M_\alpha U^*\| \\ &= \|(I - M_{\theta\alpha})\lambda^{\sigma_1}(a)M_\alpha\|\end{aligned}$$

for all $a \in C_r^*(\Gamma, \sigma_1)$ and $\alpha > 0$. Hence, for any $f \in C_c(\Gamma, \sigma_1)$, we have

$$J_{D,\theta}^{\sigma_2}(\Phi(f)) = \sup\{(\theta - 1)\alpha\|(I - M_{\theta\alpha})\Phi(f)M_\alpha\| : \alpha > 0\} = J_{D,\theta}^{\sigma_1}(f).$$

This completes the proof of the following proposition.

Proposition 3.6. *Let Γ be a discrete group endowed with a proper length function ℓ. If σ_1 and σ_2 are two cohomologous 2-cocycles on Γ, then there exists an isometrical $*$-isomorphism Φ from $C_c(\Gamma, \sigma_1)$ onto $C_c(\Gamma, \sigma_2)$ such that $J_{D,\theta}^{\sigma_2}(\Phi(f)) = J_{D,\theta}^{\sigma_1}(f)$ for any $f \in C_c(\Gamma, \sigma_1)$.*

4 Truncations

Let ℓ be a length function on a discrete group Γ, and let σ be a 2-cocycle on Γ. Let ρ be the right regular $\overline{\sigma}$-projective representation of Γ on $\ell^2(\Gamma)$, defined by

$$(\rho_t(\xi))(r) = \overline{\sigma(r,t)}\xi(rt)$$

for any $r, t \in \Gamma$ and $\xi \in \ell^2(\Gamma)$. Then ρ commutes with λ (see [9]). For any $h \in C_c(\Gamma)$, we define $\widetilde{h}(r) = h(r^{-1})$ for $r \in \Gamma$.

Proposition 4.1. *For any $h, k \in C_c(\Gamma)$ and $f \in C_c(\Gamma, \sigma)$, we have*

$$\lambda((h^* * k)f) = \sum_{t\in\Gamma} \rho_t M_{\widetilde{h}}^* \lambda(f) M_{\widetilde{k}} \rho_t^*, \tag{4.1}$$

where this sum converges for the weak operator topology. Furthermore,

$$\|\lambda((h^* * k)f)\| \leqslant \|\lambda(f)\|\|h\|_2\|k\|_2.$$

Proof. From the definition of usual convolution, we have

$$\begin{aligned}(h^* * k)(sr^{-1}) &= \sum_{t\in\Gamma} h^*(t)k(t^{-1}sr^{-1}) \\ &= \sum_{t\in\Gamma} \overline{h(t^{-1})}k(t^{-1}sr^{-1}) \\ &= \sum_{t\in\Gamma} \overline{h(t^{-1}s^{-1})}k(t^{-1}r^{-1})\end{aligned}$$

for $r, s \in \Gamma$. It follows that for any $\xi, \eta \in C_c(\Gamma)$, we have

$$\begin{aligned}&\langle \lambda((h^* * k)f)(\xi), \eta\rangle \\ &\quad= \sum_{s\in\Gamma} (\lambda((h^* * k)f)(\xi))(s)\overline{\eta(s)} \\ &\quad= \sum_{s\in\Gamma}\sum_{r\in\Gamma} (h^* * k)(sr^{-1})f(sr^{-1})\sigma(sr^{-1}, r)\xi(r)\overline{\eta(s)} \\ &\quad= \sum_{s\in\Gamma}\sum_{r\in\Gamma}\sum_{t\in\Gamma} \overline{h(t^{-1}s^{-1})}k(t^{-1}r^{-1})f(sr^{-1})\sigma(sr^{-1}, r)\xi(r)\overline{\eta(s)} \\ &\quad= \sum_{s\in\Gamma}\sum_{r\in\Gamma}\sum_{t\in\Gamma} \overline{h(t^{-1}s^{-1})}k(r^{-1})f(str^{-1})\sigma(str^{-1}, rt^{-1})\xi(rt^{-1})\overline{\eta(s)} \\ &\quad= \sum_{s\in\Gamma}\sum_{r\in\Gamma}\sum_{t\in\Gamma} f(sr^{-1})k(r^{-1})\sigma(sr^{-1}, rt^{-1})\xi(rt^{-1})\overline{h(s^{-1})\eta(st^{-1})} \\ &\quad= \sum_{t\in\Gamma}\sum_{s\in\Gamma}\left(\sum_{r\in\Gamma} f(sr^{-1})\sigma(sr^{-1}, r)k(r^{-1})\sigma(rt^{-1}, t)\xi(rt^{-1})\right) \\ &\qquad\times \overline{(h(s^{-1})\sigma(st^{-1}, t)\eta(st^{-1}))} \\ &\quad= \sum_{t\in\Gamma}\sum_{s\in\Gamma}\left(\sum_{r\in\Gamma} f(sr^{-1})\sigma(sr^{-1}, r)k(r^{-1})(\rho_t^*(\xi))(r)\right)\overline{(M_{\widetilde{h}}\rho_t^*(\eta))(s)} \\ &\quad= \sum_{t\in\Gamma}\sum_{s\in\Gamma} (\lambda(f)M_{\widetilde{k}}\rho_t^*(\xi))(s)\overline{(M_{\widetilde{h}}\rho_t^*(\eta))(s)} \\ &\quad= \sum_{t\in\Gamma} \langle \lambda(f)M_{\widetilde{k}}\rho_t^*(\xi), M_{\widetilde{h}}\rho_t^*(\eta)\rangle \\ &\quad= \sum_{t\in\Gamma} \langle \rho_t M_{\widetilde{h}}^*\lambda(f)M_{\widetilde{k}}\rho_t^*(\xi), \eta\rangle.\end{aligned}$$

Since

$$\begin{aligned}\sum_{t\in\Gamma} \|M_{\widetilde{k}}\rho_t^*(\xi)\|_2^2 &= \sum_{t\in\Gamma}\sum_{r\in\Gamma} |(M_{\widetilde{k}}\rho_t^*(\xi))(r)|^2 \\ &= \sum_{t\in\Gamma}\sum_{r\in\Gamma} |k(r^{-1})\sigma(rt^{-1}, t)\xi(rt^{-1})|^2 \\ &= \sum_{r\in\Gamma} |k(r^{-1})|^2\|\xi\|_2^2 \\ &= \|k\|_2^2\|\xi\|_2^2,\end{aligned}$$

the same steps employed for $M_{\widetilde{h}}\rho_t^*(\eta)$ and the Cauchy-Schwarz inequality lead to

$$\sum_{t\in\Gamma} |\langle \lambda(f)M_{\widetilde{k}}\rho_t^*(\xi), M_{\widetilde{h}}\rho_t^*(\eta)\rangle| \leqslant \|\lambda(f)\|\|h\|_2\|k\|_2\|\xi\|_2\|\eta\|_2.$$

This implies the convergence of the series (4.1) in the proposition for the weak operator topology and the stated norm inequality. □

Proposition 4.2. *For any $h,k\in C_c(\Gamma)$ and $f\in C_c(\Gamma,\sigma)$, we have*

$$L_D((h^**k)f)\leqslant \|h\|_2\|k\|_2L_D(f).$$

Proof. Since $(h^**k)f$ has finite support, $[D,\lambda((h^**k)f)]$ is a bounded operator. For any $\xi,\eta\in C_c(\Gamma)$, we have

$$\langle[D,\lambda((h^**k)f)](\xi),\eta\rangle=\langle\lambda((h^**k)f)(\xi),D(\eta)\rangle-\langle\lambda((h^**k)f)D(\xi),\eta\rangle.$$

By Proposition 4.1, we obtain

$$\begin{aligned}&\langle[D,\lambda((h^**k)f)](\xi),\eta\rangle\\&\quad=\sum_{t\in\Gamma}(\langle\rho_tM_{\widetilde{h}}^*\lambda(f)M_{\widetilde{k}}\rho_t^*(\xi),D(\eta)\rangle-\langle\rho_tM_{\widetilde{h}}^*\lambda(f)M_{\widetilde{k}}\rho_t^*D(\xi),\eta\rangle).\end{aligned}$$

For any $t\in\Gamma$, let $\tau_t:C_c(\Gamma)\mapsto C_c(\Gamma)$ be the right translation given by

$$\tau_t(f)(s)=f(st),\quad s\in\Gamma,\quad f\in C_c(\Gamma).$$

Thus, for any $t,r\in\Gamma$ and $\xi\in\ell^2(\Gamma)$, we have

$$\begin{aligned}(\rho_tM_{\widetilde{k}}\rho_t^*(\xi))(r)&=(M_{\widetilde{k}}\rho_t^*(\xi))(rt)\overline{\sigma(r,t)}\\&=\widetilde{k}(rt)(\rho_t^*(\xi))(rt)\overline{\sigma(r,t)}\\&=\widetilde{k}(rt)\xi(r)\\&=\tau_t(\widetilde{k})(r)\xi(r)\\&=(M_{\tau_t(\widetilde{k})}(\xi))(r).\end{aligned}$$

Following the same approach for $M_{\widetilde{h}}$, we have

$$\rho_tM_{\widetilde{k}}\rho_t^*=M_{\tau_t(\widetilde{k})}\quad\text{and}\quad\rho_tM_{\widetilde{h}}\rho_t^*=M_{\tau_t(\widetilde{h})}$$

for any $t\in\Gamma$. Then we have

$$\begin{aligned}&\langle[D,\lambda((h^**k)f)](\xi),\eta\rangle\\&\quad=\sum_{t\in\Gamma}(\langle D\rho_tM_{\widetilde{h}}^*\rho_t^*\lambda(f)\rho_tM_{\widetilde{k}}\rho_t^*(\xi),\eta\rangle-\langle\rho_tM_{\widetilde{h}}^*\rho_t^*\lambda(f)\rho_tM_{\widetilde{k}}\rho_t^*D(\xi),\eta\rangle)\\&\quad=\sum_{t\in\Gamma}(\langle D\lambda(f)M_{\tau_t(\widetilde{k})}(\xi),M_{\tau_t(\widetilde{h})}(\eta)\rangle-\langle\lambda(f)DM_{\tau_t(\widetilde{k})}(\xi),M_{\tau_t(\widetilde{h})}(\eta)\rangle)\\&\quad=\sum_{t\in\Gamma}\langle[D,\lambda(f)]M_{\tau_t(\widetilde{k})}(\xi),M_{\tau_t(\widetilde{h})}(\eta)\rangle,\end{aligned}$$

since λ commutes with ρ and $M_hD=DM_h$ for any $h\in\ell^\infty(\Gamma)$. Because

$$\begin{aligned}\sum_{t\in\Gamma}\|M_{\tau_t(\widetilde{k})}(\xi)\|_2^2&=\sum_{t\in\Gamma}\sum_{r\in\Gamma}|\tau_t(\widetilde{k})(r)|^2|\xi(r)|^2\\&=\sum_{r\in\Gamma}\left(\sum_{t\in\Gamma}|\tau_t(\widetilde{k})(r)|^2\right)|\xi(r)|^2\\&=\sum_{r\in\Gamma}\|k\|_2^2|\xi(r)|^2\\&=\|k\|_2^2\|\xi\|_2^2,\end{aligned}$$

similarly for the term $M_{\tau_t(\widetilde{h})}(\eta)$, we have

$$|\langle[D,\lambda((h^**k)f)]\xi,\eta\rangle|\leqslant L_D(f)\|h\|_2\|k\|_2\|\xi\|_2\|\eta\|_2.$$

This completes the proof. □

For the remainder of the section, we fix a real number $\theta > 1$ and assume that ℓ is a proper length function.

Proposition 4.3. *Let $\alpha > 0$. Suppose $h \in C_c(\Gamma)$ with support in $\Gamma \backslash B(\theta\alpha)$ and $k \in C_c(\Gamma)$ with support in $B(\alpha)$. Then, for any $f \in C_c(\Gamma, \sigma)$, we have*

$$\|\lambda((h^* * k)f)\| \leqslant \frac{1}{(\theta-1)\alpha}\|h\|_2\|k\|_2 J_{D,\theta}(f).$$

Proof. For any $\xi, \eta \in \ell^2(\Gamma)$, we have

$$\begin{aligned}|\langle\lambda((h^* * k)f)(\xi),\eta\rangle| &= \left|\sum_{t\in\Gamma}\langle\rho_t M_{\widetilde{h}}^*\lambda(f)M_{\widetilde{k}}\rho_t^*(\xi),\eta\rangle\right| \\ &= \left|\sum_{t\in\Gamma}\langle\rho_t M_{\widetilde{h}}^*(I-M_{\theta\alpha})\lambda(f)M_\alpha M_{\widetilde{k}}\rho_t^*(\xi),\eta\rangle\right|\end{aligned}$$

by Proposition 4.1, and thus

$$\begin{aligned}&|\langle\lambda((h^* * k)f)(\xi),\eta\rangle| \\ &\quad\leqslant \frac{1}{(\theta-1)\alpha}J_{D,\theta}(f)\sum_{t\in\Gamma}\|M_{\widetilde{k}}\rho_t^*(\xi)\|_2\|M_{\widetilde{h}}\rho_t^*(\eta)\|_2 \\ &\quad\leqslant \frac{1}{(\theta-1)\alpha}\left(\sum_{t\in\Gamma}\|M_{\widetilde{k}}\rho_t^*(\xi)\|_2^2\right)^{\frac{1}{2}}\left(\sum_{t\in\Gamma}\|M_{\widetilde{h}}\rho_t^*(\eta)\|_2^2\right)^{\frac{1}{2}}J_{D,\theta}(f) \\ &\quad= \frac{1}{(\theta-1)\alpha}\|h\|_2\|k\|_2\|\xi\|_2\|\eta\|_2 J_{D,\theta}(f)\end{aligned}$$

for any $\xi, \eta \in C_c(\Gamma)$. Hence, $\|\lambda((h^* * k)f)\| \leqslant \frac{1}{(\theta-1)\alpha}\|h\|_2\|k\|_2 J_{D,\theta}(f)$. □

Proposition 4.4. *Let $h, k \in C_c(\Gamma)$ and $f \in C_c(\Gamma, \sigma)$. Then*

$$J_{D,\theta}((h^* * k)f) \leqslant \|h\|_2\|k\|_2 J_{D,\theta}(f).$$

Proof. For any $\alpha > 0$ and any $\xi, \eta \in \ell^2(\Gamma)$, we have

$$\begin{aligned}&|\langle(I-M_{\theta\alpha})\lambda((h^* * k)f)M_\alpha(\xi),\eta\rangle| \\ &\quad= |\langle\lambda((h^* * k)f)M_\alpha(\xi),(I-M_{\theta\alpha})(\eta)\rangle| \\ &\quad= \left|\sum_{t\in\Gamma}\langle\lambda(f)M_{\widetilde{k}}\rho_t^*M_\alpha(\xi),M_{\widetilde{h}}\rho_t^*(I-M_{\theta\alpha})(\eta)\rangle\right|\end{aligned}$$

by Proposition 4.1. Since for any $t \in \Gamma$, ρ_t commutes with $\lambda(f)$, from the proof of Proposition 4.2, we have

$$\rho_t M_{\widetilde{k}}\rho_t^* M_\alpha = M_\alpha\rho_t M_{\widetilde{k}}\rho_t^*$$

and

$$\rho_t M_{\widetilde{h}}\rho_t^* M_{\theta\alpha} = M_{\theta\alpha}\rho_t M_{\widetilde{h}}\rho_t^*.$$

Thus, we obtain

$$\begin{aligned}&|\langle(I-M_{\theta\alpha})\lambda((h^* * k)f)M_\alpha(\xi),\eta\rangle| \\ &\quad= \left|\sum_{t\in\Gamma}\langle(I-M_{\theta\alpha})\lambda(f)M_\alpha\rho_t M_{\widetilde{k}}\rho_t^*(\xi),\rho_t M_{\widetilde{h}}\rho_t^*(\eta)\rangle\right| \\ &\quad\leqslant \sum_{t\in\Gamma}|\langle(I-M_{\theta\alpha})\lambda(f)M_\alpha\rho_t M_{\widetilde{k}}\rho_t^*(\xi),\rho_t M_{\widetilde{h}}\rho_t^*(\eta)\rangle| \\ &\quad\leqslant \frac{1}{(\theta-1)\alpha}\sum_{t\in\Gamma}\|\rho_t M_{\widetilde{k}}\rho_t^*(\xi)\|_2\|\rho_t M_{\widetilde{h}}\rho_t^*(\eta)\|_2 J_{D,\theta}(f)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(\theta-1)\alpha} J_{D,\theta}(f) \sum_{t\in\Gamma} \|M_{\widetilde{k}}\rho_t^*(\xi)\|_2 \|M_{\widetilde{h}}\rho_t^*(\eta)\|_2 \\
&\leqslant \frac{1}{(\theta-1)\alpha} \|h\|_2 \|k\|_2 J_{D,\theta}(f) \|\xi\|_2 \|\eta\|_2.
\end{aligned}
$$

The proof is complete. □

For a set E, we denote $|E|$ for its cardinality.

Corollary 4.5. *Let $\alpha > 0$. Suppose $E \subset B(\alpha)$ and $F \subset \Gamma \backslash B(\theta\alpha)$, and set $k = \chi_E$ and $h = \chi_F$. Then, for any $f \in C_c(\Gamma, \sigma)$, we have*

$$
\|\lambda((h^* * k)f)\| \leqslant \frac{1}{(\theta-1)\alpha} |E|^{1/2} |F|^{1/2} J_{D,\theta}(f)
$$

and

$$
J_{D,\theta}((h^* * k)f) \leqslant |E|^{1/2} |F|^{1/2} J_{D,\theta}(f).
$$

Proof. The conclusion follows from Propositions 4.3 and 4.4. □

For $\gamma > \beta > 0$ let $A(\beta, \gamma)$ denote the annulus

$$
B(\gamma) \backslash B(\beta) = \{r \in \Gamma : \beta < \ell(r) \leqslant \gamma\}.
$$

Corollary 4.6. *For given $\gamma > \beta \geqslant \theta\alpha > 0$, let $k = |B(\alpha)|^{-1} \chi_{B(\alpha)}$ and $h = \chi_{A(\beta,\gamma)}$, and let $g = h^* * k$. Then, for any $f \in C_c(\Gamma, \sigma)$ we have*

$$
\|\lambda(gf)\| \leqslant \frac{1}{(\theta-1)\alpha} (|B(\alpha)|^{-1} |B(\gamma)|)^{1/2} J_{D,\theta}(f)
$$

and

$$
J_{D,\theta}(gf) \leqslant (|B(\alpha)|^{-1} |B(\gamma)|)^{1/2} J_{D,\theta}(f).
$$

Proof. The result follows from Corollary 4.5. □

Lemma 4.7. *For given $\gamma > \beta \geqslant \theta\alpha > 0$, let $k = |B(\alpha)|^{-1} \chi_{B(\alpha)}$ and $h = \chi_{A(\beta,\gamma)}$, and let $g = h^* * k$. We have*

(i) $0 \leqslant g \leqslant 1$;

(ii) *if $g(r) \neq 0$, then $r \in A(\beta - \alpha, \gamma + \alpha)$; and*

(iii) *if $r \in A(\beta + \alpha, \gamma - \alpha)$, then $g(r) = 1$.*

Proof. From the definition of the usual convolution, we have

$$
\begin{aligned}
g(r) &= (h^* * k)(r) \\
&= \sum_{s\in\Gamma} h^*(rs^{-1}) k(s) \\
&= \sum_{s\in\Gamma} \chi_{A(\beta,\gamma)}(sr^{-1}) |B(\alpha)|^{-1} \chi_{B(\alpha)}(s)
\end{aligned}
$$

for any $r \in \Gamma$. Thus, we have $0 \leqslant g \leqslant 1$.

If $g(r) \neq 0$, there is an $s \in \Gamma$ such that $s \in B(\alpha)$ and $sr^{-1} \in A(\beta, \gamma)$. Hence, we have

$$
\ell(r) = \ell(s^{-1} s r^{-1}) \leqslant \ell(s^{-1}) + \ell(sr^{-1}) \leqslant \alpha + \gamma
$$

and

$$
\beta - \alpha < \ell(sr^{-1}) - \ell(s) \leqslant \ell(r^{-1}) = \ell(r).
$$

If $r \in A(\beta + \alpha, \gamma - \alpha)$, then $\beta + \alpha < \ell(r) \leqslant \gamma - \alpha$. For any $s \in B(\alpha)$, we have

$$
\beta = \beta + \alpha - \alpha
$$

$$
\begin{aligned}
&< \ell(r) - \ell(s^{-1}) \leqslant \ell(rs) \\
&\leqslant \ell(r) + \ell(s) = \gamma - \alpha + \alpha = \gamma,
\end{aligned}
$$

and thus $g(r) = 1$. □

Proposition 4.8. *Suppose $\gamma > \beta \geqslant (\theta+1)\alpha > 0$, and $f \in C_c(\Gamma, \sigma)$ vanishes identically on both the annuli $A(\beta - 2\alpha, \beta)$ and $A(\gamma, \gamma + 2\alpha)$. Then*

$$\|\lambda(f\chi_{A(\beta,\gamma)})\| \leqslant \frac{1}{(\theta-1)\alpha}(|B(\alpha)|^{-1}|B(\gamma+\alpha)|)^{1/2} J_{D,\theta}(f)$$

and

$$J_{D,\theta}(f\chi_{A(\beta,\gamma)}) \leqslant (|B(\alpha)|^{-1}|B(\gamma+\alpha)|)^{1/2} J_{D,\theta}(f).$$

Proof. Since $\gamma > \beta \geqslant (\theta+1)\alpha > 0$, we have

$$\gamma + \alpha > \beta - \alpha \geqslant \theta\alpha > 0.$$

Let $k = |B(\alpha)|^{-1}\chi_{B(\alpha)}$ and $h = \chi_{A(\beta-\alpha,\gamma+\alpha)}$, and let $g = h^* * k$. Because $f \in C_c(\Gamma, \sigma)$ vanishes identically on both the annuli $A((\beta-\alpha)-\alpha, (\beta-\alpha)+\alpha)$ and $A((\gamma+\alpha)-\alpha, (\gamma+\alpha)+\alpha)$, by Lemma 4.7 we have

$$gf = (h^* * k)f = \chi_{A(\beta,\gamma)} f,$$

and hence by Corollary 4.6 we get

$$\|\lambda(f\chi_{A(\beta,\gamma)})\| \leqslant \frac{1}{(\theta-1)\alpha}(|B(\alpha)|^{-1}|B(\gamma+\alpha)|)^{1/2} J_{D,\theta}(f)$$

and

$$J_{D,\theta}(f\chi_{A(\beta,\gamma)}) \leqslant (|B(\alpha)|^{-1}|B(\gamma+\alpha)|)^{1/2} J_{D,\theta}(f).$$

This completes the proof. □

5 Bounded dilation

Definition 5.1. Let Γ be a discrete group, let ℓ be a proper length function on Γ and let $\theta > 1$. We say that Γ has the *property of bounded θ-dilation* with respect to ℓ if there exists a constant $C_\ell < \infty$ such that

$$|B(\theta\alpha)| \leqslant C_\ell |B(\alpha)| \quad \text{for all} \quad \alpha \geqslant 1.$$

Let ℓ be a proper length function on a discrete group Γ with the property of bounded θ-dilation. Then, for any $\beta \geqslant 1$, we get

$$|B(\theta^k\beta)| \leqslant C_\ell^k |B(\beta)| \tag{5.1}$$

for each nonnegative integer k. If $1 \leqslant \beta \leqslant \alpha$, let k be the positive integer that satisfies

$$\theta^{k-1}\beta \leqslant \alpha < \theta^k\beta.$$

Then,

$$|B(\alpha)| \leqslant |B(\theta^k\beta)| \leqslant C_\ell^k |B(\beta)|$$

and $k - 1 \leqslant \log_\theta(\alpha/\beta)$. So

$$|B(\alpha)| \leqslant C_\ell^{1+\log_\theta(\alpha/\beta)} |B(\beta)|. \tag{5.2}$$

When $\beta = 1$, we see that ℓ has polynomial growth.

Let K_θ be the smallest positive integer such that

$$\theta^{3K} - 2\theta^{2K} - 1 > 0 \quad \text{and} \quad \frac{\theta}{\theta+1}\theta^{2K} - \theta^K - 2 > 0$$

for all $K \geqslant K_\theta$. Fix an integer $K \geqslant K_\theta$, and denote $R = \theta^K$. For any integers $m, n \geqslant 0$, we set

$$\widetilde{B}(n) = B(R^n) \quad \text{and} \quad \widetilde{A}(m,n) = A(R^m, R^n).$$

For $n \geqslant 1$, we set $k_n = |\widetilde{B}(n-1)|^{-1}\chi_{\widetilde{B}(n-1)}$ and $h_n = \chi_{\widetilde{A}(n,n+1)}$, and $g_n = h_n^* * k_n$. Then, $h_n^* = h_n$ and the support of g_n is contained in $A(R^n - R^{n-1}, R^{n+1} + R^{n-1})$. From the inequality (5.1), we obtain

$$|\widetilde{B}(n-1)|^{-1}|\widetilde{B}(n+1)| \leqslant C_\ell^{2K}.$$

Lemma 5.2. *For any $f \in C_c(\Gamma, \sigma)$ and $n \geqslant 1$, we have*

$$\|\lambda(g_n f)\| \leqslant C_1 R^{-n} J_{D,\theta}(f),$$

where $C_1 = RC_\ell^K/(\theta - 1)$.

Proof. By Corollary 4.6, we have

$$\begin{aligned}\|\lambda(g_n f)\| &\leqslant \frac{1}{(\theta-1)R^{n-1}}(|B(R^{n-1})|^{-1}|B(R^{n+1})|)^{1/2} J_{D,\theta}(f) \\ &\leqslant \frac{1}{\theta-1} R^{1-n} C_\ell^K J_{D,\theta}(f)\end{aligned}$$

for any $f \in C_c(\Gamma, \sigma)$ and $n \geqslant 1$. Set

$$C_1 = \frac{RC_\ell^K}{\theta - 1},$$

and we have the conclusion. □

Proposition 5.3. *For any integers $n, m \geqslant 1$ with $|n - m| \geqslant 2$, the supports of g_n and g_m are disjoint.*

Proof. Without loss of generality, we may assume that $n > m$. If $g_m(x) \neq 0$, then $\ell(x) \leqslant R^{m+1} + R^{m-1}$, while if $g_n(x) \neq 0$, then $R^n - R^{n-1} < \ell(x)$. Since $R \geqslant \theta^{K_\theta}$ and $n - m \geqslant 2$, we have

$$\begin{aligned}R^n - R^{n-1} - R^{m+1} - R^{m-1} &= R^{m-1}(R^{(n-m+1)} - R^{n-m} - R^2 - 1) \\ &\geqslant R^{m-1}(R^3 - 2R^2 - 1) > 0,\end{aligned}$$

which proves that the supports of g_n and g_m are disjoint. □

From this proposition, we see that for all $n \geqslant 1$, the supports of g_{2n} and $g_{2(n+1)}$ are disjoint. Thus, for any integer $N \geqslant 1$ and $f \in C_c(\Gamma, \sigma)$, we set

$$p_N^f = \sum_{n \geqslant N} (g_{2n} f).$$

Proposition 5.4. *For any integer $N \geqslant 1$ and $f \in C_c(\Gamma, \sigma)$, we have*

$$\|\lambda(p_N^f)\| \leqslant \frac{\theta^2}{\theta^2 - 1} C_1 R^{-2N} J_{D,\theta}(f).$$

Proof. For any $f \in C_c(\Gamma, \sigma)$, by Lemma 5.2 we have

$$\begin{aligned}\|\lambda(p_N^f)\| &= \bigg\| \sum_{n \geqslant N} \lambda(g_{2n} f) \bigg\| \\ &\leqslant \sum_{n \geqslant N} \|\lambda(g_{2n} f)\| \\ &\leqslant C_1 \sum_{n \geqslant N} R^{-2n} J_{D,\theta}(f) \\ &\leqslant \frac{1}{1 - \theta^{-2K_\theta}} C_1 R^{-2N} J_{D,\theta}(f)\end{aligned}$$

$$\begin{aligned}&\leqslant \frac{1}{1-\theta^{-2}} C_1 R^{-2N} J_{D,\theta}(f)\\ &= \frac{\theta^2}{\theta^2-1} C_1 R^{-2N} J_{D,\theta}(f)\end{aligned}$$

for any integer $N \geqslant 1$. □

Proposition 5.5. *For any integer $N \geqslant 1$, we have*

$$J_{D,\theta}(p_N^f) \leqslant C_2 J_{D,\theta}(f)$$

for any $f \in C_c(\Gamma, \sigma)$, where $C_2 = \frac{2R\theta^2}{\theta^2-1} C_1$.

Proof. For any $\alpha > 0$, let N_α be the smallest positive integer n such that

$$R^{2n+1} + R^{2n-1} \geqslant (\theta - 1)\alpha.$$

Then, for any $n < N_\alpha$, the annulus $A(R^{2n} - R^{2n-1}, R^{2n+1} + R^{2n-1})$ is contained in $B((\theta-1)\alpha)$. If $\xi \in C_c(\Gamma)$ has its support in $B(\alpha)$, then by the definition of the twisted convolution for any $n < N_\alpha$, the support of $\lambda(g_{2n} f)\xi$ is contained in $B(\theta\alpha)$, and thus

$$(I - M_{\theta\alpha})\lambda(g_{2n} f)\xi = 0.$$

Thus, for $n < N_\alpha$,

$$(I - M_{\theta\alpha})\lambda(g_{2n} f) M_\alpha = 0.$$

Consequently, by Lemma 5.2,

$$\begin{aligned}\|(I - M_{\theta\alpha})\lambda(p_N^f) M_\alpha\| &= \left\| \sum_{n \geqslant N_\alpha} (I - M_{\theta\alpha})\lambda(g_{2n} f) M_\alpha \right\|\\ &\leqslant \sum_{n \geqslant N_\alpha} \|\lambda(g_{2n} f)\|\\ &\leqslant \sum_{n \geqslant N_\alpha} C_1 R^{-2n} J_{D,\theta}(f)\\ &\leqslant \frac{\theta^2}{\theta^2-1} C_1 R^{-2N_\alpha} J_{D,\theta}(f).\end{aligned}$$

Now, from the definition of N_α, we have

$$(\theta - 1)\alpha \leqslant R^{2N_\alpha+1} + R^{2N_\alpha-1} \leqslant 2R^{2N_\alpha+1},$$

since $R \geqslant \theta > 1$. Thus, $R^{2N_\alpha} \geqslant (\theta-1)\alpha/(2R)$, and we obtain

$$\|(1 - M_{\theta\alpha})\lambda(p_N^f) M_\alpha\| \leqslant \frac{\theta^2}{\theta^2-1} \frac{2R}{(\theta-1)\alpha} C_1 J_{D,\theta}(f).$$

Finally, we get

$$J_{D,\theta}(p_N^f) \leqslant \frac{2R\theta^2}{\theta^2-1} C_1 J_{D,\theta}(f) = C_2 J_{D,\theta}(f).$$

This completes the proof. □

For any integer $N \geqslant 1$ and $f \in C_c(\Gamma, \sigma)$, let

$$q_N^f = f - p_N^f.$$

If $n \geqslant N$ and

$$R^{2n} + R^{2n-1} < \ell(r) \leqslant R^{2n+1} - R^{2n-1},$$

then $g_{2n}(r) = 1$ by Lemma 4.7(iii), and thus $q_N^f(r) = 0$. Denote

$$\beta_n = R^{2(n-1)+1} - R^{2(n-1)-1} \quad \text{and} \quad \gamma_n = R^{2n} + R^{2n-1}.$$

Set $\alpha_n = \frac{1}{2(\theta+1)} R^{2n-1}$. Then we have

$$(\theta+1)\alpha_n < \beta_n = R^{2n-3}(R^2-1) < \gamma_n = R^{2n} + R^{2n-1}.$$

Since

$$\begin{aligned}\beta_n - 2\alpha_n - (R^{2(n-1)} + R^{2(n-1)-1}) &= R^{2n-3}(R^2 - R - 2) - \frac{1}{\theta+1}R^{2n-1} \\ &= R^{2n-3}\left(\frac{\theta}{\theta+1}R^2 - R - 2\right) > 0,\end{aligned}$$

q_N^f vanishes on $A(\beta_n - 2\alpha_n, \beta_n)$. Because

$$\begin{aligned}\gamma_n + 2\alpha_n - (R^{2n+1} - R^{2n-1}) &= R^{2n} + 2R^{2n-1} + \frac{1}{\theta+1}R^{2n-1} - R^{2n+1} \\ &= R^{2n-1}\left(R + 2 + \frac{1}{\theta+1} - R^2\right) \\ &= -R^{2n-1}\left(\left(\frac{\theta}{\theta+1}R^2 - R - 2\right) + \frac{1}{\theta+1}(R^2-1)\right) \\ &< 0,\end{aligned}$$

q_N^f vanishes on $A(\gamma_n, \gamma_n + 2\alpha_n)$. Let us denote

$$A_n = A(\beta_n, \gamma_n).$$

By the inequality (5.2), we get

$$\begin{aligned}|B(\alpha_n)|^{-1}|B(\gamma_n + \alpha_n)| &\leqslant C_\ell^{1+\log_\theta((\gamma_n+\alpha_n)/\alpha_n)} \\ &= C_\ell^{1+\log_\theta(2(\theta+1)R+2\theta+3)}.\end{aligned}$$

Finally, by Proposition 4.8, we have

$$\|\lambda(q_N^f \chi_{A_n})\| \leqslant C_3 R^{-2n} J_{D,\theta}(q_N^f), \tag{5.3}$$

where $C_3 = \frac{2(\theta+1)}{\theta-1} R C_\ell^{\frac{1}{2}+\frac{1}{2}\log_\theta(2(\theta+1)R+2\theta+3)}$.

Lemma 5.6. *For any $f \in C_c(\Gamma, \sigma)$ and for each $n \geqslant 2$, we have*

$$\|\lambda(q_N^f \chi_{A_n})\| \leqslant C_4 R^{-2n} J_{D,\theta}(f),$$

where $C_4 = (1 + C_2)C_3$.

Proof. From Proposition 5.5, we obtain

$$\begin{aligned}J_{D,\theta}(q_N^f) &\leqslant J_{D,\theta}(f) + J_{D,\theta}(p_N^f) \\ &\leqslant (1 + C_2) J_{D,\theta}(f).\end{aligned}$$

Therefore, by the inequality (5.3), we have

$$\begin{aligned}\|\lambda(q_N^f \chi_{A_n})\| &\leqslant C_3 R^{-2n} J_{D,\theta}(q_N^f) \\ &\leqslant C_3 R^{-2n}(1 + C_2) J_{D,\theta}(f).\end{aligned}$$

Set $C_4 = (1 + C_2)C_3$, and the conclusion follows. □

For any $f \in C_c(\Gamma, \sigma)$ and for any integer $N \geqslant 2$, we set

$$\rho_N^f = \sum_{n \geqslant N} (q_N^f \chi_{A_n}).$$

If $\ell(r) > R^{2N} + R^{2N-1}$, we have $\rho_N^f(r) = q_N^f(r)$, and it follows that $f - (p_N^f + \rho_N^f)$ is supported in $B(R^{2N} + R^{2N-1})$.

Proposition 5.7. *For any $f \in C_c(\Gamma, \sigma)$ and for any integer $N \geqslant 2$, we have*

$$\|\lambda(\rho_N^f)\| \leqslant \frac{\theta^2}{\theta^2 - 1} C_4 R^{-2N} J_{D,\theta}(f).$$

Proof. For any integer $N \geqslant 2$, by Lemma 5.6 we have

$$\begin{aligned}\|\lambda(\rho_N^f)\| &= \left\| \sum_{n \geqslant N} \lambda(q_N^f \chi_{A_n}) \right\| \\ &\leqslant \sum_{n \geqslant N} \|\lambda(q_N^f \chi_{A_n})\| \\ &\leqslant \sum_{n \geqslant N} C_4 R^{-2n} J_{D,\theta}(f) \\ &\leqslant \frac{\theta^2}{\theta^2 - 1} C_4 R^{-2N} J_{D,\theta}(f)\end{aligned}$$

for any $f \in C_c(\Gamma, \sigma)$. □

Corollary 5.8. *For any $f \in C_c(\Gamma, \sigma)$ and for any integer $N \geqslant 2$, we have*

$$J_{D,\theta}(\rho_N^f) \leqslant \frac{2\theta^2}{\theta^2 - 1} C_4 J_{D,\theta}(f).$$

Proof. For any $\alpha > 0$, let N_α be the smallest positive integer n such that

$$R^{2n} + R^{2n-1} \geqslant (\theta - 1)\alpha.$$

If $\xi \in C_c(\Gamma)$ has its support in $B(\alpha)$, then for any $n < N_\alpha$, the support of $\lambda(q_N^f \chi_{A_n})\xi$ is contained in $B(\theta\alpha)$, and thus

$$(I - M_{\theta\alpha})\lambda(q_N^f \chi_{A_n})\xi = 0.$$

Thus, for $n < N_\alpha$, we have

$$(I - M_{\theta\alpha})\lambda(q_N^f \chi_{A_n})M_\alpha = 0.$$

By Lemma 5.6, we have

$$\begin{aligned}\|(I - M_{\theta\alpha})\lambda(\rho_N^f)M_\alpha\| &= \left\| \sum_{n \geqslant N_\alpha} (I - M_{\theta\alpha})\lambda(q_N^f \chi_{A_n})M_\alpha \right\| \\ &\leqslant \sum_{n \geqslant N_\alpha} \|\lambda(q_N^f \chi_{A_n})\| \\ &\leqslant \sum_{n \geqslant N_\alpha} R^{-2n} C_4 J_{D,\theta}(f) \\ &\leqslant \frac{\theta^2}{\theta^2 - 1} C_4 R^{-2N_\alpha} J_{D,\theta}(f).\end{aligned}$$

Now, from the definition of N_α, we have

$$(\theta - 1)\alpha \leqslant R^{2N_\alpha} + R^{2N_\alpha - 1} \leqslant 2R^{2N_\alpha}$$

since $R \geqslant \theta > 1$. Thus, we have $R^{2N_\alpha} \geqslant (\theta - 1)\alpha/2$. So we obtain

$$\|(I - M_{\theta\alpha})\lambda(\rho_N^f)M_\alpha\| \leqslant \frac{2\theta^2}{\theta^2 - 1} C_4 \frac{1}{(\theta - 1)\alpha} J_{D,\theta}(f).$$

Finally,

$$\begin{aligned} J_{D,\theta}(\rho_N^f) &= \sup\{(\theta - 1)\alpha\|(I - M_{\theta\alpha})\lambda(\rho_N^f)M_\alpha\| : \alpha > 0\} \\ &\leqslant \frac{2\theta^2}{\theta^2 - 1} C_4 J_{D,\theta}(f). \end{aligned}$$

This completes the proof. □

6 Leibniz Lip-norms

A $*$-*seminorm* L on a unital C^*-normed algebra A is a seminorm such that $L(a^*) = L(a)$ for any $a \in A$. A seminorm L on a unital C^*-normed algebra A is said to be *lower semicontinuous* if for every $\alpha \in \mathbb{R}$ with $\alpha > 0$, the set $\{a \in A : L(a) \leqslant \alpha\}$ is norm-closed in A. Equivalently, for any sequence $\{a_n\}$ in A that converges in norm to $a \in A$, we have

$$L(a) \leqslant \liminf_{n \to \infty} L(a_n).$$

A seminorm L on a unital C^*-normed algebra A is said to be *Leibniz* if for all $a, b \in \mathcal{A} = \{a \in A : L(a) < \infty\}$, we have

$$L(ab) \leqslant L(a)\|b\| + \|a\|L(b).$$

Recall in [18, 20, 22] that a *Lip-norm* on a C^*-algebra A with identity 1_A is a seminorm L on A which is permitted to take the value $+\infty$, and satisfies

(i) $L(a) = L(a^*)$ for all $a \in A$;

(ii) $L(1_A) = 0$;

(iii) the topology, induced by the metric

$$\rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : a \in A, L(a) \leqslant 1\}, \quad \mu, \nu \in \mathcal{S}(A)$$

on the state space $\mathcal{S}(A)$ of A, coincides with the weak $*$-topology.

Definition 6.1 (See [22])**.** A C^*-*metric algebra* is a pair (A, L) consisting of a unital C^*-algebra A and a Leibniz Lip-norm L on A.

Let A be a unital C^*-normed algebra, and let L be a seminorm on A (with the value $+\infty$ allowed) with $L(1_A) = 0$. Denote

$$\mathcal{L}_1 = \{a \in A : L(a) \leqslant 1\}.$$

Let $\overline{\mathcal{L}_1}$ be the closure of $\mathcal{L}_1$ in $\bar{A}$, the completion of A, and let $\bar{L}$ denote the corresponding "Minkowski functional" on $\bar{A}$. More precisely, $\bar{L}$ is defined by

$$\bar{L}(a) = \inf\{\beta > 0 : a \in \beta\overline{\mathcal{L}_1}\}$$

for $a \in \bar{A}$. We call $\bar{L}$ the *closure* of L (see [16]). It is clear that $a \in \overline{\mathcal{L}_1}$ if and only if $\bar{L}(a) \leqslant 1$, i.e., $\bar{L}$ is lower semicontinuous on $\bar{A}$. For any $\varepsilon > 0$ and $a \in A$ with $L(a) < \infty$, we have

$$a \in (L(a) + \varepsilon)\mathcal{L}_1 \subseteq (L(a) + \varepsilon)\overline{\mathcal{L}_1}.$$

Hence, $\bar{L}(a) \leqslant L(a) + \varepsilon$, and thus $\bar{L}(a) \leqslant L(a)$.

Proposition 6.2. *If L is a Leibniz $*$-seminorm on a unital C^*-normed algebra A, then $\bar{L}$ is also a Leibniz $*$-seminorm on $\bar{A}$.*

Proof. For any $\varepsilon > 0$ and any $a, b \in \bar{A}$ with $\bar{L}(a) < \infty$ and $\bar{L}(b) < \infty$, we have

$$\bar{L}\bigg(\frac{a}{(\bar{L}(a)+\varepsilon)}\bigg) \leqslant 1 \quad \text{and} \quad \bar{L}\bigg(\frac{b}{(\bar{L}(b)+\varepsilon)}\bigg) \leqslant 1.$$

Thus, there are sequences $\{a_n\}$ and $\{b_n\}$ in A with $L(a_n) \leqslant 1$ and $L(b_n) \leqslant 1$ for every n such that

$$\lim_{n\to\infty} a_n = \frac{a}{(\bar{L}(a)+\varepsilon)} \quad \text{and} \quad \lim_{n\to\infty} b_n = \frac{b}{(\bar{L}(b)+\varepsilon)}.$$

Since L is Leibniz, it follows that

$$\begin{aligned} L(a_n b_n) &\leqslant \|a_n\| L(b_n) + \|b_n\| L(a_n) \\ &\leqslant \|a_n\| + \|b_n\| \\ &\to \frac{\|a\|}{(\bar{L}(a)+\varepsilon)} + \frac{\|b\|}{(\bar{L}(b)+\varepsilon)}. \end{aligned}$$

From the lower semicontinuity of $\bar{L}$, we have

$$\begin{aligned} \bar{L}(ab) &\leqslant (\bar{L}(a)+\varepsilon)(\bar{L}(b)+\varepsilon) \liminf_{n\to\infty} \bar{L}(a_n b_n) \\ &\leqslant (\bar{L}(a)+\varepsilon)(\bar{L}(b)+\varepsilon) \liminf_{n\to\infty} L(a_n b_n) \\ &\leqslant \|a\|(\bar{L}(b)+\varepsilon) + \|b\|(\bar{L}(a)+\varepsilon) \end{aligned}$$

for any $\varepsilon > 0$. Thus, we have

$$\bar{L}(ab) \leqslant \|a\|\bar{L}(b) + \|b\|\bar{L}(a),$$

i.e., $\bar{L}$ is Leibniz.

For any $a \in \bar{A}$ with $\bar{L}(a) < \infty$ and any $\varepsilon > 0$, we can see that $\bar{L}(a/(\bar{L}(a)+\varepsilon)) \leqslant 1$, and thus there is a sequence $\{a_n\}$ in A with

$$\lim_{n\to\infty} a_n = \frac{a}{(\bar{L}(a)+\varepsilon)}$$

and $L(a_n) = L(a_n^*) \leqslant 1$ for every n. Hence, we have

$$\bar{L}\bigg(\frac{a^*}{(\bar{L}(a)+\varepsilon)}\bigg) \leqslant \liminf_{n\to\infty} \bar{L}(a_n^*) \leqslant \liminf_{n\to\infty} L(a_n^*) \leqslant 1,$$

i.e., $\bar{L}(a^*) \leqslant \bar{L}(a) + \varepsilon$ for any $\varepsilon > 0$, and thus $\bar{L}(a^*) \leqslant \bar{L}(a)$. Using the procedure for a^*, we obtain $\bar{L}(a) \leqslant \bar{L}(a^*)$, and thus $\bar{L}(a) = \bar{L}(a^*)$.

While for $a \in \bar{A}$ with $\bar{L}(a) = \infty$, we must have that $\bar{L}(a^*) = \infty$; otherwise, from $\bar{L}(a^*) < \infty$, we will obtain $\bar{L}(a) = \bar{L}(a^*) < \infty$, a contradiction. Thus, we prove that $\bar{L}$ is a $*$-seminorm on $\bar{A}$. □

Now, let ℓ be a length function on a discrete group Γ, and let σ be a 2-cocycle on Γ.

Proposition 6.3. *L_D is a lower semicontinuous Leibniz $*$-seminorm on $C_c(\Gamma, \sigma)$.*

Proof. Since D is a self-adjoint operator and λ is a $*$-isomorphism, for any $a \in C_c(\Gamma, \sigma)$, we have

$$L_D(a^*) = \|[D, \lambda(a^*)]\| = \|[D, \lambda(a)]^*\| = \|[D, \lambda(a)]\| = L_D(a),$$

i.e., L_D is a $*$-seminorm on $C_c(\Gamma, \sigma)$.

For any $a, b \in C_c(\Gamma, \sigma)$, we have

$$\begin{aligned} L_D(ab) &= \|[D, \lambda(ab)]\| \\ &= \|\lambda(a)[D, \lambda(b)] + [D, \lambda(a)]\lambda(b)\| \\ &\leqslant \|a\| L_D(b) + \|b\| L_D(a). \end{aligned}$$

Thus, L_D is Leibniz.

Let $\{a_n\}$ be a sequence in $C_c(\Gamma,\sigma)$ with

$$\lim_{n\to\infty} a_n = a \in C_c(\Gamma,\sigma)$$

and $L_D(a_n) \leqslant \alpha$ for some $\alpha > 0$ and all n. For any $\xi,\eta \in \ell^2(\Gamma)$, we have

$$\begin{aligned}\langle [D,\lambda(a)]\xi,\eta\rangle &= \langle \lambda(a)\xi, D\eta\rangle - \langle D\xi, \lambda(a)\eta\rangle \\ &= \lim_{n\to\infty} \langle \lambda(a_n)\xi, D\eta\rangle - \langle D\xi, \lambda(a_n)\eta\rangle \\ &= \lim_{n\to\infty} \langle [D,\lambda(a_n)]\xi,\eta\rangle.\end{aligned}$$

Since

$$|\langle [D,\lambda(a_n)]\xi,\eta\rangle| \leqslant \alpha\|\xi\|\|\eta\|$$

for all n, we have

$$L_D(a) = \|[D,\lambda(a)]\| \leqslant \alpha,$$

and thus L_D is lower semicontinuous. □

For any $a \in C_c(\Gamma,\sigma)$ with $\bar{L}_D(a) < \infty$ and $\varepsilon > 0$, we have $\bar{L}_D(a/(\bar{L}_D(a)+\varepsilon)) \leqslant 1$. By definition, there is a sequence $\{a_n\}$ in $\mathcal{L}_{D,1}$ that converges to $a/(\bar{L}_D(a)+\varepsilon)$. From the lower semicontinuity of L_D, we see that

$$L_D\left(\frac{a}{(\bar{L}_D(a)+\varepsilon)}\right) \leqslant \liminf_{n\to\infty} L_D(a_n) \leqslant 1,$$

i.e., $L_D(a) \leqslant \bar{L}_D(a)+\varepsilon$ for any $\varepsilon > 0$, and thus, $L_D(a) \leqslant \bar{L}_D(a)$. For any $a \in C_c(\Gamma,\sigma)$, we obtain $\bar{L}_D(a) \leqslant L_D(a)$. Hence, for any $a \in C_c(\Gamma,\sigma)$, we have $L_D(a) = \bar{L}(a)$.

From Propositions 6.3 and 6.2, we have the following proposition.

Proposition 6.4. *$\bar{L}_D$ is a lower semicontinuous Leibniz $*$-seminorm on $C_r^*(\Gamma,\sigma)$.*

From this proposition, we can endow the state space $\mathcal{S}(C_r^*(\Gamma,\sigma))$ with an extended metric

$$\rho_{\bar{L}_D} : \mathcal{S}(C_r^*(\Gamma,\sigma)) \times \mathcal{S}(C_r^*(\Gamma,\sigma)) \mapsto [0,\infty]$$

as follows:

$$\rho_{\bar{L}_D}(\mu,\nu) = \sup\{|\mu(a)-\nu(a)| : a \in C_r^*(\Gamma,\sigma), \bar{L}_D(a) \leqslant 1\}$$

for all $\mu,\nu \in \mathcal{S}(C_r^*(\Gamma,\sigma))$. Since

$$\{a \in C_r^*(\Gamma,\sigma) : \bar{L}_D(a) \leqslant 1\} = \overline{\{a \in C_c(\Gamma,\sigma) : L_D(a) \leqslant 1\}},$$

we have

$$\begin{aligned}\rho_{\bar{L}_D}(\mu,\nu) &= \sup\{|\mu(a)-\nu(a)| : a \in C_r^*(\Gamma,\sigma), \bar{L}_D(a) \leqslant 1\} \\ &= \sup\{|\mu(a)-\nu(a)| : a \in C_c(\Gamma,\sigma), L_D(a) \leqslant 1\} \\ &= \rho_{L_D}(\mu,\nu),\end{aligned}$$

where

$$\rho_{L_D}(\mu,\nu) = \sup\{|\mu(a)-\nu(a)| : a \in C_c(\Gamma,\sigma), L_D(a) \leqslant 1\}$$

for all $\mu,\nu \in \mathcal{S}(C_r^*(\Gamma,\sigma))$.

Lemma 6.5. *Let ℓ be a proper length function on a discrete group Γ, and let σ be a 2-cocycle on Γ. Then, for any $\alpha > 0$, there exists a constant $\beta > 0$ such that*

$$\sum_{r\in\Gamma} |f(r)| \leqslant \beta L_D(f)$$

for any $f \in W$ with support in $B(\alpha)$.

Proof. For any $f \in C_c(\Gamma, \sigma)$, we have

$$\left\| \sum_{r\in\Gamma} \ell(r) f(r) \sigma(r,e) \delta_r \right\|_2 = \|[D, \lambda(f)](\delta_e)\|_2 \leqslant \|[D,\lambda(f)]\| = L_D(f)$$

by the formula (3.1); hence,

$$\sum_{r\in\Gamma} \ell(r)^2 |f(r)|^2 = \left\| \sum_{r\in\Gamma} \ell(r) f(r)\sigma(r,e)\delta_r \right\|_2^2 \leqslant (L_D(f))^2.$$

Now, suppose that $f \in W$ and f is supported on $B(\alpha)$. We have

$$\sum_{r\in\Gamma} |f(r)| \leqslant \left(\sum_{\ell(r)\leqslant\alpha} |f(r)|^2 \ell(r)^2 \right)^{\frac{1}{2}} \left(\sum_{0<\ell(r)\leqslant\alpha} \frac{1}{\ell(r)^2} \right)^{\frac{1}{2}} \leqslant L_D(f) \left(\sum_{0<\ell(r)\leqslant\alpha} \frac{1}{\ell(r)^2} \right)^{\frac{1}{2}} < \infty$$

since ℓ is proper. Set $\beta = (\sum_{0<\ell(r)\leqslant\alpha} \frac{1}{\ell(r)^2})^{\frac{1}{2}}$, and we have the conclusion. □

Lemma 6.6. *Let ℓ be a proper length function on a discrete group Γ with the property of bounded θ-dilation. Then, for any 2-cocycle σ on Γ, there exists a real number $\gamma > 0$ such that*

$$\|\lambda(f)\| \leqslant \gamma L_D(f)$$

for any $f \in W$.

Proof. Let N be an integer with $N \geqslant 2$. By Propositions 5.4, 5.7 and 3.5 we have

$$\|\lambda(p_N^f)\| \leqslant \frac{\theta^2}{\theta^2-1} C_1 R^{-2N} J_{D,\theta}(f) \leqslant \frac{\theta^2}{\theta^2-1} C_1 R^{-2N} L_D(f)$$

and

$$\|\lambda(\rho_N^f)\| \leqslant \frac{\theta^2}{\theta^2-1} C_4 R^{-2N} J_{D,\theta}(f) \leqslant \frac{\theta^2}{\theta^2-1} C_4 R^{-2N} L_D(f)$$

for any $f \in C_c(\Gamma, \sigma)$, respectively. For any $f \in C_c(\Gamma,\sigma)$, since $f - (p_N^f + \rho_N^f)$ is supported in $B(R^{2N} + R^{2N-1}) = B(\gamma_N)$, we have

$$\begin{aligned} \|\lambda(f - p_N^f - \rho_N^f)\| &\leqslant \sum_{\ell(r)\leqslant\gamma_N} |(f - p_N^f - \rho_N^f)(r)| \\ &= \sum_{\ell(r)\leqslant\gamma_N} \left| \left(q_N^f - \sum_{n\geqslant N} (q_N^f \chi_{A_n}) \right)(r) \right| \\ &\leqslant \sum_{\ell(r)\leqslant\gamma_N} |q_N^f(r)| \\ &\leqslant \sum_{\ell(r)\leqslant\gamma_N} |f(r)| \\ &\leqslant \beta_N L_D(f) \end{aligned}$$

for some constant $\beta_N > 0$ by Lemma 6.5 and the definitions of q_N^f and ρ_N^f. Thus, we have

$$\begin{aligned}\|\lambda(f)\| &\leqslant \|\lambda(f - p_N^f - \rho_N^f)\| + \|\lambda(\rho_N^f)\| + \|\lambda(p_N^f)\| \\ &\leqslant \beta_N L_D(f) + \frac{\theta^2}{\theta^2-1} C_4 R^{-2N} L_D(f) + \frac{\theta^2}{\theta^2-1} C_1 R^{-2N} L_D(f) \\ &= \left(\beta_N + \frac{\theta^2}{\theta^2-1} C_4 R^{-2N} + \frac{\theta^2}{\theta^2-1} C_1 R^{-2N}\right) L_D(f)\end{aligned}$$

for any $f \in C_c(\Gamma, \sigma)$. Set

$$\gamma = \beta_N + \frac{\theta^2}{\theta^2-1} C_4 R^{-2N} + \frac{\theta^2}{\theta^2-1} C_1 R^{-2N}.$$

This completes the proof of the lemma. □

Proposition 6.7. *Let ℓ be a proper length function on a discrete group Γ with the property of bounded θ-dilation. Then, for any 2-cocycle σ on Γ, the diameter of the metric space $(\mathcal{S}(C_r^*(\Gamma, \sigma)), \rho_{\bar{L}_D})$ is finite, and in particular, $\rho_{\bar{L}_D}$ is a metric on the state space $\mathcal{S}(C_r^*(\Gamma, \sigma))$.*

Proof. By Lemma 6.6, there exists a constant $\gamma > 0$ such that

$$\|\lambda(f)\| \leqslant \gamma L_D(f)$$

for any $f \in W$. Now, for any $f \in C_c(\Gamma, \sigma)$, we have $f - f(e)\overline{\sigma(e,e)}\delta_e \in W$, and thus

$$\|\lambda(f - f(e)\overline{\sigma(e,e)}\delta_e)\| \leqslant \gamma L_D(f - f(e)\overline{\sigma(e,e)}\delta_e) = \gamma L_D(f),$$

i.e.,

$$\|\tilde{f}\|^{\sim} \leqslant \gamma L_D(f),$$

where $\|\cdot\|^{\sim}$ is the quotient norm on the quotient space $C_c(\Gamma, \sigma)/\mathbb{C}\delta_e$ with respect to the norm $\|\cdot\|$ on $C_c(\Gamma, \sigma)$. Hence, for any $\mu, \nu \in \mathcal{S}(C_r^*(\Gamma, \sigma))$ and $f \in C_c(\Gamma, \sigma)$ with $L_D(f) \leqslant 1$, we have

$$|\mu(f) - \nu(f)| = |(\mu - \nu)(f)| \leqslant 2\|\tilde{f}\|^{\sim} \leqslant 2\gamma.$$

Finally,

$$\begin{aligned}\rho_{\bar{L}_D}(\mu, \nu) &= \rho_{L_D}(\mu, \nu) \\ &= \sup\{|\mu(f) - \nu(f)| : f \in C_c(\Gamma, \sigma), L_D(f) \leqslant 1\} \leqslant 2\gamma,\end{aligned}$$

i.e., the diameter of the metric space $(\mathcal{S}(C_r^*(\Gamma, \sigma)), \rho_{\bar{L}_D})$ is finite. □

A *Lipschitz seminorm* [9, 13, 16, 22] on a C^*-algebra A with identity 1_A is a $*$-seminorm L on A which is permitted to take the value $+\infty$, and satisfies

(i) $L(a) = 0$ if and only if $a \in \mathbb{C}1_A$;

(ii) the set $\mathcal{A} = \{a \in A : L(a) < \infty\}$ is a dense $*$-subalgebra of A.

The following characterization of the Lip-norm is given in [13, Proposition 1.3]. We give a somewhat different proof here.

Proposition 6.8. *Let L be a Lipschitz seminorm on a unital C^*-algebra A, and let μ be a state of A. Then, L is a Lip-norm if and only if the set*

$$\{a \in A : L(a) \leqslant 1 \text{ and } \mu(a) = 0\}$$

is a totally bounded subset of A for the norm.

Proof. Suppose that L is a Lip-norm on A. Then, $(\mathcal{S}(A), \rho_L)$ is a compact metric space, and its topology coincides with the weak $*$-topology. For any $\nu \in \mathcal{S}(A)$ and $a \in E = \{a \in A : L(a) \leqslant 1, \mu(a) = 0\}$, we have

$$|\hat{a}(\nu)| = |\nu(a)| = |\nu(a) - \mu(a)|$$

$$\begin{aligned}&\leqslant \rho_L(\mu,\nu)L(a)\\&\leqslant \mathrm{diam}(\mathcal{S}(A),\rho_L)<\infty,\end{aligned}$$

and thus $\hat{E}$ is a bounded subset of the unital C^*-algebra $C(\mathcal{S}(A))$ of complex-valued continuous functions on $\mathcal{S}(A)$. Since for any $a \in E$, we have

$$\begin{aligned}|\hat{a}(\nu_1)-\hat{a}(\nu_2)|&=|\nu_1(a)-\nu_2(a)|\\&\leqslant \rho_L(\nu_1,\nu_2)L(a)\leqslant \rho_L(\nu_1,\nu_2)\end{aligned}$$

for all $\nu_1,\nu_2\in\mathcal{S}(A)$, hence $\hat{E}$ is a family of equicontinuous functions on $(\mathcal{S}(A),\rho_L)$. By the Arzelà-Ascoli theorem, $\hat{E}$ is a totally bounded subset of $C(\mathcal{S}(A))$. From the Kadison representation theorem, the canonical map

$$a\in A_{sa}\mapsto \hat{a}\in \mathrm{Aff}(\mathcal{S}(A))\subset C(\mathcal{S}(A)),$$

where $\mathrm{Aff}(\mathcal{S}(A))$ is the set of all real-valued affine continuous functions on $\mathcal{S}(A)$, is a unital order isomorphism, and hence, an isometry. For any $a\in A$, we have

$$\begin{aligned}\|\hat{a}\|_\infty\leqslant\|a\|&\leqslant\|a_1\|+\|a_2\|\\&=\|\widehat{a_1}\|_\infty+\|\widehat{a_2}\|_\infty\\&\leqslant\|\hat{a}\|_\infty+\|\hat{a}\|_\infty=2\|\hat{a}\|_\infty,\end{aligned}$$

where $a_1=\frac{a+a^*}{2}$ and $a_2=\frac{a-a^*}{2i}$. For any $\varepsilon>0$, since $\hat{E}$ is totally bounded, there exist $a_1,a_2,\ldots,a_m\in E$ such that for any $\hat{a}\in\hat{E}$, there is an a_i such that $\|\hat{a}-\hat{a}_i\|_\infty<\varepsilon/2$. It follows that for any $c\in E$, there is an a_i such that $\|\hat{c}-\hat{a}_i\|_\infty<\frac{\varepsilon}{2}$; thus, $\|c-a_i\|\leqslant 2\|\hat{c}-\hat{a}_i\|_\infty<\varepsilon$. This implies that E is a totally bounded subset of A for the norm.

For the proof in the other direction, one can refer to [13]. □

Proposition 6.9. *Let ℓ be a proper length function on a discrete group Γ with the property of bounded θ-dilation, and let σ be a 2-cocycle on Γ. Then, $\bar{L}_D$ is a Lipschitz seminorm on $C_r^*(\Gamma,\sigma)$.*

Proof. By the definition of the length function, we can find $[D,\overline{\sigma(e,e)}\lambda_e]=0$, which gives

$$\bar{L}_D(\alpha(\overline{\sigma(e,e)}\delta_e))=L_D(\alpha(\overline{\sigma(e,e)}\delta_e))=0$$

for any $\alpha\in\mathbb{C}$.

Suppose $a\in C_r^*(\Gamma,\sigma)$ with $\bar{L}_D(a)=0$. Then $\bar{L}_D(na)=0$ for all $n\in\mathbb{N}$. By Proposition 6.7, we have

$$\begin{aligned}|\mu(na)-\nu(na)|&\leqslant\rho_{\bar{L}_D}(\mu,\nu)\\&\leqslant\mathrm{diam}(\mathcal{S}(C_r^*(\Gamma,\sigma)),\rho_{\bar{L}_D})<\infty.\end{aligned}$$

Hence,

$$|\mu(a)-\nu(a)|\leqslant\frac{1}{n}\mathrm{diam}(\mathcal{S}(C_r^*(\Gamma,\sigma)),\rho_{\bar{L}_D})\to 0$$

for all $\mu,\nu\in\mathcal{S}(C_r^*(\Gamma,\sigma))$ and all $n\in\mathbb{N}$. It follows that $\mu(a)=\nu(a)$ for all $\mu,\nu\in\mathcal{S}(C_r^*(\Gamma,\sigma))$. Now, we fix a $\mu_0\in\mathcal{S}(C_r^*(\Gamma,\sigma))$. Then we have

$$\mu(a-\mu_0(a)(\overline{\sigma(e,e)}\delta_e))=\mu(a)-\mu_0(a)=0$$

for all $\mu\in\mathcal{S}(C_r^*(\Gamma,\sigma))$. Since the state space $\mathcal{S}(C_r^*(\Gamma,\sigma))$ separates the elements of the unital C^*-algebra $C_r^*(\Gamma,\sigma)$, we have $a=\mu_0(a)(\overline{\sigma(e,e)}\delta_e)\in\mathbb{C}(\overline{\sigma(e,e)}\delta_e)$.

By Proposition 6.4, $\bar{L}_D$ is a Leibniz $*$-seminorm on $C_r^*(\Gamma,\sigma)$. Thus, the set

$$\{a\in C_r^*(\Gamma,\sigma):\bar{L}_D(a)<\infty\}$$

is a $*$-subalgebra of $C_r^*(\Gamma,\sigma)$. Since for any $f\in C_c(\Gamma,\sigma)$, $\bar{L}_D(f)=L_D(f)=\|[D,f]\|<\infty$, we obtain

$$C_c(\Gamma,\sigma)\subset\{a\in C_r^*(\Gamma,\sigma):\bar{L}_D(a)<\infty\}.$$

Finally, $\{a \in C_r^*(\Gamma,\sigma) : \bar{L}_D(a) < \infty\}$ is a dense $*$-subalgebra of $C_r^*(\Gamma,\sigma)$. Therefore, $\bar{L}_D$ is a Lipschitz seminorm on $C_r^*(\Gamma,\sigma)$. □

Theorem 6.10. *Let ℓ be a proper length function on a discrete group Γ with the property of bounded θ-dilation, and let σ be a 2-cocycle on Γ. Then, the seminorm $\bar{L}_D$ is a Leibniz Lip-norm on the twisted reduced group C^*-algebra $C_r^*(\Gamma,\sigma)$, i.e., the pair $(C_r^*(\Gamma,\sigma),\bar{L}_D)$ is a C^*-metric algebra.*

Proof. By Proposition 6.4 $\bar{L}_D$ is a Leibniz $*$-seminorm on $C_r^*(\Gamma,\sigma)$, so we just need to prove that $\bar{L}_D$ is a Lip-norm on $C_r^*(\Gamma,\sigma)$. However, by Propositions 6.8 and 6.9, it is sufficient to show that the set

$$B_1 = \{a \in C_r^*(\Gamma,\sigma) : \bar{L}_D(a) \leqslant 1, \mathrm{tr}(\lambda(a)) = \langle \lambda(a)(\delta_e), \delta_e\rangle = 0\}$$

is totally bounded for the norm on $C_r^*(\Gamma,\sigma)$. Since

$$\{a \in C_r^*(\Gamma,\sigma) : \bar{L}_D(a) \leqslant 1\} = \overline{\{a \in C_c(\Gamma,\sigma) : L_D(a) \leqslant 1\}},$$

we just need to demonstrate that the set

$$\begin{aligned} B_{L_D} &= \{f \in W : L_D(f) \leqslant 1\} \\ &= \{f \in C_c(\Gamma,\sigma) : \mathrm{tr}(\lambda(f)) = \langle \lambda(f)(\delta_e), \delta_e\rangle = 0, L_D(f) \leqslant 1\} \end{aligned}$$

is totally bounded for the operator norm.

By Propositions 3.5, 5.4 and 5.7, for any $f \in B_{L_D}$ and $N \geqslant 2$, we have

$$\max\{\|\lambda(p_N^f)\|, \|\lambda(\rho_N^f)\|\} \leqslant \frac{\theta^2}{\theta^2-1} R^{-2N} \max\{C_1, C_4\}.$$

For any $\varepsilon > 0$, we fix $R \geqslant \theta^{K_\theta}$, and choose N large enough such that

$$\frac{\theta^2}{\theta^2-1} R^{-2N} \max\{C_1, C_4\} < \frac{\varepsilon}{4}.$$

For this N, we have

$$\|\lambda(p_N^f + \rho_N^f)\| < \frac{\varepsilon}{2}.$$

Thus we have

$$\|\lambda(f) - \lambda(f - (p_N^f + \rho_N^f))\| < \frac{\varepsilon}{2}.$$

Now, for any $f \in C_c(\Gamma,\sigma)$, by the construction of ρ_N^f and p_N^f, we have $f - (p_N^f + \rho_N^f)$ supported in $B(R^{2N} + R^{2N-1})$. As the proof in Lemma 6.6, we can see that the set $\{f - (p_N^f + \rho_N^f) : f \in B_{L_D}\}$ is bounded. Since ℓ is a proper length function, the set

$$\{f - (p_N^f + \rho_N^f) : f \in B_{L_D}\}$$

is contained in a finite-dimensional subspace of $C_c(\Gamma,\sigma)$; hence, it is totally bounded. It follows that there is a finite set

$$\{f_i - (p_N^{f_i} + \rho_N^{f_i}) : f_i \in B_{L_D}, 1 \leqslant i \leqslant m\},$$

such that for any $f - (p_N^f + \rho_N^f)$ with $f \in B_{L_D}$, there is an $f_i - (p_N^{f_i} + \rho_N^{f_i})$ for some $1 \leqslant i \leqslant m$ satisfying

$$\|\lambda(f - (p_N^f + \rho_N^f)) - \lambda(f_i - (p_N^{f_i} + \rho_N^{f_i}))\| < \frac{\varepsilon}{2}.$$

Now, for any $f \in B_{L_D}$, there is an $f_i - (p_N^{f_i} + \rho_N^{f_i})$ for some $1 \leqslant i \leqslant m$ such that

$$\begin{aligned} \|\lambda(f) - \lambda(f_i - (p_N^{f_i} + \rho_N^{f_i}))\| &\leqslant \|\lambda(f) - \lambda(f - (p_N^f + \rho_N^f))\| \\ &\quad + \|\lambda(f - (p_N^f + \rho_N^f)) - \lambda(f_i - (p_N^{f_i} + \rho_N^{f_i}))\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, B_{L_D} is totally bounded. This completes the proof of the theorem. □

In particular, we have the following refined result of Christ and Rieffel [2, Theorem 1.4] on reduced group C^*-algebras for the seminorm L_D induced by a length function with the property of bounded doubling.

Corollary 6.11. *Let ℓ be a length function on a discrete group Γ with the property of bounded doubling. Then, the seminorm $\bar{L}_D$ is a Leibniz Lip-norm on the reduced group C^*-algebra $C_r^*(\Gamma)$, i.e., $(C_r^*(\Gamma), \bar{L}_D)$ is a C^*-metric algebra. In particular, the pair $(C_r^*(\Gamma), \bar{L}_D)$ is a compact quantum metric space.*

Proof. This is just the trivial 2-cocycle $\sigma \equiv 1$ and $\theta = 2$ case for $C_r^*(\Gamma, \sigma)$ by Theorem 6.10. □

Let A and B be two unital C^*-algebras with Lip-norms L_A and L_B, respectively. A map $\Phi : A \mapsto B$ is said to be *Lipschitz* if there exists a constant $\gamma \geqslant 0$ such that

$$L_B(\Phi(a)) \leqslant \gamma L_A(a)$$

for all $a \in A$. When Φ is invertible and both Φ and Φ^{-1} are Lipschitz, we say that Φ is *bi-Lipschitz*. If

$$L_B(\Phi(a)) = L_A(a)$$

for all $a \in A$, then we say that Φ is *Lipschitz isometric* [7, 22]. For two compact quantum metric spaces (C^*-metric algebras) (A, L_A) and (B, L_B), if there is a $*$-isomorphism Φ from A onto B such that Φ and Φ^{-1} are Lipschitz isometric, we say that (A, L_A) and (B, L_B) are *Lipschitz isometric* (see [9]).

Let σ_1 and σ_2 be two cohomologous 2-cocycles on Γ. Let λ^{σ_1} be the left regular σ_1-projective representation of $C_c(\Gamma, \sigma_1)$ on $\ell^2(\Gamma)$, and let λ^{σ_2} be the left regular σ_2-projective representation of $C_c(\Gamma, \sigma_2)$ on $\ell^2(\Gamma)$. The map Φ defined by the equation (3.2) gives an isometrical $*$-isomorphism between $C_c(\Gamma, \sigma_1)$ and $C_c(\Gamma, \sigma_2)$, and hence induces a $*$-isomorphism from $C_r^*(\Gamma, \sigma_1)$ onto $C_r^*(\Gamma, \sigma_2)$.

Theorem 6.12. *Let ℓ be a proper length function on a discrete group Γ with the property of bounded θ-dilation. If σ_1 and σ_2 are two cohomologous 2-cocycles on Γ, then C^*-metric algebras $(C_r^*(\Gamma, \sigma_1), \bar{L}_D)$ and $(C_r^*(\Gamma, \sigma_2), \bar{L}_D)$ are Lipschitz isometric. Thus, the compact quantum metric space structures $(C_r^*(\Gamma, \sigma), \bar{L}_D)$ depend only on the cohomology class of σ.*

Proof. By Theorem 6.10, we see that $(C_r^*(\Gamma, \sigma_1), \bar{L}_D)$ and $(C_r^*(\Gamma, \sigma_2), \bar{L}_D)$ are C^*-metric algebras.

For any $a \in C_c(\Gamma, \sigma_1)$, we have

$$\begin{aligned} L_D(\Phi(a)) &= \|[D, U\lambda^{\sigma_1}(a)U^*]\| \\ &= \|U[D, \lambda^{\sigma_1}(a)]U^*\| \\ &= \|[D, \lambda^{\sigma_1}(a)]\| = L_D(a). \end{aligned}$$

For any $\varepsilon > 0$ and $a \in C_r^*(\Gamma, \sigma_1)$ with $\bar{L}_D(a) < \infty$, we have

$$\bar{L}_D\left(\frac{a}{(\bar{L}_D(a) + \varepsilon)}\right) \leqslant 1.$$

Thus, there is a sequence $\{a_n\}$ in $C_c(\Gamma, \sigma_1)$ with

$$\lim_{n\to\infty} a_n = \frac{a}{(\bar{L}_D(a) + \varepsilon)}$$

and $L_D(a_n) \leqslant 1$ for every n. Hence, we have

$$\begin{aligned} \bar{L}_D\left(\Phi\left(\frac{a}{(\bar{L}_D(a) + \varepsilon)}\right)\right) &\leqslant \liminf_{n\to\infty} \bar{L}_D(\Phi(a_n)) \\ &= \liminf_{n\to\infty} L_D(\Phi(a_n)) \\ &= \liminf_{n\to\infty} L_D(a_n) \leqslant 1, \end{aligned}$$

i.e., $\bar{L}_D(\Phi(a)) \leqslant \bar{L}_D(a) + \varepsilon$ for any $\varepsilon > 0$, so $\bar{L}_D(\Phi(a)) \leqslant \bar{L}_D(a)$.

Similarly, for any $b \in C_c(\Gamma, \sigma_2)$, we have $L_D(\Phi^{-1}(b)) = L_D(b)$. Thus, for any $b \in C_r^*(\Gamma, \sigma_2)$ with $\bar{L}_D(b) < \infty$, we have

$$\bar{L}_D(\Phi^{-1}(b)) \leqslant \bar{L}_D(b).$$

So for any $a \in C_r^*(\Gamma, \sigma_1)$ with $\bar{L}_D(a) < \infty$, we have

$$\bar{L}_D(\Phi(a)) \leqslant \bar{L}_D(a) = \bar{L}_D(\Phi^{-1}(\Phi(a))) \leqslant \bar{L}_D(\Phi(a)),$$

i.e., $\bar{L}_D(\Phi(a)) = \bar{L}_D(a)$. Similarly, for any $b \in C_r^*(\Gamma, \sigma_2)$ with $\bar{L}_D(b) < \infty$, we have

$$\bar{L}_D(\Phi^{-1}(b)) = \bar{L}_D(b).$$

For any $a \in C_r^*(\Gamma, \sigma_1)$ with $\bar{L}_D(a) = \infty$, we must have $\bar{L}_D(\Phi(a)) = \infty$; otherwise, from $\bar{L}_D(\Phi(a)) < \infty$ and

$$\bar{L}_D(a) = \bar{L}_D(\Phi^{-1}(\Phi(a))) \leqslant \bar{L}_D(\Phi(a)) < \infty,$$

we will obtain $\bar{L}_D(a) < \infty$, which leads to a contradiction. Similarly, for any $b \in C_r^*(\Gamma, \sigma_2)$ with $\bar{L}_D(b) = \infty$, we obtain $\bar{L}_D(\Phi^{-1}(b)) = \infty$.

It now follows that Φ and Φ^{-1} are Lipschitz isometric. Therefore, Φ is a Lipschitz isometric map from $C_r^*(\Gamma, \sigma_1)$ onto $C_r^*(\Gamma, \sigma_2)$. □

Recall that a unital subalgebra B of a unital algebra A is said to be *spectrally stable* in A if for any $b \in B$, the spectrum of b as an element of B is the same as its spectrum as an element of A, or equivalently, any b that is invertible in A is also invertible in B. From Theorem 6.10, we pose the question below.

Question 6.13. Let ℓ be a proper length function on a discrete group Γ with the property of bounded θ-dilation, and let σ be a 2-cocycle on Γ. Is there a dense and spectrally stable $*$-subalgebra $\mathcal{A}$ of $C_r^*(\Gamma, \sigma)$ such that the pair $(C_r^*(\Gamma, \sigma), \bar{L}_D)$ is a C^*-metric algebra, where $L_D(a) = \|[D, \lambda(a)]\|$ for any $a \in \mathcal{A}$?

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