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An adaptive C^0IPG method for the Helmholtz transmission eigenvalue problem

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Abstract The interior penalty methods using C^0 Lagrange elements (C^0IPG) developed in the recent decade for the fourth order problems are an interesting topic at present. In this paper, we discuss the adaptive proporty of C^0IPG method for the Helmholtz transmission eigenvalue problem. We give the a posteriori error indicators for primal and dual eigenfunctions, and prove their reliability and efficiency. We also give the a posteriori error indicator for eigenvalues and design a C^0IPG adaptive algorithm. Numerical experiments show that this algorithm is efficient and can get the optimal convergence rate.

Keywords transmission eigenvalues, interior penalty Galerkin method, Lagrange elements, a posteriori error estimates, adaptive algorithm

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1 Introduction

The transmission eigenvalues can be used to obtain estimates for the material properties of the scattering object (see [11, 12, 37]), and have theoretical importance in the uniqueness and reconstruction in inverse scattering theory (see [18]). In recent years, the computation of transmission eigenvalues has attracted the attention of many researchers. The first numerical treatment of the transmission eigenvalue problem appeared in [19], where three finite element methods, including the Argyris, continuous and mixed methods, are proposed for the Helmholtz transmission eigenvalues, and has been further developed by [2, 14, 23, 25, 27, 28, 32, 38] and [26, 30, 39, 42, 44–46], etc.

C^0 interior penalty Galerkin (C^0IPG) method, developed in the recent decade (see [5, 22]), is a new class of Galerkin methods for fourth order problems. The researches for C^0IPG methods have been an interesting topic at present. There exist many researches for fourth order elliptic equations (see [5, 9, 22, 24, 29]) and for eigenvalue problems (see [6, 7, 23, 31, 41, 43]) by C^0IPG methods.

The a posteriori error estimates and adaptive finite element methods are always the main streams of scientific and engineering computing. The idea of the a posteriori error estimates was first introduced by Babuska and Rheinboldt [4] in 1978. Up to now, many excellent works have been summarized in the

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books such as [1, 36, 40]. In addition, a posteriori error estimates of residual type of C^0IPG method of fourth order elliptic equations also have been summarized in [5].

Inspired by the works mentioned above, in this paper, based on the weak formulation proposed in [44, 45], we propose a new C^0IPG discrete scheme (see (2.16)) and discuss the a posteriori error estimates and adaptive algorithm of C^0IPG method for the Helmholtz transmission eigenvalue problem. We give the a posteriori error indicators for primal and dual eigenfunctions and eigenvalues. We prove that the indicators for both primal and dual eigenfunctions are reliable and efficient, and analyze the reliability of the indicator for eigenvalues. Based on the given indicators, we design an adaptive algorithm. Numerical experiments show that this algorithm is efficient and can get the optimal convergence rate. Compared with adaptive C^1 conforming finite element algorithm in [25], the adaptive C^0IPG algorithm is simpler to be constructed and implemented numerically.

In this paper, regarding the basic theory of finite element methods, we refer to [3, 8, 17, 34, 36].

Throughout this paper, the letter C (with or without subscripts) denotes a positive constant independent of mesh size h, which may not be the same constant in different places. For simplicity, we use the symbol $a \lesssim b$ to denote that $a \leqslant Cb$ and the symbol $a \approx b$ to denote $a \lesssim b \lesssim a$.

2 A C^0IPG discrete scheme

Consider the Helmholtz transmission eigenvalue problem: Find $k \in \mathbb{C}$, $w, \sigma \in L^2(\Omega)$, $w - \sigma \in H^2(\Omega)$ such that

$$\Delta w + k^2 n w = 0 \quad \text{in} \quad \Omega, \tag{2.1}$$

$$\Delta \sigma + k^2 \sigma = 0 \quad \text{in} \quad \Omega, \tag{2.2}$$

$$w - \sigma = 0 \quad \text{on} \quad \partial\Omega, \tag{2.3}$$

$$\frac{\partial w}{\partial \gamma} - \frac{\partial \sigma}{\partial \gamma} = 0 \quad \text{on} \quad \partial\Omega, \tag{2.4}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded simply connected inhomogeneous medium, γ is the unit outward normal to $\partial\Omega$ and the index of refraction $n = n(x)$ is positive.

Let $W^{s,p}(\Omega)$ denote the usual Sobolev space with norm $\|\cdot\|_{s,p}$, $H^s(\Omega) = W^{s,2}(\Omega)$, and $\|\cdot\|_{s,2} = \|\cdot\|_s$, $H^0(\Omega) = L^2(\Omega)$ with the inner product $(u, v)_0 = \int_\Omega u\bar{v}dx$. Denote $H_0^2(\Omega) = \{v \in H^2(\Omega) : v\,|_{\partial\Omega} = \frac{\partial v}{\partial \gamma}\,|_{\partial\Omega} = 0\}$. Let $H^{-1}(\Omega)$ be the "negative space" with norm $\|v\|_{-1}$.

Define Hilbert space $\boldsymbol{H} = H_0^2(\Omega) \times L^2(\Omega)$ with norm $\|(v, z)\|_{\boldsymbol{H}} = \|v\|_2 + \|z\|_0$, and define $\boldsymbol{H}^1 = H_0^1(\Omega) \times H^{-1}(\Omega)$ with norm $\|(v, z)\|_{\boldsymbol{H}^1} = \|v\|_1 + \|z\|_{-1}$. Since $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ compactly and $H^2(\Omega) \hookrightarrow H^1(\Omega)$ compactly, $\boldsymbol{H} \hookrightarrow \boldsymbol{H}^1$ compactly.

In this paper, we suppose that $n \in W^{1,\infty}(\Omega)$ satisfying the condition $1 + \delta \leqslant n(x)$ in Ω for some constant $\delta > 0$. The argument is the same if $0 < n(x) \leqslant 1 - \varrho$ in Ω $(\varrho > 0)$ holds. Denote

$$\varpi(x) = \frac{1}{n(x) - 1}.$$

From [13, 35], we know that (2.1)–(2.4) can be written as the following equivalent weak formulation: Find $k \in \mathbb{C}$, $u \in H_0^2(\Omega)$ such that

$$(\varpi \Delta u, \Delta v)_0 = k^2 (\nabla u, \nabla(n \varpi v))_0 + k^2 (\nabla(\varpi u), \nabla v)_0 - k^4 (n \varpi u, v)_0, \quad \forall\, v \in H_0^2(\Omega).$$

Introduce an auxiliary variable $\omega = k^2 u$, and let $\lambda = k^2$. Then we arrive at a linear weak formulation (see [44, 45]): Find $\lambda \in \mathbb{C}$, $(u, \omega) \in \boldsymbol{H} \setminus \{0\}$ such that

$$A((u, \omega), (v, z)) = \lambda B((u, \omega), (v, z)), \quad \forall\, (v, z) \in \boldsymbol{H}, \tag{2.5}$$

where

$$A((u, \omega), (v, z)) = ((\varpi - \mu)\Delta u, \Delta v)_0 + \mu \int_\Omega D^2 u : D^2 \bar{v} dx + (\omega, z)_0 \tag{2.6}$$

with constant $\mu > 0$, $\varpi - \mu \geqslant 0$ and $D^2u : D^2\bar{v}$ is the inner product of the Hession matrices of u and $\bar{v}$. In addition,

$$B((u,\omega),(v,z)) = (\nabla(\varpi u), \nabla v)_0 + (\nabla u, \nabla(n\varpi v))_0 - (\omega, n\varpi v)_0 + (u,z)_0.$$

It is obvious that $A(\cdot,\cdot)$ is a self-adjoint, continuous sesquilinear form on $\boldsymbol{H}\times\boldsymbol{H}$,

$$A((v,z),(v,z)) \gtrsim \|(v,z)\|_{\boldsymbol{H}}^2,$$

and for any given $(f,g) \in \boldsymbol{H}^1$, $B((f,g),(v,z))$ is a continuous linear form on $\boldsymbol{H}$,

$$|B((f,g),(v,z))| \lesssim \|(f,g)\|_{\boldsymbol{H}^1}\|(v,z)\|_{\boldsymbol{H}^1}, \quad \forall\,(v,z) \in \boldsymbol{H}^1.$$

We use $A(\cdot,\cdot)$ and $\|\cdot\|_A = A(\cdot,\cdot)^{\frac{1}{2}}$ as an inner product and norm on $\boldsymbol{H}$, respectively.

The source problem associated with (2.5) is as follows: Find $(\psi,\varphi) \in \boldsymbol{H}$ such that

$$A((\psi,\varphi),(v,z)) = B((f,g),(v,z)), \quad \forall\,(v,z) \in \boldsymbol{H}. \tag{2.7}$$

From Lax-Milgram theorem we know that (2.7) has one and only one solution. Therefore, we define the corresponding solution operator $T : \boldsymbol{H}^1 \to \boldsymbol{H}$ by

$$A(T(f,g),(v,z)) = B((f,g),(v,z)), \quad \forall\,(v,z) \in \boldsymbol{H}. \tag{2.8}$$

Then (2.5) has the equivalent operator form

$$T(u,\omega) = \lambda^{-1}(u,\omega). \tag{2.9}$$

From (2.8) we have

$$\|T(f,g)\|_{\boldsymbol{H}} \lesssim \|(f,g)\|_{\boldsymbol{H}^1}, \quad \forall\,(f,g) \in \boldsymbol{H}^1. \tag{2.10}$$

Thus we know that $T : \boldsymbol{H} \to \boldsymbol{H}$ is compact, and $T : \boldsymbol{H}^1 \to \boldsymbol{H}^1$ is compact.

Consider the dual problem of (2.5): Find $\lambda^* \in \mathbb{C}$, $(u^*,\omega^*) \in \boldsymbol{H}\setminus\{0\}$ such that

$$A((v,z),(u^*,\omega^*)) = \overline{\lambda^*}B((v,z),(u^*,\omega^*)), \quad \forall\,(v,z) \in \boldsymbol{H}. \tag{2.11}$$

The source problem associated with (2.11) is as follows: Find $(\psi^*,\varphi^*) \in \boldsymbol{H}$ such that

$$A((v,z),(\psi^*,\varphi^*)) = B((v,z),(f,g)), \quad \forall\,(v,z) \in \boldsymbol{H}. \tag{2.12}$$

Define the corresponding solution operator $T^* : \boldsymbol{H}^1 \to \boldsymbol{H}$ by

$$A((v,z),T^*(f,g)) = B((v,z),(f,g)), \quad \forall\,(v,z) \in \boldsymbol{H}. \tag{2.13}$$

Then (2.11) has the equivalent operator form

$$T^*(u^*,\omega^*) = \lambda^{*-1}(u^*,\omega^*).$$

From (2.8) and (2.13) we know that T^* is the adjoint operator of T in the sense of inner product $A(\cdot,\cdot)$. So the primal and dual eigenvalues are connected via $\lambda = \overline{\lambda^*}$ (see [44]).

Denote $\mathbb{S} = (\frac{2d}{1+d}, 2]$. We need the following regularity assumption:

$R(\Omega)$: For any $\xi \in H^{-1}(\Omega)$, there exists $\psi \in W^{3,p_0}(\Omega)$ satisfying

$$\Delta(\varpi\Delta\psi) = \xi \quad \text{in} \quad \Omega, \quad \psi = \frac{\partial\psi}{\partial\gamma} = 0 \quad \text{on} \quad \partial\Omega,$$

and $\|\psi\|_{3,p_0} \lesssim_\Omega \|\xi\|_{-1}$, where $p_0 \in \mathbb{S}$, C_Ω denotes the prior constant dependent on the $n(x)$ and Ω but independent of the right-hand side ξ of the equation.

Let π_h be a shape-regular mesh; for any element $\kappa \in \pi_h$, let h_κ denote diameter of κ, and $h = \max_{\kappa\in\pi_h} h_\kappa$. In addition, let

$$S^h = \{v \in C(\bar{\Omega}) \cap H_0^1(\Omega) : v\,|_\kappa \in P_m, \forall\, \kappa \in \pi_h\},$$

where P_m is the set of all polynomials in d variables of degree $\leqslant m$ ($m \geqslant 2$). Let $\boldsymbol{H}_h = S^h \times S^h$. Then $\boldsymbol{H}_h \subset \boldsymbol{H}^1$ but $\boldsymbol{H}_h \not\subset \boldsymbol{H}$.

Let $p \in \mathbb{S}$. From the trace theorem with scaling we have the following trace inequality:

$$\int_{\partial\ell} |w|^2 ds \lesssim h_\kappa^{d-\frac{2d}{p}-1} \|w\|_{0,p,\kappa}^2 + h_\kappa^{2+d-\frac{2d}{p}-1} |w|_{1,p,\kappa}^2, \quad \forall\, \kappa \in \pi_h. \tag{2.14}$$

Let $\mathcal{E}$ denote the set of all $(d-1)$-faces in π_h ($d = 2, 3$). We decompose $\mathcal{E} = \mathcal{E}^i \cup \mathcal{E}^b$ where $\mathcal{E}^i$ and $\mathcal{E}^b$ refer to interior faces and faces on the boundary $\partial\Omega$, respectively. For each $\ell \in \mathcal{E}^i$, we choose an arbitrary unit normal vector γ_ℓ and denote the two triangles sharing this face by κ_- and κ_+, where γ_ℓ points from κ_- to κ_+. We set the jump and average on ℓ as

$$\begin{aligned}
&[[\partial v/\partial\gamma_\ell]] = \nabla(v\,|_{\kappa_+}) \cdot \gamma_\ell - \nabla(v\,|_{\kappa_-}) \cdot \gamma_\ell, \quad \{\{\partial^2 v/\partial\gamma_\ell^2\}\} = \frac{1}{2}((\partial^2 v/\partial\gamma_\ell^2)\,|_{\kappa_-} + (\partial^2 v/\partial\gamma_\ell^2)\,|_{\kappa_+}),\\
&\{\{(\varpi-\mu)\Delta v\}\} = \frac{1}{2}((\varpi-\mu)\Delta v\,|_{\kappa_-} + (\varpi-\mu)\Delta v\,|_{\kappa_+})
\end{aligned}$$

with $\partial^2 v/\partial\gamma_\ell^2 = \gamma_\ell \cdot (D^2 v)\gamma_\ell$.

For any $\ell \in \mathcal{E}^b$ which is a face of κ, we take γ_ℓ to be the unit normal vector pointing towards the outside of Ω and set

$$[[\partial v/\partial\gamma_\ell]] = -\gamma_\ell \cdot \nabla(v\,|_\kappa), \quad \{\{\partial^2 v/\partial\gamma_\ell^2\}\} = (\partial^2 v/\partial\gamma_\ell^2)\,|_\kappa, \quad \{\{(\varpi-\mu)\Delta v\}\} = (\varpi-\mu)\Delta v\,|_\kappa.$$

Define piecewise Sobolev space

$$W^{3,p}(\Omega,\pi_h) = \{v \in C(\bar{\Omega}) \cap H_0^1(\Omega) : v\,|_\kappa \in W^{3,p}(\kappa),\ \forall\, \kappa \in \pi_h\}, \quad p \in \mathbb{S}.$$

Referring to [5, 23, 43], we define

$$\begin{aligned}
A_h((u,\omega),(v,z)) = &\sum_{\kappa\in\pi_h} \int_\kappa (\varpi-\mu)\Delta u \Delta\overline{v} dx + \mu \int_\kappa D^2 u : D^2\overline{v} dx\\
&+ \sum_{\ell\in\mathcal{E}} \int_\ell \{\{(\varpi-\mu)\Delta u\}\}[[\partial\overline{v}/\partial\gamma_\ell]] + \{\{(\varpi-\mu)\Delta v\}\}[[\partial\overline{u}/\partial\gamma_\ell]] ds\\
&+ \mu \sum_{\ell\in\mathcal{E}} \int_\ell \left\{\left\{\frac{\partial^2 u}{\partial\gamma_\ell^2}\right\}\right\}[[\partial\overline{v}/\partial\gamma_\ell]] + \{\{\partial^2 v/\partial\gamma_\ell^2\}\}[[\partial\overline{u}/\partial\gamma_\ell]] ds\\
&+ \sigma \sum_{\ell\in\mathcal{E}} \frac{1}{\hat{\ell}} \int_\ell [[\partial u/\partial\gamma_\ell]][[\partial\overline{v}/\partial\gamma_\ell]] ds + \sum_{\kappa\in\pi_h} \int_\kappa \omega\overline{z} dx, \quad \forall\, u, v \in W^{3,p}(\Omega,\pi_h),
\end{aligned} \tag{2.15}$$

where $\sigma > 1$ is the penalty parameter, and $\hat{\ell} = h_\ell$ is the diameter of ℓ.

We give the following C^0IPG discrete scheme of (2.5): Find $\lambda_h \in \mathbb{C}$, $(u_h, \omega_h) \in \boldsymbol{H}_h \setminus \{0\}$ such that

$$A_h((u_h,\omega_h),(v,z)) = \lambda_h B((u_h,\omega_h),(v,z)), \quad \forall\, (v,z) \in \boldsymbol{H}_h. \tag{2.16}$$

We define the mesh-dependent norms $\|\cdot\|_h$ and $|||\cdot|||_h$ on $W^{3,p}(\Omega,\pi_h) \times L^2(\Omega)$ as

$$\|(u,\omega)\|_h^2 = \sum_{\kappa\in\pi_h} |u|_{2,\kappa}^2 + \sigma \sum_{\ell\in\mathcal{E}} \frac{1}{\hat{\ell}} \|[[\partial u/\partial\gamma_\ell]]\|_{0,\ell}^2 + \sum_{\kappa\in\pi_h} \|\omega\|_{0,\kappa}^2, \tag{2.17}$$

$$|||(u,\omega)|||_h^2 = \|(u,\omega)\|_h^2 + \frac{1}{\sigma} \sum_{\ell\in\mathcal{E}} \|\{\{\Delta u\}\}\|_{0,\ell}^2 \hat{\ell} + \frac{1}{\sigma} \sum_{\ell\in\mathcal{E}} \|\{\{\partial^2 u/\partial\gamma_\ell^2\}\}\|_{0,\ell}^2 \hat{\ell}. \tag{2.18}$$

By the trace inequality (2.14) with $p=2$ and the inverse estimates, we have

$$\|\Delta v\|_{0,\ell} \lesssim \hat{\ell}^{-\frac{1}{2}}|v|_{2,\kappa}, \quad \|\{\{\partial^2 v/\partial\gamma_\ell^2\}\}\|_{0,\ell} \lesssim \hat{\ell}^{-\frac{1}{2}}|v|_{2,\kappa}, \quad \forall\, v \in S^h. \tag{2.19}$$

So on $\boldsymbol{H}_h$ the two norms $\|\cdot\|_h$ and $|||\cdot|||_h$ are equivalent.

For any $(u,\omega),(v,z) \in W^{3,p}(\Omega,\pi_h)\times L^2(\Omega)$, by the Schwarz inequality we can deduce

$$\begin{aligned}
&|A_h((u,\omega),(v,z))|\\
&\lesssim \sum_{\kappa\in\pi_h}\|\Delta u\|_{0,\kappa}\|\Delta\overline{v}\|_{0,\kappa} + \sum_{\kappa\in\pi_h}|u|_{2,\kappa}|\overline{v}|_{2,\kappa}\\
&\quad + \sum_{\ell\in\mathcal{E}}\left(\sqrt{\frac{\hat{\ell}}{\sigma}}\|\{\{\Delta u\}\}\|_{0,\ell}\sqrt{\frac{\sigma}{\hat{\ell}}}\|[[\partial\overline{v}/\partial\gamma_\ell]]\|_{0,\ell} + \sqrt{\frac{\hat{\ell}}{\sigma}}\|\{\{\Delta v\}\}\|_{0,\ell}\sqrt{\frac{\sigma}{\hat{\ell}}}\|[[\partial\overline{u}/\partial\gamma_\ell]]\|_{0,\ell}\right)\\
&\quad + \sum_{\ell\in\mathcal{E}}\left(\sqrt{\frac{\hat{\ell}}{\sigma}}\|\{\{\partial^2 u/\partial\gamma_\ell^2\}\}\|_{0,\ell}\sqrt{\frac{\sigma}{\hat{\ell}}}\|[[\partial\overline{v}/\partial\gamma_\ell]]\|_{0,\ell} + \sqrt{\frac{\hat{\ell}}{\sigma}}\|\{\{\partial^2 v/\partial\gamma_\ell^2\}\}\|_{0,\ell}\sqrt{\frac{\sigma}{\hat{\ell}}}\|[[\partial\overline{u}/\partial\gamma_\ell]]\|_{0,\ell}\right)\\
&\quad + \sigma\sum_{\ell\in\mathcal{E}}\frac{1}{\sqrt{\hat{\ell}}}\|[[\partial u/\partial\gamma_\ell]]\|_{0,\ell}\frac{1}{\sqrt{\hat{\ell}}}\|[[\partial\overline{v}/\partial\gamma_\ell]]\|_{0,\ell} + \sum_{\kappa\in\pi_h}\|\omega\|_{0,\kappa}\|\overline{z}\|_{0,\kappa}\\
&\lesssim |||(u,\omega)|||_h|||(v,z)|||_h.
\end{aligned}\tag{2.20}$$

In addition, for any $(u_h,\omega_h),(v,z)\in\boldsymbol{H}_h$, we have

$$|A_h((u_h,\omega_h),(v,z))| \lesssim \|(u_h,\omega_h)\|_h\|(v,z)\|_h. \tag{2.21}$$

By (2.19), we know there exists a constant C_1 such that for any $u_h\in S_h$,

$$\|\{\{(\varpi-\mu)\Delta u_h\}\}\|_{0,\ell} \leqslant \frac{C_1}{2}\hat{\ell}^{-\frac{1}{2}}|u_h|_{2,\kappa}, \quad \|\{\{\partial^2 u_h/\partial\gamma_\ell^2\}\}\|_{0,\ell} \leqslant \frac{C_1}{2}\hat{\ell}^{-\frac{1}{2}}|u_h|_{2,\kappa},$$

and referring to [23,24], when σ is large enough, for $\forall\,(u_h,\omega_h)\in\boldsymbol{H}_h$, we deduce by the Young's inequality

$$\begin{aligned}
A_h((u_h,\omega_h),(u_h,\omega_h)) &\geqslant \mu\sum_{\kappa\in\pi_h}|u_h|_{2,\kappa}^2 - \sqrt{\mu}\left(\sum_{\kappa\in\pi_h}|u_h|_{2,\kappa}^2\right)^{\frac{1}{2}}\frac{C_1}{\sqrt{\mu}}\left(\sum_{\ell\in\mathcal{E}}\frac{1}{\hat{\ell}}\|[[\partial u_h/\partial\gamma_\ell]]\|_{0,\ell}^2\right)^{\frac{1}{2}}\\
&\quad + \sigma\sum_{\ell\in\mathcal{E}}\frac{1}{\hat{\ell}}\|[[\partial u_h/\partial\gamma_\ell]]\|_{0,\ell}^2 + \sum_{\kappa\in\pi_h}\|\omega_h\|_{0,\kappa}^2\\
&\geqslant \mu\sum_{\kappa\in\pi_h}|u_h|_{2,\kappa}^2 - \frac{1}{2}\left(\mu\left(\sum_{\kappa\in\pi_h}|u_h|_{2,\kappa}^2\right) + \frac{C_1^2}{\mu}\left(\sum_{\ell\in\mathcal{E}}\frac{1}{\hat{\ell}}\|[[\partial u_h/\partial\gamma_\ell]]\|_{0,\ell}^2\right)\right)\\
&\quad + \sigma\sum_{\ell\in\mathcal{E}}\frac{1}{\hat{\ell}}\|[[\partial u_h/\partial\gamma_\ell]]\|_{0,\ell}^2 + \sum_{\kappa\in\pi_h}\|\omega_h\|_{0,\kappa}^2\\
&\geqslant \frac{\mu}{2}\sum_{\kappa\in\pi_h}|u_h|_{2,\kappa}^2 + \left(\sigma - \frac{C_1^2}{2\mu}\right)\sum_{\ell\in\mathcal{E}}\frac{1}{\hat{\ell}}\|[[\partial u_h/\partial\gamma_\ell]]\|_{0,\ell}^2 + \sum_{\kappa\in\pi_h}\|\omega_h\|_{0,\kappa}^2\\
&\gtrsim \|(u_h,\omega_h)\|_h^2.
\end{aligned}\tag{2.22}$$

Consider the C^0IPG discrete scheme of (2.7): Find $(\psi_h,\varphi_h)\in\boldsymbol{H}_h$ such that

$$A_h((\psi_h,\varphi_h),(v,z)) = B((f,g),(v,z)), \quad \forall\,(v,z)\in\boldsymbol{H}_h. \tag{2.23}$$

We introduce the corresponding solution operator $T_h:\boldsymbol{H}^1\to\boldsymbol{H}_h$ satisfying

$$A_h(T_h(f,g),(v,z)) = B((f,g),(v,z)), \quad \forall\,(v,z)\in\boldsymbol{H}_h. \tag{2.24}$$

Then (2.16) has the operator form $T_h(u_h,\omega_h) = \lambda_h^{-1}(u_h,\omega_h)$.

The C^0IPG discrete scheme of (2.11) is given by: Find $\lambda_h^* \in \mathbb{C}$, $(u_h^*, \omega_h^*) \in \boldsymbol{H}_h \setminus \{0\}$ such that

$$A_h((v,z),(u_h^*,\omega_h^*)) = \overline{\lambda_h^*} B((v,z),(u_h^*,\omega_h^*)), \quad \forall\,(v,z) \in \boldsymbol{H}_h. \tag{2.25}$$

Define the solution operator $T_h^* : \boldsymbol{H}^1 \to \boldsymbol{H}_h$ satisfying

$$A_h((v,z),T_h^*(f,g)) = B((v,z),(f,g)), \quad \forall\,(v,z) \in \boldsymbol{H}_h. \tag{2.26}$$

Thus (2.25) has the equivalent operator form $T_h^*(u_h^*,\omega_h^*) = \lambda_h^{*-1}(u_h^*,\omega_h^*)$.

It can be proved that T_h^* is the adjoint operator of T_h in the sense of inner product $A_h(\cdot,\cdot)$. In fact, $\forall\,(u,\omega),(v,z) \in \boldsymbol{H}_h$, from (2.24) and (2.26) we have

$$A_h(T_h(u,\omega),(v,z)) = B((u,\omega),(v,z)) = A_h((u,\omega),T_h^*(v,z)).$$

Hence, the primal and dual eigenvalues are connected via $\lambda_h = \overline{\lambda_h^*}$.

In this paper, we suppose that $\{\lambda_j\}$ and $\{\lambda_{j,h}\}$ are enumerations of the eigenvalues of (2.5) and (2.16), respectively according to the same sort rule, each repeated as many times as its multiplicity, and $\lambda = \lambda_i$ is the i-th eigenvalue with the algebraic multiplicity q and the ascent α, $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+q-1}$, and $\lambda_h = \lambda_{i,h}$. When $\|T_h - T\|_{\boldsymbol{H}^1} \to 0$ as $h \to 0$, $\lambda_{j,h} \to 0$ as $h \to 0$ for $j = i, i+1, \ldots, i+q-1$.

Let P be the spectral projection associated with T and λ. Then $\mathrm{ran}(P) = \mathrm{null}((\lambda^{-1} - T)^\alpha)$ is the space of generalized eigenfunctions associated with λ and T, where ran denotes the range and null denotes the null space. Let P_h be the spectral projection associated with T_h and the eigenvalues $\lambda_{i,h}, \ldots, \lambda_{i+q-1,h}$. Then $\mathrm{ran}(P_h)$ is the space spanned by all generalized eigenfunctions corresponding to all eigenvalues $\lambda_{i,h}, \ldots, \lambda_{i+q-1,h}$. In view of the adjoint problems (2.11) and (2.25), the definitions of P^*, $\mathrm{ran}(P^*)$, P_h^* and $\mathrm{ran}(P_h^*)$ are analogous to P, $\mathrm{ran}(P)$, P_h and $\mathrm{ran}(P_h)$, respectively (see [3]).

The error estimate of the C^0IPG method for eigenvalue problems is based on the error estimate of the C^0IPG method for the corresponding source problems. Next, using argument as in [43] we will prove the a priori error estimates for the source problem (2.7).

From [43, Lemma 3.1], we know that (2.7) admits a unique solution $(\psi,\varphi) \in (W^{3,p_0}(\Omega) \cap H_0^2(\Omega)) \times H_0^1(\Omega)$ and

$$\|(\psi,\varphi)\|_{W^{3,p_0}\times H_0^1} \lesssim C_R \|(f,g)\|_{\boldsymbol{H}^1}, \quad \forall\,(f,g) \in \boldsymbol{H}^1,$$

where $p_0 \in \mathbb{S}$, and C_R denotes the prior constant.

Denote $A((u,\omega),(v,z)) \equiv a(u,v) + (\omega,z)_0$, $A_h((u,\omega),(v,z)) \equiv a_h(u,v) + (\omega,z)_0$, $\|(u,\omega)\|_h \equiv \|u\|_h^2 + \|\omega\|_0^2$, $|||(u,\omega)|||_h \equiv |||u|||_h^2 + \|\omega\|_0^2$, and $B'(f,v) = \int_\Omega \nabla f \cdot \nabla \bar{v} dx$. Define the auxiliary operator $K : H_0^1(\Omega) \to H_0^2(\Omega)$ by

$$a(Kf,v) = B'(f,v), \quad \forall\,v \in H_0^2(\Omega). \tag{2.27}$$

Then for any $f \in H_0^1(\Omega)$, it is valid that $Kf \in W^{3,p_0}(\Omega)$ and

$$\|Kf\|_{3,p_0} \lesssim \|f\|_1. \tag{2.28}$$

Referring to [43, (3.7)–(3.9)], we can deduce

$$A_h((\psi,\varphi),(v,z)) = B((f,g),(v,z)), \quad \forall\,(v,z) \in \boldsymbol{H}_h, \tag{2.29}$$

$$a_h(Kf,v) = B'(f,v), \quad \forall\,v \in S^h. \tag{2.30}$$

From (2.29) and (2.23) we get

$$A_h((\psi,\varphi) - (\psi_h,\varphi_h),(v,z)) = 0, \quad \forall\,(v,z) \in \boldsymbol{H}_h. \tag{2.31}$$

Define the operator $I_h(\psi,\varphi) = (I_h^1\psi, I_h^2\varphi)$, where $I_h^1 : H_0^1(\Omega) \cap C^0(\overline{\Omega}) \to S^h$ is the Lagrange nodal interpolation operator and $I_h^2 : L^2(\Omega) \to S^h$ is defined by

$$(\varphi - I_h^2\varphi, z)_0 = 0, \quad \forall\,z \in S^h.$$

From [43, Lemma 3.3], for any $(\psi,\varphi)\in W^{m+1,p}(\Omega)\times W^{m-1,2}(\Omega)$, the following estimates hold:

$$|||(\psi,\varphi)-I_h(\psi,\varphi)|||_h \lesssim h^{m-1+(\frac{1}{2}-\frac{1}{p})d}(\|\psi\|_{m+1,p,\Omega}+\|\varphi\|_{m-1,\Omega}), \tag{2.32}$$

$$\|(\psi,\varphi)-I_h(\psi,\varphi)\|_{\boldsymbol{H}^1} \lesssim h^{m+(\frac{1}{2}-\frac{1}{p})d}(\|\psi\|_{m+1,p,\Omega}+\|\varphi\|_{m-1,\Omega}). \tag{2.33}$$

From a Poincaré-Friedrichs inequality (see [10]), we get

$$\|(v,z)\|_{\boldsymbol{H}^1}=\|v\|_1+\|z\|_{-1}\lesssim\|v\|_h+\|z\|_0\lesssim\|(v,z)\|_h,\quad\forall\,(v,z)\in\boldsymbol{H}_h.$$

Let $(v,z)=T_h(f,g)$ in (2.24). We get

$$\|T_h(f,g)\|_h\lesssim\|(f,g)\|_{\boldsymbol{H}^1},\quad\forall\,(f,g)\in\boldsymbol{H}^1.$$

Lemma 2.1. *Let (ψ,φ) and (ψ^*,φ^*) be the solutions of* (2.7) *and* (2.12), *respectively, and let (ψ_h,φ_h) and (ψ_h^*,φ_h^*) be the C^0IPG approximation solutions of* (2.7) *and* (2.12), *respectively. Assume that (ψ,φ), $(\psi^*,\varphi^*)\in W^{m+1,p}(\Omega)\times H^{m-1}(\Omega)$ $(p\in\mathbb{S})$. Then*

$$|||(\psi,\varphi)-(\psi_h,\varphi_h)|||_h\lesssim h^{m-1+(\frac{1}{2}-\frac{1}{p})d}(\|\psi\|_{m+1,p}+\|\varphi\|_{m-1}), \tag{2.34}$$

$$|||(\psi^*,\varphi^*)-(\psi_h^*,\varphi_h^*)|||_h\lesssim h^{m-1+(\frac{1}{2}-\frac{1}{p})d}(\|\psi^*\|_{m+1,p}+\|\varphi^*\|_{m-1}). \tag{2.35}$$

Furthermore, assume $R(\Omega)$ holds. Then

$$\|(\psi,\varphi)-(\psi_h,\varphi_h)\|_{\boldsymbol{H}^1}\lesssim h^{m+(1-\frac{1}{p}-\frac{1}{p_0})d}(\|\psi\|_{m+1,p}+\|\varphi\|_{m-1}), \tag{2.36}$$

$$\|(\psi^*,\varphi^*)-(\psi_h^*,\varphi_h^*)\|_{\boldsymbol{H}^1}\lesssim h^{m+(1-\frac{1}{p}-\frac{1}{p_0})d}(\|\psi^*\|_{m+1,p}+\|\varphi^*\|_{m-1}). \tag{2.37}$$

Proof. From (2.22), (2.31), (2.20) and (2.32), we deduce

$$\begin{aligned}\|I_h(\psi,\varphi)-(\psi_h,\varphi_h)\|_h^2&\lesssim A_h(I_h(\psi,\varphi)-(\psi_h,\varphi_h),I_h(\psi,\varphi)-(\psi_h,\varphi_h))\\&=A_h(I_h(\psi,\varphi)-(\psi,\varphi),I_h(\psi,\varphi)-(\psi_h,\varphi_h))\\&\lesssim h^{m-1+d(\frac{1}{2}-\frac{1}{p})}(\|\psi\|_{m+1,p}+\|\varphi\|_{m-1})\|I_h(\psi,\varphi)-(\psi_h,\varphi_h)\|_h.\end{aligned}$$

Thus we get

$$\begin{aligned}|||(\psi,\varphi)-(\psi_h,\varphi_h)|||_h&\leqslant|||(\psi,\varphi)-I_h(\psi,\varphi)|||_h+|||I_h(\psi,\varphi)-(\psi_h,\varphi_h)|||_h\\&\lesssim h^{m-1+d(\frac{1}{2}-\frac{1}{p})}(\|\psi\|_{m+1,p}+\|\varphi\|_{m-1}),\end{aligned}$$

which is the desired result (2.34). By the same argument we can prove (2.35).

Denote $e=\psi-\psi_h$. From (2.27), (2.30), (2.31) with $z=0$, (2.20), (2.34), (2.32) and (2.28), we deduce

$$\begin{aligned}|B'(e,e)|&=|a_h(Ke,e)|=|a_h(e,Ke-I_h^1Ke)|\lesssim|||e|||_h|||Ke-I_h^1Ke|||_h\\&\lesssim h^{m-1+(\frac{1}{2}-\frac{1}{p})d}\|\psi\|_{m+1,p}h^{1+(\frac{1}{2}-\frac{1}{p_0})d}\|Ke\|_{3,p_0}\lesssim h^{m+(1-\frac{1}{p}-\frac{1}{p_0})d}\|\psi\|_{m+1,p}\|e\|_1,\end{aligned}$$

i.e.,

$$\|e\|_1\lesssim h^{m+(1-\frac{1}{p}-\frac{1}{p_0})d}\|\psi\|_{m+1,p}. \tag{2.38}$$

From (2.7) and (2.23) we have $\varphi=f\in H_0^1(\Omega)$ and $(\varphi-\varphi_h,z)_0=0$, $\forall\,z\in S^h$. So

$$\|\varphi-\varphi_h\|_{-1}\lesssim h^m\|\varphi\|_{m-1}. \tag{2.39}$$

From (2.38) and (2.39) we get the desired result (2.36). By the same argument we can prove (2.37). The proof is completed. □

Based on Lemma 2.1, using the argument as [43, Theorems 3.3 and 3,4] we can prove the following a priori error estimates for the eigenvalue problem.

Theorem 2.2. *Assume that $R(\Omega)$ holds and $n \in W^{1,\infty}(\Omega) \cap H^2(\Omega)$. Then*

$$\left| \left(\frac{1}{q} \sum_{j=i}^{i+q-1} \lambda_{j,h}^{-1} \right)^{-1} - \lambda \right| \lesssim \|(T - T_h)\,|_{\mathrm{ran}(P)}\|_{\boldsymbol{H}^1}.$$

Assume $\mathrm{ran}(P) \subset W^{m+1,p}(\Omega) \times H^{m-1}(\Omega)$ $(p \in \mathbb{S})$. *Then*

$$\|(T - T_h)\,|_{\mathrm{ran}(P)}\|_{\boldsymbol{H}^1} \lesssim h^{m+(1-\frac{1}{p}-\frac{1}{p_0})d}.$$

Furthermore, assume that (u_h, ω_h) is an eigenfunction corresponding to λ_h and $\|(u_h, \omega_h)\|_h = 1$. Then there exists eigenfunction (u, ω) corresponding to λ such that

$$\|(u_h, \omega_h) - (u, \omega)\|_{\boldsymbol{H}^1} \lesssim h^{\frac{m}{\alpha}+(1-\frac{1}{p}-\frac{1}{p_0})\frac{d}{\alpha}}, \tag{2.40}$$

$$|||(u_h, \omega_h) - (u, \omega)|||_h \lesssim h^{\frac{m-1}{\alpha}+(\frac{1}{2}-\frac{1}{p})\frac{d}{\alpha}}. \tag{2.41}$$

In addition, when the ascent α of the eigenvalue λ equals 1, for $(u^*, \omega^*) \in \mathrm{ran}(P^*)$ with $\|(u^*, \omega^*)\|_h = 1$, there exists $(u_h^*, \omega_h^*) \in \mathrm{ran}(P_h^*)$ such that

$$\|(u^*, \omega^*) - (u_h^*, \omega_h^*)\|_{\boldsymbol{H}^1} \lesssim h^{m+(1-\frac{1}{p}-\frac{1}{p_0})d}, \tag{2.42}$$

$$|||(u^*, \omega^*) - (u_h^*, \omega_h^*)|||_h \lesssim h^{m-1+(\frac{1}{2}-\frac{1}{p})d}; \tag{2.43}$$

for $(u_h^*, \omega_h^*) \in \mathrm{ran}(P_h^*)$ with $\|(u_h^*, \omega_h^*)\|_h = 1$, there exists $(u^*, \omega^*) \in \mathrm{ran}(P^*)$ such that

$$\|(u_h^*, \omega_h^*) - (u^*, \omega^*)\|_{\boldsymbol{H}^1} \lesssim h^{m+(1-\frac{1}{p}-\frac{1}{p_0})d}, \tag{2.44}$$

$$|||(u_h^*, \omega_h^*) - u^*, \omega^*)|||_h \lesssim h^{m-1+(\frac{1}{2}-\frac{1}{p})d}, \tag{2.45}$$

$$|\lambda_h - \lambda| \lesssim h^{2m-2+2(\frac{1}{2}-\frac{1}{p})d}. \tag{2.46}$$

3 A posteriori error analysis of C^0IPG discrete scheme for the source problem (2.7)

In 2012, Brenner [5] proposed and analyzed the a posteriori error estimates of C^0IPG methods for biharmonic equation. Based on [5], in this section we discuss a posteriori error estimates of C^0IPG discrete scheme (2.23) for the source problem (2.7) in $\mathbb{R}^2$.

Denote

$$F = F(f, g) = -\Delta(\varpi f) - n\varpi\Delta f - n\varpi g \quad \text{in} \quad \kappa,$$

where $f, g \in W^{3,p}(\Omega, \pi_h)$, and denote

$$\eta_\kappa(F, \psi_h) = h_\kappa^2 \|F - \Delta(\varpi\Delta\psi_h)\|_{0,\kappa}, \quad \forall\, \kappa \in \pi_h, \tag{3.1}$$

$$\eta_{\ell,1}(\psi_h) = \frac{\sigma}{\hat{\ell}^{\frac{1}{2}}} \|[[\partial\psi_h/\partial\gamma_\ell]]\|_{0,\ell}, \quad \forall\, \ell \in \mathcal{E}, \tag{3.2}$$

$$\eta_{\ell,2}(\psi_h) = \mu\hat{\ell}^{\frac{1}{2}} \|[[\partial^2\psi_h/\partial\gamma_\ell^2]]\|_{0,\ell}, \quad \forall\, \ell \in \mathcal{E}^i, \tag{3.3}$$

$$\eta_{\ell,3}(\psi_h) = \hat{\ell}^{\frac{3}{2}} \|[[\partial(\varpi\Delta\psi_h)/\partial\gamma_\ell]]\|_{0,\ell}, \quad \forall\, \ell \in \mathcal{E}^i, \tag{3.4}$$

$$\eta_{\ell,4}(\psi_h) = \hat{\ell}^{\frac{1}{2}} \|[[(\varpi - \mu)\Delta\psi_h]]\|_{0,\ell}, \quad \forall\, \ell \in \mathcal{E}^i. \tag{3.5}$$

Then the residual-based error indicator η_h is defined by

$$\eta_h^2(F, \psi_h, \varphi_h, \kappa) = \eta_\kappa^2(F, \psi_h) + \sum_{\ell \in \mathcal{E}^b \cap \partial\kappa} \eta_{\ell,1}^2(\psi_h) + \frac{1}{2} \sum_{\ell \in \mathcal{E}^i \cap \kappa} \{\eta_{\ell,1}^2(\psi_h) + \eta_{\ell,2}^2(\psi_h) + \eta_{\ell,3}^2(\psi_h) + \eta_{\ell,4}^2(\psi_h)\}$$

$$+\sum_{\ell\in\mathcal{E}\cap\partial\kappa}\|1+2\varpi\|_{0,\ell}h_\ell^4\eta_{\ell,1}^2(f)+\|f-\varphi_h\|_{0,\kappa}^2+h^4\sum_{\ell\in\mathcal{E}\cap\partial\kappa}\eta_{\ell,1}^2(\varphi_h), \tag{3.6}$$

$$\eta_h^2(F,\psi_h,\varphi_h,\Omega)=\sum_{\kappa\in\pi_h}\eta_h^2(F,\psi_h,\varphi_h,\kappa). \tag{3.7}$$

Let $P_j(\Omega,\pi_h)$ be the space of piecewise polynomial functions of degree $\leqslant j$ and $\widetilde{g}\in P_j(\Omega,\pi_h)$ denote the L^2 orthogonal projection of g. In addition, denote

$$\widehat{F}=-\Delta(\widetilde{\varpi f})-\widetilde{n\varpi\Delta}f-\widetilde{n\varpi}g,\quad \widehat{\eta}_\kappa(F,\psi_h)=h_\kappa^2\|\widehat{F}-\Delta(\widetilde{\varpi}\Delta\psi_h)\|_{0,\kappa},\quad \forall\,\kappa\in\pi_h,$$
$$\widehat{\eta}_{\ell,3}(\psi_h)=\hat{\ell}^{\frac{3}{2}}\|[[\partial(\widetilde{\varpi}\Delta\psi_h)/\partial\gamma_\ell]]\|_{0,\ell},\quad \forall\,\ell\in\mathcal{E}^i,\quad \widehat{\eta}_{\ell,4}(\psi_h)=\hat{\ell}^{\frac{1}{2}}\|[[(\widetilde{\varpi-\mu})\Delta\psi_h]]\|_{0,\ell},\quad \forall\,\ell\in\mathcal{E}^i.$$

The data oscillations are defined by

$$\mathrm{Osc}_j(F)=\bigg(\sum_{\kappa\in\pi_h}h_\kappa^4\|F-\widehat{F}\|_{0,\kappa}^2\bigg)^{\frac{1}{2}},$$
$$\mathrm{Osc}_j(\Delta(\varpi\Delta\psi_h))=\bigg(\sum_{\kappa\in\pi_h}h_\kappa^4\|\Delta(\widetilde{\varpi}\Delta\psi_h)-\Delta(\varpi\Delta\psi_h)\|_{0,\kappa}^2\bigg)^{\frac{1}{2}},$$
$$\mathrm{Osc}_j(\eta_{\ell,3})=\bigg(\sum_{\kappa\in\pi_h}\sum_{\ell\in\mathcal{E}^i\cap\partial\kappa}(\eta_{\ell,3}(\psi_h)-\widehat{\eta}_{\ell,3}(\psi_h))^2\bigg)^{\frac{1}{2}},$$
$$\mathrm{Osc}_j(\eta_{\ell,4})=\bigg(\sum_{\kappa\in\pi_h}\sum_{\ell\in\mathcal{E}^i\cap\partial\kappa}(\eta_{\ell,4}(\psi_h)-\widehat{\eta}_{\ell,4}(\psi_h))^2\bigg)^{\frac{1}{2}},$$
$$\mathrm{OSC}_j(F,\psi_h)=\mathrm{Osc}_j(F)+\mathrm{Osc}_j(\eta_{\ell,3})+\mathrm{Osc}_j(\eta_{\ell,4})+\mathrm{Osc}_j(\Delta(\varpi\Delta\psi_h).$$

3.1 Reliability analysis

Using the argument in [5, Theorem 7], we can prove the following theorem.

Theorem 3.1. *Let (ψ,φ) and (ψ_h,φ_h) be the solutions of* (2.7) *and* (2.23)*, respectively. Assume that $R(\Omega)$ holds and $n\in W^{1,\infty}(\Omega)\cap H^2(\Omega)$. Then*

$$\|(\psi,\varphi)-(\psi_h,\varphi_h)\|_h\lesssim\eta_h(F,\psi_h,\varphi_h,\Omega). \tag{3.8}$$

Proof. Brenner [5, (4.4)] introduced the enriching operator $E_h:S^h\to H^2(\Omega)$ and proved

$$\sum_{\kappa\in\pi_h}(h_\kappa^{-4}\|v-E_hv\|_{0,\kappa}^2+h_\kappa^{-2}|v-E_hv|_{1,\kappa}^2+|v-E_hv|_{2,\kappa}^2)\lesssim\sum_{\kappa\in\pi_h}\frac{1}{\hat{\ell}}\|[[\partial v/\partial\gamma_\ell]]\|_{0,\ell}^2,\quad \forall\,v\in S^h. \tag{3.9}$$

Denote $\boldsymbol{E}_h(u_h,\omega_h)=(E_hu_h,E_h\omega_h)$. Due to (2.17) we need to bound $\sigma\sum_{\ell\in\mathcal{E}}\frac{1}{\hat{\ell}}\|[[\partial(\psi-\psi_h)/\partial\gamma_\ell]]\|_{0,\ell}^2$ and $\sum_{\kappa\in\pi_h}\|\psi-\psi_h\|_{2,\kappa}^2+\sum_{\kappa\in\pi_h}\|\varphi-\varphi_h\|_{0,\kappa}^2$.

Since $\sigma>1$, from (3.2) we get

$$\sigma\sum_{\ell\in\mathcal{E}}\frac{1}{\hat{\ell}}\|[[\partial(\psi-\psi_h)/\partial\gamma_\ell]]\|_{0,\ell}^2=\sigma\sum_{\ell\in\mathcal{E}}\frac{1}{\hat{\ell}}\|[[\partial\psi_h/\partial\gamma_\ell]]\|_{0,\ell}^2\leqslant\sum_{\ell\in\mathcal{E}}\eta_{\ell,1}^2. \tag{3.10}$$

From (3.9) and (3.2) we have

$$\begin{aligned}
&\sum_{\kappa\in\pi_h}\|\psi-\psi_h\|_{2,\kappa}^2+\sum_{\kappa\in\pi_h}\|\varphi-\varphi_h\|_{0,\kappa}^2\\
&\quad\leqslant 2\sum_{\kappa\in\pi_h}(\|\psi-E_h\psi_h\|_{2,\kappa}^2+\|\psi_h-E_h\psi_h\|_{2,\kappa}^2)+2\sum_{\kappa\in\pi_h}(\|\varphi-E_h\varphi_h\|_{0,\kappa}^2+\|\varphi_h-E_h\varphi_h\|_{0,\kappa}^2)\\
&\quad\lesssim\|(\psi,\varphi)-\boldsymbol{E}_h(\psi_h,\varphi_h)\|_{\boldsymbol{H}}^2+\sum_{\ell\in\mathcal{E}}\frac{1}{\hat{\ell}}\eta_{\ell,1}^2(\psi_h)+h^4\sum_{\ell\in\mathcal{E}}\eta_{\ell,1}^2(\varphi_h).
\end{aligned} \tag{3.11}$$

By duality we have

$$\|(\psi,\varphi)-\boldsymbol{E}_h(\psi_h,\varphi_h)\|_{\boldsymbol{H}} \approx \sup_{(v,z)\in\boldsymbol{H}\setminus\{0\}} \frac{A((\psi,\varphi)-\boldsymbol{E}_h(\psi_h,\varphi_h),(v,z))}{\|(v,z)\|_{\boldsymbol{H}}}. \tag{3.12}$$

Denote

$$A_\kappa((u,\omega),(v,z)) = \int_\kappa (\varpi-\mu)\Delta u\Delta\overline{v}dx + \mu\int_\kappa D^2u : D^2\overline{v}dx + \int_\kappa \omega\overline{z}dx. \tag{3.13}$$

From (2.6), (3.13), (2.7) and (2.23) we get

$$\begin{aligned}
&A((\psi,\varphi)-\boldsymbol{E}_h(\psi_h,\varphi_h),(v,z))\\
&\quad=\sum_{\kappa\in\pi_h} A_\kappa((\psi_h,\varphi_h)-\boldsymbol{E}_h(\psi_h,\varphi_h),(v,z)) - \sum_{\kappa\in\pi_h} A_\kappa((\psi_h,\varphi_h),(v,z)-I_h(v,z))\\
&\qquad+A((\psi,\varphi),(v,z)) - \sum_{\kappa\in\pi_h} A_\kappa((\psi_h,\varphi_h),I_h(v,z))\\
&\quad=\sum_{\kappa\in\pi_h} A_\kappa((\psi_h,\varphi_h)-\boldsymbol{E}_h(\psi_h,\varphi_h),(v,z)) - \sum_{\kappa\in\pi_h} A_\kappa((\psi_h,\varphi_h),(v,z)-I_h(v,z))\\
&\qquad+A_h((\psi_h,\varphi_h),I_h(v,z)) - \sum_{\kappa\in\pi_h} A_\kappa((\psi_h,\varphi_h),I_h(v,z)) + B((f,g),(v,z)-I_h(v,z))\\
&\quad\equiv I_1 - I_2 + I_3 - I_4 + I_5.
\end{aligned} \tag{3.14}$$

We have

$$\begin{aligned}
I_2 &= \sum_{\kappa\in\pi_h} A_\kappa((\psi_h,\varphi_h),(v,z)-I_h(v,z))\\
&= \sum_{\kappa\in\pi_h}\int_\kappa (\varpi-\mu)\Delta\psi_h\Delta\overline{(v-I_h^1v)}dx + \mu\int_\kappa D^2\psi_h : D^2\overline{(v-I_h^1v)}dx + \sum_{\kappa\in\pi_h}\int_\kappa \varphi_h\overline{(z-I_h^2z)}dx\\
&\equiv J_1 + J_2 + J_3,
\end{aligned} \tag{3.15}$$

and by Green's formula we have

$$\begin{aligned}
J_1 &= \sum_{\kappa\in\pi_h} -\int_\kappa \nabla((\varpi-\mu)\Delta\psi_h)\overline{\nabla(v-I_h^1v)}dx + \sum_{\kappa\in\pi_h}\int_{\partial\kappa}(\varpi-\mu)\Delta\psi_h\overline{\partial(v-I_h^1v)/\partial\gamma}ds\\
&= \sum_{\kappa\in\pi_h}\int_\kappa \Delta((\varpi-\mu)\Delta\psi_h)\overline{(v-I_h^1v)}dx - \sum_{\kappa\in\pi_h}\int_{\partial\kappa}\nabla((\varpi-\mu)\Delta\psi_h)\overline{(v-I_h^1v)\cdot\gamma}ds\\
&\quad+\sum_{\kappa\in\pi_h}\int_{\partial\kappa}(\varpi-\mu)\Delta\psi_h\overline{\partial(v-I_h^1v)/\partial\gamma}ds\\
&= \sum_{\kappa\in\pi_h}\int_\kappa \Delta((\varpi-\mu)\Delta\psi_h)\overline{(v-I_h^1v)}dx + \sum_{\ell\in\mathcal{E}}\int_\ell [[\nabla((\varpi-\mu)\Delta\psi_h)\cdot\gamma]]\overline{(v-I_h^1v)}ds\\
&\quad-\sum_{\ell\in\mathcal{E}}\int_\ell \{\{(\varpi-\mu)\Delta\psi_h\}\}\overline{[[\partial(v-I_h^1v)/\partial\gamma_\ell]]}ds - \sum_{\ell\in\mathcal{E}^i}\int_\ell [[(\varpi-\mu)\Delta\psi_h]]\overline{\{\{\partial(v-I_h^1v)/\partial\gamma_\ell\}\}}ds.
\end{aligned}$$

Also by Green's formula (see also [5, (7.10)]) we have

$$\begin{aligned}
J_2 = \mu\Bigg[&\sum_{\kappa\in\pi_h}\int_\kappa (\Delta^2\psi_h)\overline{(v-I_h^1v)}dx + \sum_{\ell\in\mathcal{E}^i}\int_\ell \left[\left[\frac{\partial(\Delta\psi_h)}{\partial\gamma_\ell}\right]\right]\overline{(v-I_h^1v)}ds\\
&+\sum_{\ell\in\mathcal{E}}\int_\ell \{\{\partial^2\psi_h/\partial\gamma_\ell^2\}\}\overline{[[\partial I_h^1v/\partial\gamma_\ell]]}ds - \sum_{\ell\in\mathcal{E}^i}\int_\ell [[\partial^2\psi_h/\partial\gamma_\ell^2]]\overline{\{\{\partial(v-I_h^1v)/\partial\gamma_\ell\}\}}ds\\
&-\sum_{\ell\in\mathcal{E}^i}\int_\ell [[\partial^2\psi_h/(\partial\gamma_\ell\partial t_\ell)]]\overline{\partial(v-I_h^1v)/\partial t_\ell}ds\Bigg].
\end{aligned}$$

By (2.15) we get

$$
\begin{aligned}
I_3 - I_4 = &\sum_{\ell\in\mathcal{E}}\int_\ell \{\{(\varpi-\mu)\Delta\psi_h\}\}\overline{[[\partial I_h^1 v/\partial\gamma_\ell]]} + \{\{(\varpi-\mu)\Delta I_h^1 v\}\}\overline{[[\partial\psi_h/\partial\gamma_\ell]]}ds\\
&+\mu\sum_{\ell\in\mathcal{E}}\int_\ell \{\{\partial^2\psi_h/\partial\gamma_\ell^2\}\}\overline{[[\partial I_h^1 v/\partial\gamma_\ell]]} + \{\{\partial^2 I_h^1 v/\partial\gamma_\ell^2\}\}\overline{[[\partial\psi_h/\partial\gamma_\ell]]}ds\\
&+\sigma\sum_{\ell\in\mathcal{E}}\frac{1}{\hat{\ell}}\int_\ell [[\partial\psi_h/\partial\gamma_\ell]]\overline{[[\partial I_h^1 v/\partial\gamma_\ell]]}ds. \qquad (3.16)
\end{aligned}
$$

From Green's formula we get

$$
\begin{aligned}
B((f,g),(v,z)) &= (\nabla(\varpi f),\nabla v)_0 + (\nabla f,\nabla(n\varpi v))_0 - (g,n\varpi v)_0 + (f,z)_0\\
&= -\sum_\kappa\int_\kappa \Delta(\varpi f)\overline{v}dx - \sum_\kappa\int_\kappa n\varpi\Delta f\overline{v}dx - (n\varpi g,v)_0 + (f,z)_0\\
&\quad+\sum_\kappa\int_{\partial\kappa}\partial(\varpi f)/\partial\gamma\overline{v}ds + \sum_\kappa\int_{\partial\kappa} n\varpi\frac{\partial f}{\partial\gamma}\overline{v}ds\\
&\equiv \sum_\kappa\int_\kappa F\overline{v}dx + (f,z)_0 + \sum_{\ell\in\mathcal{E}}\int_\ell [[\partial(\varpi f)/\partial\gamma]]\overline{v}ds + \sum_{\ell\in\mathcal{E}}\int_\ell \left[\left[n\varpi\frac{\partial f}{\partial\gamma}\right]\right]\overline{v}ds,
\end{aligned}
$$

so

$$
\begin{aligned}
I_5 &= B((f,g),(v,z)-I_h(v,z))\\
&= \sum_\kappa\int_\kappa F(\overline{v-I_h^1 v})dx + (f,z-I_h^2 z)_0 + \sum_{\ell\in\mathcal{E}}\int_\ell \left[\left[(1+2\varpi)\frac{\partial f}{\partial\gamma}\right]\right](\overline{v-I_h^1 v})ds. \qquad (3.17)
\end{aligned}
$$

Substituting (3.15), (3.16) and (3.17) into (3.14), we obtain

$$
\begin{aligned}
&A((\psi,\varphi)-\boldsymbol{E}_h(\psi_h,\varphi_h),(v,z))\\
&= I_1 + \sum_{\kappa\in\pi_h}\int_\kappa (F-\Delta(\varpi\Delta\psi_h))\overline{(v-I_h^1 v)} + (f-\varphi_h)\overline{(z-I_h^2 z)}dx\\
&\quad-\sum_{\ell\in\mathcal{E}}\int_\ell [[\nabla((\varpi-\mu)\Delta\psi_h)\cdot\gamma]]\overline{(v-I_h^1 v)}ds + \sum_{\ell\in\mathcal{E}}\int_\ell \{\{(\varpi-\mu)\Delta\psi_h\}\}\overline{[[\partial(v-I_h^1 v)/\partial\gamma_\ell]]}ds\\
&\quad+\sum_{\ell\in\mathcal{E}^i}\int_\ell [[(\varpi-\mu)\Delta\psi_h]]\overline{\{\{\partial(v-I_h^1 v)/\partial\gamma_\ell\}\}}ds - \mu\sum_{\ell\in\mathcal{E}^i}\int_\ell \left[\left[\frac{\partial(\Delta\psi_h)}{\partial\gamma_\ell}\right]\right]\overline{(v-I_h^1 v)}ds\\
&\quad-\mu\sum_{\ell\in\mathcal{E}}\int_\ell \{\{\partial^2\psi_h/\partial\gamma_\ell^2\}\}\overline{[[\partial I_h^1 v/\partial\gamma_\ell]]}ds + \mu\sum_{\ell\in\mathcal{E}^i}\int_\ell [[\partial^2\psi_h/\partial\gamma_\ell^2]]\overline{\{\{\partial(v-I_h^1 v)/\partial\gamma_\ell\}\}}ds\\
&\quad+\mu\sum_{\ell\in\mathcal{E}^i}\int_\ell [[\partial^2\psi_h/\partial\gamma_\ell\partial t_\ell]]\overline{(\partial(v-I_h^1 v)/\partial t_\ell)}ds + \sum_{\ell\in\mathcal{E}}\int_\ell \{\{(\varpi-\mu)\Delta\psi_h\}\}[[\partial\overline{I_h^1 v}/\partial\gamma_\ell]]ds\\
&\quad+\sum_{\ell\in\mathcal{E}}\int_\ell \{\{(\varpi-\mu)\Delta I_h^1 v\}\}\overline{[[\partial\psi_h/\partial\gamma_\ell]]}ds + \mu\sum_{\ell\in\mathcal{E}}\int_\ell \{\{\partial^2\psi_h/\partial\gamma_\ell^2\}\}\overline{[[\partial I_h^1 v/\partial\gamma_\ell]]}ds\\
&\quad+\mu\sum_{\ell\in\mathcal{E}}\int_\ell \{\{\partial^2 I_h^1 v/\partial\gamma_\ell^2\}\}\overline{[[\partial\psi_h/\partial\gamma_\ell]]}ds + \sigma\sum_{\ell\in\mathcal{E}}\frac{1}{\hat{\ell}}\int_\ell [[\partial\psi_h/\partial\gamma_\ell]]\overline{[[\partial I_h^1 v/\partial\gamma_\ell]]}ds\\
&\quad+\sum_{\ell\in\mathcal{E}}\int_\ell \left[\left[(1+2\varpi)\frac{\partial f}{\partial\gamma}\right]\right](\overline{v-I_h^1 v})ds\\
&\equiv I_1 + G_2 + G_3 + \cdots + G_{15}. \qquad (3.18)
\end{aligned}
$$

By (3.13), the Schwarz inequality, (3.9) and (3.2) we get

$$
|I_1| = \left|\sum_{\kappa\in\pi_h} A_\kappa((\psi_h,\varphi_h)-\boldsymbol{E}_h(\psi_h,\varphi_h),(v,z))\right|
$$

$$\lesssim \sum_{\kappa\in\pi_h}(\|\varpi\|_{0,\infty,\kappa}|\psi_h - E_h\psi_h|_{2,\kappa} + \|\varphi_h - E_h\varphi_h\|_{0,\kappa})\|(v,z)\|_{\boldsymbol{H}}$$
$$\lesssim \sum_{\ell\in\mathcal{E}}\int_\ell(\|\varpi\|^2_{0,\infty,\kappa}\eta_{\ell,1}(\psi_h)^2 + \eta_{\ell,1}(\varphi_h)^2)^{\frac{1}{2}}\|(v,z)\|_{\boldsymbol{H}}.$$

By (3.1) we get

$$|G_2| \lesssim \left(\sum_{\kappa\in\pi_h} h_\kappa^4\|F - \Delta(\varpi\Delta\psi_h)\|^2_{0,\kappa}\right)^{\frac{1}{2}}\left(\sum_{\kappa\in\pi_h} h_\kappa^{-4}\|v - I_h^1 v\|^2_{0,\kappa}\right)^{\frac{1}{2}} + \|f-\varphi_h\|_0\|z - I_h^2 z\|_0$$
$$\lesssim \left(\sum_{\kappa\in\pi_h}\eta_\kappa^2\right)^{\frac{1}{2}}|v|_2 + \|f-\varphi_h\|_0\|z\|_0.$$

By (3.4) we get

$$|G_3 + G_6| \lesssim \left(\sum_{\ell\in\mathcal{E}^i}\hat{\ell}^3\|[[\nabla(\varpi\Delta\psi_h)\cdot\gamma]]\|^2_{0,\ell}\right)^{\frac{1}{2}}\left(\sum_{\ell\in\mathcal{E}^i}\hat{\ell}^{-3}\|v - I_h^1 v\|^2_{0,\ell}\right)^{\frac{1}{2}} \lesssim \left(\sum_{\ell\in\mathcal{E}^i}\eta^2_{\ell,3}\right)^{\frac{1}{2}}|v|_2.$$

We see $G_4 + G_{10} = 0$, $G_7 + G_{12} = 0$, and by (3.5) we get

$$|G_5| \lesssim \left(\sum_{\ell\in\mathcal{E}^i}\hat{\ell}[[(\varpi-\mu)\Delta\psi_h]]^2\right)^{\frac{1}{2}}|v|_2 \lesssim \left(\sum_{\ell\in\mathcal{E}^i}\eta^2_{\ell,4}\right)^{\frac{1}{2}}|v|_2.$$

By (3.3) we get $|G_8| \lesssim (\sum_{\ell\in\mathcal{E}^i}\eta_{\ell,2}(\psi_h)^2)^{\frac{1}{2}}|v|_2$. By (3.2), the trace theorem with scaling and a standard inverse estimate, we deduce

$$|G_9| \lesssim \mu\left(\sum_{\ell\in\mathcal{E}^i}\hat{\ell}\|[[\partial^2\psi_h/(\partial\gamma_\ell\partial t_\ell)]]\|^2_{0,\ell}\right)^{\frac{1}{2}}\left(\sum_{\ell\in\mathcal{E}^i}\hat{\ell}^{-1}\|\partial(v - I_h^1 v)/\partial t_\ell\|^2_{0,\ell}\right)^{\frac{1}{2}} \lesssim \mu\left(\sum_{\ell\in\mathcal{E}^i}\eta_{\ell,1}(\psi_h)^2\right)^{\frac{1}{2}}|v|_2.$$

By (3.2) we get

$$|G_{11}| \lesssim \frac{1}{\sigma}\left(\sum_{\ell\in\mathcal{E}}\|\varpi-\mu\|_{0,\infty,\ell}\eta_{\ell,1}(\psi_h)^2\right)^{\frac{1}{2}}|v|_2, \quad |G_{13}| \lesssim \frac{\mu}{\sigma}\left(\sum_{\ell\in\mathcal{E}}\eta_{\ell,1}(\psi_h)^2\right)^{\frac{1}{2}}|v|_2,$$
$$|G_{14}| \lesssim \left(\sum_{\ell\in\mathcal{E}}\eta_{\ell,1}(\psi_h)^2\right)^{\frac{1}{2}}|v|_2, \quad |G_{15}| \lesssim \left(\sum_{\ell\in\mathcal{E}}\|1+2\varpi\|_{0,\infty,\ell}h_\ell^4\eta_{\ell,1}(f)^2\right)^{\frac{1}{2}}|v|_2.$$

Substituting these estimates into (3.18), we obtain

$$A((\psi,\varphi) - \boldsymbol{E}_h(\psi_h,\varphi_h),(v,z)) \lesssim \eta_h(F,\psi_h,\varphi_h,\Omega)\|(v,z)\|_{\boldsymbol{H}}. \tag{3.19}$$

Combining (3.10)–(3.12) and (3.20) we obtain (3.8). □

3.2 Efficiency analysis

Using the argument in [5, Theorem 8], we estimate the each terms in (3.6). In the following analysis, $v\in S^h$ and $\kappa\in\pi_h$ are arbitrary, $\mathcal{J}_\ell$ is the set of the elements in π_h that share the common edge ℓ, where $\ell\in\mathcal{E}_h^i$ is arbitrary, $\ell = \kappa_-\cap\kappa_+$.

3.2.1 *Estimate for* $\eta_k^2(F,v) = h_\kappa^4\|F - \Delta(\varpi\Delta v)\|^2_{0,\kappa}$

Let $\zeta\in P_6(\kappa)$ be the real bubble function, which vanishes to the first order on $\partial\kappa$ and equals 1 at the center of κ. It follows from scaling that

$$|\zeta|_{2,\kappa} \leqslant Ch_\kappa^{-2}\|\zeta\|_{0,\kappa} \leqslant Ch_\kappa^{-1}, \quad \|\zeta\|_{0,\infty,\kappa} \leqslant C.$$

By the equivalence of norms on finite dimensional spaces, we have

$$C_1 \int_\kappa |\hat{F} - \Delta(\widetilde{\varpi}\Delta v)|^2 \zeta^2 dx \leqslant \|\hat{F} - \Delta(\widetilde{\varpi}\Delta v)\|_{0,\kappa}^2 \leqslant C_2 \int_\kappa |\hat{F} - \Delta(\widetilde{\varpi}\Delta v)|^2 \zeta dx. \tag{3.20}$$

Let $z = (\hat{F} - \Delta(\widetilde{\varpi}\Delta v))\zeta$. We extend z to H_0^2 trivially. From (2.7) with $z = 0$, (3.20), the Green's formula and a standard inverse estimate we derive

$$\begin{aligned}\|\hat{F} - \Delta(\widetilde{\varpi}\Delta v)\|_{0,\kappa}^2 &\lesssim \int_\kappa (\hat{F} - \Delta(\widetilde{\varpi}\Delta v))\overline{z} dx \\ &= C\bigg[\int_\kappa (F - \Delta(\varpi\Delta v))\overline{z} dx + \int_\kappa (\hat{F} - \Delta(\widetilde{\varpi}\Delta v))\overline{z} dx - \int_\kappa (F - \Delta(\varpi\Delta v))\overline{z} dx\bigg] \\ &= C\bigg[\int_\kappa \varpi(\Delta\psi - \Delta v)\overline{\Delta z} dx + \int_\kappa ((\hat{F} - F) - (\Delta(\widetilde{\varpi}\Delta v) - \Delta(\varpi\Delta v)))\overline{z} dx\bigg] \\ &\lesssim [h_\kappa^{-2}|\psi - v|_{2,\kappa} + \|\hat{F} - F\|_{0,\kappa} + \|\Delta(\widetilde{\varpi}\Delta v) - \Delta(\varpi\Delta v)\|_{0,\kappa}]\|z\|_{0,\kappa},\end{aligned}$$

which implies

$$h_\kappa^2 \|\hat{F} - \Delta(\widetilde{\varpi}\Delta v)\|_{0,\kappa} \lesssim |\psi - v|_{2,\kappa} + h_\kappa^2 (\|\hat{F} - F\|_{0,\kappa} + \|\Delta(\widetilde{\varpi}\Delta v) - \Delta(\varpi\Delta v)\|_{0,\kappa}),$$

and by the triangle inequality, we deduce

$$h_\kappa^4 \|F - \Delta(\varpi\Delta v)\|_{0,\kappa}^2 \lesssim |\psi - v|_{2,\kappa}^2 + h_\kappa^4 \|\hat{F} - F\|_{0,\kappa}^2 + h_\kappa^4 \|\Delta(\widetilde{\varpi}\Delta v) - \Delta(\varpi\Delta v)\|_{0,\kappa}^2. \tag{3.21}$$

Summing up (3.21) over all the triangles in π_h we find

$$\sum_{\kappa\in\pi_h} h_\kappa^4 \|F - \Delta(\varpi\Delta v)\|_{0,\kappa}^2 \lesssim \sum_{\kappa\in\pi_h} |\psi - v|_{2,\kappa}^2 + \mathrm{Osc}_j(F)^2 + \mathrm{Osc}_j(\Delta(\varpi\Delta v))^2. \tag{3.22}$$

3.2.2 *Estimate for* $\eta_{\ell,4} = \hat{\ell}\|[[\varpi\Delta v]]\|_{0,\ell}^2 \geqslant \hat{\ell}\|[[(\varpi - \mu)\Delta v]]\|_{0,\ell}^2$

We construct a real bubble function $\beta \in P_{m-2+j}(\mathbb{R}^2)$ on $\kappa_- \cup \kappa_+$, which is the polynomial that equals the jump $[[\widetilde{\varpi}\Delta v]]$ on the edge ℓ and which is constant on the lines perpendicular to ℓ. We define $\zeta_1 \in P_{m-1+j}(\kappa_- \cup \kappa_+)$ to be the polynomial that satisfies

$$\zeta_1 = 0 \quad \text{on} \quad \ell \quad \text{and} \quad \partial\zeta_1/\partial\gamma_\ell = \beta. \tag{3.23}$$

By a direct calculation and standard inverse estimates, we have

$$\hat{\ell}^{-1}|\zeta_1|_{L_2(\kappa_-\cup\kappa_+)} + \|\zeta_1\|_{L_\infty(\kappa_-\cup\kappa_+)} \lesssim \hat{\ell}^{\frac{1}{2}}\|[[\widetilde{\varpi}\Delta v]]\|_{0,\ell}^2. \tag{3.24}$$

Next, we define $\zeta_2 \in P_8(\kappa_- \cup \kappa_+)$ that satisfies the following conditions: (i) ζ_2 vanishes to the first order on $(\partial\kappa_- \cup \partial\kappa_+)\backslash\ell$, and (ii) ζ_2 equals 1 at the midpoint of ℓ. It follows from scaling that

$$\hat{\ell}^{-1}|\zeta_2|_{L_2(\kappa_-\cup\kappa_+)} + \|\zeta_2\|_{L_\infty(\kappa_-\cup\kappa_+)} \leqslant C, \tag{3.25}$$

$$\|[[\widetilde{\varpi}\Delta v]]\|_{0,\ell}^2 \lesssim \int_\ell |[[\widetilde{\varpi}\Delta v]]|^2 \overline{\zeta_2} ds. \tag{3.26}$$

From (3.23) and (3.26) we get

$$\|[[\widetilde{\varpi}\Delta v]]\|_{0,\ell}^2 \lesssim \int_\ell [[\widetilde{\varpi}\Delta v]]\overline{\beta\zeta_2} ds = C\int_\ell [[\widetilde{\varpi}\Delta v]]\overline{(\partial(\zeta_1\zeta_2)/\partial\gamma_\ell)} ds. \tag{3.27}$$

We extend $\zeta_1\zeta_2$ to $H_0^2(\Omega)$ trivially. From (2.7) with $z = 0$ and the Green's formula we derive

$$\int_\ell [[\varpi\Delta v]]\overline{(\partial(\zeta_1\zeta_2)/\partial\gamma_\ell)} ds = \sum_{\kappa\in\mathcal{J}_\ell}\bigg(- \int_\kappa \varpi\Delta v\overline{\Delta(\zeta_1\zeta_2)} dx + \int_\kappa \Delta(\varpi\Delta v)\overline{(\zeta_1\zeta_2)} dx\bigg)$$

$$= \sum_{\kappa\in\mathcal{J}_\ell}\int_\kappa \varpi\Delta(\psi-v)\overline{\Delta(\zeta_1\zeta_2)}dx - \int_\Omega (F-\Delta(\varpi\Delta v))\overline{(\zeta_1\zeta_2)}dx. \tag{3.28}$$

Combining (3.27) and (3.28), we find by the Schwarz inequality and a standard inverse estimate

$$\begin{aligned}
&\|[[\widetilde{\varpi}\Delta v]]\|_{0,\ell}^2\\
&\lesssim \int_\ell [[\varpi\Delta v]]\overline{(\partial(\zeta_1\zeta_2)/\partial\gamma_\ell)}ds + \left(\int_\ell [[\widetilde{\varpi}\Delta v]]\overline{(\partial(\zeta_1\zeta_2)/\partial\gamma_\ell)}ds - \int_\ell [[\varpi\Delta v]]\overline{(\partial(\zeta_1\zeta_2)/\partial\gamma_\ell)}ds\right)\\
&\lesssim \sum_{\kappa\in\mathcal{J}_\ell}\int_\kappa \varpi\Delta(\psi-v)\overline{\Delta(\zeta_1\zeta_2)}dx - \int_\kappa (F-\varpi\Delta v)\overline{(\zeta_1\zeta_2)}dx + \|[[\widetilde{\varpi}\Delta v]]-[[\varpi\Delta v]]\|_{0,\ell}\|\partial(\zeta_1\zeta_2)/\partial\gamma_\ell\|_{0,\ell}\\
&\lesssim \left[\sum_{\kappa\in\mathcal{J}_\ell}(h_\kappa^{-2}|\psi-v|_{2,\kappa}+\|F-\varpi\Delta v\|_{0,\kappa}) + \hat{\ell}^{-\frac{3}{2}}\|[[\widetilde{\varpi}\Delta v]]-[[\varpi\Delta v]]\|_{0,\ell}\right]\|\zeta_1\zeta_2\|_{0,\kappa}.
\end{aligned} \tag{3.29}$$

From (3.24) and (3.25) we know $\|\zeta_1\zeta_2\|_{L_2(\kappa_-\cup\kappa_+)} \lesssim \hat{\ell}^{\frac{3}{2}}\|[[\widetilde{\varpi}\Delta v]]\|_{0,\ell}$, which together with (3.29) implies

$$\hat{\ell}\|[[\widetilde{\varpi}\Delta v]]\|_{0,\ell}^2 \lesssim \sum_{\kappa\in\mathcal{J}_\ell}(|\psi-v|_{2,\kappa}^2 + h_\kappa^4\|F-\varpi\Delta v\|_{0,\kappa}^2) + \hat{\ell}\|[[\widetilde{\varpi}\Delta v]]-[[\varpi\Delta v]]\|_{0,\ell}^2.$$

From the triangle inequality, we get

$$\hat{\ell}\|[[\varpi\Delta v]]\|_{0,\ell}^2 \lesssim \sum_{\kappa\in\mathcal{J}_\ell}(|\psi-v|_{2,\kappa}^2 + h_\kappa^4\|F-\varpi\Delta v\|_{0,\kappa}^2) + \hat{\ell}\|[[\widetilde{\varpi}\Delta v]]-[[\varpi\Delta v]]\|_{0,\ell}^2.$$

In view of (3.21),

$$\begin{aligned}
\hat{\ell}\|[[\varpi\Delta v]]\|_{0,\ell}^2 \lesssim& \sum_{\kappa\in\mathcal{J}_\ell}(|\psi-v|_{2,\kappa}^2 + h_\kappa^4\|\hat{F}-F\|_{0,\kappa}^2\\
&+ h_\kappa^4\|\Delta(\widetilde{\varpi}\Delta v)-\Delta(\varpi\Delta v)\|_{0,\kappa}^2) + \hat{\ell}\|[[\widetilde{\varpi}\Delta v]]-[[\varpi\Delta v]]\|_{0,\ell}^2.
\end{aligned} \tag{3.30}$$

Thus we get

$$\sum_{\ell\in\mathcal{E}_h^i}\hat{\ell}\|[[\varpi\Delta v]]^2\|_{0,\ell} \lesssim \sum_{\kappa\in\pi_h}|\psi-v|_{2,\kappa}^2 + \mathrm{Osc}_j(F)^2 + \mathrm{Osc}_j(\Delta(\varpi\Delta v))^2 + \mathrm{Osc}_j(\eta_{\ell,3})^2. \tag{3.31}$$

3.2.3 *Estimate for* $\eta_{\ell,2}(v) = \mu\hat{\ell}^{\frac{1}{2}}\|[[\partial^2 v/\partial\gamma_\ell^2]]\|_{0,\ell}^2$

We construct $\beta\in P_{m-2}(\mathbb{R}^2)$, which is the polynomial that equals the jump $[[\partial^2 v/\partial\gamma_\ell^2]]$ on the edge ℓ and which is constant on the lines perpendicular to ℓ. We define $\zeta_1\in P_{m-1}(\kappa_-\cup\kappa_+)$ to be the polynomial that satisfies

$$\zeta_1 = 0 \quad \text{on} \quad \ell \quad \text{and} \quad \partial\zeta_1/\partial\gamma_\ell = \beta. \tag{3.32}$$

By a direct calculation and the standard inverse estimates, we have

$$\hat{\ell}^{-1}|\zeta_1|_{L_2(\kappa_-\cup\kappa_+)} + \|\zeta_1\|_{L_\infty(\kappa_-\cup\kappa_+)} \lesssim \hat{\ell}^{\frac{1}{2}}\|[[(\partial^2 v/\partial\gamma_\ell^2)\Delta v]]\|_{0,\ell}^2. \tag{3.33}$$

Let ζ_2 be defined as in Subsection 3.2.2. It follows from scaling that

$$\hat{\ell}^{-1}|\zeta_2|_{L_2(\kappa_-\cup\kappa_+)} + \|\zeta_2\|_{L_\infty(\kappa_-\cup\kappa_+)} \leqslant C, \tag{3.34}$$

$$\|[[\partial^2 v/\partial\gamma_\ell^2]]\|_{0,\ell}^2 \lesssim \int_\ell |[[\partial^2 v/\partial\gamma_\ell^2]]|^2\overline{\zeta_2}ds. \tag{3.35}$$

From (3.35) and (3.32) we get

$$\|[[\partial^2 v/\partial\gamma_\ell^2]]\|_{0,\ell}^2 \lesssim \int_\ell [[\partial^2 v/\partial\gamma_\ell^2]]\overline{\beta\zeta_2}ds = C\int_\ell [[\partial^2 v/\partial\gamma_\ell^2]]\overline{(\partial(\zeta_1\zeta_2)/\partial\gamma_\ell)}ds. \tag{3.36}$$

We extend $\zeta_1\zeta_2$ to $H_0^2(\Omega)$ trivially. From Green's formula and (2.7) with $z=0$ we derive

$$\begin{aligned}
&\mu\int_\ell [[\partial^2 v/\partial\gamma_\ell^2]]\overline{\partial(\zeta_1\zeta_2)/\partial\gamma_\ell}ds\\
&=\mu\sum_{\kappa\in\mathcal{J}_\ell}\left(-\int_\kappa \nabla^2 v:\overline{\nabla^2(\zeta_1\zeta_2)}dx+\int_\kappa(\Delta^2 v)\overline{(\zeta_1\zeta_2)}dx\right)\\
&=\sum_{\kappa\in\mathcal{J}_\ell}\mu\int_\kappa \nabla^2(\psi-v):\overline{\nabla^2(\zeta_1\zeta_2)}dx-\mu\int_\Omega\nabla^2\psi:\overline{\nabla^2(\zeta_1\zeta_2)}dx-((\varpi-\mu)\Delta\psi,\Delta(\zeta_1\zeta_2))_0\\
&\quad+((\varpi-\mu)\Delta\psi,\Delta(\zeta_1\zeta_2))_0+\mu\sum_{\kappa\in\mathcal{J}_\ell}\int_\kappa\Delta^2 v\overline{(\zeta_1\zeta_2)}dx\\
&=\sum_{\kappa\in\mathcal{J}_\ell}\mu\int_\kappa \nabla^2(\psi-v):\overline{\nabla^2(\zeta_1\zeta_2)}dx-\int_\Omega F\zeta_1\zeta_2dx+((\varpi-\mu)\Delta\psi,\Delta(\zeta_1\zeta_2))_0\\
&\quad+\mu\sum_{\kappa\in\mathcal{J}_\ell}\int_\kappa\Delta v\overline{\Delta(\zeta_1\zeta_2)}dx-\mu\int_\ell[[\Delta v]]\overline{(\partial(\zeta_1\zeta_2)/\partial\gamma_\ell)}ds\\
&=\sum_{\kappa\in\mathcal{J}_\ell}\mu\int_\ell \nabla^2(\psi-v):\overline{\nabla^2(\zeta_1\zeta_2)}dx-\sum_{\kappa\in\mathcal{J}_\ell}\int_\kappa F\overline{(\zeta_1\zeta_2)}dx+\sum_{\kappa\in\mathcal{J}_\ell}\int_\kappa(\varpi-\mu)\Delta(\psi-v)\overline{\Delta(\zeta_1\zeta_2)}dx\\
&\quad+\sum_{\kappa\in\mathcal{J}_\ell}\int_\kappa\Delta(\varpi\Delta v)\overline{(\zeta_1\zeta_2)}dx+\int_\ell[[\varpi\Delta v]]\overline{(\partial(\zeta_1\zeta_2)/\partial\gamma_\ell)}ds-\mu\int_\ell[[\Delta v]]\overline{(\partial(\zeta_1\zeta_2)/\partial\gamma_\ell)}ds. \qquad (3.37)
\end{aligned}$$

Combining (3.36) and (3.37), we find by the Schwarz inequality and a standard inverse estimate

$$\begin{aligned}
\mu\|[[\partial^2 v/\partial\gamma_\ell^2]]\|_{0,\ell}^2\lesssim&\sum_{\kappa\in\mathcal{J}_\ell}h_\kappa^{-2}|\psi-v|_{2,\kappa}\|\zeta_1\zeta_2\|_{0,\kappa}+\sum_{\kappa\in\mathcal{J}_\ell}\|F-\Delta(\varpi\Delta v)\|_{0,\kappa}\|\zeta_1\zeta_2\|_{0,\kappa}\\
&+\hat\ell^{-\frac32}\|[[(\varpi-\mu)\Delta v]]\|_{0,\ell}\|\zeta_1\zeta_2\|_{0,\kappa}. \qquad (3.38)
\end{aligned}$$

From (3.33) and (3.34) we know $\|\zeta_1\zeta_2\|_{L_2(\kappa_-\cup\kappa_+)}\lesssim\hat\ell^{\frac32}\|[[\partial^2 v/\partial\gamma_\ell^2]]\|_{0,\ell}$, which together with (3.38) implies

$$\hat\ell\|[[\partial^2 v/\partial\gamma_\ell^2]]\|_{0,\ell}^2\lesssim\sum_{\kappa\in\mathcal{J}_\ell}(|\psi-v|_{2,\kappa}^2+h_\kappa^4\|F-\Delta(\varpi\Delta v)\|_{0,\kappa}^2)+\hat\ell\|[[(\varpi-\mu)\Delta v]]\|_{0,\ell}^2.$$

Substituting (3.21) and (3.30) into the above inequality and summing up over all the triangles we can get

$$\sum_{\ell\in\mathcal{E}_h^i}\hat\ell\|[[\partial^2 v/\partial\gamma_\ell^2]]\|_{0,\ell}^2\lesssim\sum_{\kappa\in\pi_h}|\psi-v|_{2,\kappa}^2+\mathrm{Osc}_j(F)^2+\mathrm{Osc}_j(\Delta(\varpi\Delta v))^2+\mathrm{Osc}_j(\eta_{\ell,4})^2. \qquad (3.39)$$

3.2.4 *Estimate for* $\eta_{\ell,3}(v)=\hat\ell^3\|[[\partial(\varpi\Delta v)/\partial\gamma_\ell]]\|_{0,\ell}^2$

We define $\zeta_3\in P_{m-3+j}(\kappa_-\cup\kappa_+)$ that satisfies $\zeta_3=[[\partial(\widetilde\varpi\Delta v)/\partial\gamma_\ell]]$ on ℓ and ζ_3 is constant on the lines perpendicular to ℓ. By a direct calculation, we have

$$\|\zeta_3\|_{L_2(\kappa_-\cup\kappa_+)}\lesssim\hat\ell^{\frac12}\|[[\partial(\widetilde\varpi\Delta v)/\partial\gamma_\ell]]\|_{0,\ell}. \qquad (3.40)$$

Let $\zeta_2\in P_8(\kappa_-\cup\kappa_+)$ be defined as in Subsection 3.2.2. It follows from the equivalence of norms on finite dimensional spaces and scaling that

$$\|[[\partial(\widetilde\varpi\Delta v)/\partial\gamma_\ell]]\|_{0,\ell}^2\lesssim\int_\ell[[\partial(\widetilde\varpi\Delta v)/\partial\gamma_\ell]]\overline{(\zeta_2\zeta_3)}ds. \qquad (3.41)$$

We extend $\zeta_2\zeta_3$ to $H_0^2(\Omega)$ trivially. From Green's formula and (2.7) with $z=0$, we derive

$$\begin{aligned}
&\int_\ell[[\partial(\varpi\Delta v)/\partial\gamma_\ell]]\overline{(\zeta_2\zeta_3)}ds\\
&=\sum_{\kappa\in\mathcal{J}_\ell}\left(\int_\kappa\varpi\Delta v\overline{\Delta(\zeta_2\zeta_3)}dx-\int_\kappa\Delta(\varpi\Delta v)\overline{(\zeta_2\zeta_3)}dx\right)+\int_\ell([[\varpi\Delta v]]\overline{(\partial(\zeta_2\zeta_3)/\partial\gamma_\ell)})ds
\end{aligned}$$

$$= \sum_{\kappa\in\mathcal{J}_\ell}\int_\kappa \varpi\Delta(v-\psi)\overline{\Delta(\zeta_2\zeta_3)}dx + \int_\Omega (F-\Delta(\varpi\Delta v))\overline{(\zeta_2\zeta_3)}dx + \int_\ell ([[\varpi\Delta v]]\overline{(\partial(\zeta_2\zeta_3)/\partial\gamma_\ell)})ds. \quad (3.42)$$

From (3.41), (3.42), the Schwarz inequality and standard inverse estimates we derive

$$\begin{aligned}
&\|[[\partial(\tilde{\varpi}\Delta v)/\partial\gamma_\ell]]\|_{0,\ell}^2 \\
&\lesssim \int_\ell [[\partial(\varpi\Delta v)/\partial\gamma_\ell]]\overline{(\zeta_2\zeta_3)}ds + \left(\int_\ell [[\partial(\tilde{\varpi}\Delta v)/\partial\gamma_\ell]]\overline{(\zeta_2\zeta_3)}ds - \int_\ell [[\partial(\varpi\Delta v)/\partial\gamma_\ell]]\overline{(\zeta_2\zeta_3)}ds\right) \\
&\lesssim \sum_{\kappa\in\mathcal{J}_\ell}\int_\kappa \varpi\Delta(v-\psi)\overline{\Delta(\zeta_2\zeta_3)}dx + \int_\Omega (F-\Delta(\varpi\Delta v))\overline{(\zeta_2\zeta_3)}dx + \int_\ell [[\varpi\Delta v]]\overline{(\partial(\zeta_2\zeta_3)/\partial\gamma_\ell)}ds \\
&\quad + \int_\ell ([[\partial(\tilde{\varpi}\Delta v)/\partial\gamma_\ell]] - [[\partial(\varpi\Delta v)/\partial\gamma_\ell]])\overline{(\zeta_2\zeta_3)}ds \\
&\lesssim \sum_{\kappa\in\mathcal{J}_\ell}(|\psi-v|_{2,\kappa}|\zeta_2\zeta_3|_{2,\kappa} + \|F-\Delta(\varpi\Delta v)\|_{0,\kappa}\|\zeta_2\zeta_3\|_{0,\kappa}) + \|[[\varpi\Delta v]]\|_{0,\ell}\|\partial(\zeta_2\zeta_3)/\partial\gamma_\ell\|_{0,\ell} \\
&\quad + \|[[\partial(\tilde{\varpi}\Delta v)/\partial\gamma_\ell]] - [[\partial(\varpi\Delta v)/\partial\gamma_\ell]]\|_{0,\ell}\|\zeta_2\zeta_3\|_{0,\ell} \\
&\lesssim \Bigg[\sum_{\kappa\in\mathcal{J}_\ell}(h_\kappa^{-2}|\psi-v|_{2,\kappa} + \|F-\Delta(\varpi\Delta v)\|_{0,\kappa}) + \hat{\ell}^{-\frac{3}{2}}\|[[\varpi\Delta v]]\|_{0,\ell} \\
&\quad + \hat{\ell}^{-\frac{1}{2}}\|[[\partial(\tilde{\varpi}\Delta v)/\partial\gamma_\ell]] - [[\partial(\varpi\Delta v)/\partial\gamma_\ell]]\|_{0,\ell}\Bigg]\|\zeta_2\zeta_3\|_{0,\kappa}. \qquad (3.43)
\end{aligned}$$

From (3.34) and (3.40) we have $\|\zeta_2\zeta_3\|_{0,\kappa} \lesssim \hat{\ell}^{\frac{1}{2}}\|[[\partial(\tilde{\varpi}\Delta v)/\partial\gamma_\ell]]\|_{0,\ell}$, which together with (3.43) implies

$$\begin{aligned}
\hat{\ell}^3\|[[\partial(\varpi\Delta v)/\partial\gamma_\ell]]\|_{0,\ell}^2 &\lesssim \sum_{\kappa\in\mathcal{J}_\ell}(|\psi-v|_{2,\kappa}^2 + \hat{\ell}^4\|F-\Delta(\varpi\Delta v)\|_{0,\kappa}^2) \\
&\quad + \hat{\ell}\|[[\varpi\Delta v]]\|_{0,\ell}^2 + \hat{\ell}^3\|[[\partial(\tilde{\varpi}\Delta v)/\partial\gamma_\ell]] - [[(\varpi\Delta v)/\partial\gamma_\ell]]\|_{0,\ell}^2. \qquad (3.44)
\end{aligned}$$

Substituting (3.21) and (3.30) into the above inequality and summing up over all the triangles we can get

$$\begin{aligned}
&\sum_{\ell\in\mathcal{E}_h^i}\hat{\ell}^3\|[[\partial(\varpi\Delta v)/\partial\gamma_\ell]]\|_{0,\ell}^2 \\
&\quad\lesssim \sum_{\kappa\in\mathcal{J}_\ell}|\psi-v|_{2,\kappa}^2 + \mathrm{Osc}_j^2(F) + \mathrm{Osc}_j(\Delta(\varpi\Delta v))^2 + \mathrm{Osc}_j^2(\eta_{\ell,4}) + \mathrm{Osc}_j^2(\eta_{\ell,3}). \qquad (3.45)
\end{aligned}$$

Theorem 3.2. *Under the condition of Theorem* 3.1*, we have*

$$\eta_h(F,\psi_h,\varphi_h,\Omega) \lesssim \|(\psi,\varphi)-(\psi_h,\varphi_h)\|_h + \mathrm{OSC}_j(F,\psi_h).$$

Proof. Substituting (3.22), (3.31), (3.39) and (3.45) into (3.7) and neglecting high order small quantity, we get

$$\begin{aligned}
\eta_h(F,\psi_h,\varphi_h,\Omega) &\lesssim \sum_{\kappa\in\pi_h}|\psi-v|_{2,\kappa}^2 + \sum_{\ell\in\mathcal{E}_h}\frac{\sigma^2}{\hat{\ell}}\|[[\partial(\psi-\psi_h)/\partial\gamma_\ell]]\|_{0,\ell}^2 + \mathrm{OSC}_j(F,\psi_h) \\
&\lesssim \|(\psi,\varphi)-(\psi_h,\varphi_h)\|_h^2 + \mathrm{OSC}_j(F,\psi_h).
\end{aligned}$$

The proof is completed. □

4 A posteriori error analysis for the eigenvalue problem (2.5)

Now, we analyze the a posteriori error of the C^0IPG eigenpair $(\lambda_h, u_h, \omega_h)$.

Consider the source problem (2.7) associated with (2.5) with $(f,g) = \lambda_h(u_h,\omega_h)$. Then its generalized solution $(\psi,\varphi) = \lambda_h T(u_h,\omega_h)$ and the C^0IPG approximation $(\psi_h,\varphi_h) = \lambda_h T_h(u_h,\omega_h) = (u_h,\omega_h)$. Let

$v = 0$ in (2.16). We get $\omega_h = \lambda_h u_h$. Thus, in (3.6), we have

$$\bigg(\sum_{\ell\in\mathcal{E}} \|1+2\varpi\|_{0,\ell}^2 \hat{\ell}^4 \eta_{\ell,1}(f)^2\bigg)^{\frac{1}{2}} = \bigg(\sum_{\ell\in\mathcal{E}} \|1+2\varpi\|_{0,\ell}^2 \hat{\ell}^4 \eta_{\ell,1}(\lambda_h u_h)^2\bigg)^{\frac{1}{2}} \approx \bigg(\sum_{\ell\in\mathcal{E}} \hat{\ell}^4 \eta_{\ell,1}(u_h)^2\bigg)^{\frac{1}{2}},$$
$$h^4 \sum_{\ell\in\mathcal{E}\cap\partial\kappa} \eta_{\ell,1}^2(\varphi_h) = h^4 \sum_{\ell\in\mathcal{E}\cap\partial\kappa} \eta_{\ell,1}^2(\lambda_h u_h) \approx \sum_{\ell\in\mathcal{E}\cap\partial\kappa} h^4 \eta_{\ell,1}^2(u_h),$$
$$\|f-\varphi_h\|_{0,\kappa}^2 = \|\lambda_h u_h - \omega_h\|_{0,\kappa}^2 = 0.$$

Hence, from (3.6)–(3.8) and (3.32) we obtain

$$\eta_h^2(F,u_h,\omega_h,\kappa) = \eta_\kappa^2(F,u_h) + \sum_{\ell\in\mathcal{E}^b\cap\partial\kappa} \eta_{\ell,1}^2(u_h) + \frac{1}{2}\sum_{\ell\in\mathcal{E}^i\cap\kappa} \{\eta_{\ell,1}^2(u_h) + \eta_{\ell,2}^2(u_h) + \eta_{\ell,3}^2(u_h) + \eta_{\ell,4}^2(u_h)\} + O\bigg(\sum_{\ell\in\mathcal{E}\cap\partial\kappa} h^4\eta_{\ell,1}^2(u_h)\bigg),$$
$$\eta_h^2(F,u_h,\omega_h,\Omega) = \sum_{\kappa\in\pi_h} \eta_h^2(F,u_h,\omega_h,\kappa),$$
$$\|\lambda_h T(u_h,\omega_h) - \lambda_h T_h(u_h,\omega_h)\|_h \lesssim \eta_h(F,u_h,\omega_h,\Omega), \tag{4.1}$$
$$\eta_h(F,u_h,\omega_h,\Omega) \lesssim \|\lambda_h T(u_h,\omega_h) - \lambda_h T_h(u_h,\omega_h)\|_h + \mathrm{OSC}_j(F,u_h), \tag{4.2}$$

where $f = \lambda_h u_h$, $g = \lambda_h \omega_h$ in F.

Note that $\sum_{\ell\in\mathcal{E}\cap\partial\kappa} h^4 \eta_{\ell,1}^2(u_h)$ is higher-order small than $\sum_{\ell\in\mathcal{E}^b\cap\partial\kappa} \eta_{\ell,1}^2(u_h) + \frac{1}{2}\sum_{\ell\in\mathcal{E}^i\cap\kappa} \eta_{\ell,1}^2(u_h)$. So it can be neglected in actual numerical computation.

The following lemma states a crucial property of eigenvlaue and eigenfunction approximation.

Lemma 4.1. *Let (λ,u,ω) and (λ^*,u^*,ω^*) be the eigenpairs of* (2.5) *and* (2.11)*, respectively. Then for any $(v,z),(v^*,z^*)\in \boldsymbol{H}_h$, when $B((v,z),(v^*,z^*))\neq 0$ it is valid that*

$$\frac{A_h((v,z),(v^*,z^*))}{B((v,z),(v^*,z^*))} - \lambda = \frac{A_h((u,\omega)-(v,z),(u^*,\omega^*)-(v^*,z^*))}{B((v,z),(v^*,z^*))} - \lambda\frac{B((u,\omega)-(v,z),(u^*,\omega^*)-(v^*,z^*))}{B((v,z),(v^*,z^*))}. \tag{4.3}$$

Proof. See [43, Lemma 3.5]. □

Referring to [44, Lemma 4.1] we can deduce the following theorem.

Theorem 4.2. *Assume that λ and λ_h are the i-th eigenvalues of* (2.5) *and* (2.16)*, respectively, (u_h,ω_h) is an eigenfunction corresponding to λ_h with $\|(u_h,\omega_h)\|_h = 1$, and the ascent α of λ is equal to* 1*, and assume that $R(\Omega)$ holds and $n\in W^{1,\infty}(\Omega)\cap H^2(\Omega)$. Let $(\bar{u}_h,\bar{\omega}_h)$ be the orthogonal projection of (u_h,ω_h) to $\mathrm{ran}(P_h^*)$ in the sense of inner product $A_h(\cdot,\cdot)$, and*

$$(u_h^*,\omega_h^*) = (\bar{u}_h,\bar{\omega}_h)/\|(\bar{u}_h,\bar{\omega}_h)\|_h. \tag{4.4}$$

Then there exist $(u,\omega)\in\mathrm{ran}(P)$ and $(u^*,\omega^*)\in\mathrm{ran}(P^*)$ such that $(u_h,\omega_h)-(u,\omega)$ and $(u_h^*,\omega_h^*)-(u^*,\omega^*)$ satisfy (2.40)–(2.41) *and* (2.44)–(2.45) *respectively, and*

$$|\lambda_h-\lambda| \lesssim |||(u_h,\omega_h)-(u,\omega)|||_h |||(u_h^*,\omega_h^*)-(u^*,\omega^*)|||_h + \|(u_h,\omega_h)-(u,\omega)\|_{\boldsymbol{H}^1}\|(u_h^*,\omega_h^*)-(u^*,\omega^*)\|_{\boldsymbol{H}^1}. \tag{4.5}$$

Proof. From $\alpha = 1$, we know $\mathrm{ran}(P^*)$ is the space of eigenfunctions associated with λ^*. Choose $(u,\omega)\in\mathrm{ran}(P)$ such that (2.40)–(2.41) hold. Define

$$f((v,z)) = A(P(v,z),(u,\omega)), \quad \forall\,(v,z)\in\boldsymbol{H}.$$

Since for all $(v,z)\in\boldsymbol{H}$ one has

$$|f((v,z))| = |A(P(v,z),(u,\omega))| \leqslant \|P(v,z)\|_A\|(u,\omega)\|_A \lesssim \sqrt{\lambda}\|P(v,z)\|_{\boldsymbol{H}^1} \lesssim \|P\|_{\boldsymbol{H}^1}\|(v,z)\|_A,$$

f is a linear and bounded functional on $\boldsymbol{H}$ and $\|f\|_A \lesssim \|P\|_{\boldsymbol{H}^1}$. Using the Riesz theorem, we know that there exists $(u^*, \omega^*) \in \boldsymbol{H}$ satisfying $\|(u^*, \omega^*)\|_A = \|f\|_A$ and

$$A((v,z),(u^*,\omega^*)) = A(P(v,z),(u,\omega)). \tag{4.6}$$

For any $(v,z) \in \boldsymbol{H}$, notice $P(I-P)(v,z) = 0$. Then

$$\begin{aligned} A((v,z),(\lambda^{*-1} - T^*)(u^*,\omega^*)) &= A((\lambda^{-1} - T)(v,z),(u^*,\omega^*)) \\ &= A((\lambda^{-1} - T)P(v,z),(u^*,\omega^*)) + A((\lambda^{-1} - T)(I-P)(v,z),(u^*,\omega^*)) \\ &= 0, \end{aligned}$$

i.e., $(\lambda^{*-1} - T^*)(u^*,\omega^*) = 0$; hence $(u^*,\omega^*) \in \mathrm{ran}(P^*)$. By (4.6) we have

$$\begin{aligned} \lambda B((u,\omega),(u^*,\omega^*)) &= A((u,\omega),(u^*,\omega^*)) = A(P(u,\omega),(u,\omega)) \\ &= A((u,\omega),(u,\omega)) \approx A_h((u_h,\omega_h),(u_h,\omega_h)) \approx 1. \end{aligned} \tag{4.7}$$

Then, there exists $(\bar{u}_h^*, \bar{\omega}_h^*) \in \mathrm{ran}(P_h^*)$ such that $(\bar{u}_h^*, \bar{\omega}_h^*) - (u^*,\omega^*)$ satisfies (2.42), and from (2.40), (2.42) and (4.7), when h is small enough, there is a positive constant C_0 independent of h such that

$$|B((u_h,\omega_h),(\bar{u}_h^*,\bar{\omega}_h^*))| \geqslant C_0.$$

Since $(\bar{u}_h, \bar{\omega}_h)$ is the orthogonal projection of (u_h,ω_h) to $\mathrm{ran}(P_h^*)$ in the sense of inner product $A_h(\cdot,\cdot)$,

$$\begin{aligned} |B((u_h,\omega_h),(u_h^*,\omega_h^*))| &= \left| \frac{1}{\lambda_h} A_h((u_h,\omega_h),(u_h^*,\omega_h^*)) \right| \\ &\geqslant \left| \frac{1}{\lambda_h} A_h\left((u_h,\omega_h), \frac{(\bar{u}_h^*,\bar{\omega}_h^*)}{\|(\bar{u}_h^*,\bar{\omega}_h^*)\|_h} \right) \right| \geqslant \frac{1}{\|(\bar{u}_h^*,\bar{\omega}_h^*)\|_h} |B((u_h,\omega_h),(\bar{u}_h^*,\bar{\omega}_h^*))| \gtrsim C_0. \end{aligned}$$

In (4.3), choose $(v,z) = (u_h,\omega_h)$ and $(v^*,z^*) = (u_h^*,\omega_h^*)$, and choose (u^*,ω^*) such that $(u_h^*,\omega_h^*) - (u^*,\omega^*)$ satisfies (2.44)–(2.45). Noting that $\lambda_h = A((u_h,\omega_h),(u_h^*,\omega_h^*))/B((u_h,\omega_h),(u_h^*,\omega_h^*))$, we obtain (4.5). □

Remark 4.3. When λ is a simple eigenvalue, $\mathrm{ran}(P_h^*)$ is a one-dimensional space spanned by the eigenfunction (u_h^*,ω_h^*) of (2.25) with the mesh size h. When the multiplicity $q > 1$ of λ, in actual computation we can use the two sided Arnoldi algorithm to compute both left and right eigenfunctions of (2.16) at the same time, and obtain (u_h,ω_h) and (u_h^*,ω_h^*).

The following lemma shows that the a posterior error for eigenfunctions can be derived from that for the boundary value problem.

Lemma 4.4. *Let $(\lambda_h,(u_h,\omega_h))$ be the i-th eigenpair of* (2.16) *with $\|(u_h,\omega_h)\|_h = 1$, λ be the i-th eigenvalue of* (2.5)*, and the ascent α of λ be equal to* 1*. Then there exists an eigenfunction (u,ω) corresponding to λ, such that*

$$\|(u_h,\omega_h) - (u,\omega)\|_h = \lambda_h \|T(u_h,\omega_h) - T_h(u_h,\omega_h)\|_h + R_1, \tag{4.8}$$

where $|R_1| \lesssim \|(T - T_h)(u_h,\omega_h)\|_{\boldsymbol{H}^1}$.

Proof. Using the argument as in [15, Proposition 5.3] we can deduce

$$\|(u_h,\omega_h) - (u,\omega)\|_{\boldsymbol{H}^1} \lesssim \|(T - T_h)(u_h,\omega_h)\|_{\boldsymbol{H}^1}. \tag{4.9}$$

A simple calculation shows

$$\begin{aligned} B((T - T_h)(u_h,\omega_h),(u^*,\omega^*)) &= B(T(u_h,\omega_h),(u^*,\omega^*)) - B(T_h(u_h,\omega_h),(u^*,\omega^*)) \\ &= \lambda^{-1} A(T(u_h,\omega_h),(u^*,\omega^*)) - B(T_h(u_h,\omega_h),(u^*,\omega^*)) \\ &= \lambda^{-1} B((u_h,\omega_h),(u^*,\omega^*)) - \lambda_h^{-1} B((u_h,\omega_h),(u^*,\omega^*)) \\ &= (\lambda^{-1} - \lambda_h^{-1}) B((u_h,\omega_h),(u^*,\omega^*)), \end{aligned}$$

where (u^*,ω^*) satisfies Theorem 4.2. Then the above equality implies

$$|\lambda_h-\lambda|\lesssim\|(T-T_h)(u_h,\omega_h)\|_{\boldsymbol{H}^1}. \tag{4.10}$$

Combining (4.9) and (4.10) we get

$$|\lambda_h-\lambda|+\|(u_h,\omega_h)-(u,\omega)\|_{\boldsymbol{H}^1}\lesssim\|(T-T_h)(u_h,\omega_h)\|_{\boldsymbol{H}^1}. \tag{4.11}$$

From (2.9), (2.10) and (4.11) we have

$$\begin{aligned}\|(u,\omega)-\lambda_hT(u_h,\omega_h)\|_h&=\|\lambda T(u,\omega)-\lambda_hT(u_h,\omega_h)\|_h\\&\lesssim\|\lambda(u,\omega)-\lambda_h(u_h,\omega_h)\|_{\boldsymbol{H}^1}\lesssim\|(T-T_h)(u_h,\omega_h)\|_{\boldsymbol{H}^1}.\end{aligned} \tag{4.12}$$

Denote

$$\|(u_h,\omega_h)-(u,\omega)\|_h=\lambda_h\|(T-T_h)(u_h,\omega_h)\|_h+R_1. \tag{4.13}$$

From the triangle inequality and (4.12) we deduce

$$\begin{aligned}|R_1|&=|\|(u_h,\omega_h)-(u,\omega)\|_h-\lambda_h\|(T-T_h)(u_h,\omega_h)\|_h|\\&=|\|(u_h,\omega_h)-(u,\omega)\|_h-\|\lambda_hT(u_h,\omega_h)-(u_h,\omega_h)\|_h\ |\\&\leqslant\|(u,\omega)-\lambda_hT(u_h,\omega_h)\|_h\lesssim\|(T-T_h)(u_h,\omega_h)\|_{\boldsymbol{H}^1}.\end{aligned} \tag{4.14}$$

Due to (4.13) and (4.14), (4.8) is obtained. □

Theorem 4.5. *Let $(\lambda_h,(u_h,\omega_h))$ be the i-th eigenpair of* (2.16) *with $\|(u_h,\omega_h)\|_h=1$, and λ be the i-th eigenvalue of* (2.5)*. Assume that $R(\Omega)$ holds, $n\in W^{1,\infty}(\Omega)\cap H^2(\Omega)$ and $h\ll 1$. Then there exists an eigenfunction (u,ω) corresponding to λ, such that*

$$\|(u_h,\omega_h)-(u,\omega)\|_h\lesssim\eta_h(F,u_h,\omega_h,\Omega), \tag{4.15}$$

$$\eta_h(F,u_h,\omega_h,\Omega)\lesssim\|(u_h,\omega_h)-(u,\omega)\|_h+\mathrm{OSC}_j(F,u_h). \tag{4.16}$$

Proof. Combining (4.11) with (4.1) we get (4.15). Combining (4.11) with (4.2) and neglecting the higher-order small quantity R_1 we get (4.16). □

For the dual problem (2.11), denote

$$F^*=F^*(f,g)=-\varpi\Delta f-\Delta(n\varpi f)-n\varpi g.$$

Using the same argument as in Theorem 4.5 we can prove the following theorem.

Theorem 4.6. *Let $(\lambda_h^*,(u_h^*,\omega_h^*))$ be the i-th eigenpair of* (2.25) *with $\|(u_h^*,\omega_h^*)\|_h=1$, and λ^* be the i-th eigenvalue of* (2.11)*. Assume that $R(\Omega)$ holds, $n\in W^{1,\infty}(\Omega)\cap H^2(\Omega)$ and $h\ll 1$. Then there exists an eigenfunction (u^*,ω^*) corresponding to λ^*, such that*

$$\|(u_h^*,\omega_h^*)-(u^*,\omega^*)\|_h\lesssim\eta_h(F^*,u_h^*,\omega_h^*,\Omega),$$
$$\eta_h(F^*,u_h^*,\omega_h^*,\Omega)\lesssim\|(u_h^*,\omega_h^*)-(u^*,\omega^*)\|_h+\mathrm{OSC}_j(F^*,u_h^*),$$

where $f=\lambda_h^*u_h^*$, $g=\lambda_h^*\omega_h^*$ in F^*.

Theorem 4.7. *Assume that the conditions of Theorem* 4.2 *hold and $h\ll 1$. Then the following estimate holds*:

$$|\lambda_h-\lambda|\lesssim\eta_h^2(F,u_h,\omega_h,\Omega)+\eta_h^2(F^*,u_h^*,\omega_h^*,\Omega)+R_2, \tag{4.17}$$

where $R_2=\sum_{\kappa\in\pi_h}h_\kappa^{2\alpha}\|(u,\omega)-I_h(u,\omega)\|^2_{H^{2+\alpha}(\kappa)}+\sum_{\kappa\in\pi_h}h_\kappa^{2\alpha}\|(u^*,\omega^*)-I_h(u^*,\omega^*)\|^2_{H^{2+\alpha}(\kappa)}$.

Proof. Thanks to Poincaré-Friedrichs inequalities in [10], we have $\|(u_h,\omega_h)-(u,\omega)\|_{\boldsymbol{H}^1} \lesssim |||(u_h,\omega_h) - (u,\omega)|||_h$ and $\|(u_h^*,\omega_h^*)-(u^*,\omega^*)\|_{\boldsymbol{H}^1} \lesssim |||(u_h^*,\omega_h^*)-(u^*,\omega^*)|||_h$. Thus from (4.5) we get

$$|\lambda_h-\lambda| \lesssim |||(u_h,\omega_h)-(u,\omega)|||_h|||(u_h^*,\omega_h^*)-(u^*,\omega^*)|||_h. \tag{4.18}$$

Due to (2.20), the triangle inequality, (2.21), (2.22) and the interpolation estimate, we deduce

$$\begin{aligned}|||(u_h,\omega_h)-(u,\omega)|||_h^2 &\leqslant (|||(u_h,\omega_h)-I_h(u,\omega)|||+|||(u,\omega)-I_h(u,\omega)|||_h)^2\\ &\lesssim (\|(u_h,\omega_h)-I_h(u,\omega)\|_h+|||(u,\omega)-I_h(u,\omega)|||_h)^2\\ &\lesssim (\|(u_h,\omega_h)-(u,\omega)\|_h+|||(u,\omega)-I_h(u,\omega)|||_h)^2\\ &\lesssim \eta_h^2(F,u_h,\omega_h,\Omega)+\sum_{\kappa\in\pi_h} h_\kappa^{2\alpha}\|(u,\omega)-I_h(u,\omega)\|_{H^{2+\alpha}(\kappa)}^2.\end{aligned}$$

Similarly, we can get

$$|||(u_h^*,\omega_h^*)-(u^*,\omega^*)|||_h^2 \lesssim \eta_h^2(F,u_h^*,\omega_h^*,\Omega^*)+\sum_{\kappa\in\pi_h} h_\kappa^{2\alpha}\|(u^*,\omega^*)-I_h(u^*,\omega^*)\|_{H^{2+\alpha}(\kappa)}^2.$$

Submitting the above two estimates into (4.18), we get (4.17). □

Remark 4.8. From Theorems 4.6 and 4.7, we know the indicator $\eta_h^2(F,u_h,\omega_h,\Omega)+\eta_h^2(F^*,u_h^*,\omega_h^*,\Omega)$ of the eigenfunction error $\|(u_h,\omega_h)-(u,\omega)\|_h^2+\|(u_h^*,\omega_h^*)-(u^*,\omega^*)\|_h^2$ is reliable and efficient up to data oscillation, so Algorithm 1 in Section 5 can generate a good graded mesh, which makes approximation eigenfunctions can get the optimal convergent rate h^{m-1} in $\|\cdot\|_h$. Thus we are able to expect to get $R_2 \lesssim h^{2(m-1)}$, and thereby from (4.17) we have $|\lambda_h-\lambda| \lesssim h^{2(m-1)}$. Therefore, we think that $\eta_h^2(F,u_h,\omega_h,\Omega)+\eta_h^2(F^*,u_h^*,\omega_h^*,\Omega)$ can be viewed as the indicator of λ_h. The numerical experiments in Section 5 show this indicator of λ_h is reliable and efficient. In addition, λ_h can achieve the optimal convergent rate.

5 Adaptive algorithms and numerical experiments

Using the a posteriori error estimates and consulting the existing standard algorithms (see, e.g., [20,25]), we present the following algorithm:

Algorithm 1

Require: Choose the parameter $\sigma,\mu,0<\theta<1$;

1: set $l=0$ and pick any initial mesh π_{h_l} with the mesh size h_l;
2: solve (2.16) on π_{h_l} for discrete solution $(\lambda_{h_l},(u_{h_l},\omega_{h_l}))$ with $\|(u_{h_l},\omega_{h_l})\|_h=1$ and find $(u_{h_l}^*,\omega_{h_l}^*)\in \mathrm{ran}(P_{h_l}^*)$ by (4.4) (also see Remark 4.3);
3: compute the local indicators $\eta_{h_l}^2(F,u_{h_l},\omega_{h_l},\kappa)+\eta_{h_l}^2(F^*,u_{h_l}^*,\omega_{h_l}^*,\kappa)$;
4: construct $\hat{\pi}_{h_l}\subset\pi_{h_l}$ by **Marking Strategy E**;
5: refine π_{h_l} to get a new mesh $\pi_{h_{l+1}}$ by procedure **refine**;
6: set $l=l+1$ and goto Step 2.

Marking Strategy E

Require: Given parameter $0<\theta<1$;

1: construct a minimal subset $\widehat{\pi}_{h_l}$ of π_{h_l} by selecting some elements in π_{h_l} such that

$$\sum_{\kappa\in\widehat{\pi}_{h_l}}(\eta_{h_l}^2(F,u_{h_l},\omega_{h_l},\kappa)+\eta_{h_l}^2(F^*,u_{h_l}^*,\omega_{h_l}^*,\kappa)) \geqslant \theta(\eta_{h_l}^2(F,u_{h_l},\omega_{h_l},\Omega)+\eta_{h_l}^2(F^*,u_{h_l}^*,\omega_{h_l}^*,\Omega));$$

2: mark all the elements in $\widehat{\pi}_{h_l}$.

The above marking strategy was introduced in [21] (see also [33]).

We compute the transmission eigenvalues on the unit square domain with a slit $[0,1]^2\setminus[0.5,1]$ and the L-shaped domain $[-1,1]^2\setminus[0,1]\times[-1,0]$ using Algorithm 1 with $m=2,3$. All the initial meshes are made up of congruent triangles. In addition, the mesh sizes take $h_0=\frac{\sqrt{2}}{32}$ and $h_0=\frac{\sqrt{2}}{16}$ for the domain with a slit and the L-shaped domain, respectively. $\theta = 0.25$ and $\theta = 0.5$ for $m = 2$ and $m = 3$, respectively. We use MATLAB2012a and the iFEM package (see [16]) on an HP-Z230 workstation (CPU 3.6 GHZ and RAM 32 GB).

We use the sparse solver *eigs* to solve (2.16) and (2.25) for eigenvalues. Before showing the results, some symbols need to be explained:

$k_j = \sqrt{\lambda_j}$;

λ_{j,h_l}: the j-th eigenvalue derived from the l-th iteration using Algorithm 1, $k_{j,h_l} = \sqrt{\lambda_{j,h_l}}$;

DOF: the number of degrees of freedom.

The accurate eigenvalues for the problems on the two above domains are unknown. For the domain with a slit, we take $k_1 \approx 2.80677803, k_2 \approx 2.98066000$ for $n = 16$, and take $k_1 \approx 4.14438323, k_7 \approx 5.57000885 - 1.31142340i$ for $n = 8 + x - y$. For the L-shaped domain, we take $k_1 \approx 1.47609911, k_2 \approx 1.56972499$ for $n = 16$, and take $k_1 \approx 2.30212024, k_5 \approx 2.92423162 - 0.56458999i$ for $n = 8 + x - y$. All of them with high accuracy are obtained by Algorithm 1. By computation we also know that the first ten smallest eigenvalues are all simple.

We present some adaptive refined mesh in Figure 1, and the curves of the error of the numerical eigenvalues in Figures 2–5. From Figure 1, we can see that the singularities of the eigenfunctions for the two domains mainly center on the corner points.

From Figures 2–5, we see that the curves of the indicator are parallel to the curves of the error of $\lambda_{j,h}$, which shows the posteriori error estimators are reliable and efficient for all the cases; we also see that the accuracy of the numerical eigenvalues on adaptive meshes, better than that on uniform meshes, can get the optimal convergence order $O(\mathrm{DOF}^{-m+1})$, $m = 2, 3$.

However, from Figures 2–5, we also see that there exists the fluctuation in the results on adaptive meshes when DOF is large enough. This is probably the consequence of the performance of linear algebra routine on this problem. To treat such problems to get higher accurate approximation much more careful design of the routine is needed.

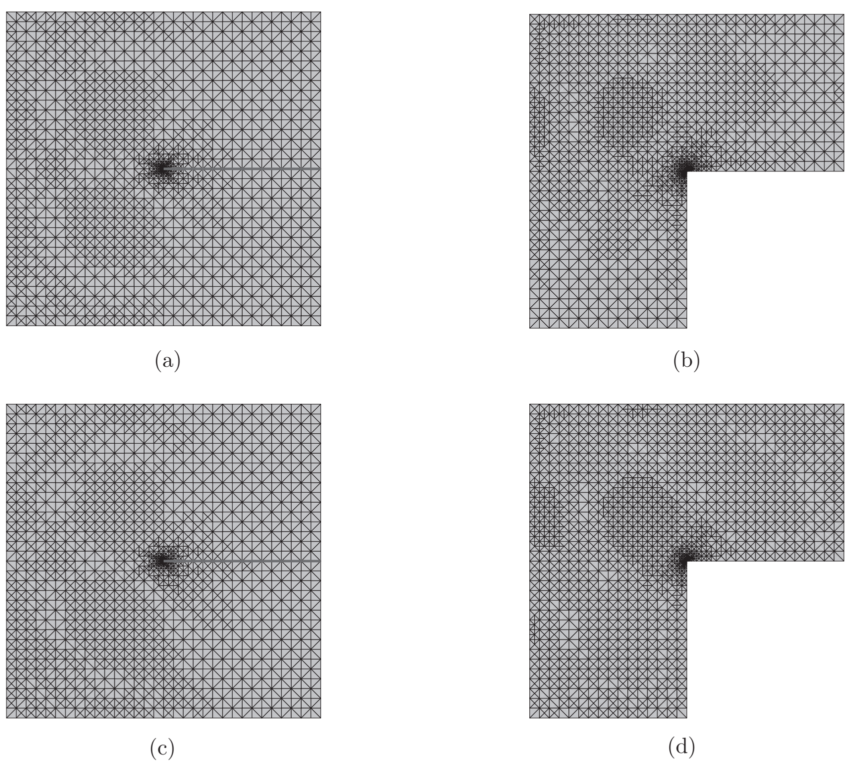

Figure 1 Adaptive meshes for the smallest eigenvalue on the domain with a slit with DOF = 28,688, $n = 16$, $m = 3$ (a), on the L-shaped domain with DOF = 29,972, $n = 16$, $m = 3$ (b), on the domain with a slit with DOF = 29,564, $n = 8 + x - y$, $m = 3$ (c) and on the L-shaped domain with DOF = 32,954, $4n = 8 + x - y$, $m = 3$ (d)

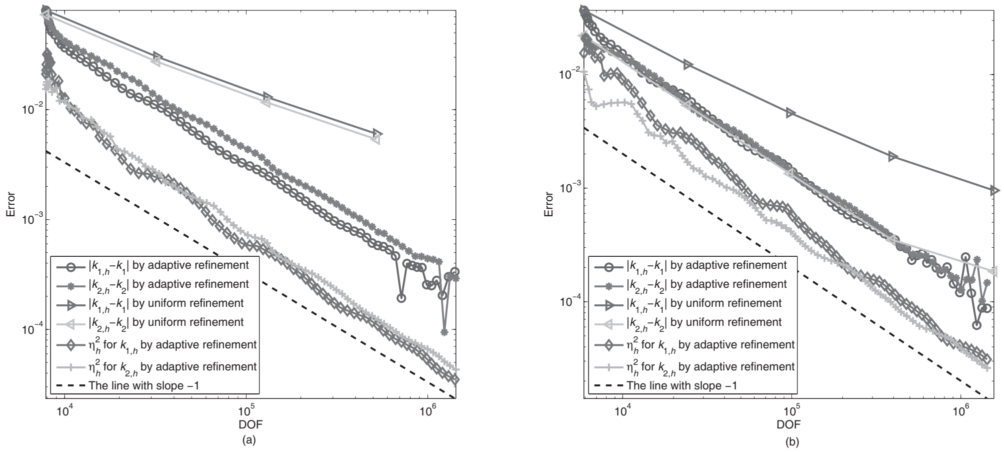

Figure 2 The convergence rates of eigenvalues for the domain with a slit (a) and for the L-shaped domain (b) when $n = 16, m = 2, \sigma = 30, \mu = \frac{1}{15}$

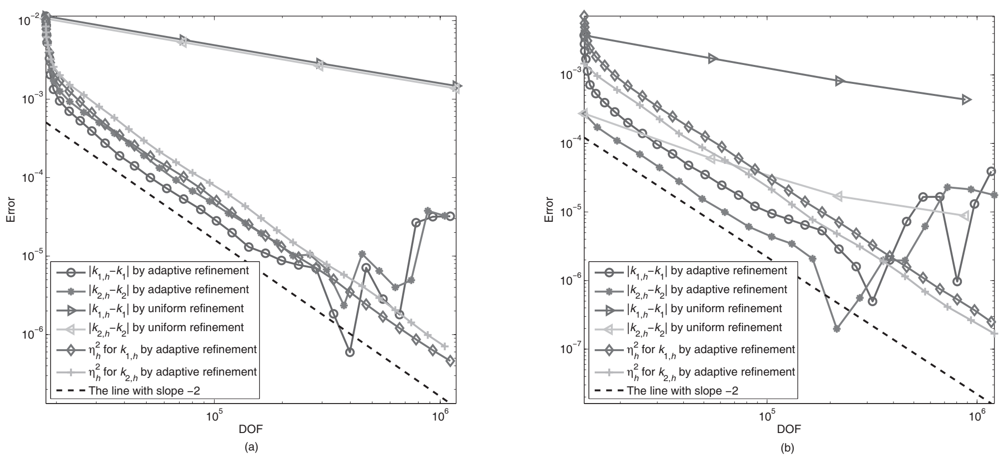

Figure 3 The convergence rates of eigenvalues for the domain with a slit (a) and for the L-shaped domain (b) when $n = 16, m = 3, \sigma = 30, \mu = \frac{1}{15}$

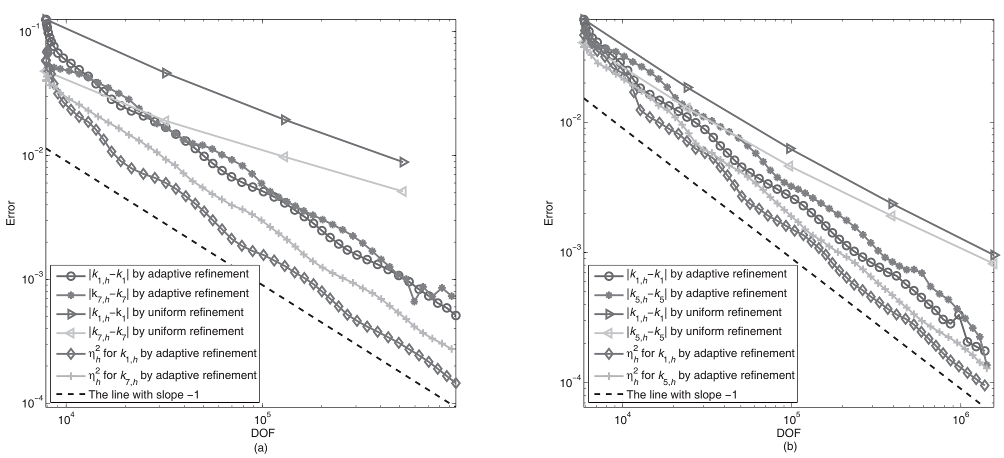

Figure 4 The convergence rates of eigenvalues for the domain with a slit (a) and for the L-shaped domain (b) when $n = 8 + x - y, m = 2, \sigma = 20, \mu = \frac{1}{9}$

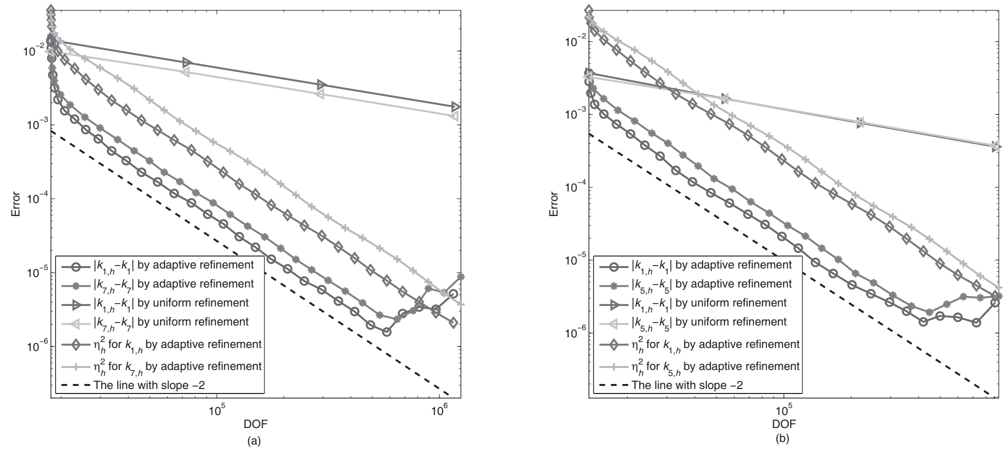

Figure 5 The convergence rates of eigenvalues for the domain with a slit (a) and for the L-shaped domain (b) when $n = 8 + x - y, m = 3, \sigma = 20, \mu = \frac{1}{9}$

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