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Randomized statistical inference: A unified statistical inference frame of frequentist, fiducial, and Bayesian inference

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Abstract We propose randomized inference (RI), a new statistical inference approach. RI may be realized through a randomized estimate (RE) of a parameter vector, which is a random vector that takes values in the parameter space with a probability density function (PDF) that depends on the sample or sufficient statistics, such as the posterior distributions in Bayesian inference. Based on the PDF of an RE of an unknown parameter, we propose a framework for both the vertical density representation (VDR) test and the construction of a confidence region. This approach is explained with the aid of examples. For the equality hypothesis of multiple normal means without the condition of variance homogeneity, we present an exact VDR test, which is shown as an extension of one-way analysis of variance (ANOVA). In the case of two populations, the PDF of the Welch statistics is given by using the RE. Furthermore, through simulations, we show that the empirical distribution function, the approximated t, and the RE distribution function of Welch statistics are almost equal. The VDR test of the homogeneity of variance is shown to be more efficient than both the Bartlett test and the revised Bartlett test. Finally, we discuss the prospects of RI.

Keywords confidence distribution, pivot, randomized inference, vertical density representation, VDR test

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1 Introduction

The parameter θ, which is regarded as the central objective of the frequentist statistical inference, is an unknown constant, and this parameter may take any value within the specified parameter space Θ. The frequentist inference focuses on the true value of the parameter by employing data analysis that works on the basis of the true model and has sensible properties regardless of the true value of θ.

The Bayesian attitude toward the parameter θ is different. Specifically, the parameter is regarded as a realized value of a random variable ϑ with the PDF $f_\vartheta(\theta)$, where $\theta \in \Theta$; the prior distribution contains information about the parameter θ before we observe a sample $X = x$. After obtaining x, one combines

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the information contained in x with the prior distribution and obtains the posterior distribution $f_\vartheta(\theta, x)$, which is the conditional distribution of ϑ, given that $X = x$. The Bayesian inference emphasizes the process for obtaining the posterior distribution and making inference to the parameter θ based on the posterior distribution.

In the 1930s, Fisher [3, 4] attempted to find a distribution over Θ straightly instead of using a prior distribution, and he called it the fiducial distribution. He gave examples to show how to find the fiducial distribution. Both Bayesian inference and fiducial inference consider parameter θ as a random variable, and this diverges from the viewpoint of the frequentist inference (see [1, 6] for more comments).

Instead of considering θ as a random variable, we can infer the parameter θ using a random variable $W = W_\theta$ with the PDF $f(\cdot\,; x)$ over the parameter space Θ, where the observed sample x is a parameter of the distribution. The function $f(\cdot\,; x)$ may be a posterior or a fiducial density function, and could also be a density function of other types. W is called an RE of the parameter θ (see [16]). The inference can then be performed based on the RE in a frequentist way. Such a frame maintains the concept of a parameter being an unknown constant, and it is possible that both Bayesian and fiducial inferences exist in the frequentist inference system. In fact, the frequentist inference is also a source of the RE (see Subsection 2.1).

For illustrating the main idea, we recall the classical example of a normal distribution. Let $\boldsymbol{x} = (x_1, \ldots, x_n)'$ be the observed sample from the normal distribution $N(\mu, \sigma^2)$ and $\boldsymbol{X} = (X_1, \ldots, X_n)'$ be the corresponding random sample. We denote the sample variance computed from $\boldsymbol{x}$ by s_n^2 and its random version as S_n^2. The pivot of μ is

$$h(\bar{x}, s_n^2; \mu) = \frac{\sqrt{n}(\bar{x} - \mu)}{s_n}, \quad h(\bar{X}, S_n^2; \mu) = \frac{\sqrt{n}(\bar{X} - \mu)}{S_n} \sim dt(\cdot, n-1),$$

where $dt(\cdot\,, n-1)$ is the PDF of the Student's t distribution with $n-1$ degrees of freedom. Given the significance level α, it is well known that

$$1 - \alpha = \mathrm{P}\left(\bar{X} - t_{n-1}\left(1 - \frac{\alpha}{2}\right)\frac{S_n}{\sqrt{n}} \leqslant \mu \leqslant \bar{X} - t_{n-1}\left(\frac{\alpha}{2}\right)\frac{S_n}{\sqrt{n}}\right). \tag{1.1}$$

$[\bar{X} - t_{n-1}(1 - \frac{\alpha}{2})\frac{S_n}{\sqrt{n}},\ \bar{X} - t_{n-1}(\frac{\alpha}{2})\frac{S_n}{\sqrt{n}}]$ is a random interval that covers parameter μ with the probability $1 - \alpha$. Accordingly, when considering the hypothesis

$$H_0 : \mu = \mu_0 \quad \text{vs.} \quad H_1 : \mu \neq \mu_0,$$

we reject H_0 with the significance level α if

$$\mu_0 \in [\underline{\mu}, \bar{\mu}] = \left[\bar{x} - t_{n-1}\left(1 - \frac{\alpha}{2}\right)\frac{s_n}{\sqrt{n}}, \bar{x} - t_{n-1}\left(\frac{\alpha}{2}\right)\frac{s_n}{\sqrt{n}}\right]$$

does not hold.

The above statements are shown in Figure 1 (see also [5]). It shows also that

$$T_{n-1} \sim dt(\cdot\,, n-1), \quad T_{n-1} \in \left[t_{n-1}\left(\frac{\alpha}{2}\right), t_{n-1}\left(1 - \frac{\alpha}{2}\right)\right] \Leftrightarrow \mu \in [\underline{\mu}, \bar{\mu}].$$

This can be formulized as follows:

$$\frac{\sqrt{n}(\bar{x} - \mu)}{s_n} = T_{n-1}, \quad T_{n-1} \sim dt(\cdot, n-1). \tag{1.2}$$

Then, the fiducial inference takes μ as a random variable and $\bar{x}$ and s_n^2 as constants. This is also a way to use a pivot.

The inconformity of μ being both a constant and a random variable could be removed by using the RE of μ. We consider that parameter μ is an unknown constant while replacing μ in (1.2) with a random variable W, which is defined as the RE of μ. This implies the following:

$$\frac{\sqrt{n}(\bar{x} - W)}{s_n} = T_{n-1}, \quad T_{n-1} \sim dt(\cdot, n-1). \tag{1.3}$$

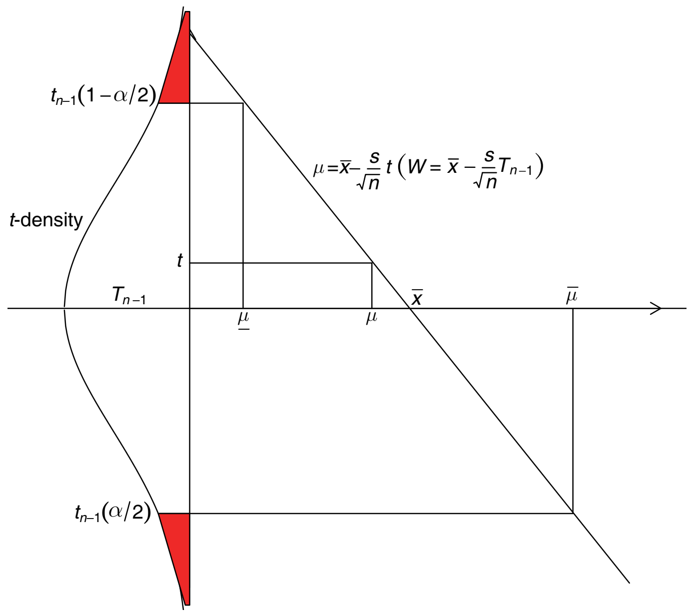

Figure 1 (Color online) Statistical inference of μ for $N(\mu,\sigma^2)$

Equation (1.3) is a way to define an RE by using a pivot. The basic idea is just from Fisher [3]. From (1.3), we find that the PDF of W is $\frac{\sqrt{n}}{s_n}dt(\frac{\sqrt{n}(\bar{x}-\mu)}{s_n}, n-1)$. The RI is based on this distribution. Here, we use the notations of the R software.

The rest of this paper is organized as follows. In Section 2, the basic concepts and theorems on RI are presented. Several applications of RI are discussed in Section 3. In Section 4, we present the concluding remarks and suggest some future topics for RI.

2 Randomized inference

In this section, we discuss the RI approach in detail and extend all the statements of Section 1 to general cases. For convenience, we first formulate the model and recall the concept of the RE. Then, we present the RI including the VDR test, the confidence region, and several basic properties.

2.1 Formulation

Consider a distribution family with the PDF $f(\boldsymbol{x};\boldsymbol{\eta},\boldsymbol{\lambda}),\ \boldsymbol{x}\in\Re^p$, where $\boldsymbol{\eta}=(\eta_1,\ldots,\eta_l)'\in\mathcal{N}\subseteq\Re^l$ is the parameter of interest; $\boldsymbol{\lambda}\in\Lambda\subseteq\Re^{s-l}$ is an unknown nuisance parameter. Let $\mathbb{X}_n=(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n)'$ and $\mathcal{X}_n=(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)'$ denote a random sample from $f(\cdot;\boldsymbol{\eta},\boldsymbol{\lambda})$ and its observed value, respectively, and let the sample space be $\mathscr{X}_n=\times_{i=1}^n\Re^p$. The PDF of $\mathbb{X}_n$ is

$$f_n(\mathcal{X}_n;\boldsymbol{\eta},\boldsymbol{\lambda})=\prod_{i=1}^n f(\boldsymbol{x}_i;\boldsymbol{\eta},\boldsymbol{\lambda}),\quad \mathcal{X}_n\in\mathscr{X}_n.$$

This is a general parametric model. One of basic problems is how to test the hypothesis

$$H_0:\boldsymbol{\eta}=\boldsymbol{\eta}_0\quad\text{vs.}\quad H_1:\boldsymbol{\eta}\neq\boldsymbol{\eta}_0.\tag{2.1}$$

This is equivalent to constructing a confidence region of $\boldsymbol{\eta}$.

2.2 Randomized estimate

Definition 2.1 (RE [16])**.** Let $\boldsymbol{W}$ be a random vector over $\mathcal{N}$ with a known PDF $f_w(\cdot;\mathcal{X}_n),\ \mathcal{X}_n\in\mathscr{X}_n$ (or $f_w(\cdot;\boldsymbol{T})$, where $\boldsymbol{T}=\boldsymbol{T}(\mathcal{X}_n)$ is a sufficient statistic of $\boldsymbol{\eta}$) that takes the sample space (or the range space of the sufficient statistic) as its parameter space of the distribution family. Then, we call $\boldsymbol{W}$ an RE of $\boldsymbol{\eta}$.

The expectation $\boldsymbol{w}_n(\mathcal{X}_n) = \mathrm{E}\boldsymbol{W} = \int_{\mathcal{N}} \boldsymbol{w} f_w(\boldsymbol{w}; \mathcal{X}_n) d\boldsymbol{w}$, if it exists, is a statistic and can be viewed as a point estimate of $\boldsymbol{\eta}$. We call it as the expectation estimate corresponding to $\boldsymbol{W}$. If

$$f_w(\hat{\boldsymbol{\eta}}_n; \mathcal{X}_n) = \max_{\boldsymbol{\eta} \in \mathcal{N}} f_w(\boldsymbol{\eta}; \mathcal{X}_n),$$

we call $\hat{\boldsymbol{\eta}}_n$ a randomized mass estimate.

If $\mathrm{E}(\boldsymbol{w}_n(\mathbb{X}_n)) = \int_{\mathscr{X}_n} \boldsymbol{w}_n(\mathcal{X}) f_n(\mathcal{X}; \boldsymbol{\eta}, \boldsymbol{\lambda}) d\mathcal{X} = \boldsymbol{\eta}$, we call $\boldsymbol{W}$ an unbiased RE (URE). If $\boldsymbol{w}_n(\mathbb{X}_n)$ is the uniformly minimum variance unbiased estimator, then we call $\boldsymbol{W}$ a uniformly minimum variance unbiased randomized estimate (UMVURE) . If

$$\lim_{n\to\infty} \mathrm{P}(|\boldsymbol{w}_n(\mathbb{X}_n) - \boldsymbol{\eta}| \geqslant \epsilon) = 0, \quad \forall\, \epsilon > 0$$

holds for any $\boldsymbol{\eta} \in \mathcal{N}$, then we call $\boldsymbol{W}$ a consistent RE (CRE). Many RE concepts could be defined but have been omitted here.

Suppose that $A = A(\mathcal{X}_n) \subseteq \mathcal{N}$ and

$$\mathrm{P}(\boldsymbol{W} \in A(\mathcal{X}_n)) = \int_A f_w(\boldsymbol{w}; \mathcal{X}_n) d\boldsymbol{w} = 1 - \alpha, \quad \forall\, \mathcal{X}_n \in \mathscr{X}_n.$$

$A(\mathbb{X}_n)$ is a random set. If

$$\mathrm{P}(\boldsymbol{\eta} \in A(\mathbb{X}_n)) = 1 - \alpha, \tag{2.2}$$

we call $\boldsymbol{W}$ an equivalent RE (ERE). This concept specifies a way of constructing a confidence region using ERE.

Example 2.2. Let $\boldsymbol{x}$ be a sample of size n ($n > 2$) from $N(\mu, \sigma^2)$, and both μ and σ^2 are unknown. With the notation in Section 1, denote

$$W = \bar{x} - \frac{s_n}{\sqrt{n}} T_{n-1}, \quad T_{n-1} \sim dt(\cdot, n-1),$$
$$W_1 = \bar{x} - \frac{s_n}{\sqrt{n}} Z, \quad Z \sim N(0, 1).$$

Then it is easy to see that both W and W_1 are URE, CRE, and UMVURE of μ, but W is an ERE and W_1 is not. If σ^2 is known, neither W nor W_1 is an ERE of μ, while $\bar{x} - \frac{\sigma}{\sqrt{n}} Z$ is an ERE.

2.3 VDR test

Now, consider the test of the hypothesis (2.1). We propose the VDR test, which is based on the PDF $f_W(\cdot; \mathcal{X}_n)$ of an RE [16]. From the likelihood principle, when $f_W(\boldsymbol{\eta}_0; \mathcal{X}_n)$ is too small, we should reject H_0. Let $Z = f_W(\boldsymbol{W}; \mathcal{X}_n)$, and we call it a test random variable. Note that Z is always one-dimensional regardless of the dimension of $\boldsymbol{W}$. The PDF of Z is as follows (see [11] or [12]):

$$f_Z(z; \mathcal{X}_n) = -z \frac{\partial L_l(D(z; \mathcal{X}_n))}{\partial z},$$

where L_k is the Lebesgue measure on $\mathfrak{R}^k$ and $D(z, \mathcal{X}_n) = \{\boldsymbol{\eta} : f_W(\boldsymbol{\eta}; \mathcal{X}_n) \geqslant z, \boldsymbol{\eta} \in \mathcal{N}\}$. Let $Q_Z(\alpha; \mathcal{X}_n)$ be the α quantile of Z that satisfies

$$\mathrm{P}(Z \leqslant Q_Z(\alpha; \mathcal{X}_n)) = \int_0^{Q_Z(\alpha; \mathcal{X}_n)} f_Z(z; \mathcal{X}_n) dz = \alpha. \tag{2.3}$$

The VDR test rejects $H_0 : \boldsymbol{\eta} = \boldsymbol{\eta}_0$ if $f_W(\boldsymbol{\eta}_0; \mathcal{X}_n) \leqslant Q_Z(\alpha; \mathcal{X}_n)$.

A $1 - \alpha$ VDR confidence region denoted by $CR(1 - \alpha; \mathcal{X}_n)$ is defined as

$$CR(1 - \alpha; \mathcal{X}_n) = \{\boldsymbol{\eta} : f_W(\boldsymbol{\eta}; \mathcal{X}_n) > Q_{\boldsymbol{Z}}(\alpha; \mathcal{X}_n)\}. \tag{2.4}$$

It is obvious from (2.3) that $\mathrm{P}(\boldsymbol{W} \in CR(1 - \alpha; \mathcal{X}_n)) = 1 - \alpha$. If $\boldsymbol{W}$ is an ERE, then $CR(1 - \alpha; \mathcal{X}_n)$ is also a $1 - \alpha$ confidence region in the sense of the frequentist inference.

Remark 2.3. Martin and Liu [8] introduced an interesting inference framework for asserting the uncertainty of the unknown parameter. Similar to this study, they employed an unobserved auxiliary random variable, which led to a data-dependent probability measure; however, both the basic idea about the parameter information and the inference procedure were very different from those used herein.

2.4 Pivotal quantity for a vector parameter

From the above subsection, we see that an ERE is helpful for finding a confidence region. In this subsection, we illustrate how to obtain an ERE through a pivot.

Definition 2.4 (Pivot [7])**.** Let $\boldsymbol{h}(\mathcal{X}_n;\boldsymbol{\eta})$ be a function from $\mathscr{X}_n \times \mathcal{N}$ to $\mathcal{N}_{\boldsymbol{h}}$. If this function satisfies the conditions that (i) for a given $\mathcal{X}_n$, $\boldsymbol{h}(\mathcal{X}_n;\cdot)$ is a one-to-one mapping from $\mathcal{N}$ to $\mathcal{N}_{\boldsymbol{h}}$ and (ii) for a given $\boldsymbol{\eta}$, the distribution function $F_{\boldsymbol{H}}(\cdot)$ of $\boldsymbol{H} = \boldsymbol{h}(\mathbb{X}_n;\boldsymbol{\eta})$ does not involve any parameters, then $\boldsymbol{h}(\cdot\,;\cdot)$ is called a pivot of the parameter $\boldsymbol{\eta}$.

Example 2.5. The example of pivots. We consider the normal population $N_p(\boldsymbol{\mu},\Sigma)$ and denote

$$\boldsymbol{h}(\mathcal{X};\boldsymbol{\mu},\Sigma) = \begin{pmatrix} \boldsymbol{h}_1(\mathcal{X};\boldsymbol{\mu},\Sigma) \\ \boldsymbol{h}_2(\mathcal{X};\boldsymbol{\mu},\Sigma) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n \hat{\Sigma}^{-\frac{1}{2}}(\boldsymbol{x}_i-\boldsymbol{\mu}) \\ \sum_{i=1}^n \Sigma^{-\frac{1}{2}}(\boldsymbol{x}_i-\boldsymbol{\mu})(\boldsymbol{x}_i-\boldsymbol{\mu})'\Sigma^{-\frac{1}{2}} \end{pmatrix}.$$

Then

$$\boldsymbol{h}(\mathbb{X};\boldsymbol{\mu},\Sigma) \sim \begin{pmatrix} t_p(n-1) \\ W_p(n,I_p) \end{pmatrix},$$

where

$$\hat{\Sigma} = \frac{1}{n-1}\sum_{i=1}^n (\boldsymbol{x}_i-\bar{\boldsymbol{x}})(\boldsymbol{x}_i-\bar{\boldsymbol{x}})',$$

$t_p(n-1)$ is the p-dimensional t distribution with $n-1$ degrees of freedom, and $W_p(n,I_p)$ is the Wishart distribution with n degrees of freedom and the scale matrix I_p. This means $\boldsymbol{h}(\mathcal{X};\boldsymbol{\mu},\Sigma)$ is a pivot of $(\mu,\ \Sigma)$.

It is difficult to infer a vector parameter directly from its pivots, but we can achieve this using an RE defined through a pivot. The RE $\boldsymbol{W}$ of $\boldsymbol{\eta}$ is defined by

$$\boldsymbol{h}(\mathcal{X}_n;\boldsymbol{W}) = \boldsymbol{H}, \quad \boldsymbol{H} \sim f_{\boldsymbol{H}}(\cdot). \tag{2.5}$$

The PDF of $\boldsymbol{W}$ is

$$f_{\boldsymbol{W}}(\cdot\,;\mathcal{X}_n) = f_{\boldsymbol{H}}(\boldsymbol{h}(\mathcal{X}_n;\cdot))\left|\det\left(\frac{\partial \boldsymbol{h}(\mathcal{X}_n;\cdot)}{\partial\,\cdot}\right)\right| = f_{\boldsymbol{H}}(\boldsymbol{h}(\mathcal{X}_n;\cdot))J(\mathcal{X}_n;\cdot).$$

We call (2.5) an RE equation of $\boldsymbol{\eta}$. The idea of obtaining the distribution is from Fisher [4]. The RE defined through a pivot is an ERE.

Lemma 2.6. *Let $\boldsymbol{W}$ be the RE of $\boldsymbol{\eta}$ defined by* (2.5) *through $\boldsymbol{h}(\mathcal{X}_n;\boldsymbol{u})$, and $f_{\boldsymbol{W}}(\cdot\,;\mathcal{X}_n)$ be the PDF of $\boldsymbol{W}$. If $C\subset\mathcal{N}$ satisfies*

$$\mathrm{P}(\boldsymbol{W}\in C) = \int_C f_{\boldsymbol{W}}(\boldsymbol{u};\mathcal{X}_n)d\boldsymbol{u} = 1-\alpha, \tag{2.6}$$

then C is a $1-\alpha$ confidence region in the sense of the frequentist inference.

Proof. Assume that $C_h = \{\boldsymbol{z} : \boldsymbol{z} = \boldsymbol{h}(\mathcal{X}_n,\boldsymbol{\eta}),\ \boldsymbol{\eta}\in C\}\subset\mathcal{N}_{\boldsymbol{h}}$. Then from the definition of a pivot, we have

$$C = \{\boldsymbol{\eta} : \boldsymbol{h}(\mathcal{X}_n,\boldsymbol{\eta})\in C_h,\ \boldsymbol{\eta}\in\mathcal{N}\}, \tag{2.7}$$

and thus,

$$1-\alpha = \mathrm{P}(\boldsymbol{W}\in C) = \int_C f_{\boldsymbol{W}}(\boldsymbol{u};\mathcal{X}_n)d\boldsymbol{u} = \int_C J(\mathcal{X}_n;\boldsymbol{u})f_{\boldsymbol{H}}(\boldsymbol{h}(\mathcal{X}_n;\boldsymbol{u}))d\boldsymbol{u}$$

$$= \int_{C_h} f_{\boldsymbol{H}}(\boldsymbol{z})d\boldsymbol{z} = \mathrm{P}(\boldsymbol{H} \in C_h). \tag{2.8}$$

Set $\mathscr{X}^c(\alpha, \boldsymbol{\eta}) = \{\mathcal{X}^* : \boldsymbol{h}(\mathcal{X}^*, \boldsymbol{\eta}) \in C_h\}$ and

$$C_C(1-\alpha, \mathcal{X}_n) = \{\boldsymbol{\eta}_0 : \mathcal{X}_n \in \mathscr{X}^c(\alpha, \boldsymbol{\eta}_0)\} = \{\boldsymbol{\eta}_0 : \boldsymbol{h}(\mathcal{X}_n, \boldsymbol{\eta}_0) \in C_h\}. \tag{2.9}$$

Then from (2.8), it holds that

$$\mathrm{P}(\boldsymbol{\eta}_0 \in C_C(1-\alpha, \mathbb{X}_n)) = \mathrm{P}(\boldsymbol{h}(\mathbb{X}_n, \boldsymbol{\eta}_0) \in C_h) = 1-\alpha.$$

Therefore, $C_C(1-\alpha, \mathcal{X}_n)$ is a $1-\alpha$ confidence region of $\boldsymbol{\eta}$. By comparing (2.9) with (2.7), we have $C_C(1-\alpha, \mathcal{X}) = C$. □

Lemma 2.7. *Suppose that $f(\cdot)$ is a PDF over $\mathcal{N}$ and*

$$C_f(1-\alpha) = \{\boldsymbol{\eta} : f(\boldsymbol{\eta}) \geqslant h(\alpha)\}, \quad \int_{C_f(1-\alpha)} f(\boldsymbol{\eta})d\boldsymbol{\eta} = 1-\alpha.$$

Then, the following expression holds:

$$L_l(C_f(1-\alpha)) = \min\left\{L_l(C) : \int_C f(\boldsymbol{\eta})d\boldsymbol{\eta} \geqslant 1-\alpha\right\},$$

where L_k is Lebesgue measure on $\mathfrak{R}^k$.

Proof. Let C satisfy $\int_C f(\boldsymbol{\eta})d\boldsymbol{\eta} \geqslant 1-\alpha$, and let

$$C^* = C_f(1-\alpha) \cap C, \quad C_1 = C_f(1-\alpha) \setminus C^*, \quad C_2 = C \setminus C^*.$$

Then, we have

$$f(\boldsymbol{z}) \geqslant h(\alpha), \quad \forall\, \boldsymbol{z} \in C_1, \quad f(\boldsymbol{z}) \leqslant h(\alpha), \quad \forall\, \boldsymbol{z} \in C_2.$$

Therefore,

$$\begin{aligned}
&\int_{C^*} f(\boldsymbol{z})d\boldsymbol{z} + h(\alpha)L_l(C_1)\\
&\quad\leqslant \int_{C^*} f(\boldsymbol{z})d\boldsymbol{z} + \int_{C_1} f(\boldsymbol{z})d\boldsymbol{z} = \int_{C_f(1-\alpha)} f(\boldsymbol{z})d\boldsymbol{z} = 1-\alpha\\
&\quad\leqslant \int_{C} f(\boldsymbol{z})d\boldsymbol{z} = \int_{C^*} f(\boldsymbol{z})d\boldsymbol{z} + \int_{C_2} f(\boldsymbol{z})d\boldsymbol{z}\\
&\quad\leqslant \int_{C^*} f(\boldsymbol{z})d\boldsymbol{z} + h(\alpha)L_l(C_2).
\end{aligned}$$

Since $h(\alpha) > 0$, the above inequality implies $L_l(C_1) \leqslant L_l(C_2)$. Therefore,

$$L_l(C_f(1-\alpha)) = L_l(C^*) + L_l(C_1) \leqslant L_l(C^*) + L_l(C_2) = L_l(C).$$

The conclusion is thus obtained. □

Lemma 2.7 means that the VDR confidence region defined by (2.4) has the minimal Lebesgue measure.

2.5 RE defined by confidence distribution

For some models, the interest parameter does not have any pivots. An example of this is the parameter p of the binomial distribution $B(n,p)$. However, p still has an RE, which is given by a random variable with a confidence distribution (CD) (see [10]).

Suppose $X \sim B(n,p)$ and x is an observation of X. Then, a γ upper confidence limit $\bar{p}$ of p satisfies

$$\gamma = \sum_{i=x+1}^{n} \binom{n}{i} \bar{p}^i (1-\bar{p})^{n-i} = F_n(\bar{p}; x).$$

The right-hand side is a function of the upper confidence limit $\bar{p}$, and it is solely a CD function. Now, let us suppose that the distribution function of the random variable W is $F_n(\cdot\,;x)$. Then, W is an RE of p. Its PDF is

$$\begin{aligned} f_n(w;x) &= \frac{\partial F_n(w;x)}{\partial w} = x\begin{pmatrix} n \\ x \end{pmatrix} w^x(1-w)^{n-x} \\ &= dbe(w, x+1, n-x+1), \quad 0 \leqslant w \leqslant 1. \end{aligned}$$

The corresponding expectation estimate $\mathrm{E}W = \frac{x+1}{n+2}$ is solely the Bayesian estimate with the uniform prior distribution. This example clearly shows that the distribution of W plays the role of a posterior distribution.

Generally, when the interested parameter is one-dimensional, one can always use a CD to define an RE similarly; this is called a CD random variable in [10]. Conversely, an RE also leads to a CD function. However, it is difficult to use a CD to deal with multiple parameters (see [10]). For more interesting applications of CD, please refer to [13–15].

2.6 Basic theorem of RE for composite parameters

Composite parameters are functions of natural parameters. For example, the failure rate is a natural parameter in an exponential distribution and reliability is a composite parameter. Generally, it is easy to find a pivot for natural parameters with closed forms and thus find an RE. The following theorem is useful for the construction of REs for composite parameters.

Theorem 2.8 (Theorem of REs for composite parameters). *Suppose that $\mathcal{X}$ is a sample drawn from $f(\cdot;\boldsymbol{\eta},\boldsymbol{\lambda})$, $\boldsymbol{\eta}=(\eta_1,\ldots,\eta_l)'\in\mathcal{N}$, $\boldsymbol{h}(\mathcal{X};\boldsymbol{\eta})=(h_1(\mathcal{X};\boldsymbol{\eta}),\ldots,h_l(\mathcal{X};\boldsymbol{\eta}))'$ is a pivot of $\boldsymbol{\eta}$, and the PDF of $\boldsymbol{h}(\mathbb{X},\boldsymbol{\eta})$ is $f_{\boldsymbol{h}}(\cdot)$. The RE $\boldsymbol{W}=(W_1,\ldots,W_l)'$ of $\boldsymbol{\eta}$ satisfies*

$$\boldsymbol{h}(\mathcal{X};\boldsymbol{W})=\boldsymbol{H}, \quad \boldsymbol{H}\sim f_{\boldsymbol{H}}(\cdot).$$

$\boldsymbol{\gamma}=(\gamma_1,\ldots,\gamma_k)'=\boldsymbol{\phi}(\boldsymbol{\eta})=(\phi_1(\boldsymbol{\eta}),\ldots,\phi_k(\boldsymbol{\eta}))'$ is a one-to-one smooth function from $\mathcal{N}$ to $\Phi=\{\boldsymbol{\gamma}:\boldsymbol{\gamma}=\boldsymbol{\phi}(\boldsymbol{\eta}),\,\boldsymbol{\eta}\in\mathcal{N}\}\subseteq\Re^k$ $(k\geqslant l)$, an l-dimensional surface in $\Re^k$. Set

$$\mathcal{J}=\mathcal{J}(\boldsymbol{\eta})=\left(\frac{\partial\boldsymbol{\phi}(\boldsymbol{\eta})}{\partial\boldsymbol{\eta}'}\right)=\begin{pmatrix} \frac{\partial\phi_1(\boldsymbol{\eta})}{\partial\eta_1} & \cdots & \frac{\partial\phi_1(\boldsymbol{\eta})}{\partial\eta_l} \\ \frac{\partial\phi_2(\boldsymbol{\eta})}{\partial\eta_1} & \cdots & \frac{\partial\phi_2(\boldsymbol{\eta})}{\partial\eta_l} \\ \vdots & \ddots & \vdots \\ \frac{\partial\phi_k(\boldsymbol{\eta})}{\partial\eta_1} & \cdots & \frac{\partial\phi_k(\boldsymbol{\eta})}{\partial\eta_l} \end{pmatrix}.$$

If $rk(\mathcal{J})=l$, $\forall\,\boldsymbol{\eta}\in\mathcal{N}$, then

$$\boldsymbol{V}=\boldsymbol{\phi}(W_1,\ldots,W_l)=\boldsymbol{\phi}(\boldsymbol{W}) \tag{2.10}$$

is an ERE of $\boldsymbol{\gamma}$. The PDF of $\boldsymbol{V}$ is

$$f_V(\boldsymbol{v})=C^{-1}|\det(\mathcal{J}'(\boldsymbol{\phi}^{-1}(\boldsymbol{v}))\mathcal{J}(\boldsymbol{\phi}^{-1}(\boldsymbol{v})))|^{-\frac{1}{2}}f_{\boldsymbol{H}}(\boldsymbol{\phi}^{-1}(\boldsymbol{v})), \quad \forall\,\boldsymbol{v}\in\Phi,$$

where

$$C=\int_{\Phi}|\det(\mathcal{J}'(\boldsymbol{\phi}^{-1}(\boldsymbol{v}))\mathcal{J}(\boldsymbol{\phi}^{-1}(\boldsymbol{v})))|^{-\frac{1}{2}}f_{\boldsymbol{H}}(\boldsymbol{\phi}^{-1}(\boldsymbol{v}))L_l(d\boldsymbol{v},\Phi).$$

Proof. Let $\boldsymbol{h}^*(\mathcal{X};\boldsymbol{\gamma})=\boldsymbol{h}(\mathcal{X};\boldsymbol{\phi}^{-1}(\boldsymbol{\gamma}))$. Then,

$$\boldsymbol{h}^*(\mathbb{X};\boldsymbol{\gamma})=\boldsymbol{h}(\mathbb{X};\phi^{-1}(\boldsymbol{\gamma}))\sim f_{\boldsymbol{H}}(\cdot).$$

Thus, $\boldsymbol{h}^*(\mathcal{X},\boldsymbol{\gamma})$ is a pivot of $\boldsymbol{\gamma}$; the RE $\boldsymbol{V}$ of $\boldsymbol{\gamma}$ given by

$$\boldsymbol{H}=\boldsymbol{h}^*(\mathcal{X},\boldsymbol{V})=\boldsymbol{h}(\mathcal{X},\boldsymbol{\phi}^{-1}(\boldsymbol{V}))$$

is an ERE. This RE is equivalent to $\boldsymbol{\phi}(\boldsymbol{W})$. □

Corollary 2.9. *Suppose that $\boldsymbol{W}$ is the RE of $\boldsymbol{\eta}$ given in Theorem 2.8. Let $\gamma = \phi(\eta_1, \dots, \eta_l) \in \Lambda \subseteq \Re$, where $\phi(\cdot)$ is a multivariate continuously differentiable function and*

$$\sum_{i=1}^{l} \left(\frac{\partial \phi}{\partial \eta_i}\right)^2 > 0.$$

Then, an ERE of γ is given by

$$V = \phi(W_1, \dots, W_l) = \phi(\boldsymbol{W}). \tag{2.11}$$

Proof. Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_{l-1}, \gamma)' = (\eta_1, \dots, \eta_{l-1}, \phi(\eta_1, \dots, \eta_l))' = \boldsymbol{\phi}(\boldsymbol{\eta})$. Then,

$$\left|\det\left(\frac{\partial \boldsymbol{\gamma}}{\partial \boldsymbol{\eta}}\right)\right| \neq 0, \quad \forall \boldsymbol{\eta} \in \mathcal{N}$$

and $\boldsymbol{\phi}(\cdot)$ satisfies the conditions of Theorem 2.8. Hence,

$$\boldsymbol{V} = (V_1, \dots, V_l, V)' = \boldsymbol{\phi}(\boldsymbol{W}) = (W_1, \dots, W_{l-1}, \phi(\boldsymbol{W}))$$

is an ERE of $\boldsymbol{\gamma}$ and $V = \phi(\boldsymbol{W})$ is an ERE of γ. □

Theorem 2.8 shows that an ERE is equivariant under parameter transformations. When composite parameters do not have any closed-form pivot, we can find an ERE using Theorem 2.8 and Corollary 2.9. Thus, we can perform the VDR test for the hypothesis

$$H_0 : \boldsymbol{\gamma} = \boldsymbol{\gamma}_0 \quad \text{vs.} \quad H_1 : \boldsymbol{\gamma} \neq \boldsymbol{\gamma}_0,$$

and construct a confidence region of $\boldsymbol{\gamma}$.

3 Examples of randomized inference

We will list some applications of RI in this section.

3.1 Classical problems for normal distribution

Let $\boldsymbol{x} = (x_1, \dots, x_n)'$ be an observation of a random sample $\boldsymbol{X} = (X_1, \dots, X_n)'$ from $N(\mu, \sigma^2)$. Consider the following hypotheses:

$$H_0 : \mu = \mu_0 \quad \text{vs.} \quad H_1 : \mu \neq \mu_0 \tag{3.1}$$

and

$$H_0 : \sigma^2 = \sigma_0^2 \quad \text{vs.} \quad H_1 : \sigma^2 \neq \sigma_0^2. \tag{3.2}$$

The pivot of μ is

$$h(\boldsymbol{x}, \mu) = \frac{\sqrt{n}(\bar{x} - \mu)}{s_n}, \quad h(\boldsymbol{X}, \mu) = \frac{\sqrt{n}(\bar{X} - \mu)}{S_n} \sim dt(\cdot, n-1).$$

Let W be the RE of μ given by

$$\frac{\sqrt{n}(\bar{x} - W)}{s_n} = T_{n-1}, \quad T_{n-1} \sim dt(\cdot, n-1).$$

The PDF of W is

$$f_W(u, \bar{x}, s_n^2) = \frac{\sqrt{n}}{s_n} dt\left(\frac{\sqrt{n}}{s_n}(\bar{x} - u), n-1\right).$$

The VDR test random variable is taken as

$$Z = \frac{s_n}{\sqrt{n}} f_W(W, \bar{x}, s_n^2) = dt\left(\frac{\sqrt{n}}{s_n}(\bar{x} - W), n-1\right) = dt(T_{n-1}, n-1).$$

Let $Q_Z(\alpha, n)$ be the α quantile of Z. Then,

$$\begin{aligned}\alpha &= \mathrm{P}(Z \leqslant Q_Z(\alpha, n)) = \mathrm{P}(dt(T_{n-1}, n-1) \leqslant Q_Z(\alpha, n))\\ &= \mathrm{P}\bigg(T_{n-1} \leqslant qt\bigg(\frac{\alpha}{2}, n-1\bigg)\bigg) + \mathrm{P}\bigg(T_{n-1} \geqslant qt\bigg(1-\frac{\alpha}{2}, n-1\bigg)\bigg).\end{aligned}$$

This means that $Q_Z(\alpha, n) = dt(qt(\frac{\alpha}{2}, n-1), n-1) = dt(qt(1-\frac{\alpha}{2}, n-1), n-1)$. Therefore, the rule of the VDR test for the hypothesis (3.1) is that *if*

$$\frac{s_n}{\sqrt{n}} f_W(\mu_0, \bar{x}, s_n^2) \leqslant Q_Z(\alpha, n) \quad \textit{or equivalently} \quad \bigg|\frac{\sqrt{n}(\bar{x}-\mu_0)}{s_n}\bigg| \geqslant qt\bigg(1-\frac{\alpha}{2}, n-1\bigg),$$

then reject the null hypothesis. This rule is exactly the same as that used for the frequentist inference.

All the above steps show the RI procedure by employing the RE. We can find the VDR test of the hypothesis (3.2) in the same manner. The RE of σ^2 is $V = \frac{(n-1)s_n^2}{\chi_{n-1}^2}$, $\chi_{n-1}^2 \sim \mathrm{dchisq}(\cdot\,, n-1)$. The PDF of V is

$$f_V(v, s_n^2, n-1) = \frac{(n-1)s_n^2}{v^2} \mathrm{dchisq}\bigg(\frac{(n-1)s_n^2}{v}, n-1\bigg).$$

The test random variable of the hypothesis (3.2) is

$$Z_1 = f_V(V, s_n^2, n-1) = \frac{(n-1)s_n^2}{V^2} \mathrm{dchisq}\bigg(\frac{(n-1)s_n^2}{V}, n-1\bigg).$$

Let $Q_1(\alpha, s_n^2, n-1)$ be the α quantile of Z_1. Then,

$$\begin{aligned}\alpha &= \mathrm{P}(Z_1 \leqslant Q_1(\alpha, s_n^2, n-1))\\ &= \mathrm{P}((\chi_{n-1}^2)^2 \mathrm{dchisq}(\chi_{n-1}^2, n-1) \leqslant Q_1(\alpha, n-1))\\ &= \mathrm{P}(\chi_{n-1}^2 \leqslant v_1) + \mathrm{P}(\chi_{n-1}^2 \geqslant v_2),\end{aligned}$$

where $Q_1(\alpha, n-1) = (n-1)s_n^2 Q_1(\alpha, s_n^2, n-1)$, v_1 and v_2 satisfy

$$Q_1(\alpha, n-1) = v_1^2 \mathrm{dchisq}(v_1, n-1) = v_2^2 \mathrm{dchisq}(v_2, n-1), \quad v_1 < v_2.$$

The rule for the VDR test of the hypothesis (3.2) is that *if*

$$f_V(\sigma_0^2, s_n^2, n-1) \leqslant Q_1(\alpha, s_n^2, n-1),$$

or equivalently

$$\sigma_0^2 \geqslant \frac{(n-1)s_n^2}{v_1} \quad \textit{or} \quad \sigma_0^2 \leqslant \frac{(n-1)s_n^2}{v_2},$$

then reject the null hypothesis of (3.2).

Furthermore, a $1-\alpha$ confidence interval of σ^2 is $[\frac{(n-1)s_n^2}{v_2}, \frac{(n-1)s_n^2}{v_1}]$. Note that the length of the interval is less than that of the conventional confidence interval $[\frac{(n-1)s_n^2}{\mathrm{qchisq}(1-\frac{\alpha}{2}, n-1)}, \frac{(n-1)s_n^2}{\mathrm{qchisq}(\frac{\alpha}{2}, n-1)}]$.

3.2 Empirical Bayesian procedure of RI

Certainly, an RE may be introduced from the posterior distribution in Bayesian analysis, i.e., a random variable with the posterior distribution can be viewed as an RE. In this subsection, we show by means of examples that the RE's distribution could be taken as the prior distribution in the empirical Bayesian procedure.

Example 3.1. Suppose $\boldsymbol{x} = (x_1, \ldots, x_n)'$ is an observed sample from $N(\mu, \sigma^2)$.

Case 1. σ^2 is known. An RE of μ is given by $W = \bar{x} - \frac{\sigma}{\sqrt{n}} Z$, $Z \sim N(0,1)$, and hence, $W \sim N(\bar{x}, \frac{\sigma^2}{n})$. If there is a historical sample $\boldsymbol{y} = (y_1, \ldots, y_m)'$, then we may take $N(\bar{y}, \frac{\sigma^2}{m})$, which is the distribution of μ's RE corresponding to $\boldsymbol{y}$, as a prior distribution of μ in the Bayesian inference.

Case 2. σ^2 is unknown. A similar RE of μ is $W = \bar{x} - \frac{s_n}{\sqrt{n}} T_{n-1}$, $T_{n-1} \sim dt(\cdot, n-1)$. Based on a historical sample $\boldsymbol{y}$ of size m, we may take $dt(\frac{\sqrt{m}(\bar{y}-\cdot)}{s_m})$ as a prior distribution of μ. However, it is not a conjugate prior distribution.

Case 3. An RE of σ^2 is $W = \frac{(n-1)s_n^2}{\chi^2_{n-1}}$, $\chi^2_{n-1} \sim \mathrm{dchisq}(\cdot, n-1)$, and its PDF is

$$f_W(w) = \frac{1}{(n-1)s_n^2}\left(\frac{(n-1)s_n^2}{w}\right)^2 \mathrm{dchisq}\left(\frac{(n-1)s_n^2}{w}\right).$$

Similarly, if one has a historical sample $\boldsymbol{y} = (y_1, \ldots, y_m)'$, the distribution of the corresponding RE of σ^2 can be taken as a prior distribution.

Example 3.2. Assume that the lifetime of a color TV follows an exponential distribution with the following PDF:

$$p(t;\theta) = \frac{1}{\theta}\mathrm{e}^{-\frac{t}{\theta}}, \quad t > 0,$$

where $\theta > 0$ is the mean lifetime. Consider a type-II censoring lifetime test with n TV sets, which was stopped after r sets failed. The observed failure times are $t_{(1)} \leqslant t_{(2)} \leqslant \cdots \leqslant t_{(r)}$, and the PDF of the censored sample $\boldsymbol{T}_r = (T_{(1)}, \ldots, T_{(r)})'$ is

$$p(\boldsymbol{t};\theta) = \frac{n!}{(n-r)!}\frac{1}{\theta^r}\mathrm{e}^{-\frac{s_r}{\theta}},$$

where $s_r = \sum_{i=1}^r t_{(i)} + (n-r)t_{(r)}$ and $\boldsymbol{t} = (t_{(1)}, \ldots, t_{(r)})'$. $h(\boldsymbol{t};\theta) = \frac{s_r}{\theta}$ is a pivot of θ, and

$$h(\boldsymbol{T};\theta) = \frac{S_r}{\theta} \sim \Gamma(r,1), \tag{3.3}$$

where $\Gamma(r,1)$ is the Γ distribution with the shape parameter r and scale 1. Thus, an RE W of θ can be defined as

$$W = \frac{s_r}{\Gamma_{r,1}}, \quad \Gamma_{r,1} \sim \Gamma(r,1).$$

It is an ERE and has the density

$$f_W(w;s_r) = \frac{s_r^r}{(r-1)!} w^{-(r+1)}\mathrm{e}^{-\frac{s_r}{w}} = d\mathrm{I}\Gamma(w;r,s_r).$$

If there is a historical data set $(r',\ s_{r'})$, one may take $d\mathrm{I}\Gamma(\cdot\,;r',s_{r'})$ as a prior distribution. This is a common type of prior distributions in the Bayesian method. The prior distribution is objective, with no subjective elements [7]. The posterior distribution is

$$\pi(\theta \,|\, s_{r'}, s_r) = \frac{(s_{r'}+s_r)^{r'+r}}{(r'+r-1)!}\theta^{r'+r+1}\mathrm{e}^{-\frac{s_{r'}+s_r}{\theta}}.$$

It combines the information on θ from both the historical data and the current sample.

3.3 Multiple population comparison problem

Testing the equality of the means or the homogeneity of the variances for multiple normal distributions is difficult in frequentist inference only because of the lack of a closed-form pivot. In this section, we propose RI solutions that are quite different from the classical solutions.

Let $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{in_i})'$ and $\boldsymbol{X}_i = (X_{i1}, \ldots, X_{in_i})'$ be an observed sample and a random sample, respectively, drawn from $N(\mu_i, \sigma_i^2)$ with the sample size n_i, where $i = 1, \ldots, p$, and $\boldsymbol{X}_i,\ i = 1, \ldots, p$ be independent of each other. Next, we will study the following problems:

1. Testing the equality of the mean parameters

$$H_0: \mu_1 = \cdots = \mu_p \quad \text{vs.} \quad H_1: \mu_1 = \cdots = \mu_p \quad \text{is not true.} \tag{3.4}$$

This has been solved only when all variances are equal, known as the analysis of variance for one factor. When $p = 2$ and $\sigma_1^2 \neq \sigma_2^2$, we have the famous Behrens-Fisher problem.

2. Testing the homogeneity of the variances

$$H_0 : \sigma_1^2 = \cdots = \sigma_p^2 \quad \text{vs.} \quad H_1 : \sigma_1^2 = \cdots = \sigma_p^2 \quad \text{is not true.} \tag{3.5}$$

There are many ways to test (3.5). For the case of $p = 2$, the F test is an exact test. When $p > 2$, the Hartley test can be performed when all sample sizes are equal. The Bartlett test is for all cases, and the revised Bartlett test is for small sample sizes. However, these are all asymptotic tests, whereas the VDR test is an exact test.

We will use the following notations:

$$\begin{aligned}
&\boldsymbol{\sigma}^2 = (\sigma_1^2, \ldots, \sigma_p^2)', \quad \boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)', \\
&\bar{x}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}, \quad s_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2, \quad i = 1, \ldots, p, \\
&\bar{\boldsymbol{x}} = (\bar{x}_1, \ldots, \bar{x}_p)', \quad \boldsymbol{s}^2 = (s_1^2, \ldots, s_p^2)', \\
&\boldsymbol{n} = (n_1, \ldots, n_p)', \quad \boldsymbol{f} = \boldsymbol{n} - \mathbf{1}_p = (n_1 - 1, \ldots, n_p - 1)', \\
&\mathcal{N} = \mathrm{diag}(\boldsymbol{n}), \quad \mathcal{F} = \mathrm{diag}(\boldsymbol{f}), \\
&\mathcal{S} = \mathrm{diag}(\boldsymbol{s}^2), \quad \mathbf{1}_p = (1, \ldots, 1)' \in \mathfrak{R}^p.
\end{aligned}$$

3.3.1 *Testing the hypothesis* (3.4)

Let W_i be the RE of μ_i defined by

$$h(\bar{x}_i, s_i^2; W_i) = \frac{\sqrt{n_i}(\bar{x}_i - W_i)}{s_i} = T_{n_i - 1}, \quad T_{n_i - 1} \sim dt(\cdot, n_i - 1), \quad i = 1, \ldots, p,$$

and denote $\boldsymbol{W} = (W_1, \ldots, W_p)'$. Then, from the above equations, we have

$$\mathcal{N}^{\frac{1}{2}} \mathcal{S}^{-\frac{1}{2}} (\bar{\boldsymbol{x}} - \boldsymbol{W}) = \boldsymbol{T}_{\boldsymbol{n}-1}, \quad \boldsymbol{T}_{\boldsymbol{n}-1} = (T_{n_1 - 1}, \ldots, T_{n_p - 1})'. \tag{3.6}$$

Take a constant vector $\boldsymbol{r} = (r_1, \ldots, r_p)'$ with $\sum_{i=1}^p r_i = 1$, $r_i > 0$ for all i, and let $\bar{\mu} = \bar{\mu}(\boldsymbol{r}) = \boldsymbol{r}'\boldsymbol{\mu}$, $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_p)' = \boldsymbol{\mu} - \mathbf{1}_p \bar{\mu}$. Hence, the test of the hypothesis (3.4) is equivalent to the test of

$$H_0 : \boldsymbol{\nu} = \mathbf{0}_p \quad \text{vs.} \quad H_1 : \boldsymbol{\nu} \neq \mathbf{0}_p. \tag{3.7}$$

If we set

$$\begin{aligned}
\Phi &= \left\{ \boldsymbol{u} = (u_1, \ldots, u_{p+1})' : \sum_{i=1}^p r_i u_i = 0, u_{p+1} \in \mathfrak{R} \right\} \\
&= \{ \boldsymbol{u} = (u_1, \ldots, u_p)' : \boldsymbol{r}'\boldsymbol{u} = 0,\ \boldsymbol{u} \in \mathfrak{R}^p \} \times \mathfrak{R} \equiv \Phi_0 \times \mathfrak{R},
\end{aligned}$$

then the function $\boldsymbol{u}(\boldsymbol{v}) = ((\boldsymbol{v} - \mathbf{1}_p \bar{v})', \bar{v})'$ is a one-to-one map from $\mathfrak{R}^p$ to Φ. We have

$$\begin{aligned}
&\frac{\partial ((\boldsymbol{v} - \mathbf{1}_p \bar{v})', \bar{v})'}{\partial \boldsymbol{v}'} = \begin{pmatrix} I_p - \mathbf{1}_p \boldsymbol{r}' \\ \boldsymbol{r}' \end{pmatrix}, \\
&J(((\boldsymbol{v} - \mathbf{1}_p \bar{v})', \bar{v})' \to \boldsymbol{v}) = \left| \det \left((I_p - \boldsymbol{r}\mathbf{1}_p', \boldsymbol{r}) \begin{pmatrix} I_p - \mathbf{1}_p \boldsymbol{r}' \\ \boldsymbol{r}' \end{pmatrix} \right) \right|^{\frac{1}{2}} \\
&\qquad = p - 1 - \left(1 - \frac{1}{\sqrt{\boldsymbol{r}'\boldsymbol{r}}} \right)^2 = c = \text{constant}.
\end{aligned} \tag{3.8}$$

Table 1 Empirical significance level given by simulations

Weight	Sizes of		Nominal levels						
$\boldsymbol{r}$	samples	Variances	0.0100	0.0500	0.1000	0.5000	0.9000	0.9500	0.9900
$\boldsymbol{n}/n$	$10, 20, 25, 37$	$1, 3, 6, 7$	0.0079	0.0478	0.0953	0.4908	0.8920	0.9460	0.9908
$\mathbf{1}_4/4$	$10, 20, 25, 37$	$1, 3, 6, 7$	0.0094	0.0461	0.0920	0.4954	0.8959	0.9489	0.9902
$\boldsymbol{n}/n$	$10, 20, 30$	$1, 2, 3$	0.0083	0.0439	0.0938	0.4852	0.8974	0.9468	0.9892
$\mathbf{1}_3/3$	$10, 20, 30$	$1, 2, 3$	0.0070	0.0422	0.0925	0.4947	0.8964	0.9463	0.9906

Let $\bar{W}$ and $\boldsymbol{V}$ be the REs of $\bar{\mu} = \boldsymbol{r}'\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, respectively. Then,

$$\begin{aligned} &\boldsymbol{W} = \bar{\boldsymbol{x}} - \mathcal{S}^{\frac{1}{2}}\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{\boldsymbol{n}-1}, \\ &\bar{W} = \boldsymbol{r}'\boldsymbol{W} = \boldsymbol{r}'\bar{\boldsymbol{x}} - \boldsymbol{r}'\mathcal{S}^{\frac{1}{2}}\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{\boldsymbol{n}-1}, \\ &\boldsymbol{V} = \boldsymbol{W} - \mathbf{1}_p\bar{W} = (I_p - \mathbf{1}_p\boldsymbol{r}')\bar{\boldsymbol{x}} - (I_p - \mathbf{1}_p\boldsymbol{r}')\mathcal{S}^{\frac{1}{2}}\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{\boldsymbol{n}-1}. \end{aligned} \tag{3.9}$$

Note that $\operatorname{rank}(I_p - \mathbf{1}_p\boldsymbol{r}') = p - 1$. The joint PDF of $(I_p - \mathbf{1}_p\boldsymbol{r}')\mathcal{S}^{\frac{1}{2}}\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{\boldsymbol{n}-1}$ and $\boldsymbol{r}'\mathcal{S}^{\frac{1}{2}}\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{\boldsymbol{n}-1}$ is

$$c^{-1}\prod_{i=1}^{p} dt\bigg(\frac{\sqrt{n_i}}{s_i}(v_i + v), n_i - 1\bigg)$$

according to Theorem 2.8. If we denote the PDF of $(I_p - \mathbf{1}_p\boldsymbol{r}')\mathcal{S}^{\frac{1}{2}}\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{\boldsymbol{n}-1}$ by $f_{\boldsymbol{T}}(\cdot\,, \boldsymbol{s}^2, \boldsymbol{n})$, then

$$f_{\boldsymbol{T}}(\boldsymbol{v}, \boldsymbol{s}^2, \boldsymbol{n}) = c^{-1}\int_{-\infty}^{\infty}\prod_{i=1}^{p} dt\bigg(\frac{\sqrt{n_i}}{s_i}(v_i + v), n_i - 1\bigg)dv, \quad \boldsymbol{r}'\boldsymbol{v} = 0. \tag{3.10}$$

The PDF of $\boldsymbol{V}$ is

$$\begin{aligned} f_{\boldsymbol{V}}(\boldsymbol{v}, \boldsymbol{s}^2, \boldsymbol{n}) &= f_{\boldsymbol{T}}((\bar{\boldsymbol{x}} - \mathbf{1}_p\boldsymbol{r}'\bar{\boldsymbol{x}}) - \boldsymbol{v}, \boldsymbol{s}^2, \boldsymbol{n}), \\ &= c^{-1}\int_{-\infty}^{\infty}\prod_{i=1}^{p} dt\bigg(\frac{\sqrt{n_i}}{s_i}((\bar{x}_i - \boldsymbol{r}'\bar{\boldsymbol{x}}) - v_i + v), n_i - 1\bigg)dv, \quad \sum_{j=1}^{p} r_j v_j = 0. \end{aligned} \tag{3.11}$$

The test variable is $Z = cf_{\boldsymbol{V}}(\boldsymbol{V}, \boldsymbol{s}^2, \boldsymbol{n})$. From (3.9), it holds that

$$Z = cf_{\boldsymbol{T}}((I_p - \mathbf{1}_p\boldsymbol{r}')\mathcal{S}^{\frac{1}{2}}\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{\boldsymbol{n}-1}, \boldsymbol{s}^2, \boldsymbol{n}).$$

The α quantile of Z, denoted by $Q_Z(\alpha, \boldsymbol{s}^2, \boldsymbol{n})$, can be obtained via simulation. The VDR test rule of the hypothesis (3.7) is that *if*

$$\int_{-\infty}^{\infty}\prod_{i=1}^{p} dt\bigg(\frac{\sqrt{n_i}}{s_i}((\bar{x}_i - \bar{x}) + v), n_i - 1\bigg)dv \leqslant Q_Z(\alpha, \boldsymbol{s}^2, \boldsymbol{n}),$$

then reject the null hypothesis, where $\bar{x} = \bar{x}(\boldsymbol{r}) = \boldsymbol{r}'\bar{\boldsymbol{x}}$.

In general, $\boldsymbol{r}$ should be chosen according to the alternative hypothesis. We can choose $\boldsymbol{r} = \frac{\boldsymbol{n}}{n}$, where $n = \sum_{i=1}^{p} n_i$, for the alternative hypothesis of (3.4).

We performed two group simulations with 10,000 repetitions. The first group had four populations, with the variances 1, 3, 6 and 7 respectively, whereas the second group had three populations, with the variances 1, 2 and 3, respectively. The empirical significance levels under the various nominal levels are listed in Table 1. All results are very close to the nominal levels.

3.3.2 *ANOVA for a single factor*

Suppose that $\sigma_1^2 = \cdots = \sigma_p^2 = \sigma^2$. This is the case of the variance analysis for a single factor. The pivots of $\boldsymbol{\mu}$ and σ^2 are given by

$$\boldsymbol{h}(\mathcal{X}, \boldsymbol{\mu}) = (h(\bar{x}_1, s_n^2; \mu_1), \ldots, h(\bar{x}_p, s_n^2; \mu_p))'$$

$$= \left(\frac{\sqrt{n_1}(\bar{x}_1-\mu_1)}{s_n},\ldots,\frac{\sqrt{n_p}(\bar{x}_p-\mu_p)}{s_n}\right)' = \mathcal{N}^{\frac{1}{2}}\frac{(\bar{\boldsymbol{x}}-\boldsymbol{\mu})}{s_n},$$

$$h_1(\mathcal{X},\sigma^2) = \frac{(n-p)s_n^2}{\sigma^2},$$

where $n=\sum_{i=1}^p n_i$ and $s_n^2 = \frac{1}{n-p}\sum_{i=1}^p (n_i-1)s_i^2$. It is well known that

$$\begin{pmatrix} \boldsymbol{h}(\mathbb{X},\boldsymbol{\mu}) \\ h_1(\mathbb{X},\sigma^2) \end{pmatrix} \overset{d}{=} \begin{pmatrix} \frac{\boldsymbol{Z}}{\sqrt{\frac{\chi^2_{n-p}}{n-p}}} \\ \chi^2_{n-p} \end{pmatrix},$$

where $\boldsymbol{Z}\sim N_p(\mathbf{0}_p, I_p)$ and $\chi^2_{n-p}\sim \text{dchisq}(\cdot, n-p)$ are independent of each other. We specify

$$\boldsymbol{T}_{p,\boldsymbol{n}} = (T_{1,n},\ldots,T_{p,n})' = \frac{\boldsymbol{Z}}{\sqrt{\frac{\chi^2_{n-p}}{n-p}}} = \left(\frac{Z_1}{\sqrt{\frac{\chi^2_{n-p}}{n-p}}},\ldots,\frac{Z_p}{\sqrt{\frac{\chi^2_{n-p}}{n-p}}}\right)'.$$

Its PDF is

$$\begin{aligned} dt_p(\boldsymbol{t},\boldsymbol{n}) &= \int_0^\infty \text{dnorm}_p\left(\boldsymbol{z},\frac{v}{n-p}I_p\right)\text{dchisq}(v,n-p)dv \\ &= \frac{\Gamma(\frac{n-p+1}{2})}{((n-p)\pi)^{\frac{p}{2}}\Gamma(\frac{n-2p+1}{2})}\left(1+\frac{\boldsymbol{t}'\boldsymbol{t}}{n-p}\right)^{-\frac{n}{2}}. \end{aligned} \tag{3.12}$$

The RE equations of $\boldsymbol{W}$, $\boldsymbol{r}'\boldsymbol{W}$ and $\boldsymbol{V}$ are

$$\begin{aligned} &\boldsymbol{x}-\boldsymbol{W} = s_n\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{p,\boldsymbol{n}}, \\ &\boldsymbol{r}'\boldsymbol{x}-\boldsymbol{r}'\boldsymbol{W} = s_n\boldsymbol{r}'\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{p,\boldsymbol{n}}, \\ &(I_p-\mathbf{1}_p\boldsymbol{r}')\boldsymbol{x}-\boldsymbol{V} = s_n(I_p-\mathbf{1}_p\boldsymbol{r}')\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{p,\boldsymbol{n}}, \\ &\boldsymbol{V} = \boldsymbol{W}-\mathbf{1}_p\boldsymbol{r}'\boldsymbol{W}. \end{aligned} \tag{3.13}$$

From (3.12), the PDF of

$$\boldsymbol{U} = (I_p-\mathbf{1}_p\boldsymbol{r}')\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{p,\boldsymbol{n}}$$

can be expressed as

$$\begin{aligned} f_{\boldsymbol{U}}(\boldsymbol{u},\boldsymbol{n}) &= \prod_{i=1}^p n_i^{\frac{1}{2}}\int_{-\infty}^\infty dt_p(\mathcal{N}^{\frac{1}{2}}(\boldsymbol{u}+\mathbf{1}s),\boldsymbol{n})ds \\ &= C\int_{-\infty}^\infty\left(1+\frac{(\boldsymbol{u}+\mathbf{1}_ps)'\mathcal{N}(\boldsymbol{u}+\mathbf{1}_ps)}{n-p}\right)^{-\frac{n}{2}}ds \end{aligned}$$

according to Theorem 2.8. Let $\boldsymbol{r}=\frac{\boldsymbol{n}}{n}$ and $\boldsymbol{r}'\bar{\boldsymbol{x}} = \frac{1}{n}\sum_{i=1}^p\sum_{j=1}^{n_i}x_{ij} = \bar{x}$. Then, it is easy to show that

$$f_{\boldsymbol{U}}(\boldsymbol{u},\boldsymbol{n}) = C\left(1+\frac{\boldsymbol{u}'\mathcal{N}\boldsymbol{u}}{n-p}\right)^{-\frac{n-1}{2}}.$$

Thus, the PDF of $\boldsymbol{V}$ is

$$f_{\boldsymbol{V}}(\boldsymbol{v},\boldsymbol{x},\boldsymbol{n}) = C_0^*\left(1+\frac{((I_p-\mathbf{1}_p\boldsymbol{r}')\boldsymbol{x}-\boldsymbol{v})'\mathcal{N}((I_p-\mathbf{1}_p\boldsymbol{r}')\boldsymbol{x}-\boldsymbol{v})}{(n-p)s_n^2}\right)^{-\frac{n-1}{2}}.$$

Note that

$$\begin{aligned} (I_p-\mathbf{1}_p\boldsymbol{r}')\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{p,\boldsymbol{n}} &= \mathcal{N}^{-\frac{1}{2}}(I_p-\mathcal{N}^{\frac{1}{2}}\mathbf{1}_p\boldsymbol{r}'\mathcal{N}^{-\frac{1}{2}})\boldsymbol{T}_{p,\boldsymbol{n}} \\ &= \mathcal{N}^{-\frac{1}{2}}(I_p-\boldsymbol{a}\boldsymbol{a}')\boldsymbol{T}_{p,\boldsymbol{n}}, \end{aligned}$$

$$\boldsymbol{a} = \left(\sqrt{\frac{n_1}{n}}, \dots, \sqrt{\frac{n_p}{n}}\right)'.$$

The test variable is

$$Z = \frac{1}{C_0^*} f_{\boldsymbol{V}}(\boldsymbol{V}, \boldsymbol{x}, \boldsymbol{n}).$$

From the RE equation of $\boldsymbol{V}$ and the above equation, the following holds:

$$\begin{aligned} Z &= \left(1 + \frac{(\bar{\boldsymbol{x}} - \mathbf{1}_p \boldsymbol{r}'\bar{\boldsymbol{x}} - \boldsymbol{V})'\mathcal{N}(\bar{\boldsymbol{x}} - \mathbf{1}_p \boldsymbol{r}'\bar{\boldsymbol{x}} - \boldsymbol{V})}{(n-p)s_n^2}\right)^{-\frac{n-1}{2}} \\ &= \left(1 + \frac{(s_n(I_p - \mathbf{1}_p\boldsymbol{r}')\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{p,\boldsymbol{n}})'\mathcal{N}(s_n(I_p - \mathbf{1}_p\boldsymbol{r}')\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{p,\boldsymbol{n}})}{(n-p)s_n^2}\right)^{-\frac{n-1}{2}} \\ &= \left(1 + \frac{\boldsymbol{T}'_{p,\boldsymbol{n}}(I_p - \boldsymbol{a}\boldsymbol{a}')\boldsymbol{T}_{p,\boldsymbol{n}}}{n-p}\right)^{-\frac{n-1}{2}}. \end{aligned}$$

Based on the definition of $\boldsymbol{T}_{p,\boldsymbol{n}}$ and the fact that $I_p - \boldsymbol{a}\boldsymbol{a}'$ is idempotent, we have

$$\frac{n-p}{p-1}\frac{\boldsymbol{T}'_{p,\boldsymbol{n}}(I_p - \boldsymbol{a}\boldsymbol{a}')\boldsymbol{T}_{p,\boldsymbol{n}}}{n-p} = \frac{\frac{\boldsymbol{Z}'(I_p - \boldsymbol{a}\boldsymbol{a}')\boldsymbol{Z}}{p-1}}{\frac{\chi^2_{n-p}}{n-p}} \sim dF(\cdot, p-1, n-p).$$

Let $Q_Z(\alpha, \boldsymbol{n})$ be the α quantile of Z. Then,

$$\begin{aligned} \alpha = \mathrm{P}(Z \leqslant Q_Z(\alpha, \boldsymbol{n})) &= \mathrm{P}\left(\left(1 + \frac{\boldsymbol{T}'_{p,\boldsymbol{n}}(I_p - \boldsymbol{a}\boldsymbol{a}')\boldsymbol{T}_{p,\boldsymbol{n}}}{n-p}\right)^{-\frac{n-1}{2}} \leqslant Q_Z(\alpha, \boldsymbol{n})\right) \\ &= \mathrm{P}\left(\frac{\boldsymbol{T}'_{p,\boldsymbol{n}}(I_p - \boldsymbol{a}\boldsymbol{a}')\boldsymbol{T}_{p,\boldsymbol{n}}}{p-1} > \frac{n-p}{p-1}\left(\frac{1}{(Q_Z(\alpha, \boldsymbol{n}))^{\frac{2}{n-1}}} - 1\right)\right). \end{aligned}$$

Hence,

$$F_{p-1,n-p}(1-\alpha) = \frac{n-p}{p-1}\left(\frac{1}{(Q_Z(\alpha, \boldsymbol{n}))^{\frac{2}{n-1}}} - 1\right).$$

The VDR test rule is that *if*

$$\frac{1}{C_0^*} f_{\boldsymbol{V}}(\mathbf{0}_p, \boldsymbol{x}, \boldsymbol{n}) = \left(1 + \frac{((I_p - \mathbf{1}_p\boldsymbol{r}')\boldsymbol{x})'\mathcal{N}((I_p - \mathbf{1}_p\boldsymbol{r}')\boldsymbol{x})}{(n-p)s_n^2}\right)^{-\frac{n-1}{2}} \leqslant Q_Z(\alpha, \boldsymbol{n}),$$

which is equivalent to

$$\frac{n-p}{p-1}\frac{((I_p - \mathbf{1}_p\boldsymbol{r}')\boldsymbol{x})'\mathcal{N}((I_p - \mathbf{1}_p\boldsymbol{r}')\boldsymbol{x})}{(n-p)s_n^2} = \frac{\frac{\sum_{i=1}^p n_i(\bar{x}_i - \bar{x})^2}{p-1}}{s_n^2} \geqslant F_{p-1,n-p}(1-\alpha),$$

then reject the null hypothesis. This means that the VDR test is equivalent to the single-factor variance analysis.

3.3.3 *Testing the equality of means for the case of $p = 2$*

When $p = 2$, the test of $\mu_1 = \mu_2$ is the famous Behrens-Fisher problem. The difficulty lies in having no closed-form pivot for $\mu_1 - \mu_2$. In this case, a well-known test is the Welch test [5], which is given by

$$Wh(\mathcal{X}) = \frac{\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}. \tag{3.14}$$

When $\mu_1 = \mu_2$, $Wh(\mathbb{X})$'s distribution function is approximated by

$$F_W(w) = pt(w, l), \tag{3.15}$$

where

$$l = \frac{(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2})^2}{\frac{s_1^4}{n_1^2(n_1-1)} + \frac{s_2^4}{n_2^2(n_2-1)}}.$$

Next, we discuss the RI with the Welch statistics. By dividing the two sides of the last equation in (3.9) by $b = \sqrt{\sum_{i=1}^p \frac{s_i^2}{n_i}}$, we obtain

$$\frac{(I_p - \mathbf{1}_p \boldsymbol{r}')\bar{\boldsymbol{x}}}{b} - \frac{\boldsymbol{V}}{b} = (I_p - \mathbf{1}_p \boldsymbol{r}')\frac{\mathcal{S}^{\frac{1}{2}}}{b}\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{\boldsymbol{n}-1}. \tag{3.16}$$

If we set

$$\boldsymbol{b}^2 = (b_1^2, \ldots, b_p^2)' = \left(\frac{s_1^2}{b^2}, \ldots, \frac{s_p^2}{b^2}\right)',$$

then the PDF of $\frac{\mathcal{S}^{\frac{1}{2}}}{b}\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{\boldsymbol{n}-1}$ is

$$dt_p(\boldsymbol{v}; \boldsymbol{b}^2, \boldsymbol{n}) = \prod_{i=1}^p \sqrt{\frac{n_i}{b_i}} dt\left(\frac{\sqrt{n_i}}{b_i} v_i, n_i - 1\right),$$

and the PDF of $(I_p - \mathbf{1}_p \boldsymbol{r}')\frac{\mathcal{S}^{\frac{1}{2}}}{b}\mathcal{N}^{-\frac{1}{2}}\boldsymbol{T}_{\boldsymbol{n}-1}$ is

$$\begin{aligned} f_p(\boldsymbol{v}; \boldsymbol{b}^2, \boldsymbol{n}, \boldsymbol{r}) &= C^{-1}\int_{-\infty}^{\infty} dt_p(\boldsymbol{v} + \mathbf{1}_p v, \boldsymbol{b}^2, \boldsymbol{n}) dv \\ &= C^{-1}\int_{-\infty}^{\infty} \prod_{i=1}^p dt\left(\frac{\sqrt{n_i}}{b_i}(v_i + v), n_i - 1\right) dv. \end{aligned} \tag{3.17}$$

Let $p = 2$, $\boldsymbol{r} = (\frac{n_1}{n_1+n_2}, \frac{n_2}{n_1+n_2})'$ and $\boldsymbol{v} = (I_p - \mathbf{1}_p \boldsymbol{r}')\boldsymbol{t}$. Then, the PDF becomes

$$\begin{aligned} & f_2(\boldsymbol{t}, b_1^2, b_2^2, n_1, n_2) \\ &= \frac{\sqrt{n_1 n_2}}{b_1 b_2}\int_{-\infty}^{\infty} dt\left(\sqrt{\frac{n_1}{b_1}}(t_1 - \bar{t} + v), n_1 - 1\right) dt\left(\sqrt{\frac{n_2}{b_2}}(t_2 - \bar{t} + v), n_1 - 1\right) dv \\ &= \frac{\sqrt{n_1 n_2}}{b_1 b_2}\int_{-\infty}^{\infty} dt\left(\sqrt{\frac{n_1}{b_1}}\left(\frac{t_1 - t_2}{2} + v\right), n_1 - 1\right) dt\left(\sqrt{\frac{n_2}{b_2}}\left(\frac{t_2 - t_1}{2} + v\right), n_2 - 1\right) dv \\ &= \frac{\sqrt{n_1 n_2}}{b_1 b_2}\int_{-\infty}^{\infty} dt\left(\sqrt{\frac{n_1}{b_1}} v, n_1 - 1\right) dt\left(\sqrt{\frac{n_2}{b_2}}(t_2 - t_1 + v), n_2 - 1\right) dv \\ &= f_1(t_2 - t_1, b_1, b_2, n_1, n_2). \end{aligned} \tag{3.18}$$

$f_1(t_1 - t_2, b_1, b_2, n_1, n_2)$ is a symmetric function of $t_1 - t_2$. Let Z be the test random variable and $Q_Z(\alpha, n_1, n_2)$ be the corresponding critical value. Put $u = t_1 - t_2$. Consequently, we have

$$\begin{aligned} \alpha &= \mathrm{P}(Z \leqslant Q_Z(\alpha, n_1, n_2)) \\ &= 1 - \int_{\{f_1(u, b_1, b_2, n_1, n_2) > Q_Z(\alpha, n_1, n_2)\}} f_1(u, b_1, b_2, n_1, n_2) du \\ &= 1 - \int_{-q(\alpha, n_1, n_2)}^{q(\alpha, n_1, n_2)} f_1(u, b_1, b_2, n_1, n_2) du \\ &= 2\left(1 - \int_{-\infty}^{q(\alpha, n_1, n_2)} f_1(u, b_1, b_2, n_1, n_2) du\right) \\ &= 2\left(1 - \frac{\sqrt{n_1 n_2}}{s_1 s_2}\int_{-\infty}^{\infty} dt\left(\sqrt{\frac{n_1}{b_1}} v, n_1 - 1\right) dv \int_{-\infty}^{-q(\alpha, n_1, n_2)} dt\left(\sqrt{\frac{n_2}{b_2}}(u + v), n_2 - 1\right)\right) du \\ &= 2\left(1 - \frac{\sqrt{n_1 n_2}}{b_1 b_2}\int_{-\infty}^{\infty} dt\left(\sqrt{\frac{n_1}{b_1}} v, n_1 - 1\right) pt\left(\sqrt{\frac{n_2}{b_2}}(-q(\alpha, n_1, n_2) + v), n_2 - 1\right) dv\right), \end{aligned} \tag{3.19}$$

Table 2 Comparison of $F_{\boldsymbol{V}}(\cdot), F_n(\cdot)$ and pt via simulation with 10,000 repetitions

Method	Sizes of samples	Variances	v = −4.0	−3.5	−3.0	−2.5	−2.0	−1.5
$F_{\boldsymbol{V}}(v)$	10	1	0.000240	0.000979	0.003439	0.010857	0.030788	0.077350
$F_n(v)$	21	4	0.000600	0.000900	0.002800	0.009700	0.026000	0.072600
$pt(v,l)$			0.000183	0.000830	0.003228	0.009613	0.028182	0.072619
Method	Sizes of samples	Variances	v = −1.0	−0.5	0.0	0.5	1.0	1.5
$F_{\boldsymbol{V}}(v)$	10	1	0.168903	0.313445	0.499873	0.685520	0.832433	0.922926
$F_n(v)$	21	4	0.164400	0.315600	0.504200	0.696600	0.843400	0.932800
$pt(v,l)$			0.163185	0.310886	0.499871	0.689101	0.836070	0.926604
Method	Sizes of samples	Variances	v = 2.0	2.5	3.0	3.5	4.0	
$F_{\boldsymbol{V}}(v)$	10	1	0.967861	0.988992	0.996442	0.998786	0.999597	
$F_n(v)$	21	4	0.971400	0.989500	0.996900	0.999000	0.999800	
$pt(v,l)$			0.972354	0.990453	0.996865	0.998964	0.999628	

where

$$\begin{aligned} Q_Z(\alpha,n_1,n_2) &= f_1(q(\alpha,n_1,n_2),b_1,b_2,n_1,n_2) \\ &= f_2(-q(\alpha,n_1,n_2),b_1,b_2,n_1,n_2), \quad q(\alpha,n_1,n_2)>0. \end{aligned}$$

The simulation results are listed in Table 2 for the case that $n_1=10$, $n_2=21$, $\sigma_1^2=1$ and $\sigma_2^2=4$. The results show that

$$F_n(v)\simeq F_{\boldsymbol{V}}(v)\simeq pt(v,l), \tag{3.20}$$

where $F_n(\cdot)$, $F_{\boldsymbol{V}}(\cdot)$ and $pt(\cdot,l)$ are the empirical distribution functions of $Wh(\mathbb{X})$, RE, and t approximation, respectively.

3.3.4 *Testing the homogeneity of variances*

The RE of σ_i^2 is

$$W_i \;=\; f_i s_i^2\chi_{f_i}^{-2},\ \chi_{f_1}^2\sim \mathrm{dchisq}(\cdot,f_i),\ \ i=1,\dots,p, \tag{3.21}$$

and the RE of $\boldsymbol{\sigma}^2$ is

$$\boldsymbol{W}=(W_1,\dots,W_p)'=(f_1s_1^2\chi_{f_1}^{-2},\dots,f_ps_p^2\chi_{f_p}^{-2})'=\mathcal{FS}\boldsymbol{\chi}_{\boldsymbol{f}}^{-2}. \tag{3.22}$$

Hence, the PDF of $\boldsymbol{W}$ is

$$\begin{aligned} f_{\boldsymbol{W}}(\boldsymbol{w};\boldsymbol{f},\boldsymbol{s}^2) &= \det{}^{-1}(\mathcal{FS})f_{\boldsymbol{\chi}_{\boldsymbol{f}}^{-2}}((\mathcal{FS})^{-1}\boldsymbol{w},\boldsymbol{f}) \\ &= \left(\prod_{i=1}^p \frac{f_i^{-1}s_i^{-2}}{2^{\frac{f_i}{2}}\Gamma(\frac{f_i}{2})}\right)\prod_{i=1}^p\left(\frac{f_is_i^2}{w_i}\right)^{\frac{f_i}{2}+1}\mathrm{e}^{-\frac12\sum_{i=1}^p\frac{f_is_i^2}{w_i}}, \quad \boldsymbol{w}\in\Re^{p+}. \end{aligned}$$

We now perform parameter transformation to represent the hypothesis of homogeneous variances in a simple hypothesis way. Let

$$\begin{aligned} \nu &= \ln(\bar{G}(\boldsymbol{\sigma}^2))=\ln\left(\prod_{i=1}^p\sigma_i^2\right)^{\frac1p}=\frac1p\sum_{i=1}^p\ln(\sigma_i^2)\in\Re, \\ \boldsymbol{\nu} &= (\nu_1,\dots,\nu_p)'=(\ln(\sigma_1^2)-\nu,\dots,\ln(\sigma_p^2)-\nu)'\in\mathcal{L}_0, \end{aligned}$$

where

$$\mathcal{L}_0=\left\{\boldsymbol{u}=(u_1,\dots,u_p)':\sum_{i=1}^p u_i=0,\ \boldsymbol{u}\in\Re^p\right\}.$$

Testing the homogeneity of variances is equivalent to testing the hypothesis

$$H_0 : \boldsymbol{\nu} = \mathbf{0}_p \quad \text{vs.} \quad H_1 : \boldsymbol{\nu} \neq \mathbf{0}_p. \tag{3.23}$$

The REs of ν and $\boldsymbol{\nu}$ are given by

$$\begin{aligned} W_\nu &= \frac{1}{p}\sum_{i=1}^{p}\ln((n_i-1)s_i^2) + \frac{1}{p}\sum_{i=1}^{p}\ln(\chi_{n_i-1}^{-2}) = \overline{\ln(\mathcal{F}\boldsymbol{s}^2)} + \overline{\ln(\boldsymbol{\chi}_{\boldsymbol{n}-1}^{-2})}, \\ \boldsymbol{W_\nu} &= (\ln(\boldsymbol{\chi}_{\boldsymbol{n}-1}^{-2}) - \overline{\ln(\boldsymbol{\chi}_{\boldsymbol{n}-1}^{-2})}\mathbf{1}_p) + (\ln(\mathcal{F}\boldsymbol{s}^2) - \overline{\ln(\mathcal{F}\boldsymbol{s}^2)}\mathbf{1}_p), \end{aligned} \tag{3.24}$$

respectively. The PDF of $\boldsymbol{V} = \ln(\boldsymbol{\chi}_{\boldsymbol{n}-1}^{-2})$ is

$$f_{\ln(\boldsymbol{\chi}^{-2})}(\boldsymbol{v},\boldsymbol{f}) = \left(\prod_{i=1}^{p}\frac{1}{2^{\frac{f_i}{2}}\Gamma(\frac{f_i}{2})}\right)\mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{p}f_iv_i}\mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{p}\mathrm{e}^{-v_i}}.$$

The PDF of $\boldsymbol{V}^* = \boldsymbol{V} - \bar{V}\mathbf{1}_p$ is

$$\begin{aligned} f_0(\boldsymbol{v};\boldsymbol{f}) &= \int_{-\infty}^{\infty} f_{\ln(\boldsymbol{\chi}^{-2})}(\boldsymbol{v}+\mathbf{1}_pv,\boldsymbol{f})dv \\ &= C_0\prod_{i=1}^{p}\left(\frac{\mathrm{e}^{-v_i}}{\sum_{j=1}^{p}\mathrm{e}^{-v_j}}\right)^{\frac{1}{2}f_i}, \quad \forall\,\boldsymbol{v}\in\mathcal{L}_0. \end{aligned} \tag{3.25}$$

From (3.25), the PDF of $\boldsymbol{W_\nu}$ can be expressed as

$$\begin{aligned} f_{\boldsymbol{\nu}}(\boldsymbol{v};\boldsymbol{f},\boldsymbol{s}^2) &= f_0\left(\boldsymbol{v} - \ln\left(\frac{\mathcal{F}\boldsymbol{s}^2}{\bar{G}(\mathcal{F}\boldsymbol{s}^2)}\right),\boldsymbol{f}\right) \\ &= C_0\prod_{i=1}^{p}\left(\frac{\mathrm{e}^{-(v_i-\ln(\frac{f_is_i^2}{\bar{G}(\mathcal{F}\boldsymbol{s}^2)}))}}{\sum_{j=1}^{p}\mathrm{e}^{-(v_j-\ln(\frac{f_js_j^2}{\bar{G}(\mathcal{F}\boldsymbol{s}^2)}))}}\right)^{\frac{1}{2}f_i}, \quad \boldsymbol{v}\in\mathcal{L}_0, \end{aligned}$$

where

$$\bar{G}(\mathcal{F}\boldsymbol{s}^2) = \overline{\ln(\mathcal{F}\boldsymbol{s}^2)}$$

and

$$C_0 = \frac{\Gamma(\frac{1}{2}\sum_{i=1}^{p}(n_i+1))}{\prod_{i=1}^{p}\Gamma(\frac{n_i+1}{2})}.$$

If we consider the test variable as

$$\begin{aligned} Z = f_{\boldsymbol{\nu}}(\boldsymbol{W}_\nu;\boldsymbol{f},\boldsymbol{s}^2) &= C_0\prod_{i=1}^{p}\left(\frac{\mathrm{e}^{-((\boldsymbol{W}_\nu)_i-\ln(\frac{f_is_i^2}{\bar{G}(\mathcal{F}\boldsymbol{s}^2)}))}}{\sum_{i=1}^{p}\mathrm{e}^{-((\boldsymbol{W}_\nu)_i-\ln(\frac{f_is_i^2}{\bar{G}(\mathcal{F}\boldsymbol{s}^2)}))}}\right)^{\frac{1}{2}f_i} \\ &= C_0\prod_{i=1}^{p}\left(\frac{\mathrm{e}^{-V_i}}{\sum_{i=1}^{p}\mathrm{e}^{-V_i}}\right)^{\frac{1}{2}f_i} = C_0\prod_{i=1}^{p}\left(\frac{\chi_{f_i}^2}{\sum_{i=1}^{p}\chi_{f_i}^2}\right)^{\frac{1}{2}(n_i+1)-1}, \end{aligned}$$

and denote the α quantile of Z by $Q_Z(\alpha,\boldsymbol{f})$, then

$$\begin{aligned} \alpha &= \mathrm{P}(Z \leqslant Q_Z(\alpha,\boldsymbol{f})) \\ &= \mathrm{P}\left(\frac{\Gamma(\frac{1}{2}\sum_{i=1}^{p}(n_i+1))}{\prod_{i=1}^{p}\Gamma(\frac{n_i+1}{2})}\prod_{i=1}^{p}\left(\frac{\chi_{n_i-1}^2}{\sum_{i=1}^{p}\chi_{n_i-1}^2}\right)^{\frac{1}{2}(n_i+1)-1} \leqslant Q_Z(\alpha,\boldsymbol{n})\right) \\ &= \mathrm{P}\left(\mathrm{dDirichlet}\left(\boldsymbol{D},\frac{1}{2}(\boldsymbol{n}+1)\right) \leqslant Q_Z(\alpha,\boldsymbol{n})\right), \end{aligned}$$

Table 3 Empirical levels of the VDR test

Nominal significance levels	Sizes of samples							
	5, 10, 12	10, 15, 20	20, 30, 40	10, 15 25, 40	7, 10 13, 15	10, 15, 35 40, 50	5, 10, 15 18, 21, 30	10, 10, 14, 15 20, 23, 30
0.01	0.0094	0.0106	0.0108	0.0068	0.0104	0.0113	0.0099	0.0094
0.05	0.0478	0.0506	0.0506	0.0491	0.0493	0.0522	0.0482	0.0466
0.10	0.1002	0.1015	0.1020	0.0977	0.1012	0.1071	0.0983	0.0976
0.15	0.1481	0.1495	0.1509	0.1476	0.1515	0.1584	0.1472	0.1498
0.20	0.1995	0.2014	0.2009	0.1976	0.2047	0.2089	0.2015	0.1978
0.25	0.2480	0.2501	0.2499	0.2470	0.2545	0.2593	0.2510	0.2484
0.30	0.3009	0.2983	0.2954	0.3014	0.3026	0.3047	0.3027	0.2960
0.40	0.4033	0.3969	0.3964	0.4037	0.4000	0.4036	0.4005	0.4013
0.50	0.5110	0.4949	0.5033	0.5035	0.4983	0.4961	0.5036	0.4982

Table 4 Comparison of the VDR test (V) with the Bartlett test (B) and the revised Bartlett test (rB)

Test methods	Sizes of samples	Significance levels	Vector of variances							
			1, 1, 1	1, 1, 1.5	1, 1, 2	1, 1, 2.5	1, 1, 3	1, 1, 3.5	1, 1, 4	1, 1, 4.5
V	(10, 15, 20)	0.01	0.0117	0.0302	0.0958	0.2084	0.3263	0.4511	0.5757	0.6691
B		0.01	0.0117	0.0302	0.0962	0.2087	0.3268	0.4514	0.5760	0.6698
rB		0.01	0.0117	0.0303	0.0965	0.2090	0.3271	0.4517	0.5763	0.6702
V	(10, 15, 20)	0.05	0.0526	0.1167	0.2573	0.4317	0.5785	0.7003	0.7950	0.8579
B		0.05	0.0525	0.1165	0.2572	0.4314	0.5782	0.6999	0.7948	0.8577
rB		0.05	0.0526	0.1166	0.2572	0.4316	0.5783	0.7001	0.7950	0.8578
V	(10, 15, 20)	0.10	0.1013	0.2004	0.3707	0.5596	0.6963	0.7969	0.8719	0.9172
B		0.10	0.1016	0.2005	0.3711	0.5601	0.6967	0.7971	0.8719	0.9176
rB		0.10	0.1017	0.2005	0.3713	0.5601	0.6968	0.7972	0.8719	0.9177
Test methods	Sizes of samples	Significance levels	Vector of variances							
			1, 1, 1	1, 1, 1.5	1, 1, 2	1, 1, 2.5	1, 1, 3	1, 1, 3.5	1, 1, 4	1, 1, 4.5
V	(10, 25, 40)	0.01	0.0120	0.0508	0.1916	0.4054	0.6095	0.7647	0.8694	0.9303
B		0.01	0.0120	0.0511	0.1922	0.4059	0.6107	0.7650	0.8697	0.9306
rB		0.01	0.0120	0.0512	0.1924	0.4062	0.6116	0.7650	0.8698	0.9308
V	(10, 25, 40)	0.05	0.0516	0.1684	0.4129	0.6610	0.8221	0.9151	0.9616	0.9830
B		0.05	0.0517	0.1695	0.4141	0.6625	0.8230	0.9156	0.9620	0.9832
rB		0.05	0.0518	0.1696	0.4143	0.6626	0.8231	0.9156	0.9620	0.9832
V	(10, 25, 40)	0.10	0.1036	0.2658	0.5521	0.7686	0.8941	0.9567	0.9827	0.9926
B		0.10	0.1036	0.2658	0.5521	0.7686	0.8941	0.9567	0.9827	0.9926
rB		0.10	0.1036	0.2657	0.5515	0.7683	0.8941	0.9566	0.9827	0.9926

where

$$\begin{aligned}\boldsymbol{D} = (D_1, \ldots, D_p)' &\sim \text{dDirichlet}\left(\cdot, \frac{1}{2}(n_1+1), \ldots, \frac{1}{2}(n_p+1)\right)\\ &\stackrel{\text{def}}{=} \text{dDirichlet}\left(\cdot, \frac{1}{2}(\boldsymbol{n}+1)\right).\end{aligned}$$

The rule of the VDR test for the hypothesis (3.23) is that *if*

$$\text{dDirichlet}\left(\frac{\mathcal{F}\boldsymbol{s}^2}{\sum_{i=1}^p f_i s_i^2}, \frac{1}{2}(\boldsymbol{n}+1)\right) < Q_Z(\alpha, \boldsymbol{n}),$$

then reject the null hypothesis $H_0: \sigma_1^2 = \cdots = \sigma_p^2$.

When $p = 2$, the VDR test is solely the F test for two normal distributions.

Table 3 lists the empirical levels of the test obtained via simulations for p equal to $3, 4, 5, 6$ and 7 with various sample sizes. We performed 10,000 repeated samplings in each case. The underlined values in the table mark the empirical levels that show the maximum divergence from the nominal levels on the line. In general, the empirical levels are very close to the nominal levels.

A well-known test for the homogeneity of variances is the Bartlett test [7]. Table 4 summarizes the simulation results of comparing the VDR test with the Bartlett test and the revised Bartlett test given by Box (see [7]). The entries in Table 4 are the empirical powers of the tests obtained from simulations, with 10,000 repetition samplings. The results show that the performances of the three tests are very close for the simulated cases. Note that both the Bartlett test and its revised version are based on approximated distributions, whereas the VDR test is an exact test.

3.4 Testing parameters of multivariate normal distribution

When applying RI to the multivariate normal distribution $N_p(\boldsymbol{\mu}, \Sigma)$, the VDR test for the hypothesis

$$H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0 \quad \text{vs.} \quad H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$$

is simply the Hotelling test. Furthermore, the VDR test for the hypothesis

$$H_0 : \Sigma = \Sigma_0 \quad \text{vs.} \quad H_1 : \Sigma \neq \Sigma_0$$

is an exact test and the simulation results of the VDR test's level are more accurate than those of the likelihood ratio test. We omit the details here because of space considerations.

4 Prospect of randomized inference

The RI proposed in this paper is a new framework for statistical inference. The main idea is to consider a random variable, which takes values in the parameter space and depends on the observed sample, as an estimate of the constant parameter. As stated in previous sections, this idea was taken from Fisher's fiducial inference; however, we do not need to consider the parameter as a random variable. Note that RI is based on the distribution of an RE, which is a distribution that depends on the observed sample such as the posterior distribution; it acts similarly to the Bayesian inference to some extent. However, the idea of VDR is based on the likelihood principle, and as is shown in the previous sections, RI technically follows the frequentist way, except that it is based on the RE.

On the other hand, both the fiducial and posterior distributions depend on the observed samples, and can thus be employed to define an RE. Thus, RI could be seen as a unified inference framework. Many other methods, such as the generalized p-value method, are related to RI; hence, it is possible to apply RI to a great variety of problems.

4.1 RI as a tool

Parallel to CD inference and prior-free inference (see, for example, [10] and [8] among others), the RE includes a general view of statistical inference, and may be applied to wide range of problems. Some problems cannot be easily solved by using the conventional frequentist method although the problems might seem very elementary. For example, the coefficient of variation for a normal distribution is a common parameter, but it is difficult to construct a confidence interval for it. If we apply Theorem 2.8 to it, it is easy to find an RE and its PDF, and thus obtain a confidence interval.

The comparison of multiple univariate normal populations can be extended to multivariate normal distributions or to even other location and scale distribution families.

4.2 Further developments

The following are a few points pertaining to the development of RI.

1. Computation of a quantile of the test variable.

In general, we can obtain the quantile $Q_Z(\alpha, \boldsymbol{x}, \boldsymbol{n})$ of the test variable Z via simulations. It is possible to obtain an expression or an equation for it using the VDR theory. Let $f(\boldsymbol{x})$, $\boldsymbol{x} \in \Re^P$ be a PDF over $\Re^p$, and denote

$$D_{[f]} = \{(\boldsymbol{z}', z)' : 0 < z \leqslant f(\boldsymbol{z}),\ \boldsymbol{z} \in \Re^p,\ z \in \Re\}.$$

Assume that the PDF of the random vector $(\boldsymbol{Z}', Z_{p+1})'$ is

$$p(\boldsymbol{z}, z_{p+1}) = I_{D_{[f]}}((\boldsymbol{z}', z_{p+1})'),$$

i.e., $(\boldsymbol{Z}', Z_{p+1})'$ is uniformly distributed on $D_{[f]}$. Then it is not difficult to verify that the PDF of $\boldsymbol{Z}$ is $f(\cdot)$ and that the density of Z_{p+1} is $L_p(D_{[f]}(z))$. Here,

$$D_{[f]}(v) = \{\boldsymbol{z} : f(\boldsymbol{z}) \geqslant v,\ \boldsymbol{z} \in \Re^p\},$$

and $L_p(\cdot)$ is the Lebesgue measure. See the formulas in Figure 2 (see [2, 9, 12]).

Let $g(z)$ and $G(z)$ be the PDF and the distribution function, respectively, of $Z = f(\boldsymbol{Z})$, where $z \in \Re$. From Figure 2, we have

$$\begin{aligned} G(h) &= \mathrm{P}(f(\boldsymbol{Z}) \leqslant h) \\ &= \mathrm{P}(\{(\boldsymbol{z}', z_{p+1})' : z_{p+1} \leqslant h, (\boldsymbol{z}', z_{p+1})' \in D_{[f]}(h)\} \setminus D_{[f]}(h) \times (0, h)) \\ &= \int_0^h L_p(D_{[f]}(v))dv - L_p(D_{[f]}(h))h, \\ g(h) &= \frac{dG(h)}{dh} = -h\frac{L_p(D_{[f]}(h))}{dh}. \end{aligned}$$

$Q_Z(\alpha, \boldsymbol{x}, \boldsymbol{n})$ satisfies $G(Q_Z(\alpha, \boldsymbol{x}, \boldsymbol{n})) = \alpha$. In some special cases, it is possible to obtain a closed-form solution of the equation. However, in general, it is difficult to solve the equation even numerically. Any algorithm for solving the equation would be helpful.

2. New ways to determine REs.

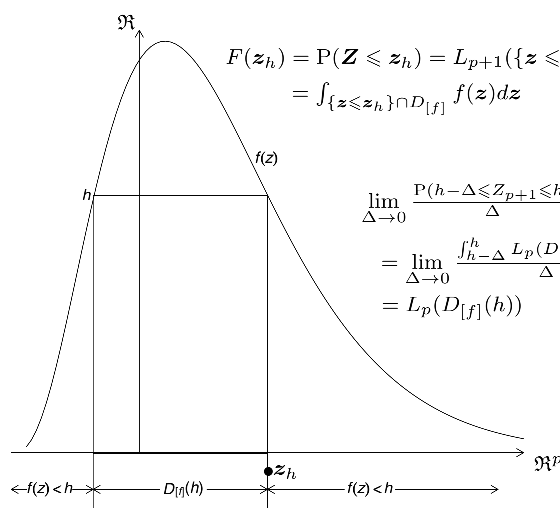

Figure 2 The VDR theorem

The key point of RI is to find an RE and its density. We have shown several methods to find an RE, including the CD method, the pivot method and Theorem 2.8. There are certainly other ways to find the REs of parameters. For example, the generalized p-value method may be employed in some cases. Furthermore, it is interesting to compare REs.

3. Applying RI to censored data.

RI can be applied to type-II censored data, as has been shown in Example 3.1. There are many kinds of censored data. Currently, the use of RI for other kinds of censored data remains a problem.

4. Test of parameters.

In Section 3, we transformed a composite hypothesis into a simple one. The obtained value space of the test parameter is often a surface in the parameter space. In this case, the VDR test variable was related to a degenerate population. This may be useful for massive data analysis.

We also note that the way to construct a test parameter is not unique. For example, another transformation to construct the test parameter for the hypothesis (3.4) is as follows. Let $A = A_{p\times p}$ be a positive definite matrix and

$$\boldsymbol{\mu} = \frac{1}{\mathbf{1}_p' A \mathbf{1}_p} \mathbf{1}_p \mathbf{1}_p' A \boldsymbol{\mu} + \left(I_p - \frac{1}{\mathbf{1}_p' A \mathbf{1}_p} \mathbf{1}_p \mathbf{1}_p' A\right)\boldsymbol{\mu} = \bar{\mu}_A + \boldsymbol{\nu}_A.$$

Then, $\mathbf{1}_p' A \boldsymbol{\nu} = 0$ and the hypothesis (3.4) is equivalent to the following hypothesis:

$$H_0 : \boldsymbol{\nu}_A = \mathbf{0}_p \quad \text{vs.} \quad H_1 : \boldsymbol{\nu}_A \neq \mathbf{0}_p.$$

5. Asymptotic theory.

The problems discussed in this paper only focus on exact inference. Extension to asymptotic concepts and theory could be established further. These are not presented in this paper considering the space constraint.

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